

**Open-quantum-systems approach to complementarity in neutral-kaon interferometry**Gustavo de Souza,<sup>1,\*</sup> J. G. G. de Oliveira, Jr.,<sup>2</sup> Adalberto D. Varizi,<sup>3</sup> Edson C. Nogueira,<sup>3</sup> and Marcos D. Sampaio<sup>3</sup><sup>1</sup>*Universidade Federal de Ouro Preto – Departamento de Matemática – ICEB, Campus Morro do Cruzeiro, s/n, 35400-000, Ouro Preto/MG, Brazil*<sup>2</sup>*Universidade Estadual de Santa Cruz – Departamento de Ciências Exatas e Tecnológicas 45662900, Ilhéus/BA, Brazil*<sup>3</sup>*Universidade Federal de Minas Gerais – Departamento de Física – ICEX, P.O. BOX 702, 30.161-970, Belo Horizonte/MG, Brazil*

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In bipartite quantum systems, entanglement correlations between the parties exerts direct influence in the phenomenon of wave-particle duality. This effect has been quantitatively analyzed in the context of two qubits by Jakob and Bergou [Opt. Commun. **283**, 827 (2010)]. Employing a description of the  $K$ -meson propagation in free space where its weak decay states are included as a second party, we study here this effect in the kaon-antikaon oscillations. We show that a new quantitative “trianity” relation holds, similar to the one considered by Jakob and Bergou. In our case, it relates the distinguishability between the decay-product states corresponding to the distinct kaon propagation modes  $K_S$ ,  $K_L$ , the amount of wave-like path interference between these states, and the amount of entanglement given by the reduced von Neumann entropy. The inequality can account for the complementarity between strangeness oscillations and lifetime information previously considered in the literature, therefore allowing one to see how it is affected by entanglement correlations. As we will discuss, it allows one to visualize clearly through the  $K^0$ - $\bar{K}^0$  oscillations the fundamental role of entanglement in quantum complementarity.

DOI: [10.1103/PhysRevA.94.062108](https://doi.org/10.1103/PhysRevA.94.062108)**I. INTRODUCTION**

In the present work, we revisit the complementarity between strangeness oscillations and lifetime information in the neutral kaon system previously studied by Bramon, Garbarino, and Hiesmayr (BGH) [1–3]. However, instead of using the Wigner–Weisskopf approach to the isolated, free-kaon propagation, we consider an open systems model in which the neutral kaon’s weak decay states are included as a second party. The interaction between the two subsystems is given by a completely positive probability-preserving quantum dynamical map. Our model coincides with that proposed by Caban *et al.* [4] (and also discussed in Bertlmann *et al.* [5]) upon partial trace of the decay products, but it has the new feature of allowing bipartite entanglement to be studied. We examine quantitatively the effects of these correlations on complementarity in the context of neutral kaon interferometry.

In this case, the quantitative duality relation of the Greenberger–Yasin type [6] considered in Refs. [1–3] must extend to a “trianity” relation incorporating a quantitative entanglement measure. We show here that a new such quantitative complementarity relation holds:

$$\mathcal{V}(\tau) \leq \sqrt{1 - \mathcal{D}^2(\tau) - \mathcal{S}^2(\tau)} \quad \forall \tau \in I, \quad (1)$$

where  $\tau$  denotes the proper time and  $I$  is the time interval relevant for the analysis (see Sec. III). This inequality is similar to that proposed by Jakob and Bergou for bipartite systems [7]. Here,  $\mathcal{D}$  denotes the distinguishability between the decay-product states corresponding to the distinct kaon propagation modes  $K_S$ ,  $K_L$ . As we will see,  $\mathcal{D}$  quantifies the increasing amount of lifetime information which becomes available (due to entanglement correlations) in the decay-states subsystem. The associated visibility  $\mathcal{V}$  quantifies the amount

of wave-like path interference between these states, while  $\mathcal{S}$  denotes the von Neumann entropy of the kaon state and measures bipartite entanglement. We will demonstrate that the quantitative complementarity relation (1) also accounts for the complementarity between strangeness oscillations and lifetime information considered by BGH. The results allow us to visualize and discuss in a clear way through the  $K^0$ - $\bar{K}^0$  oscillations the essential role played by entanglement in wave-particle duality.

**II. THE MODEL**

While there are several open quantum system models available in the literature offering completely positive probability-preserving descriptions of the composite neutral kaon plus a weak decay-products system [4,5,8–11], here we consider a model in which these two subsystems are treated as different parties. Therefore, we take the composite system state space as the tensor product  $\mathcal{H} = \mathcal{H}_Q \otimes \mathcal{H}_P$  between the kaon (quanton) Hilbert space  $\mathcal{H}_Q$  and the space of decay products states  $\mathcal{H}_P$ .

A short-lived kaon  $K_S^0$  always decays into two pions, either  $\pi^+ + \pi^-$  or  $\pi^0 + \pi^0$ . On the other hand, a long-lived kaon  $K_L^0$  has several decay modes: it can decay into three neutral pions or  $\pi^+ + \pi^- + \pi^0$ , but there are also the semileptonic decays into  $\pi^\pm + \mu^\mp + \nu_\mu$ ,  $\pi^\pm + e^\mp + \nu_e$ , and the considerably rare  $K_L^0$  decays into two pions. However, we will not consider here this last decay mode associated with charge-parity violation. In this case, the state of the decay-products subsystem can be labeled by its pion content. Thus, we take  $\mathcal{H}_P$  as the Hilbert space spanned by the (orthonormalized) vectors  $|0_\pi\rangle$ ,  $|\pi\pi\rangle$ , and  $|\bar{\pi}\bar{\pi}\rangle$ , which represent respectively states with no pions, two pions, and one or three pions.

The kaon state space is taken as the direct sum  $\mathcal{H}_Q = H_0 \oplus H_{K^0}$ , where  $H_0$  is the Hilbert space spanned by the vector  $|0_K\rangle$  representing the vacuum (absence of kaon) and  $H_{K^0}$  is the usual kaon Hilbert space spanned by the strangeness eigenstates

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$|K^0\rangle, |\bar{K}^0\rangle$ . Under our assumption of charge-parity symmetry, the neutral kaon mass eigenstates  $|K_S^0\rangle, |K_L^0\rangle$  corresponding to the short-lived and long-lived propagation modes are

$$|K_S^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle), \quad |K_L^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle), \quad (2)$$

and we have  $\langle K_S^0|K_L^0\rangle = 0$ . We assume that  $|0_K\rangle$  is normalized and orthogonal to  $|K_S^0\rangle, |K_L^0\rangle$ .

So far for the kinematic aspects. Let us turn now to dynamics. The only physically meaningful initial configurations are those with a kaon and no pion—that is, factorized initial conditions of the form  $|\Psi(0)\rangle = (\alpha|K_S^0\rangle + \beta|K_L^0\rangle)|0_\pi\rangle$ . We assume that evolution takes place entirely in the subspace  $\mathcal{W} \subset \mathcal{H}$  spanned by  $\{|K_S^0\rangle|0_\pi\rangle, |K_L^0\rangle|0_\pi\rangle, |0_K\rangle|\pi\pi\rangle, |0_K\rangle|\tilde{\pi}\tilde{\pi}\rangle\}$  and according to the quantum map

$$\begin{aligned} |0_K\rangle|0_\pi\rangle &\longrightarrow |0_K\rangle|0_\pi\rangle, \\ |K_S^0\rangle|0_\pi\rangle &\longrightarrow e^{-\frac{1}{2}\Gamma_S\tau} e^{-im_S\tau} |K_S^0\rangle|0_\pi\rangle \\ &\quad + \sqrt{1 - e^{-\Gamma_S\tau}} |0_K\rangle|\pi\pi\rangle, \\ |K_L^0\rangle|0_\pi\rangle &\longrightarrow e^{-\frac{1}{2}\Gamma_L\tau} e^{-im_L\tau} |K_L^0\rangle|0_\pi\rangle \\ &\quad + \sqrt{1 - e^{-\Gamma_L\tau}} |0_K\rangle|\tilde{\pi}\tilde{\pi}\rangle, \end{aligned} \quad (3)$$

where  $m_S$  and  $\Gamma_S = \frac{1}{\tau_S}$  ( $m_L$  and  $\Gamma_L = \frac{1}{\tau_L}$ ) are the  $K_S^0$  ( $K_L^0$ ) mass and decay width [12], and where  $p_S(\tau) \equiv 1 - e^{-\Gamma_S\tau}$  [ $p_L(\tau) \equiv 1 - e^{-\Gamma_L\tau}$ ] denotes the amplitude for the state  $|K_S^0\rangle|0_\pi\rangle$  ( $|K_L^0\rangle|0_\pi\rangle$ ) mapping at time  $\tau$  into  $|0_K\rangle|\pi\pi\rangle$  ( $|0_K\rangle|\tilde{\pi}\tilde{\pi}\rangle$ ). Moreover, we assume that all the interactions experienced by the kaon with other degrees of freedom have been included in the description above. In this case the composite system density operator  $\rho(\tau) = |\Psi(\tau)\rangle\langle\Psi(\tau)|$  for the initial  $|\Psi(0)\rangle$  can be assumed to remain pure in the course of dynamics. From Eq. (3),

$$\begin{aligned} |\Psi(\tau)\rangle &= \alpha e^{-\frac{1}{2}\Gamma_S\tau} e^{-im_S\tau} |K_S^0\rangle \otimes |0_\pi\rangle \\ &\quad + \beta e^{-\frac{1}{2}\Gamma_L\tau} e^{-im_L\tau} |K_L^0\rangle \otimes |0_\pi\rangle \\ &\quad + \alpha \sqrt{1 - e^{-\Gamma_S\tau}} |0_K\rangle \otimes |\pi\pi\rangle \\ &\quad + \beta \sqrt{1 - e^{-\Gamma_L\tau}} |0_K\rangle \otimes |\tilde{\pi}\tilde{\pi}\rangle. \end{aligned} \quad (4)$$

In the sequence, we focus our attention on kaons produced in strangeness eigenstates—say,  $|K^0\rangle$  states generated by strong reactions such as  $\pi^- p \rightarrow K^0 \Lambda$ . So we take  $\alpha = \frac{1}{\sqrt{2}} = \beta$ . Taking this into  $\rho(\tau)$ , we see that the reduced kaon state is

$$\begin{aligned} \rho_Q(\tau) = \text{Tr}_P[\rho(\tau)] &= \frac{1}{2} e^{-\Gamma_S\tau} |K_S^0\rangle\langle K_S^0| \\ &\quad + \frac{1}{2} e^{-\Gamma_L\tau} |K_L^0\rangle\langle K_L^0| \\ &\quad + \left(1 - \frac{e^{-\Gamma_S\tau}}{2} - \frac{e^{-\Gamma_L\tau}}{2}\right) |0_K\rangle\langle 0_K| \\ &\quad + \left\{ \frac{1}{2} e^{-\frac{1}{2}\Gamma_S\tau} e^{-\frac{1}{2}\Gamma_L\tau} e^{i\Delta m\tau} |K_S^0\rangle\langle K_L^0| + \text{H.c.} \right\}. \end{aligned} \quad (5)$$

This coincides with the kaon state evolution considered in Ref. [4] by Caban *et al.* [13]. In this work, the authors deduced the general form in Eq. (5) for the kaon's dynamics under the assumptions that (i) the kaon state evolution must be completely positive and probability preserving, and (ii) compatible with the Wigner–Weisskopf phenomenological prescription [14]. Therefore, these properties are also true for the reduced kaon state  $\rho_Q(\tau)$  in the present model (4). It is straightforward to check that the composite system evolution given by  $\rho(\tau)$  is also completely positive and probability preserving.

### III. RESULTS

We apply our model now to a quantitative analysis of complementarity in the  $K^0\text{-}\bar{K}^0$  system, including the duality between strangeness oscillations and lifetime information. But here we investigate the phenomenon in light of the new feature presented by the model's bipartite character: entanglement. Our goal is to examine its role on complementarity in the context of neutral kaon interferometry.

We can restrict our analysis to focus only on the proper time interval  $0 \leq \tau \leq \tau_0$ , where  $\tau_0 = 4.79\tau_S$ . The reason is that it can be verified in experiment [1] that neutral kaons decaying after  $\tau_0$  can be regarded as  $K_0^L$  kaons with negligible error probability. In other words: at  $\tau = \tau_0$  one can already consider to have complete width information on the kaon. Therefore, we assume in the sequence that  $\tau$  ranges from 0 to  $\tau_0$ .

For pure composite-system states, the degree of mixedness of a reduced party state both qualifies and quantifies entanglement. Here we will use the von Neumann entropy  $\mathcal{S}$  of the reduced pionic subsystem state. It can be readily evaluated from

$$\begin{aligned} \rho_P(\tau) = \text{Tr}_Q[\rho(\tau)] &= \frac{1 - e^{-\Gamma_S\tau}}{2} |\pi\pi\rangle\langle\pi\pi| \\ &\quad + \frac{1 - e^{-\Gamma_L\tau}}{2} |\tilde{\pi}\tilde{\pi}\rangle\langle\tilde{\pi}\tilde{\pi}| + \left( \frac{e^{-\Gamma_S\tau}}{2} + \frac{e^{-\Gamma_L\tau}}{2} \right) |0_\pi\rangle\langle 0_\pi| \\ &\quad + \left\{ \frac{1}{2} \sqrt{1 - e^{-\Gamma_S\tau}} \sqrt{1 - e^{-\Gamma_L\tau}} |\pi\pi\rangle\langle\tilde{\pi}\tilde{\pi}| + \text{H.c.} \right\}, \end{aligned} \quad (6)$$

whose eigenvalues are  $\{0, x, 1 - x\}$  for

$$x(\tau) \equiv \frac{e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau}}{2},$$

$0 \leq x \leq 1 \forall \tau$ . Direct numerical analysis reveals that

$$\mathcal{S} = x \ln x + (1 - x) \ln(1 - x)$$

is monotone increasing in  $[0, \tau_0]$  (see Fig. 1).

As entanglement correlations are dynamically generated, information about the kaon's  $\{|K_S^0\rangle, |K_L^0\rangle\}$  component leaks to the pionic subsystem. The natural quantifier of the amount of lifetime information which thus becomes available to be retrieved (through the pionic state) is the distinguishability

$$\mathcal{D}(\tau) = \frac{1}{2} \|\rho_P^{(S)}(\tau) - \rho_P^{(L)}(\tau)\|. \quad (7)$$

It is given by the trace distance between the pionic subsystem states  $\rho_P^{(S)}(\tau) = 2\langle K_S^0|\rho(\tau)|K_S^0\rangle = e^{-\tau\Gamma_S}|0_\pi\rangle\langle 0_\pi|$  and

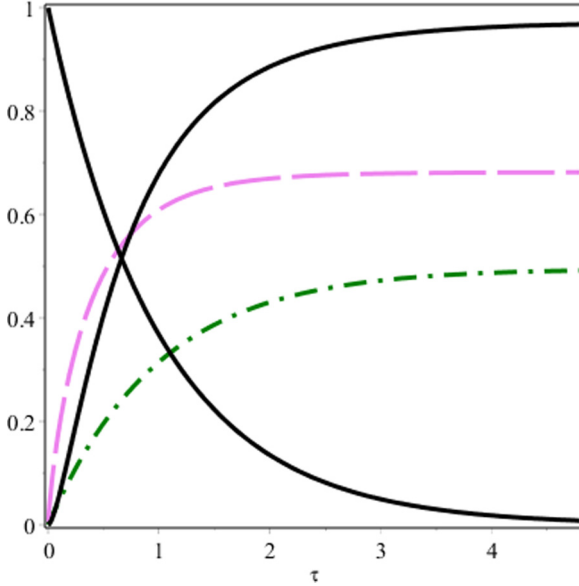


FIG. 1. The solid black lines show  $\mathcal{D}^2 + \mathcal{S}^2$  increasing towards 1 while  $\mathcal{V}^2$  decreases correspondingly, such that  $\mathcal{V}^2 \leq 1 - \mathcal{D}^2 - \mathcal{S}^2$ . Proper time ranges from  $\tau = 0$  to  $\tau = 4.79\tau_S$ . The green dot-dashed line shows  $\mathcal{D}$ . The purple dashed line shows  $\mathcal{S}$ .

$\rho_P^{(L)}(\tau) = 2\langle K_L^0 | \rho(\tau) | K_L^0 \rangle = e^{-\tau\Gamma_L} |0_\pi\rangle \langle 0_\pi|$  corresponding to the distinct kaon propagation modes  $K_S, K_L$ . Due to the generation of entanglement, we expect  $\mathcal{D}(\tau)$  to also increase monotonically in  $[0, \tau_0]$ . In fact, we found that  $\mathcal{D}(\tau) = \frac{1}{2}|e^{-\tau\Gamma_S} - e^{-\tau\Gamma_L}|$  is an increasing function of  $\tau$  in this interval. The numerical results are summarized in Fig. 1.

The quantity naturally complementary to  $\mathcal{D}$  and playing the role of interferometric visibility here is the Uhlmann fidelity

$$\mathcal{V}(\tau) = \mathcal{F}(\rho_P^{(S)}(\tau), \rho_P^{(L)}(\tau)), \quad (8)$$

where  $\mathcal{F}(\hat{\rho}, \hat{\tau}) = \text{Tr}[(\sqrt{\hat{\rho}} \hat{\tau} \sqrt{\hat{\rho}})^{1/2}]$ . The fidelity is an ‘‘overlap’’ measure generalized to arbitrary mixed states  $\hat{\rho}, \hat{\tau}$ , therefore quantifying the visibility of quantum interferences between  $\rho_P^{(S)}(\tau), \rho_P^{(L)}(\tau)$ . Moreover, it is well known to be related to the trace distance by the information-theoretic inequality

$$\mathcal{F}(\hat{\rho}, \hat{\tau}) \leq \sqrt{1 - \mathcal{D}^2(\hat{\rho}, \hat{\tau})}. \quad (9)$$

We have  $\mathcal{V}(\tau) = e^{-\Gamma\tau}$ , where  $\Gamma \equiv \frac{1}{2}(\Gamma_S + \Gamma_L)$ .

As was pointed out by Jakob and Bergou in Ref. [7], complementarity in bipartite systems must relate the single-particle properties distinguishability and visibility to the amount of entanglement. Here we have

$$\mathcal{V}^2 + \mathcal{D}^2 = e^{-\tau(\Gamma_S + \Gamma_L)} + \frac{(e^{-\Gamma_S\tau} - e^{-\Gamma_L\tau})^2}{4} = x(\tau)^2,$$

in such a way that

$$\mathcal{V}^2 + \mathcal{D}^2 + \mathcal{S}^2 = x^2 + [x \ln(x) + (1-x) \ln(1-x)]^2 \leq 1.$$

Indeed, numerical analysis of the quantities  $\mathcal{S}, \mathcal{D}$ , and  $\mathcal{V}$  shows that the inequality

$$\mathcal{V}^2 + \mathcal{D}^2 + \mathcal{S}^2 \leq 1 \quad (10)$$

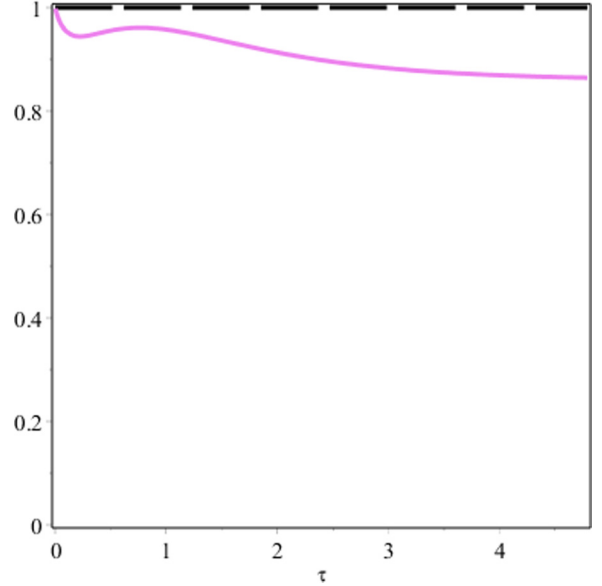


FIG. 2. The quantitative complementarity relation (10) for  $\tau \in [0, \tau_0]$ . The solid violet line shows  $\mathcal{V}^2 + \mathcal{D}^2 + \mathcal{S}^2$ . The upper bound 1 is shown dashed, in black.

holds within the relevant proper time interval  $[0, \tau_0]$  (see Fig. 2). As  $\mathcal{D}^2 + \mathcal{S}^2$  increases towards (nearly) 1 in  $[0, \tau_0]$ , the quantity  $\mathcal{V}$  must correspondingly decrease towards 0, therefore enforcing the visibility of the wave-like path interference phenomena to reduce.

### Strangeness oscillations

To see that Eq. (10) also accounts for the complementarity between strangeness oscillations and lifetime information in  $[0, \tau_0]$ , notice first that the visibility of  $K^0 \bar{K}^0$  oscillations must be defined here as the quantity  $\mathcal{V}_0(\tau)$  such that

$$2\langle \bar{K}^0 | \text{Tr}_P[\rho(\tau)] | \bar{K}^0 \rangle = F(\tau) \{1 - \mathcal{V}_0(\tau) \cos(\Delta m \tau)\}. \quad (11)$$

That is, by the oscillatory term in the probability that the initial  $|K^0\rangle$  is detected in the strangeness eigenstate  $|\bar{K}^0\rangle$  at the later time  $\tau = 0$ . Direct calculation gives

$$\mathcal{V}_0(\tau) = \frac{2e^{-\frac{1}{2}(\Gamma_S + \Gamma_L)\tau}}{e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau}}. \quad (12)$$

Next, a straightforward argument (see the appendix) shows that the ratio  $\frac{d\mathcal{V}_0}{d\tau} / \frac{d\mathcal{V}}{d\tau}$  between its derivative and that of the fidelity visibility (8) is positive in the time interval  $0 \leq \tau \leq \tau_0$ . The quantities  $\mathcal{V}, \mathcal{V}_0$  are then either both increasing or both decreasing in  $[0, \tau_0]$ . Therefore, we see from Eq. (9) that the increase of lifetime information as measured by  $\mathcal{D}(\tau)$  in fact enforces (not only  $\mathcal{V}$ , but also) the visibility  $\mathcal{V}_0$  of the strangeness oscillations to decrease in this interval.

## IV. CONCLUSIONS

Entanglement plays a crucial role in quantum-mechanical complementarity for bipartite systems. We have shown in the present work how it can be clearly illustrated and discussed in the kaon-antikaon oscillating system. We considered a

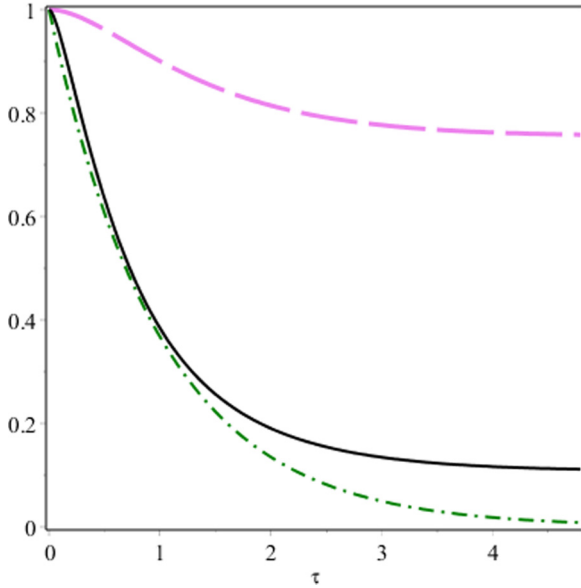


FIG. 3. Upper bounds for  $\mathcal{V}^2$  (the dot-dashed green line) given by  $1 - \mathcal{D}^2$  (dashed purple line) and by  $1 - \mathcal{D}^2 - \mathcal{S}^2$  (solid black line).

bipartite model where a single neutral kaon interacts with the environment consisting of its weak interaction decay products. From an interferometric point of view, the kaon is treated as the interfering object (quantum) and the lifetime or width information plays the role of which-way information. This is similar to the *neutral kaon interferometry* of Bramon, Garbarino, and Hiesmayr [3]. We verified that, as entanglement correlations are established between these two parties, lifetime information leaks and becomes available in the environmental state. Corresponding to the entanglement generation and acquisition of lifetime information, we saw how the visibility of which-way interference is reduced. The interplay between the single-particle properties visibility and distinguishability and entanglement was proved to be governed by a quantitative complementarity relation:

$$\mathcal{V}^2 + \mathcal{D}^2 + \mathcal{S}^2 \leq 1.$$

This inequality is similar to the one proposed by Jakob and Bergou in their analysis of wave-particle duality in bipartite systems [7].

In this direction, it is interesting to notice that the inclusion of the quantitative entanglement measure in this “trinity” relation is very important if we want to see the reduction of the interference visibility  $\mathcal{V}$  as enforced by quantum complementarity. In Fig. 3, we compare the upper bounds for  $\mathcal{V}^2$  given by  $1 - \mathcal{D}^2$  alone and by  $1 - \mathcal{D}^2 - \mathcal{S}^2$ . The upper bound including entanglement is much sharper and consistent with the actual reduction in  $\mathcal{V}$ .

We have also shown how our inequality accounts for the complementarity between strangeness oscillations and lifetime information in the time interval relevant for the analysis. This demonstrates consistency with the previous analysis of complementarity in the neutral kaon system, and with the general principle that the visibility of any quantum interference phenomenon whatsoever must reduce when which-way information becomes available [15–17].

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#### APPENDIX

Let us show that the ratio  $\frac{d\mathcal{V}_0}{d\tau} / \frac{d\mathcal{V}}{d\tau}$  between the derivatives of the visibility of strangeness oscillations [Eq. (12)] and the fidelity visibility [Eq. (8)] is positive in the time interval  $0 \leq \tau \leq \tau_0$ . Observe that this dimensionless ratio is given by

$$\frac{d\mathcal{V}_0}{d\tau} / \frac{d\mathcal{V}}{d\tau} = \frac{2}{e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau}} \left( 1 - \frac{\Gamma_S e^{-\Gamma_S\tau} + \Gamma_L e^{-\Gamma_L\tau}}{\Gamma(e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau})} \right) > 0 \quad \forall 0 \leq \tau \leq \tau_0.$$

Therefore, it is enough to show that

$$\frac{\Gamma_S e^{-\Gamma_S\tau} + \Gamma_L e^{-\Gamma_L\tau}}{\Gamma(e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau})} \leq 1 \quad \forall 0 \leq \tau \leq \tau_0.$$

To do this, notice that, since  $\Gamma_S = 579\Gamma_L$ , we have  $e^{-\Gamma_S\tau} \leq e^{-\Gamma_L\tau}$  for every  $0 \leq \tau \leq \tau_0$ . Thus, in the interval  $[0, \tau_0]$  we have

$$(\Gamma_S - \Gamma_L)e^{-\Gamma_S\tau} \leq (\Gamma_S - \Gamma_L)e^{-\Gamma_L\tau},$$

or

$$\frac{1}{2}(\Gamma_S e^{-\Gamma_S\tau} + \Gamma_L e^{-\Gamma_L\tau}) \leq \frac{1}{2}(\Gamma_L e^{-\Gamma_S\tau} + \Gamma_S e^{-\Gamma_L\tau}).$$

Adding  $\frac{1}{2}(\Gamma_S e^{-\Gamma_S\tau} + \Gamma_L e^{-\Gamma_L\tau})$  to both sides of the previous inequality gives

$$\Gamma_S e^{-\Gamma_S\tau} + \Gamma_L e^{-\Gamma_L\tau} \leq \Gamma(e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau}),$$

as desired.

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