Lamb shift in muonic ions of lithium, beryllium, and boron

A. A. Krutov, A. P. Martynenko, F. A. Martynenko, and O. S. Sukhorukova
Samara University, 443086, Moskovskoe shosse 34, Samara, Russia
(Received 1 October 2016; published 15 December 2016)

We present a precise calculation of the Lamb shift $(2P_{1/2}-2S_{1/2})$ in muonic ions $(\mu^6_3\mathrm{Li})^{2+}$, $(\mu^7_3\mathrm{Li})^{2+}$, $(\mu^9_4\mathrm{Be})^{3+}$, $(\mu^{10}_4\mathrm{Be})^{3+}$, $(\mu^{10}_5\mathrm{B})^{4+}$, $(\mu^{11}_5\mathrm{B})^{4+}$. The contributions of orders $\alpha^3\div\alpha^6$ to the vacuum polarization, nuclear structure and recoil, and relativistic effects are taken into account. Our numerical results are consistent with previous calculations and improved by additional corrections. The obtained results can be used for the comparison with future experimental data, and extraction more accurate values of nuclear charge radii.

DOI: 10.1103/PhysRevA.94.062505

I. INTRODUCTION

In recent years there has been a special interest in the physics of elementary particles related to muons. Experimental study of the muon anomalous magnetic moment revealed a certain discrepancy between theoretical and experimental results. The subsequent 2010 measurement of the Lamb shift in muonic hydrogen has led to another problem, called the proton charge radius puzzle [1–4]. After a measurement of the Lamb shift [5] in muonic deuterium, it became clear that there was a discrepancy in the values of the charge radius of the proton and deuteron determined by electronic and muonic atoms. This may mean that the muons play an important role in subatomic physics, which is not fully understood. The experimental CREMA Collaboration program includes other muon atoms, especially muonic helium ions [6], but it apparently can be extended to the study of other light muonic atoms. The transition energy $2P\to 2S$ in light muonic atoms can be precisely measured by laser spectroscopy as in muonic hydrogen. Therefore, additional theoretical study of muon bound states and a calculation of their energy levels, along with experimental investigations, can contribute to a better understanding of the essence of the problem.

The interest in muonic ions $(\mu\mathrm{Li})^{2+}$, $(\mu\mathrm{Be})^{3+}$, and $(\mu\mathrm{B})^{4+}$ is also connected to the fact that, as has been established in this case, there is a strong cancellation of two main contributions to the one-loop vacuum polarization and the structure of the nucleus [7,8]. As a result, the Lamb shift value lies in a wide range of wavelengths $150\div 1100$ nm, from ultraviolet to infrared region of the spectrum, making it possible for its study of laser spectroscopy methods. The measurement of transition frequencies gives an opportunity to obtain more exact information about nuclear size and structure. Another important conclusion arising out of this cancellation is that it plays a more significant role than previously thought, beginning with the contributions of higher order in α, as well as contributions containing large degrees of nuclear charge Z. The methodical analysis is very important to increase the accuracy of calculation of the Lamb shift [9,10]. As usual, the most important corrections in the Lamb shift are the corrections to the vacuum polarization, the nuclear structure and recoil, as well as complex combination corrections to the vacuum polarization and relativism, which we explore in this paper.

Fundamentals of calculating of energy spectra for light muonic atoms were formulated many years ago in a relativistic approach based on the Dirac equation and in nonrelativistic Schrödinger method in Refs. [11–16] (see other references in review articles, Refs. [13,15]). After recent experiments of the CREMA Collaboration, there were many works devoted to the muonic atoms, in order to overcome the difference in the magnitude of the charge radius of the proton [17–26] (see other references in Ref. [3]). They carried out an analysis of the main contribution to the Lamb shift and hyperfine structure of the spectrum, and the various corrections have quite significant numerical value. They also analyzed a number of relatively subtle effects in the fine and hyperfine structure, which, however, did not lead to any considerable change in the results (see, for example, Refs. [27–30]). The aim of the present work is to extend our previous calculations of the Lamb shift in muonic helium ions [31] to other muonic ions such as muonic lithium, muonic beryllium, and muonic boron. We consistently calculate the contributions of orders $\alpha^3\div\alpha^6$ within the framework of the quasipotential method in quantum electrodynamics [32–35]. It is important to know also the hyperfine splitting of levels for the evaluation of observed transition frequencies [8]. As in previous works we take modern numerical values of fundamental physical constants from Refs. [36,37]. Since the corrections to the structure of the nucleus play a key role, let us write the explicitly used values of the nuclear charge radii: $r(^6_3\mathrm{Li}) = (2.5890\pm 0.0390)$ fm, $r(^7_3\mathrm{Li}) = (2.4440\pm 0.0420)$ fm, $r(^9_4\mathrm{Be}) = (2.5190\pm 0.0120)$ fm, $r(^{10}_4\mathrm{Be}) = (2.3550\pm 0.0170)$ fm, $r(^{10}_5\mathrm{B}) = (2.4277\pm 0.0499)$ fm, and $r(^{11}_5\mathrm{B}) = (2.4060\pm 0.0294)$ fm.

II. EFFECTS OF VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Let us begin by recalling the basic assumptions of the quasipotential approach in the calculation of the Lamb shift. The muonic ion is described by the Schrödinger equation with the Breit Hamiltonian [38]:

$$H_B = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r} - \frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\delta(\mathbf{r}) - \frac{Z\alpha}{2m_1m_2r}\left(\mathbf{p}^2+\frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right) + \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)\times(\mathbf{L}\boldsymbol{\sigma}_1) = H_0 + \Delta V^B, \quad (1)$$

 ©2016 American Physical Society

where $H_0 = \mathbf{p}^2/2\mu - Z\alpha/r$, m_1, m_2 are the muon and nucleus masses, $\mu = m_1 m_2/(m_1+m_2)$, $\delta_I = 1$ for nucleus with half-integer spin, and $\delta_I = 0$ for nucleus with integer spin. The exact solution of the Schrödinger equation with the Hamiltonian H_0 is then used in the calculation of the shifts of $2S$ and 2P energy levels by perturbation theory.

As is well known, the basic contribution to the Lamb shift in muonic atoms is determined by the effect of electron vacuum polarization (VP) in the 1γ interaction. The potential of particle interaction corresponding to this effect has the form

$$V^C_{vp}(r) = \frac{\alpha}{3\pi}\int_1^\infty d\xi\rho(\xi)\left(-\frac{Z\alpha}{r}e^{-2m_e\xi r}\right),$$
$$\rho(\xi) = \frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}. \quad (2)$$

It gives the shift of energy levels for $2S$ and $2P$ states, which can be presented in analytical form ($k_1 = 2m_e/W$, $W = \mu Z\alpha$):

$$\Delta E_{vp}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi}\int_1^\infty \rho(\xi)d\xi\int_0^\infty xdx\left(1-\frac{x}{2}\right)^2 e^{-x(1+\frac{2m_e\xi}{W})}$$
$$= \frac{1}{12(1-k_1^2)^{5/2}}\left\{\sqrt{1-k_1^2}\left[-168k_1^6+272k_1^4-49k_1^2+6\pi\left(k_1^2-1\right)^2\left(14k_1^2+3\right)k_1-28\right]\right.$$
$$\left.+3\left(56k_1^8-128k_1^6+75k_1^4+10k_1^2-4\right)\ln\left(\frac{1-\sqrt{1-k_1^2}}{k_1}\right)\right\}, \quad (3)$$

$$\Delta E_{vp}(2P) = -\frac{\mu(Z\alpha)^2\alpha}{72\pi}\int_1^\infty \rho(\xi)d\xi\int_0^\infty x^3dxe^{-x(1+\frac{2m_e\xi}{W})}$$
$$= \frac{1}{(1-k_1^2)^{5/2}}\left\{\sqrt{1-k_1^2}\left[-120k_1^6+184k_1^4-23k_1^2+6\pi\left(k_1^2-1\right)^2\left(10k_1^2+3\right)k_1-32\right]\right.$$
$$\left.+3\left(40k_1^8-88k_1^6+45k_1^4+10k_1^2-4\right)\ln\left(\frac{1-\sqrt{1-k_1^2}}{k_1}\right)\right\}. \quad (4)$$

Expressions (3) and (4) give the following numerical values to the Lamb shift in muonic ions (see line 1 in Tables I–III):

$$\Delta E_{vp}(2P-2S) = \begin{cases} {}^6_3\mathrm{Li}: 4664.95\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 4682.38\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 9255.79\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 9270.74\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: 15356.42\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 15375.55\ \mathrm{meV} \end{cases} \quad (5)$$

We retain two significant figures after the decimal point in all obtained expressions. The order of contributions (3) and (4) is clearly extracted in front of integrals. For the calculation of muon VP contribution we use again (3) and (4), changing $m_e \to m_\mu$. Corresponding numerical values which have the order $\alpha(Z\alpha)^4$ are included in Tables I–III in line 25 [see also (60) and (61)].

The two-loop vacuum polarization effects in the one-photon interaction can be divided into two parts: loop-after-loop correction (vp-vp) and two-loop vacuum polarization operator correction which we denote further as the "2-loop vp" correction. The potential of loop-after-loop VP effect has the form [10,31]

$$V^C_{vp-vp}(r) = \frac{\alpha^2}{9\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\left(-\frac{Z\alpha}{r}\right)\frac{1}{(\xi^2-\eta^2)}$$
$$\times(\xi^2e^{-2m_e\xi r}-\eta^2e^{-2m_e\eta r}). \quad (6)$$

Calculating the matrix elements of the potential (5) in the first-order perturbation theory, we find the contribution to the Lamb shift of order $\alpha^2(Z\alpha)^2$:

$$\Delta E_{vp-vp}(2P-2S)$$
$$= \begin{cases} {}^6_3\mathrm{Li}: 14.20\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 14.28\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 33.79\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 34.06\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: 64.40\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 64.52\ \mathrm{meV} \end{cases} \quad (7)$$

There is another correction to the potential which is determined by the amplitude with two sequential electron and muon loops:

$$\Delta V_{vp-mvp}(r)$$
$$= -\frac{4(Z\alpha)\alpha^2}{45\pi^2m_1^2}\int_1^\infty\rho(\xi)d\xi\left[\pi\delta(\mathbf{r})-\frac{m_e^2\xi^2}{r}e^{-2m_e\xi r}\right]. \quad (8)$$

It gives the correction of order $\alpha^2(Z\alpha)^4$ to the Lamb shift $(2P-2S)$, which is included in Tables I–III (see line 3).

The two-loop polarization operator contribution to the potential can be presented in a form similar to (2) with more

TABLE I. Lamb shift ($2P_{1/2}-2S_{1/2}$) in muonic ions $(\mu^7_3\mathrm{Li})^{2+}$ and $(\mu^6_3\mathrm{Li})^{2+}$. In parentheses are given the results obtained by other authors, with some references to their works, which discuss the calculation of corrections of this type.

	Contribution to the splitting	$(\mu^7_3\mathrm{Li})^{2+}$ (meV)	$(\mu^6_3\mathrm{Li})^{2+}$ (meV)
1	VP contribution of order $\alpha(Z\alpha)^2$ in 1γ interaction	4682.38 (4682.4 [7])	4664.95 (4665.0 [7])
2	Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in 1γ interaction	32.54 (32.44 [7])	32.41(32.27 [7])
3	VP and MVP contribution in one-photon interaction	0.01	0.01
4	Three-loop VP contribution in one-photon interaction	0.17 [40,44]	0.17 [40,44]
5	The Wichmann-Kroll correction	−0.09	−0.09
6	Light-by-light scattering correction	0.03 [44]	0.03 [44]
7	Relativistic and VP corrections of order $\alpha(Z\alpha)^4$ in the first-order PT	−6.13	−6.07
8	Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in the first-order PT	−0.02	−0.02
9	Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in the second-order PT	5.66	5.63
10	Two-loop MVP and EVP contribution in the second-order PT	0.02	0.02
11	Relativistic and one-loop VP corrections of order $\alpha(Z\alpha)^4$ in the second-order PT	8.98	8.88
12	Three-loop VP contribution in the second-order PT of order $\alpha^3(Z\alpha)^2$	0.08	0.08
13	Three-loop VP contribution in the third-order PT of order $\alpha^3(Z\alpha)^2$	0.04 [48,49]	0.05 [48,49]
14	Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in the second-order PT	0.15	0.15
15	Nuclear structure contribution of order $(Z\alpha)^4$	-3301 ± 117 [7,15]	-3675 ± 112 [7,15]
16	Nuclear structure contribution of order $(Z\alpha)^5$ from 2γ amplitudes	177 ± 9 [16,46]	208 ± 9 [16,46]
17	Nuclear structure and VP contribution in 1γ interaction of order $\alpha(Z\alpha)^4$	−12.78	−14.21
18	Nuclear structure and VP contribution in the second-order PT of order $\alpha(Z\alpha)^4$	−20.68	−23.00
19	Nuclear structure and two-loop VP contribution in 1γ interaction of order $\alpha^2(Z\alpha)^4$	−0.11	−0.11
20	Nuclear structure and two-loop VP contribution in the second-order PT of order $\alpha^2(Z\alpha)^4$	−0.30	−0.33
21	Nuclear structure contribution of order $\alpha(Z\alpha)^5$ from 2γ amplitudes with VP insertion	2.75 ± 0.13	3.19 ± 0.14
22	Recoil correction of order $(Z\alpha)^4$	0.13 [15,17,45,50]	0.68 [15,17,45,50]
23	Recoil correction of order $(Z\alpha)^5$	−1.86 [15,17,50]	−2.15 [15,17,50]
24	Recoil correction of order $(Z\alpha)^6$	0.02 [15,17,52]	0.03 [15,17,52]
25	Muon self-energy and MVP contribution	−51.36 [15,17,53]	−50.99 [15,17,53]
26	Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$	−0.12 [15]	−0.16 [15]
27	Nuclear structure corrections of orders $(Z\alpha)^6$, $\alpha(Z\alpha)^5$	-5.51 ± 0.17 [54,55]	-6.07 ± 0.17 [54,55]
28	Muon form factor $F_1'(0)$, $F_2(0)$ contributions	−0.16 [15,56]	−0.16 [15,56]
29	Muon self-energy and VP contribution	−0.23 [7,16,57]	−0.23 [7,16,57]
30	HVP contribution	1.17 [7,58–60]	1.16 [7,58–60]
31	Nuclear polarizability	21 ± 4 [7]	15 ± 4 [7]
32	Total contribution	1531.78	1161.85

complicated spectral function $f(v)$ [39]:

$$\Delta V^C_{2\text{-loop}\,vp} = -\frac{2}{3}\frac{Z\alpha}{r}\left(\frac{\alpha}{\pi}\right)^2\int_0^1 \frac{f(v)dv}{(1-v^2)}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}, \tag{9}$$

$$f(v) = v\Bigg\{(3-v^2)(1+v^2)\left[\mathrm{Li}_2\left(-\frac{1-v}{1+v}\right)+2\mathrm{Li}_2\left(\frac{1-v}{1+v}\right)+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}-\ln\frac{1+v}{1-v}\ln v\right]$$
$$+\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]+\frac{3}{8}v(5-3v^2)\Bigg\}, \tag{10}$$

where $\mathrm{Li}_2(z)$ is the Euler dilogarithm. The potential $\Delta V^C_{2\text{-loop}\,vp}(r)$ gives the contribution to the Lamb shift ($2P-2S$) of order $\alpha^2(Z\alpha)^2$:

$$\Delta E_{2\text{-loop}\,vp}(2P-2S) = \begin{cases} {}^6_3\mathrm{Li}: 18.21\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 18.26\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 31.66\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 31.69\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: 47.54\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 47.57\ \mathrm{meV} \end{cases} \tag{11}$$

Numerical value of corrections (7) and (11), which are included in line 2 of the tables, show that at the necessary level of accuracy we should calculate three-loop VP contributions in one-photon interaction. One part of three-loop VP effects with successive loops in the scattering amplitude (loop-after-loop-after-loop, two-loop-after-loop) can be derived as potential (6).

TABLE II. Lamb shift ($2P_{1/2}-2S_{1/2}$) in muonic ions $(\mu^{10}_4\mathrm{Be})^{3+}$ and $(\mu^9_4\mathrm{Be})^{3+}$. In parentheses are given the results obtained by other authors, with some references to their works, which discuss the calculation of corrections of this type.

	Contribution to the splitting	$(\mu^{10}_4\mathrm{Be})^{3+}$ (meV)	$(\mu^9_4\mathrm{Be})^{3+}$ (meV)
1	VP contribution of order $\alpha(Z\alpha)^2$ in 1γ interaction	9270.74 [7]	9255.79 (9255.8 [7])
2	Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in 1γ interaction	65.75 [7]	65.66 (65.3 [7])
3	VP and MVP contribution in one-photon interaction	0.05	0.05
4	Three-loop VP contribution in one-photon interaction	0.42 [40,44]	0.42 [40,44]
5	The Wichmann-Kroll correction	−0.24	−0.24
6	Light-by-light scattering correction	0.07 [44]	0.07 [44]
7	Relativistic and VP corrections of order $\alpha(Z\alpha)^4$ in the first-order PT	−22.34	−22.23
8	Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in the first-order PT	−0.07	−0.07
9	Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in the second-order PT	12.73	12.70
10	Two-loop MVP and EVP contribution in the second-order PT	0.04	0.04
11	Relativistic and one-loop VP corrections of order $\alpha(Z\alpha)^4$ in the second-order PT	31.93	31.74
12	Three-loop VP contribution in the second-order PT of order $\alpha^3(Z\alpha)^2$	0.19	0.19
13	Three-loop VP contribution in the third-order PT of order $\alpha^3(Z\alpha)^2$	0.05 [48,49]	0.05 [48,49]
14	Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in the second-order PT	0.55	0.54
15	Nuclear structure contribution of order $(Z\alpha)^4$	-9826 ± 142 [7,15]	-11200 ± 107 [7,15]
16	Nuclear structure contribution of order $(Z\alpha)^5$ from 2γ amplitudes	679 ± 14 [16,46]	826 ± 12 [16,46]
17	Nuclear structure and VP contribution in 1γ interaction of order $\alpha(Z\alpha)^4$	−42.44	−48.35
18	Nuclear structure and VP contribution in the second-order PT of order $\alpha(Z\alpha)^4$	−70.68	−80.52
19	Nuclear structure and two-loop VP contribution in 1γ interaction of order $\alpha^2(Z\alpha)^4$	−0.36	−0.41
20	Nuclear structure and two-loop VP contribution in the second-order PT of order $\alpha^2(Z\alpha)^4$	−1.09	−1.25
21	Nuclear structure contribution of order $\alpha(Z\alpha)^5$ from 2γ amplitudes with VP insertion	10.64 ± 0.21	12.78 ± 0.17
22	Recoil correction of order $(Z\alpha)^4$	0.79 [15,17,45,50]	0.24 [15,17,45,50]
23	Recoil correction of order $(Z\alpha)^5$	−5.40 [15,17,50]	−5.97 [15,17,50]
24	Recoil correction of order $(Z\alpha)^6$	0.10 [15,17,52]	0.11 [15,17,52]
25	Muon self-energy and MVP contribution	−149.52 [15,17,53]	−149.00 [15,17,53]
26	Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$	−0.31 [15]	−0.39 [15]
27	Nuclear structure corrections of orders $(Z\alpha)^6$, $\alpha(Z\alpha)^5$	-31.04 ± 0.43 [54,55]	-35.29 ± 0.33 [54,55]
28	Muon form factor $F_1'(0)$, $F_2(0)$ contributions	−0.52 [15,56]	−0.51 [15,56]
29	Muon self-energy and VP contribution	−0.71 [7,16,57]	−0.71 [7,16,57]
30	HVP contribution	3.75 [7,58–60]	3.74 [7,58–60]
31	Nuclear polarizability	104 ± 21 [7]	82 ± 16 [7]
32	Total contribution	30.08	−1252.82

Corresponding contributions to the potential and the Lamb shift ($2P-2S$) are the following:

$$V^C_{vp\text{-}vp\text{-}vp}(r)=-\frac{Z\alpha}{r}\frac{\alpha^3}{(3\pi)^3}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta d\eta\int_1^\infty\rho(\zeta)d\zeta \times\left[e^{-2m_e\zeta r}\frac{\zeta^4}{(\xi^2-\zeta^2)(\eta^2-\zeta^2)}+e^{-2m_e\xi r}\frac{\xi^4}{(\zeta^2-\xi^2)(\eta^2-\xi^2)}+e^{-2m_e\eta r}\frac{\eta^4}{(\xi^2-\eta^2)(\zeta^2-\eta^2)}\right], \quad (12)$$

$$V^C_{vp\text{-2-loop }vp}=-\frac{4\mu\alpha^3(Z\alpha)}{9\pi^3}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\frac{f(\eta)d\eta}{\eta}\frac{1}{r(\eta^2-\xi^2)}(\eta^2e^{-2m_e\eta r}-\xi^2e^{-2m_e\xi r}), \quad (13)$$

$$\Delta E_{vp\text{-}vp\text{-}vp}(2P-2S)=\begin{cases}{}^6_3\mathrm{Li}: 0.04\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 0.04\ \mathrm{meV}\\ {}^9_4\mathrm{Be}: 0.11\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 0.11\ \mathrm{meV}\\ {}^{10}_5\mathrm{B}: 0.20\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 0.24\ \mathrm{meV}\end{cases}, \quad (14)$$

$$\Delta E_{vpm2\text{-loop }vp}(2P-2S)=\begin{cases}{}^6_3\mathrm{Li}: 0.12\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 0.12\ \mathrm{meV}\\ {}^9_4\mathrm{Be}: 0.27\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 0.27\ \mathrm{meV}\\ {}^{10}_5\mathrm{B}: 0.48\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 0.48\ \mathrm{meV}\end{cases}. \quad (15)$$

TABLE III. Lamb shift ($2P_{1/2} - 2S_{1/2}$) in muonic ions $(\mu_5^{11}B)^{4+}$ and $(\mu_5^{10}B)^{4+}$. In parentheses are given the results obtained by other authors, with some references to their works, which discuss the calculation of corrections of this type.

	Contribution to the splitting	$(\mu_5^{11}B)^{4+}$ (meV)	$(\mu_5^{10}B)^{4+}$ (meV)
1	VP contribution of order $\alpha(Z\alpha)^2$ in 1γ interaction	15375.55 (15376.0 [7])	15356.42 [7]
2	Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in 1γ interaction	112.09 (111.1 [7])	111.94 (111.3 [7])
3	VP and MVP contribution in one-photon interaction	0.13	0.13
4	Three-loop VP contribution in one-photon interaction	0.77 [40,44]	0.73 [40,44]
5	The Wichmann-Kroll correction	−0.50	−0.50
6	Light-by-light scattering correction	0.16, [44]	0.16, [44]
7	Relativistic and VP corrections of order $\alpha(Z\alpha)^4$ in the first-order PT	−59.73	−59.46
8	Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in the first-order PT	−0.17	−0.17
9	Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in the second-order PT	23.43	23.39
10	Two-loop MVP and EVP contribution in the second-order PT	0.10	0.10
11	Relativistic and one-loop VP corrections of order $\alpha(Z\alpha)^4$ in the second-order PT	83.67	83.29
12	Three-loop VP contribution in the second-order PT of order $\alpha^3(Z\alpha)^2$	0.35	0.35
13	Three-loop VP contribution in the third-order PT of order $\alpha^3(Z\alpha)^2$	0.31 [48,49]	0.31 [48,49]
14	Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in the second-order PT	1.44	1.44
15	Nuclear structure contribution of order $(Z\alpha)^4$	-25115 ± 618 [7,15]	-25493 ± 1059 [7,15]
16	Nuclear structure contribution of order $(Z\alpha)^5$ from 2γ amplitudes	2217 ± 82 [16,46]	2269 ± 143 [16,46]
17	Nuclear structure and VP contribution in 1γ interaction of order $\alpha(Z\alpha)^4$	−117.14	−118.86
18	Nuclear structure and VP contribution in the second-order PT of order $\alpha(Z\alpha)^4$	−200.06	−202.98
19	Nuclear structure and two-loop VP contribution in 1γ interaction of order $\alpha^2(Z\alpha)^4$	−1.00	−1.02
20	Nuclear structure and two-loop VP contribution in the second-order PT of order $\alpha^2(Z\alpha)^4$	−3.25	−3.30
21	Nuclear structure contribution of order $\alpha(Z\alpha)^5$ from 2γ amplitudes with VP insertion	34.60 ± 1.20	35.34 ± 2.07
22	Recoil correction of order $(Z\alpha)^4$	0.40 [15,17,45,50]	1.94 [15,17,45,50]
23	Recoil correction of order $(Z\alpha)^5$	−14.63 [15,17,50]	−16.03 [15,17,50]
24	Recoil correction of order $(Z\alpha)^6$	0.34 [15,17,52]	0.37 [15,17,52]
25	Muon self-energy and MVP contribution	−338.40 [15,17,53]	−337.45 [15,17,53]
26	Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$	−0.91 [15]	−1.10 [15]
27	Nuclear structure corrections of orders $(Z\alpha)^6$, $\alpha(Z\alpha)^5$	-128.14 ± 3.29 [54,55]	-130.15 ± 5.68 [54,55]
28	Muon form factor $F_1'(0)$, $F_2(0)$ contributions	−1.26 [15,56]	−1.26 [15,56]
29	Muon self-energy and VP contribution	−1.66 [7,16,57]	−1.66 [7,16,57]
30	HVP contribution	9.18 [7,58–60]	9.15 [7,58–60]
31	Nuclear polarizability	122 ± 24 [7]	103 ± 21 [7]
32	Total contribution	−8000.33	−8369.88

Another part of the diagrams corresponds to the three-loop corrections to the polarization operator. They were first calculated for the ($2P - 2S$) Lamb shift in muonic hydrogen in Refs. [40,41]. An estimate of their contribution to the Lamb shift is included in Tables I–III in a sum with (14) and (15) in line 4.

Finally, there exists another one-loop vacuum polarization correction of order $\alpha(Z\alpha)^4$ in the Lamb shift known as the Wichmann-Kroll correction [42,43]. Its calculation was discussed repeatedly in Refs. [15,31], so we restrict ourselves here by including numerical results in the final tables (line 5) as well as the whole light-by-light contribution (line 6) (see detailed calculation in Ref. [44]). Almost all of the corrections presented in this section are written in the integral form, and are therefore specific for each muon atom. The presented numerical values provide important information about the change in the value of corrections in different muonic ions.

III. RELATIVISTIC CORRECTIONS WITH THE ACCOUNT OF VACUUM POLARIZATION EFFECTS

The electron vacuum polarization effects modify not only the Coulomb potential but also all other terms of the Breit Hamiltonian. Appropriate potentials, which take into account relativistic effects and vacuum polarization effects, were built in Refs. [10,16,45,46]:

$$\Delta V_{vp}^B(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi \sum_{i=1}^{4}\Delta V_{i,vp}^B(r), \tag{16}$$

$$\Delta V_{1,vp}^B = \frac{Z\alpha}{8}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\left[4\pi\delta(\mathbf{r}) - \frac{4m_e^2\xi^2}{r}e^{-2m_e\xi r}\right], \tag{17}$$

$$\Delta V^B_{2,vp} = -\frac{Z\alpha m_e^2\xi^2}{m_1 m_2 r} e^{-2m_e\xi r}(1 - m_e\xi r), \quad (18)$$

$$\Delta V^B_{3,vp} = -\frac{Z\alpha}{2m_1 m_2} p_i \frac{e^{-2m_e\xi r}}{r}\left[\delta_{ij} + \frac{r_i r_j}{r^2}(1 + 2m_e\xi r)\right] p_j, \quad (19)$$

$$\Delta V^B_{4,vp} = \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2} + \frac{1}{2m_1 m_2}\right) e^{-2m_e\xi r}(1 + 2m_e\xi r)(\mathbf{L}\boldsymbol{\sigma}_1). \quad (20)$$

An averaging of these terms gives the corrections of order $\alpha(Z\alpha)^4$ to the Lamb shift $(2P - 2S)$:

$$\Delta E^B_{1,vp}(2P - 2S) = \begin{cases} {}^6_3\mathrm{Li}: -5.55\ \mathrm{meV}, & {}^7_3\mathrm{Li}: -5.60\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: -19.94\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: -20.02\ \mathrm{meV}, \\ {}^{10}_5\mathrm{B}: -52.76\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: -52.94\ \mathrm{meV} \end{cases} \quad (21)$$

$$\Delta E^B_{2,vp}(2P - 2S) = \begin{cases} {}^6_3\mathrm{Li}: 0.04\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 0.04\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 0.90\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 0.08\ \mathrm{meV}, \\ {}^{10}_5\mathrm{B}: 0.20\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 0.18\ \mathrm{meV} \end{cases} \quad (22)$$

$$\Delta E^B_{3,vp}(2P - 2S) = \begin{cases} {}^6_3\mathrm{Li}: 0.10\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 0.09\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 0.27\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 0.25\ \mathrm{meV}, \\ {}^{10}_5\mathrm{B}: 0.68\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 0.62\ \mathrm{meV} \end{cases} \quad (23)$$

$$\Delta E^B_{4,vp}(2P - 2S) = \begin{cases} {}^6_3\mathrm{Li}: -0.66\ \mathrm{meV}, & {}^7_3\mathrm{Li}: -0.66\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: -2.65\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: -2.65\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: -7.58\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: -7.59\ \mathrm{meV} \end{cases} \quad (24)$$

The sum of corrections (21)–(24) is included in Tables I–III (line 7). The next step to refine the results of the Lamb shift calculation is related to the two-loop corrections to the vacuum polarization in the Breit Hamiltonian. So, for example, two-loop analog of expression (17) is equal to

$$\Delta V^B_{2\text{-loop}\ vp}(r) = \frac{\alpha^2(Z\alpha)}{12\pi^2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\int_0^1 \frac{f(v)dv}{1 - v^2} \times \left[4\pi\delta(\mathbf{r}) - \frac{4m_e^2}{(1 - v^2)r} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right]. \quad (25)$$

The corresponding correction to the $(2P - 2S)$ shift is on the limit of the accuracy of our calculations. The contribution of other two-loop corrections to the Breit potential can be roughly estimated in the energy spectrum at 10% of the contribution (25) (see the summary of two-loop results in Tables I–III).

In the second-order perturbation theory (sopt) there are one-loop and two-loop electron vacuum polarization contributions of orders $\alpha^2(Z\alpha)^2$ and $\alpha(Z\alpha)^4$. To better understand the structure of these contributions, we present diagrams in Fig. 1. The general expression for corrections has the form

$$\Delta E^{vp}_{\mathrm{sopt}} = \langle\psi|\Delta V^C_{vp}\tilde{G}\Delta V^C_{vp}|\psi\rangle + 2\langle\psi|\Delta V^B\tilde{G}\Delta V^C_{vp}|\psi\rangle, \quad (26)$$

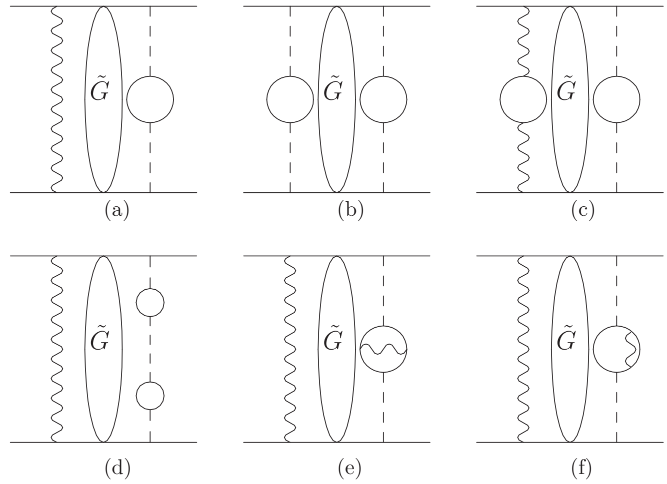

FIG. 1. Effects of one-loop and two-loop vacuum polarization in the second-order perturbation theory (sopt). Dashed line shows the Coulomb photon. $\tilde{G}$ is the reduced Coulomb Green's function (33). Wave line shows terms of the Breit potential.

where $\tilde{G}$ is the reduced Coulomb Green's function (RCGF). For the calculation of the Lamb shift contributions we use a representation of the RCGF for $2S-$ and $2P-$ states obtained in Ref. [47] (see exact expressions for $\tilde{G}_{2S}$, $\tilde{G}_{2P}$, g_{2S}, and g_{2P} in Ref. [31]). In the case of the two-loop corrections shown in Fig. 1(c), we get the integral expressions for $2S$ and $2P$ states:

$$\Delta E^{vp,vp}_{\mathrm{sopt}}(2S) = -\frac{\mu\alpha^2(Z\alpha)^2}{72\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta \times \int_0^\infty\left(1 - \frac{x}{2}\right)e^{-x(1-\frac{2m_e\xi}{W})}dx \times \int_0^\infty\left(1 - \frac{x'}{2}\right)e^{-x'(1-\frac{2m_e\eta}{W})}dx' g_{2S}(x,x'), \quad (27)$$

$$\Delta E^{vp,vp}_{\mathrm{sopt}}(2P) = -\frac{\mu\alpha^2(Z\alpha)^2}{7776\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta \times \int_0^\infty e^{-x(1+\frac{2m_e\xi}{W})}dx\int_0^\infty e^{-x'(1+\frac{2m_e\eta}{W})}dx' g_{2P}(x,x'), \quad (28)$$

which then give the following numerical results for the Lamb shift (in line 9 for pure electron VP and in line 10 for mixed electron and muon VP):

$$\Delta E^{vp,vp}_{\mathrm{sopt}}(2P - 2S) = \begin{cases} {}^6_3\mathrm{Li}: 5.63\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 5.66\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 12.70\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 12.73\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: 23.39\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 23.43\ \mathrm{meV} \end{cases} \quad (29)$$

The relations (27)–(29) show a sequence of steps in the calculation of the Lamb shift. Note also that all the integrals over the coordinates of the particles are calculated analytically. Another contribution, corresponding to the amplitude in

Fig. 1(c), is obtained by changing the perturbation potential with electron vacuum polarization to the potential with muon vacuum polarization. The order of this correction is increased by an additional factor α^2.

The second term in Eq. (26) has a similar structure [see Fig. 1(b)]. For its evaluation we can use a number of intermediate algebraic transformation. We show them in the example of one part of the Breit potential, proportional to $\mathbf{p}^4/(2\mu)^2$:

$$\begin{aligned}\langle\psi|\frac{\mathbf{p}^4}{(2\mu)^2}\sum\nolimits'_m \frac{|\psi_m\rangle\langle\psi_m|}{E_2-E_m}\Delta V^C_{vp}|\psi\rangle \\ = \langle\psi|\left(E_2+\frac{Z\alpha}{r}\right)\left(\hat{H}_0+\frac{Z\alpha}{r}\right)\sum\nolimits'_m \frac{|\psi_m\rangle\langle\psi_m|}{E_2-E_m}\Delta V^C_{vp}|\psi\rangle \\ = \langle\psi|\left(E_2+\frac{Z\alpha}{r}\right)^2\tilde{G}\Delta V^C_{vp}|\psi\rangle - \langle\psi|\frac{Z\alpha}{r}\Delta V^C_{vp}|\psi\rangle \\ + \langle\psi|\frac{Z\alpha}{r}|\psi\rangle\langle\psi|\Delta V^C_{vp}|\psi\rangle.\end{aligned} \quad (30)$$

In matrix elements (30), the integration is performed analytically over the coordinates and then numerically by the spectral parameter. Here are final numerical values of the matrix elements for four parts of the Breit potential (1) (relativistic term, contact term, relativistic-recoil term, and spin-orbit term):

$$\Delta E^{B,vp}_{sopt,1} = \begin{cases} {}^6_3\text{Li}: 18.57\ \text{meV}, & {}^7_3\text{Li}: 18.79\ \text{meV} \\ {}^9_4\text{Be}: 68.57\ \text{meV}, & {}^{10}_4\text{Be}: 68.95\ \text{meV}, \\ {}^{10}_5\text{B}: 184.95\ \text{meV}, & {}^{11}_5\text{B}: 185.78\ \text{meV} \end{cases} \quad (31)$$

$$\Delta E^{B,vp}_{sopt,2} = \begin{cases} {}^6_3\text{Li}: -8.98\ \text{meV}, & {}^7_3\text{Li}: -9.06\ \text{meV} \\ {}^9_4\text{Be}: -33.20\ \text{meV}, & {}^{10}_4\text{Be}: -33.34\ \text{meV}, \\ {}^{10}_5\text{B}: -90.10\ \text{meV}, & {}^{11}_5\text{B}: -90.42\ \text{meV} \end{cases} \quad (32)$$

$$\Delta E^{B,vp}_{sopt,3} = \begin{cases} {}^6_3\text{Li}: 0.18\ \text{meV}, & {}^7_3\text{Li}: 0.15\ \text{meV} \\ {}^9_4\text{Be}: 0.44\ \text{meV}, & {}^{10}_4\text{Be}: 0.40\ \text{meV}, \\ {}^{10}_5\text{B}: 1.08\ \text{meV}, & {}^{11}_5\text{B}: 0.98\ \text{meV} \end{cases} \quad (33)$$

$$\Delta E^{B,vp}_{sopt,4} = \begin{cases} {}^6_3\text{Li}: -0.89\ \text{meV}, & {}^7_3\text{Li}: -0.90\ \text{meV} \\ {}^9_4\text{Be}: -4.07\ \text{meV}, & {}^{10}_4\text{Be}: -4.08\ \text{meV}. \\ {}^{10}_5\text{B}: -12.64\ \text{meV}, & {}^{11}_5\text{B}: -12.67\ \text{meV} \end{cases} \quad (34)$$

Total values of one-loop VP and relativistic corrections in the second-order PT from (31)–(34) are presented in Tables I–III (line 11).

Other corrections of the second-order PT shown in Figs. 1(d)–1(f) have a similar structure. They appear after the replacements $\Delta V^C_{vp} \to \Delta V^B$ and $\Delta V^C_{vp} \to \Delta V^C_{vp,vp}$ in the basic amplitude presented in Fig. 1(c). Finally, the remaining two-loop corrections in second-order PT appear when one makes a replacement in Fig. 1(c), $\Delta V^C_{vp} \to \Delta V^B_{vp}$. In general, the calculation of the matrix elements in this case is quite similar to expressions (27) and (28). We present in the tables (line 14) only the full numerical results for this type of corrections, omitting the details of the calculation.

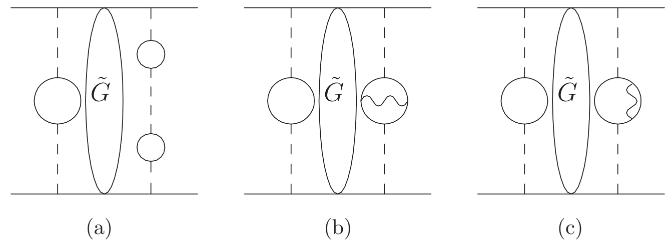

FIG. 2. Three-loop vacuum polarization corrections in the second-order perturbation theory. $\tilde{G}$ is the reduced Coulomb Green's function. Dashed line represents the Coulomb photon.

Three-loop vacuum polarization contributions in the second-order PT are presented in Fig. 2. The perturbation potentials which should be used in this case are determined by relations (2), (6), and (9). Omitting the intermediate expressions, we give only numerical values of these contributions:

$$\Delta E^{vp\text{-}vp,vp}_{sopt}(2P-2S) = \begin{cases} {}^6_3\text{Li}: 0.04\ \text{meV}, & {}^7_3\text{Li}: 0.04\ \text{meV} \\ {}^9_4\text{Be}: 0.10\ \text{meV}, & {}^{10}_4\text{Be}: 0.10\ \text{meV}, \\ {}^{10}_5\text{B}: 0.19\ \text{meV}, & {}^{11}_5\text{B}: 0.19\ \text{meV} \end{cases} \quad (35)$$

$$\Delta E^{2\text{-loop}\ vp,vp}_{sopt}(2P-2S) = \begin{cases} {}^6_3\text{Li}: 0.04\ \text{meV}, & {}^7_3\text{Li}: 0.04\ \text{meV} \\ {}^9_4\text{Be}: 0.09\ \text{meV}, & {}^{10}_4\text{Be}: 0.09\ \text{meV}. \\ {}^{10}_5\text{B}: 0.16\ \text{meV}, & {}^{11}_5\text{B}: 0.16\ \text{meV} \end{cases} \quad (36)$$

The sum of corrections (35) and (36) is in Tables I–III in line 12.

In the third order of perturbation theory (topt) there exists also three-loop VP correction of order $\alpha^3(Z\alpha)^2$ which is determined by the following relation [48,49]:

$$\begin{aligned}\Delta E^{3loop vp}_{topt} = \langle\psi_2|\Delta V^C\tilde{G}\Delta V^C\tilde{G}\Delta V^C|\psi_2\rangle \\ - \langle\psi_2|\Delta V^C|\psi_2\rangle\langle\psi_2|\Delta V^C\tilde{G}\tilde{G}\Delta V^C|\psi_2\rangle.\end{aligned} \quad (37)$$

Its numerical contribution to the Lamb shift (line 13 of the tables) is equal to

$$\Delta E^{3loop vp}_{topt}(2P-2S) = \begin{cases} {}^6_3\text{Li}: 0.05\ \text{meV}, & {}^7_3\text{Li}: 0.04\ \text{meV} \\ {}^9_4\text{Be}: 0.05\ \text{meV}, & {}^{10}_4\text{Be}: 0.05\ \text{meV}. \\ {}^{10}_5\text{B}: 0.31\ \text{meV}, & {}^{11}_5\text{B}: 0.31\ \text{meV} \end{cases} \quad (38)$$

IV. NUCLEAR STRUCTURE AND VACUUM POLARIZATION EFFECTS

The second effect of the Lamb shift, comparable in magnitude to the effect of vacuum polarization, is a nuclear structure effect. The energy levels of S states are sensitive to the nucleus finite size, owing to their spherical symmetry. In muonic ions, the Bohr radius is $\sim m_\mu/m_e \sim 207$ times smaller than the corresponding one in electron ions, which implies a greater overlap with the nucleus. Thus, the sensitivity to the finite size effect is greatly enhanced, what makes muonic

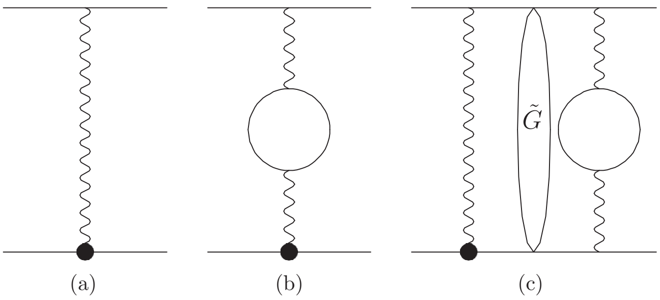

FIG. 3. Leading-order nuclear structure and vacuum polarization corrections. Thick point represents the nuclear vertex operator.

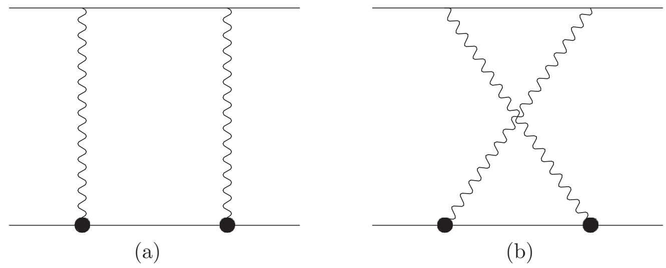

FIG. 4. Nuclear structure corrections of order $(Z\alpha)^5$. Thick point is the nuclear vertex operator.

atoms are very attractive for more accurate determination of the charge radius of the nuclei. In the leading order $(Z\alpha)^4$ it is determined by the nuclear charge radius r_N after an expansion of nuclear electric form factor as follows [Fig. 3(a)]:

$$\Delta E_{\rm str}(2P-2S) = -\frac{\mu^3(Z\alpha)^4}{12}\langle r_N^2\rangle = \begin{cases} {}^6_3\text{Li}: -3674.69\ \text{meV}, & {}^7_3\text{Li}: -3300.70\ \text{meV} \\ {}^9_4\text{Be}: -11200.03\ \text{meV}, & {}^{10}_4\text{Be}: -9825.73\ \text{meV} \\ {}^{10}_5\text{B}: -25492.50\ \text{meV}, & {}^{11}_5\text{B}: -25115.11\ \text{meV} \end{cases}, \tag{39}$$

where we take the nuclear charge radii from Ref. [37] for numerical estimates which are presented in line 15 of Tables I–III. The growth of the absolute value of the contribution (39) is due to two factors r_N and Z^4. The signs in formulas (5) and (39) are opposite; thus a significant reduction of the sum (5) and (39) occurs at a certain r_N and Z. As a result this leads to significant decrease in total value of the Lamb shift.

In the next-to-leading-order $(Z\alpha)^5$ there is nuclear structure correction which is defined by one-loop exchange diagrams (Fig. 4). Introducing only the charge form factor of the nucleus, we can represent the contribution on finite size of nucleus to the shift of S levels in the form

$$\Delta E^{2\gamma}_{\rm str}(nS) = -\frac{\mu^3(Z\alpha)^5}{\pi n^3}\delta_{l0}\int_0^\infty \frac{dk}{k}V(k), \tag{40}$$

$$\begin{aligned} V(k) = {} & \frac{2(F^2-1)}{m_1m_2} + \frac{8m_1\left[-F(0)-4m_2^2F'(0)\right]}{m_2(m_1+m_2)k} + \frac{k^2}{2m_1^3m_2^3}\left[2(F^2-1)\left(m_1^2+m_2^2\right)-F^2m_1^2\right] \\ & + \frac{\sqrt{k^2+4m_1^2}}{2m_1^3m_2\left(m_1^2-m_2^2\right)k}\left\{k^2\left[2(F^2-1)m_2^2-F^2m_1^2\right]+8m_1^4F^2+\frac{16m_1^4m_2^2(F^2-1)}{k^2}\right\} \\ & - \frac{\sqrt{k^2+4m_2^2}m_1}{2m_2^3\left(m_1^2-m_2^2\right)k}\left\{k^2[2(F^2-1)-F^2]+8m_2^2F^2+\frac{16m_2^4(F^2-1)}{k^2}\right\}, \end{aligned} \tag{41}$$

with n the usual principle quantum number. A subtraction of the point-like contribution and iteration term of quasipotential is made. To perform numerical integration in Eq. (40) we use dipole and Gaussian parameterizations for the charge form factor:

$$F_D(k^2) = \frac{\Lambda^4}{(k^2+\Lambda^2)^2}, \quad \Lambda^2 = \frac{12}{\langle r_N^2\rangle}, \quad F_G(k^2) = e^{-\frac{1}{6}k^2r_N^2}. \tag{42}$$

Numerical values of this correction for muonic atoms are the following (the results for Gaussian parameterization are in parentheses):

$$\Delta E^{2\gamma}_{\rm str}(2P-2S) = \begin{cases} {}^6_3\text{Li}: 207.83(190.63)\ \text{meV}, & {}^7_3\text{Li}: 176.67(162.05)\ \text{meV} \\ {}^9_4\text{Be}: 826.35(757.77)\ \text{meV}, & {}^{10}_4\text{Be}: 678.62(622.30)\ \text{meV} \\ {}^{10}_5\text{B}: 2268.56(2080.24)\ \text{meV}, & {}^{11}_5\text{B}: 2217.00(2032.89)\ \text{meV} \end{cases}. \tag{43}$$

We observe a significant change in the value of this contribution (approximately 10%) in the transition from the dipole to the Gaussian parametrizations. To be specific, we place in line 16 of the tables the result obtained with dipole parametrization.

To increase the accuracy of the Lamb shift calculation we have to consider corrections, which are determined by the nuclear structure effects and vacuum polarization simultaneously. In one-photon interaction corresponding contribution is represented by the amplitude in Fig. 3(b). To obtain a particle interaction operator we make an expansion of the charge form factor in momentum representation and replace the conventional Coulomb potential to the potential of the vacuum polarization. Then in coordinate representation we obtain

$$\Delta V^{vp}_{\rm str}(r)=\frac{2}{3}\pi Z\alpha\langle r_N^2\rangle\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\left[\delta(\mathbf{r})-\frac{m_e^2\xi^2}{\pi r}e^{-2m_e\xi r}\right]. \quad (44)$$

Averaging (44) over wave functions we find the following integral expressions for the corrections to the levels $2S$ and $2P$ and their numerical values in the Lamb shift:

$$\Delta E^{vp}_{\rm str}(2S)=\frac{\alpha(Z\alpha)^4\langle r_N^2\rangle\mu^3}{36\pi}\int_1^\infty\rho(\xi)d\xi\frac{8a_1^3\xi^3+11a_1^2\xi^2+8a_1\xi+2}{2(a_1\xi+1)^4},\quad a_1=\frac{2m_e}{W}, \quad (45)$$

$$\Delta E^{vp}_{\rm str}(2P)=-\frac{\alpha(Z\alpha)^4\mu^3\langle r_N^2\rangle}{72\pi}\int_1^\infty\rho(\xi)d\xi\frac{a_1^2\xi^2}{(a_1\xi+1)^4}, \quad (46)$$

$$\Delta E^{vp}_{\rm str}(2P-2S)=\begin{cases}{}^6_3\mathrm{Li}:-14.21\ \mathrm{meV}, & {}^7_3\mathrm{Li}:-12.78\ \mathrm{meV}\\ {}^9_4\mathrm{Be}:-48.35\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}:-42.44\ \mathrm{meV}\,.\\ {}^{10}_5\mathrm{B}:-118.86\ \mathrm{meV}, & {}^{11}_5\mathrm{B}:-117.14\ \mathrm{meV}\end{cases} \quad (47)$$

The same order $\alpha(Z\alpha)^4$ contribution is given by the amplitude in the second-order PT presented in Fig. 3(c):

$$\Delta E^{vp}_{\rm str,sopt}(2P-2S)=-\frac{\alpha(Z\alpha)^4\mu^3\langle r_N^2\rangle}{36\pi}\int_1^\infty\rho(\xi)d\xi$$
$$\times\frac{4(a_1\xi+1)(2a_1^2\xi^2+1)\ln(a_1\xi+1)+a_1\xi\{4a_1\xi[a_1\xi(a_1\xi+3)+1]+11\}+3}{(a_1\xi+1)^5}$$
$$=\begin{cases}{}^6_3\mathrm{Li}:-23.00\ \mathrm{meV}, & {}^7_3\mathrm{Li}:-20.68\ \mathrm{meV}\\ {}^9_4\mathrm{Be}:-80.52\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}:-70.68\ \mathrm{meV}\,.\\ {}^{10}_5\mathrm{B}:-202.98\ \mathrm{meV}, & {}^{11}_5\mathrm{B}:-200.06\ \mathrm{meV}\end{cases} \quad (48)$$

The contributions (47) and (48) are written separately in the tables (lines 17 and 18).

Bearing in mind that the quantities (47) and (48) are large, we evaluate also nuclear structure corrections with the account of two-loop vacuum polarization effects in 1γ interaction [Figs. 5(a)–5(c)]. The method of constructing the potentials is the same as in this and preceding sections. Corresponding potentials have the following form:

$$\Delta V^{vp\text{-}vp}_{\rm str}(r)=\frac{2}{3}Z\alpha\langle r_N^2\rangle\left(\frac{\alpha}{3\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\left[\pi\delta(\mathbf{r})-\frac{m_e^2}{r(\xi^2-\eta^2)}(\xi^4e^{-2m_e\xi r}-\eta^4e^{-2m_e\eta r})\right], \quad (49)$$

$$\Delta V^{\text{2-loop }vp}_{\rm str}(r)=\frac{4}{9}Z\alpha\langle r_N^2\rangle\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{1-v^2}\left[\pi\delta(\mathbf{r})-\frac{m_e^2}{r(1-v^2)}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right]. \quad (50)$$

The sum of corrections to the Lamb shift $(2P-2S)$ that are provided by (49) and (50) is equal (see line 19 in the tables):

$$\Delta E^{vp,vp}_{\rm str}(2P-2S)=\begin{cases}{}^6_3\mathrm{Li}:-0.11\ \mathrm{meV}, & {}^7_3\mathrm{Li}:-0.11\ \mathrm{meV}\\ {}^9_4\mathrm{Be}:-0.41\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}:-0.36\ \mathrm{meV}\,.\\ {}^{10}_5\mathrm{B}:-1.02\ \mathrm{meV}, & {}^{11}_5\mathrm{B}:-1.00\ \mathrm{meV}\end{cases} \quad (51)$$

There are two-loop VP corrections with nuclear structure of order $\alpha^2(Z\alpha)^4$ in the second-order PT [see Figs. 6(a)–6(d)]. They can be calculated as a contribution (45) with the replacement of one-loop vacuum polarization on the two-loop vacuum polarization. Their numerical values are included in Tables I–III (line 20). A correction by two-photon exchange diagrams with the effect of vacuum polarization plays an important role, as it is reinforced by the factor Z^5 (see Fig. 7). An analytical expression for this correction and its numerical values is defined by modified potential $V(p)$ from (41) as follows:

$$\Delta E^{2\gamma}_{{\rm str},vp}(nS)=-\frac{2\mu^3\alpha(Z\alpha)^5}{\pi^2n^3}\int_0^\infty kV(k)dk\times\int_0^1\frac{v^2\left(1-\frac{v^2}{3}\right)dv}{k^2(1-v^2)+4m_e^2}, \quad (52)$$

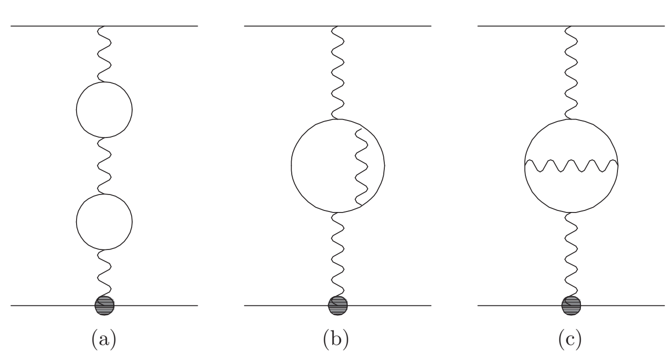

FIG. 5. Nuclear structure and two-loop vacuum polarization effects in the one-photon interaction. Thick point is the nuclear vertex operator.

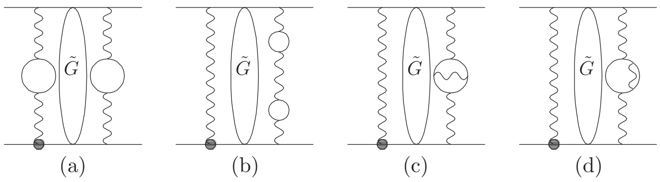

FIG. 6. Nuclear structure and two-loop vacuum polarization effects in the second-order perturbation theory. Thick point is the nuclear vertex operator. $\tilde{G}$ is the reduced Coulomb Green's function.

$$\Delta E^{2\gamma}_{str,vp}(2P-2S) = \begin{cases} {}^6_3\mathrm{Li}: 3.19(3.05)\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 2.73(2.61)\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 12.74(12.08)\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 10.80(10.05)\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: 35.07(33.37)\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 33.67(32.63)\ \mathrm{meV} \end{cases} \quad (53)$$

Since expression (52) contains the charge form factor of the nucleus, we present in Eq. (53) two numerical values of the contribution corresponding to the parametrizations in Eq. (42). In the tables (line 20), we have included only one result corresponding to the dipole parametrization. Corrections to the Lamb shift discussed in this and previous sections are such that analytical expressions for them are quite bulky, since the characteristic parameter W/m_e is large. It cannot be used as an expansion parameter. For this reason, it is more convenient to present corrections in integral form, which we do in this paper.

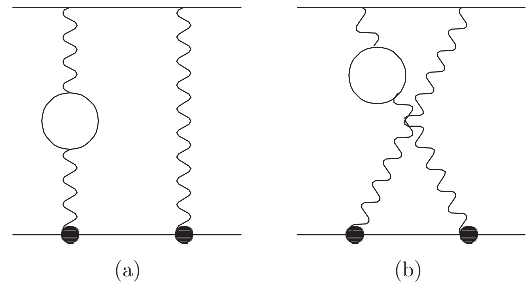

FIG. 7. Nuclear structure and electron vacuum polarization effects in the two-photon exchange diagrams. Thick point is the nuclear vertex operator.

V. RECOIL CORRECTIONS, MUON SELF-ENERGY, AND VACUUM POLARIZATION EFFECTS

There is another group of corrections, which were obtained in analytical form in the study of the Lamb shift $(2P-2S)$ in the hydrogen atom during many years. Their calculation is discussed in detail in Ref. [15]. Corresponding analytical results can be used directly for numerical estimates in muonic atoms. For the sake of completeness we present in this section the key expressions for such corrections, which have necessary order in α and the ratio of particle masses and provide significant numerical values in the Lamb shift $(2P-2S)$.

An analytical expression for the recoil correction of order α^4 was obtained after calculating the matrix elements of the Breit potential [15,45,50] (table line 22):

$$\Delta E^{(Z\alpha)^4}_{rec}(2P-2S) = \begin{cases} \frac{\mu^3(Z\alpha)^4}{48m_2^2}, & \delta_I = 1, \\ \frac{\mu^3(Z\alpha)^4}{12m_2^2}, & \delta_I = 0, \end{cases}$$
$$= \begin{cases} {}^6_3\mathrm{Li}: 0.68\ \mathrm{meV}, & {}^7_3\mathrm{Li}: 0.13\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 0.24\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: 0.79\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: 1.94\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: 0.40\ \mathrm{meV} \end{cases} \quad (54)$$

The recoil correction of order $(Z\alpha)^5$ is related with two-photon exchange amplitudes in which the nucleus is considered as a point particle [15,50]:

$$\Delta E^{(Z\alpha)^5}_{rec} = \frac{\mu^3(Z\alpha)^5}{m_1 m_2 \pi n^3}\left[\left(\frac{2}{3}\ln\frac{1}{Z\alpha}-\frac{1}{9}\right)\delta_{l0} - \frac{8}{3}\ln k_0(n,l) - \frac{7}{3}a_n - \frac{2}{m_2^2-m_1^2}\delta_{l0}\left(m_2^2\ln\frac{m_1}{\mu} - m_1^2\ln\frac{m_2}{\mu}\right)\right], \quad (55)$$

where $\ln k_0(n,l)$ is the Bethe logarithm [15,51]:

$$\ln k_0(2S) = 2.811769893120563, \quad (56)$$

$$\ln k_0(2P) = -0.030016708630213, \quad (57)$$

$$a_n = -2\left[\ln\left(\frac{2}{n}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) + 1 - \frac{1}{2n}\right]\delta_{l0} + \frac{(1-\delta_{l0})}{l(l+1)(2l+1)}. \quad (58)$$

The expression (55) gives the following numerical result (line 23 in the tables):

$$\Delta E^{(Z\alpha)^5}_{rec}(2P-2S) = \begin{cases} {}^6_3\mathrm{Li}: -2.15\ \mathrm{meV}, & {}^7_3\mathrm{Li}: -1.86\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: -5.97\ \mathrm{meV}, & {}^{10}_4\mathrm{Be}: -5.40\ \mathrm{meV}. \\ {}^{10}_5\mathrm{B}: -16.03\ \mathrm{meV}, & {}^{11}_5\mathrm{B}: -14.63\ \mathrm{meV} \end{cases} \quad (59)$$

Numerical result for recoil correction of order $(Z\alpha)^6$ is presented in Tables I–III (line 24) according to analytical formula from Refs. [15,52].

It should be noted that there is a significant value contribution, which is given by the radiative corrections to the muon line from the Dirac and Pauli form factors of the muon and muon vacuum polarization (mvp). It is appropriate to quote here the relevant analytical formulas [15,53]:

$$\Delta E_{\rm mvp,mse}(2S) = \frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left[\frac{4}{3}\ln\frac{m_1}{\mu(Z\alpha)^2} - \frac{4}{3}\ln k_0(2S) + \frac{38}{45} + \frac{\alpha}{\pi}\left(-\frac{9}{4}\zeta(3) + \frac{3}{2}\pi^2\ln 2 - \frac{10}{27}\pi^2 - \frac{2179}{648}\right) + 4\pi Z\alpha\left(\frac{427}{384} - \frac{\ln 2}{2}\right)\right], \quad (60)$$

$$\Delta E_{\rm mvp,mse}(2P) = \frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left[-\frac{4}{3}\ln k_0(2P) - \frac{m_1}{6\mu} - \frac{\alpha}{3\pi}\frac{m_1}{\mu}\left(\frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln 2 + \frac{\pi^2}{12} + \frac{197}{144}\right)\right], \quad (61)$$

which lead to numerical results (line 25 in the tables):

$$\Delta E_{\rm mse,mvp}(2P-2S) = \begin{cases} {}^6_3\text{Li}: -50.99\ \text{meV}, & {}^7_3\text{Li}: -51.36\ \text{meV} \\ {}^9_4\text{Be}: -149.00\ \text{meV}, & {}^{10}_4\text{Be}: -149.52\ \text{meV}. \\ {}^{10}_5\text{B}: -337.45\ \text{meV}, & {}^{11}_5\text{B}: -338.40\ \text{meV} \end{cases} \quad (62)$$

The radiative-recoil corrections of orders $\alpha(Z\alpha)^5$ and $(Z^2\alpha)(Z\alpha)^4$ from Tables 8 and 9 in Ref. [15] are significantly smaller (see their explicit form in Ref. [31]). We have included the numerical values in the summary, Tables I–III (line 26).

On the basis of Refs. [54,55] we give an estimate of the nuclear structure corrections of orders $(Z\alpha)^6$ and $\alpha(Z\alpha)^5$ to the Lamb shift in muonic ions. So, for the structure correction of order $(Z\alpha)^6$ we obtain:

$$\Delta E_{\rm str}^{(Z\alpha)^6}(2P-2S) = \frac{(Z\alpha)^6}{12}\mu^3\left\{r_N^2\left[\langle\ln\mu Z\alpha r\rangle + C - \frac{3}{2}\right] - \frac{1}{2}r_N^2 + \frac{1}{3}\langle r^3\rangle\left\langle\frac{1}{r}\right\rangle - I_2^{\rm rel} - I_3^{\rm rel} - \mu^2 F_{NR} + \frac{1}{40}\mu^2\langle r^4\rangle\right\}$$
$$= \begin{cases} {}^6_3\text{Li}: -7.38\ \text{meV}, & {}^7_3\text{Li}: -6.69\ \text{meV} \\ {}^9_4\text{Be}: -40.60\ \text{meV}, & {}^{10}_4\text{Be}: -35.70\ \text{meV}. \\ {}^{10}_5\text{B}: -145.27\ \text{meV}, & {}^{11}_5\text{B}: -143.04\ \text{meV} \end{cases} \quad (63)$$

where the quantities $I_{2,3}^{\rm rel}, F_{NR}$ are written explicitly in Ref. [54]. Significant growth of the numerical values in Eq. (63) in the transition from one to the other muonic ions muon is caused by two factors, Z and r_N. A summary result for structure corrections of orders $(Z\alpha)^6$ and $\alpha(Z\alpha)^5$ is presented in Tables I–III (line 27).

An amplitude in Fig. 8(b) gives the contribution to the energy spectrum, which can be expressed in terms of the slope of the Dirac form factor F_1' and the Pauli form factor F_2. Numerical values in the Lamb shift are obtained by means of two-loop corrections to form factors $F_1'(0)$ and $F_2(0)$ which were calculated in Ref. [56] (line 28 of the tables). Another contribution with vacuum polarization in Fig. 8(a) was investigated in Refs. [16,57]. It is included in final tables on line 29. The contribution of hadron vacuum polarization to the Lamb shift can be derived by means of corresponding result for muonic hydrogen [58–60] (table line 30).

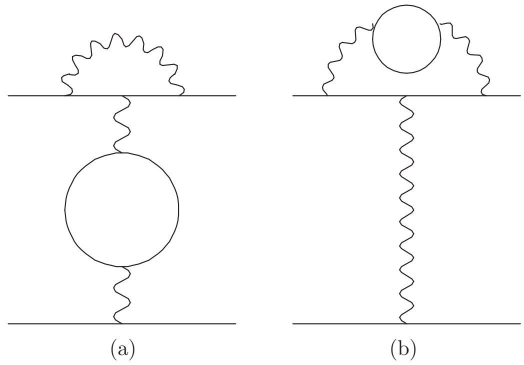

FIG. 8. Radiative corrections with the vacuum polarization effects.

VI. SUMMARY AND CONCLUSION

In this study, we perform a calculation of the Lamb shift $(2P_{1/2} - 2S_{1/2})$ in the number of muonic ions with different nuclear charge and nuclear charge radius. Various corrections with fairly high degrees of fine structure constant $\alpha^3 \div \alpha^6$ have been taken into account. All contributions that are analyzed may be divided into two groups. The first group includes the corrections specific to each muonic ion, which are presented in the integral form and calculated analytically and numerically. The second group of corrections is obtained on the basis of the known analytical expressions derived in the study of the Lamb shift in the hydrogen atom. Numerical values of all corrections are written explicitly in Tables I–III. Since for these muonic ions numerical values of the contributions previously have been hardly discussed, with a few exceptions [7], we are limited to only the key papers containing analytical results. As we wrote above in Sec. V, we use these expressions to obtain numerical contributions to the Lamb shift. The resulting total numerical values of shifts in muonic ions of lithium, beryllium, and boron can be used for comparison with future experimental data. These numerical values allow us to trace the dynamics of changes in the values of corrections during the transition from one ion to another.

It is known that the position of the energy levels of the $2S_{1/2}$ and $2P_{1/2}$ in atoms of electronic hydrogen and muonic hydrogen differs significantly. We have seen a similar change in the study of muonic ions $(\mu\mathrm{Li})^{2+}$, $(\mu\mathrm{Be})^{3+}$, and $(\mu\mathrm{B})^{4+}$: If a muonic lithium ion $2P$ level is above the $2S$ level, then for ions of muonic beryllium and boron we get the reverse arrangement of levels. This effect is due to the compensation for two basic contributions to the Lamb shift from a one-loop electronic vacuum polarization and nuclear structure of order $(Z\alpha)^4$. As a result the corrections of higher order in α, enhanced by nuclear charge degrees, become more important.

As noted above, the problem of the Lamb shift in muonic ions of lithium, beryllium, and boron was studied many years ago in Ref. [7]. One part of results in Ref. [7] was obtained with the use of nonrelativistic wave functions and treatment of the finite nuclear size and vacuum-polarization potentials as small perturbations. It is consistent with our results within a small change in the fundamental physical constants. Another part of the results in Ref. [7] was obtained by means of numerical solution of the Dirac equation and treatment of the remaining small corrections due to the muon self-energy, the higher-order Källen-Sabrey vacuum polarization term, and nuclear polarization by perturbation theory. To compare our results with such calculations in Ref. [7] we must take the sum of a few lines of our tables, corresponding to the first- and second-order perturbation theory, and in the case of effects on the structure of the nucleus to the sum of corrections of one-photon and two-photon amplitudes. Since our other corrections are numerically small, it is convenient further to compare the complete results for the Lamb shift. Our values of the charge radii of nuclei are slightly different from the values of Ref. [7], what is one of the reasons for the differences of total numerical results. Another reason is related to corrections of a higher order, accounted for in this paper. For example, we use the value of the charge radius of lithium nucleus $r({}^6_3\mathrm{Li}) = 2.5890$ fm from Ref. [37]. It is slightly different from the value 2.560 fm used in Ref. [7]. This difference gives an addition of 82 meV to our result from Table I due to structure correction of order $(Z\alpha)^4$ (39). In turn, our structure correction of order $(Z\alpha)^5$ (43) becomes smaller on 7 meV and reduces essentially the divergence from the value of the Lamb shift in Ref. [7]. The remaining difference can be referred to a difference in model assumptions for the charge form factor in our work and in Ref. [7] (uniform charge distribution), which, as noted above, provides a 10% change in results. The same situation occurs for the other nuclei.

Note also that there is another contribution to the polarizability of the nucleus, for which we use an estimate from Ref. [7]. It is expressed in terms of the (-2) moment of the total electric-dipole photoabsorption cross section σ_{-2}. In the case of light nuclei there exists a simple formula $\sigma_{-2} = 3.5kA^{5/3}\frac{\mu\mathrm{b}}{\mathrm{MeV}}$ [61] which allows us to obtain an estimate for the cross sections of lithium, beryllium, and boron. We add error margins for many contributions in the tables connected with the nuclear structure and polarizability. Other contributions in the tables are presented with the accuracy 0.01 meV. The errors of these contributions related to uncertainties in determining the fundamental physical constants are much smaller. As follows from Tables I–III, the contributions containing $\langle r_N^2\rangle$ are obtained with the errors significantly exceeding 0.01 meV. However, we indicate in the tables that the contributions are to an accuracy of 0.01 meV, as this will provide a more accurate value of the charge radii of nuclei in the presence of the relevant experimental data, as was done for muonic hydrogen and muonic deuterium.

The question of the parametrization of the charge form factors of light nuclei, too, should be studied further in order to reduce the errors connected with the determination of contribution (43). In the summary Tables I–III, we have assumed that there is a dipole parametrization for the nuclear charge form factor. If it turns out that the more correct (corresponding to the experimental data on the scattering of leptons on nuclei) is, for example, the Gaussian parametrization, then it will be necessary to recalculate only the contributions to the nuclear structure and to use other corrections obtained in this work. For this reason, it is useful to present total results for the Lamb shifts in muonic ions of Li, Be, and B in the form: $\Delta E(2P-2S) = A + B\langle r_N^2\rangle + C_{2\gamma} + D_{\mathrm{pol}} + E_{\mathrm{Friar}}$, introducing the factor of $\langle r_N^2\rangle$ outside the brackets in a number of terms. For example, in the case of ${}^6_3\mathrm{Li}$ nucleus the coefficients in meV are the following: $A = 4654.38$, $B = -1434.01$, $C_{2\gamma} = 211.19$, $D_{\mathrm{pol}} = 15$, and $E_{\mathrm{Friar}} = -6.07$, where the coefficients A and B are written out with a precision of 0.01 meV. This representation of total results identifies the main sources of uncertainty connected with the nucleus charge radius, two-photon exchange diagrams, contribution to the nucleus polarizability, and correction to the nucleus structure, obtained in Ref. [54]. Weak interaction contribution is very small and is not considered in this study (see Ref. [62]). Thus, with proper experimental accuracy we can obtain more precise values of the nuclear charge radii.

ACKNOWLEDGMENTS

This work is supported by the Russian Foundation for Basic Research (Grant No. 16-02-00554) and Ministry of Education and Science of Russia (Grant No. 1394).

[1] R. Pohl *et al.*, Nature (London) **466**, 213 (2010).
[2] A. Antognini *et al.*, Science **339**, 417 (2013).
[3] A. Antognini *et al.*, Ann. Phys. (NY) **331**, 127 (2013).
[4] R. Pohl, R. Gilman, G. A. Miller, and K. Pachucki, Annu. Rev. Nucl. Part. Sci. **63**, 175 (2013).
[5] R. Pohl *et al.*, Science **353**, 669 (2016).
[6] R. Pohl *et al.*, *Proc. 12th Int. Conf. on Low Energy Antiproton Physics* (LEAP 2016); arXiv:1609.03440 [physics.atom-ph] (unpublished).
[7] G. W. F. Drake and L. L. Byer, Phys. Rev. A **32**, 713 (1985).
[8] R. Swainson and G. W. F. Drake, Phys. Rev. A **34**, 620 (1986).
[9] R. N. Faustov and A. P. Martynenko, J. Exp. Theor. Phys. **88**, 672 (1999).
[10] A. P. Martynenko, J. Exp. Theor. Phys. **101**, 1021 (2005).
[11] E. Borie, Z. Phys. A **275**, 347 (1975).
[12] E. Borie and G. A. Rinker, Phys. Rev. A **18**, 324 (1978).
[13] E. Borie and G. A. Rinker, Rev. Mod. Phys. **54**, 67 (1982).
[14] J. L. Friar, Ann. Phys. **122**, 151 (1979).

[15] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rep. **342**, 62 (2001); *Theory of Light Hydrogenic Bound States*, Springer Tracts in Modern Physics Vol. 222 (Springer, Berlin, 2007).
[16] K. Pachucki, Phys. Rev. A **53**, 2092 (1996).
[17] E. Borie, Ann. Phys. **327**, 733 (2012).
[18] U. D. Jentschura, Ann. Phys. **326**, 500 (2011).
[19] C. E. Carlson, V. Nazaryan, and K. Griffioen, Phys. Rev. A **83**, 042509 (2011).
[20] E. Yu. Korzinin, V. G. Ivanov and S. G. Karshenboim, Phys. Rev. D **88**, 125019 (2013).
[21] J. L. Friar, Phys. Rev. C **88**, 034003 (2013).
[22] C. E. Carlson, M. Gorchtein, and M. Vanderhaeghen, Phys. Rev. A **89**, 022504 (2014).
[23] S. G. Karshenboim, V. G. Ivanov, E. Yu. Korzinin, and V. A. Shelyuto, Phys. Rev. A **81**, 060501 (2010).
[24] G. A. Miller, A. W. Thomas, J. D. Carroll, and J. Rafelski, Phys. Rev. A **84**, 020101(R) (2011).
[25] A. De Rujula, Phys. Lett. B **697**, 26 (2011).
[26] J. J. Krauth *et al.*, Ann. Phys. **366**, 168 (2016).
[27] A. V. Eskin, R. N. Faustov, A. P. Martynenko, and F. A. Martynenko, Mod. Phys. Lett. A **31**, 1650104 (2016).
[28] R. N. Faustov, A. P. Martynenko, G. A. Martynenko, and V. V. Sorokin, Phys. Rev. A **92**, 052512 (2015).
[29] F. Hagelstein and V. Pascalutsa, PoS **CD15**, 077 (2016).
[30] N. T. Huong, E. Kou, and B. Moussallam, Phys. Rev. D **93**, 114005 (2016).
[31] A. A. Krutov, A. P. Martynenko, G. A. Martynenko, and R. N. Faustov, J. Exp. Theor. Phys. **120**, 73 (2015).
[32] R. N. Faustov and A. P. Martynenko, J. Exp. Theor. Phys. **98**, 39 (2004).
[33] R. N. Faustov, A. P. Martynenko, G. A. Martynenko, and V. V. Sorokin, Phys. Rev. A **90**, 012520 (2014).
[34] R. N. Faustov, A. P. Martynenko, G. A. Martynenko, and V. V. Sorokin, Phys. Lett. B **733**, 354 (2014).
[35] A. A. Krutov and A. P. Martynenko, Phys. Rev. A **84**, 052514 (2011).
[36] P. J. Mohr, B. N. Taylor, and D. B. Newell, Rev. Mod. Phys. **84**, 1527 (2012).
[37] I. Angeli and K. P. Marinova, Atom. Data Nucl. Data Tables **99**, 69 (2013).
[38] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Butterworth-Heinemann, Oxford, 1999).
[39] G. Källen and A. Sabry, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **29**, 17 (1955).
[40] T. Kinoshita and M. Nio, Phys. Rev. Lett. **82**, 3240 (1999).
[41] T. Kinoshita and M. Nio, Phys. Rev. D **60**, 053008 (1999).
[42] E. H. Wichmann and N. M. Kroll, Phys. Rev. **101**, 843 (1956).
[43] P. J. Mohr, G. Plunien, and G. Soff, Phys. Rep. **293**, 228 (1998).
[44] S. G. Karshenboim, V. G. Ivanov, E. Yu. Korzinin, and V. A. Shelyuto, JETP Lett. **92**, 8 (2010).
[45] U. D. Jentschura, Phys. Rev. A **84**, 012505 (2011).
[46] A. P. Martynenko, Phys. Rev. A **76**, 012505 (2007).
[47] H. F. Hameka, Jour. Chem. Phys. **47**, 2728 (1967).
[48] T. Kinoshita and M. Nio, Phys. Rev. Lett. **103**, 079901(E) (2009).
[49] V. G. Ivanov, E. Yu. Korzinin, and S. G. Karshenboim, Phys. Rev. D **80**, 027702 (2009).
[50] J. R. Sapirstein and D. R. Yennie, in *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990), p. 560.
[51] G. W. F. Drake and R. A. Swainson, Phys. Rev. A **41**, 1243 (1990).
[52] M. I. Eides and H. Grotch, Phys. Rev. A **55**, 3351 (1997).
[53] M. I. Eides and H. Grotch, Phys. Rev. A **56**, R2507 (1997).
[54] J. L. Friar and G. L. Payne, Phys. Rev. A **56**, 5173 (1997).
[55] G. P. Lepage, D. R. Yennie, and G. W. Erickson, Phys. Rev. Lett. **47**, 1640 (1981).
[56] R. Barbieri, M. Caffo, and E. Remiddi, Nuovo Cimento Lett. **7**, 60 (1973)
[57] U. D. Jentschura and B. J. Wundt, Eur. Phys. J. D **65**, 357 (2011).
[58] E. Borie, Z. Phys. A **302**, 187 (1981).
[59] J. L. Friar, J. Martorell, and D. W. L. Sprung, Phys. Rev. A **59**, 4061 (1999).
[60] A. P. Martynenko and R. N. Faustov, Phys. Atom. Nucl. **64**, 1282 (2001).
[61] J. N. Orce, Phys. Rev. C **91**, 064602 (2015).
[62] M. I. Eides, Phys. Rev. A **53**, 2953 (1996); **85**, 034503 (2012)