Grassmann phase-space methods for fermions: Uncovering the classical probability structure

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The phase-space description of bosonic quantum systems has numerous applications in such fields as quantum optics, trapped ultracold atoms, and transport phenomena. Extension of this description to the case of fermionic systems leads to formal Grassmann phase-space quasiprobability distributions and master equations. The latter are usually considered as not possessing probabilistic interpretation and as not directly computationally accessible. Here, we describe how to construct c-number interpretations of Grassmann phase-space representations and their master equations. As a specific example, the Grassmann B representation is considered. We discuss how to introduce c-number probability distributions on Grassmann algebra and how to integrate them. A measure of size and proximity is defined for Grassmann numbers, and the Grassmann derivatives are introduced which are based on infinitesimal variations of function arguments. An example of c-number interpretation of formal Grassmann equations is presented.

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I. INTRODUCTION

The phase-space approach to quantum mechanics has proved to be invaluable tool in such fields as quantum optics and trapped ultracold atoms [1–4]. This approach allows one to calculate quantum observable properties as averages of classical quantities over certain quasiprobability distributions. At the same time, the full quantum evolution often takes the form of a simple Fokker-Planck equation for these quasiprobability distributions. The latter property turns phase-space techniques into a stochastic simulation tool which was used to conduct Monte Carlo calculations of a number of full many-body problems [1–4].

When extending phase-space techniques to the case of fermions, a fundamental limitation is faced due to anticommutation of fermionic canonical variables. Because of this, the corresponding canonical operators cannot have c-number eigenvalues except zero. As a consequence, it is impossible to construct c-number quasiprobability distributions for them.

There are several approaches to this problem, e.g., consider pairs of fermionic canonical operators [5–7]. However, the most formal, nonclassical, and elegant one is to change the notion of number [8]. Fermionic canonical operators can have nonzero eigenvalues if we consider these eigenvalues as anticommuting numbers, which are conventionally called Grassmann numbers (hereinafter, the term "Grassmann number" will be abbreviated as "g number"). This way it is possible to develop phase-space representations for fermions which bear remarkable analogy to bosonic ones [9]. In particular, there are Grassmann quasiprobability distributions of the same types: P, B, Q functions, s-ordered representations [9–11], and also Wigner functions [12]. Moreover, their master equations also look quite similar to the bosonic case. For example, in the case of real-time quantum dynamics with pairwise interactions, it is possible to derive a master equation which looks similar to the Fokker-Planck equation for positive-P distribution [10,11].

Nevertheless, there is an important difference: all fermionic quasiprobability distributions are g numbers. Grassmann numbers are dramatically different from c numbers: the latter are simple and the most basic things. However, the g number is not simple: it may carry the structure of a many-body correlated state. Every g number defines a hierarchy of n-point functions, just as the physical state defines a hierarchy of correlations.

Because of this complexity, Grassmann phase-space methods are usually considered as not possessing probabilistic interpretation and as not directly computationally accessible [6,7,13]. At the same time, there are published works in which c-number stochastic unravelings are constructed for Grassmann master equations [10,11]. These findings raise a number of questions. First, the possibility of stochastic unraveling means that the Grassmann representations are in fact equivalent to certain c-number quasiprobability distributions, with their own correspondence rules for observables, quantum states, and evolution equations. Second, in the work [11] there exists controversy with the earlier paper [10]. This means that the nature of this stochastic unraveling is not completely understood.

The goal of this work is to clarify these questions: to describe the classical phase-space representations which emerge in these stochastic unravelings and to find its physical interpretation. We also investigate how to apply the Grassmann phase-space representations in order to describe open fermionic systems which are coupled to reservoirs.

For the purpose of this work, we choose a particular g-number phase-space method, i.e., the Grassmann B representation [10,11,14], because currently it is the only method whose master equations were stochastically unraveled. This representation is the analog of the Drummond-Gardiner positive-P representation [15]. However, we believe that the techniques we describe can be applied to find classical probability interpretations of other g-number representations.

In Sec. II, we begin with a brief exposition of Grassmann B representation. We discuss its physical interpretation as a description of the quantum system's state as a hole excitation of a certain filled Fermi sea of states. A fermionic analog of optical equivalence theorem is derived. Next, we construct probability and stochastic calculus on Grassmann algebra. In order to accomplish this, in Sec. III we discuss such notions as the function of an arbitrary g number, the proximity and size of

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a g number, Grassmann derivatives based on an infinitesimal variation of the function argument, and probability distribution on Grassmann algebra and its integral. In Sec. IV, we discuss how to introduce c-number probability distribution into the Grassmann B representation. It turns out that there are two ways of doing it. Physically, the first way corresponds to a fixed hole excitation in a stochastically moving Fermi sea. In Sec. V, the second way is considered, which corresponds to the opposite description in terms of stochastic hole excitation on top of a fixed Fermi sea. It is shown that evolution of the emerging quasiprobability distributions is governed by the Fokker-Planck equation (for systems with pairwise interactions). The corresponding equations for the stochastic Fermi sea are found to coincide with that derived in [11]. A question of how to apply the Grassmann phase-space approach to fermionic open quantum systems coupled to a reservoir is considered in Sec. VI. We discuss the results of this work in Sec. VII.

II. GRASSMANN B REPRESENTATION

The close g-number analog of Drummond-Gardiner positive-P representation [15] is the Grassmann B representation, which was introduced in works [10,11]. In this section, we briefly review the main results about the B representation according to the literature and discuss its physical meaning.

A. Definition of representation and its physical content

Suppose we have a fermionic system with M modes (single-particle states). For each mode j, there are associated creation $\widehat{a}_j^\dagger$ and annihilation $\widehat{a}_j$ operators. The state of the quantum system is described by density operator $\widehat{\rho}$. Strictly speaking, a physical density operator should be number conserving and it can be written in the following form [9]:

$$\widehat{\rho} = \sum_{N\geqslant 0} \Psi_N(\widehat{\boldsymbol{a}}^\dagger)|0_\mathrm{p}\rangle\langle 0_\mathrm{p}|\Psi_N^\dagger(\widehat{\boldsymbol{a}}). \tag{1}$$

Here, $\widehat{\boldsymbol{a}}^\dagger = (\widehat{a}_1^\dagger,\dots,\widehat{a}_M^\dagger)$ is a vector of creation operators. The N-particle wave function $\Psi_N(\widehat{\boldsymbol{a}}^\dagger)$ is a homogeneous function of the order of N,

$$\Psi_N(\widehat{\boldsymbol{a}}^\dagger) = \sum_{i_1,\dots,i_N} \Psi(i_1,\dots,i_N)\widehat{a}_{i_1}^\dagger,\dots,\widehat{a}_{i_N}^\dagger. \tag{2}$$

However, in order to facilitate the description of superconductivity and superfluidity phenomena, a relaxed non-number-conserving form of the density operator is considered,

$$\widehat{\rho} = \sum_{l} \Psi_{\mathrm{p};l}(\widehat{\boldsymbol{a}}^\dagger)|0_\mathrm{p}\rangle\langle 0_\mathrm{p}|\Psi_{\mathrm{p};l}^\dagger(\widehat{\boldsymbol{a}}), \tag{3}$$

where now $\Psi_{\mathrm{p};l}(\widehat{\boldsymbol{a}}^\dagger)$ are either even or odd polynomials in $\widehat{a}_j^\dagger$. The polynomials of undefined parity are prohibited [9]. Hereinafter, we will deal only with density operators of this class, and we will refer to them as the "physical density operators." The subscript p in Eq. (3) means "related to particle picture."

A Bargmann coherent state is defined as [10,11]

$$|\boldsymbol{e}\rangle = \exp\left(-\sum_j e_j\widehat{a}_j^\dagger\right)|0_\mathrm{p}\rangle, \tag{4}$$

where e_j is a Grassmann number and $|0_\mathrm{p}\rangle$ is a particle vacuum (when each mode is empty). The required basic notions of Grassmann calculus and the explanation of notation we employ may be found in Appendix A. An overview of "practical" Grassmann calculus is also given in [9]. The utility of coherent states is based on the fact that annihilation and creation operators act upon them as

$$\widehat{a}_j|\boldsymbol{e}\rangle = e_j|\boldsymbol{e}\rangle,\ \ \widehat{a}_j^\dagger|\boldsymbol{e}\rangle = -\overrightarrow{\partial}_j|\boldsymbol{e}\rangle = |\boldsymbol{e}\rangle\overleftarrow{\partial}_j, \tag{5}$$

where $\overrightarrow{\partial}_j$ and $\overleftarrow{\partial}_j$ are the usual left and right Grassmann derivative operators with respect to element e_j [9].

Analogously to the bosonic Glauber-Sudarshan P representation [16,17], we can expand the density operator $\widehat{\rho}$ over the diagonal coherent state projections [10,11],

$$\widehat{\rho} = \int de_1^*\dots de_M^* de_M\dots de_1 \rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*)|\boldsymbol{e}\rangle\langle\boldsymbol{e}^*|. \tag{6}$$

Here, $\langle\boldsymbol{e}^*| = (|\boldsymbol{e}\rangle)^\dagger = \langle 0_\mathrm{p}|\exp(-\sum_j \widehat{a}_j e_j^*)$. The expansion coefficient $\rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*)$ is a g number and it is called the Grassmann B function of operator $\widehat{\rho}$. We will omit the word "Grassmann" and refer to it just as the "B function." Hereinafter, we will employ the following notation: for arbitrary (physical) operator $\widehat{o}$, its B function will be denoted as $o_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*)$. For compound operator expressions, e.g., $\widehat{a}\widehat{o}$, their B functions will be denoted as $\{\widehat{a}\widehat{o}\}_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*)$. From the results of [11], it follows that $\rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*)$ always exists, is an even g number for physical operators, and is unique. The subscript h means "related to hole picture," as we will now illustrate.

Let us discuss the physical content of the representation given by Eq. (6). There is a particle-hole duality in the description of fermion systems. This means that we can introduce a hole vacuum (when every mode is filled),

$$|0_\mathrm{h}\rangle = \widehat{a}_1^\dagger\dots\widehat{a}_M^\dagger|0_\mathrm{p}\rangle, \tag{7}$$

and with respect to this state, the density matrix assumes the form

$$\widehat{\rho} = \sum_l \Psi_{\mathrm{h};l}(\widehat{\boldsymbol{a}})|0_\mathrm{h}\rangle\langle 0_\mathrm{h}|\Psi_{\mathrm{h};l}^\dagger(\widehat{\boldsymbol{a}}^\dagger), \tag{8}$$

where again $\Psi_{\mathrm{h};l}(\widehat{\boldsymbol{a}})$ are either even or odd polynomials in $\widehat{a}_j$. The B function in Eq. (6) is related to the hole representation given by Eq. (8):

$$\rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*) = \sum_l \Psi_{\mathrm{h};l}(\boldsymbol{e})\Psi_{\mathrm{h};l}^\dagger(\boldsymbol{e}^*). \tag{9}$$

This can be proven by inserting the form (9) into the expansion given by Eq. (6) and evaluating the resulting Grassmann integrals. In other words, for a given filled sea of states $j = 1,\dots,M$, the B function is obtained by representing the density operator as a hole excitation and then by substituting each creation or annihilation operator with a corresponding g number.

This physical picture allows one to directly write B functions without calculations. For example, consider a system of free thermal fermions in a grand-canonical ensemble with chemical potential μ, inverse temperature β, and occupying M states with energies ε_k. Each mode is filled with probability

$$\langle n_j \rangle = \frac{1}{e^{\beta(\varepsilon_j-\mu)}+1}, \tag{10}$$

and a hole (represented by $e_j e_j^*$) is excited with probability $1-\langle n_j\rangle$. Therefore, we have

$$\rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*) = \prod_j [\langle n_j\rangle + (1-\langle n_j\rangle) e_j e_j^*] \tag{11}$$

for the normalized grand-canonical density operator.

We can also define the B function for the density operator in the particle picture. In order to accomplish this, we define a *hole* Bargmann coherent state,

$$|\boldsymbol{e}\rangle' = \exp\left(-\sum_j e_j \widehat{a}_j\right)|0_\mathrm{h}\rangle = \prod_{j=1}^{M}(\widehat{a}_j^\dagger - e_j)|0_\mathrm{p}\rangle, \tag{12}$$

which is the unnormalized displaced fully occupied state. It was introduced in [9] and has the properties

$$\widehat{a}_j^\dagger|\boldsymbol{e}\rangle' = e_j|\boldsymbol{e}\rangle',\ \ \widehat{a}_j|\boldsymbol{e}\rangle' = -\overrightarrow{\partial}_j|\boldsymbol{e}\rangle' = -|\boldsymbol{e}\rangle'\overleftarrow{\partial}_j. \tag{13}$$

We expand the density operator over the diagonal hole-coherent-state projections,

$$\widehat{\rho} = \int de_1^* \dots de_M^* de_M \dots de_1 \rho_\mathrm{p}(\boldsymbol{e},\boldsymbol{e}^*)|\boldsymbol{e}\rangle'\langle \boldsymbol{e}^*|'. \tag{14}$$

Here $\langle \boldsymbol{e}^*|' = (|\boldsymbol{e}\rangle')^\dagger$; the expansion coefficient $\rho_\mathrm{p}(\boldsymbol{e},\boldsymbol{e}^*)$ will also be called the B function. Again, by a direct calculation of Grassmann integrals, it can be shown that $\rho_\mathrm{p}(\boldsymbol{e},\boldsymbol{e}^*)$ corresponds to the density operator $\widehat{\rho}$ in the particle picture given by Eq. (3):

$$\rho_\mathrm{p}(\boldsymbol{e},\boldsymbol{e}^*) = \sum_l \Psi_{\mathrm{p};l}(\boldsymbol{e})\Psi_{\mathrm{p};l}^\dagger(\boldsymbol{e}^*). \tag{15}$$

We will call $\rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*)$ the "hole-B function" and $\rho_\mathrm{p}(\boldsymbol{e},\boldsymbol{e}^*)$ the "particle-B function." The particle-B function may be useful when we need to represent correlated physical states for which the particle picture is more natural. For example, for the Bardeen-Cooper-Schrieffer state

$$\widehat{\rho} = \exp\left(\sum_k \varphi_k \widehat{a}_k^\dagger \widehat{a}_{-k}^\dagger\right)|0_\mathrm{p}\rangle\langle 0_\mathrm{p}| \exp\left(\sum_k \varphi_k^* \widehat{a}_{-k}\widehat{a}_k\right), \tag{16}$$

we can directly write the particle-B function,

$$\rho_\mathrm{p}(\boldsymbol{e},\boldsymbol{e}^*) = \prod_k \exp(\varphi_k e_k e_{-k} + \varphi_k^* e_{-k}^* e_k^*), \tag{17}$$

where the g-number e_k corresponds to a mode with momentum k.

Now let us turn to the question of how to evaluate observables in the Grassmann B representation. This question was considered in [10,11] by a different approach. Here we will continue our physical interpretation. Let us consider an elementary projection observable $\widehat{o} = \widehat{b}_{i_1}^\dagger \dots \widehat{b}_{i_n}^\dagger |0_\mathrm{s}\rangle\langle 0_\mathrm{s}| \widehat{b}_{j_m}\dots\widehat{b}_{j_1}$. By subscript s we mean one of the representations, i.e., particle p or hole h. The operators $\widehat{b}_j$ are equal to the particle annihilation operators $\widehat{a}_j$ in the p picture, and to the hole annihilation operators $\widehat{a}_j^\dagger$ in the h picture. In order for $\widehat{o}$ to be physical, the parities of n and of m should coincide. The following relation for expected value of $\widehat{o}$,

$$\begin{aligned}\mathrm{Tr}[\widehat{b}_{i_1}^\dagger \dots \widehat{b}_{i_n}^\dagger |0_\mathrm{s}\rangle\langle 0_\mathrm{s}| \widehat{b}_{j_m}\dots\widehat{b}_{j_1}\widehat{\rho}] &= \int de_1^* \dots de_M^* de_M \dots de_1 \\ &\quad\times e_{j_m}\dots e_{j_1} e_{i_1}^* \dots e_{i_n}^* \rho_{\bar{\mathrm{s}}}(\boldsymbol{e},\boldsymbol{e}^*),\end{aligned} \tag{18}$$

can be proven by substituting the expansions of $\widehat{\rho}$ given by Eqs. (6) and (14) into the first line of this relation, and by evaluating the Grassmann integrals. Here the subscript $\bar{\mathrm{s}}$ means the picture which is reciprocal to s, i.e., $\bar{\mathrm{p}} = \mathrm{h}$ and $\bar{\mathrm{h}} = \mathrm{p}$. We can swap e_{j_k}'s and $e_{i_l}^*$'s in the last line to obtain

$$\begin{aligned}\mathrm{Tr}[\widehat{b}_{i_1}^\dagger \dots \widehat{b}_{i_n}^\dagger |0_\mathrm{s}\rangle\langle 0_\mathrm{s}| \widehat{b}_{j_m}\dots\widehat{b}_{j_1}\widehat{\rho}] &= \int de_1^* \dots de_M^* de_M \dots de_1 \\ &\quad\times (-e_{i_1}^*)\dots(-e_{i_n}^*) e_{j_m}\dots e_{j_1} \rho_{\bar{\mathrm{s}}}(\boldsymbol{e},\boldsymbol{e}^*).\end{aligned} \tag{19}$$

Then, for arbitrary observable we get:

$$\begin{aligned}\mathrm{Tr}[\widehat{o}\widehat{\rho}] &= \int de_1^* \dots de_M^* de_M \dots de_1 \\ &\quad\times o_\mathrm{s}(-\boldsymbol{e}^*,\boldsymbol{e})\rho_{\bar{\mathrm{s}}}(\boldsymbol{e},\boldsymbol{e}^*).\end{aligned} \tag{20}$$

In other words, in order to compute the quantum average, we (i) represent the observable and the density operator in reciprocal pictures, (ii) substitute all annihilation and creation operators with g numbers, and (iii) evaluate the Grassmann integral of their product. This prescription is a close fermionic analog of the bosonic optical equivalence theorem. Observe that in the relation (20), the operators $\widehat{o}$ and $\widehat{\rho}$ may be arbitrary physical operators.

Usually the observables are expressed not as projections, but as normally ordered polynomials, e.g., $\widehat{o} = \widehat{a}_i^\dagger \widehat{a}_j$. We can express normally ordered observables in terms of projections by writing

$$\widehat{o} = \widehat{a}_{i_1}^\dagger \dots \widehat{a}_{i_n}^\dagger \widehat{a}_{j_m}\dots \widehat{a}_{j_1} = \widehat{a}_{i_1}^\dagger \dots \widehat{a}_{i_n}^\dagger \widehat{I}_M \widehat{a}_{j_m}\dots \widehat{a}_{j_1}, \tag{21}$$

where $\widehat{I}_M$ is the identity operator in the space generated by modes $j = 1,\dots,M$. In the particle picture, this operator is defined as

$$\widehat{I}_M = \prod_j [|0_j\rangle\langle 0_j| + \widehat{a}_j^\dagger |0_j\rangle\langle 0_j|\widehat{a}_j], \tag{22}$$

where 0_j is the particle vacuum for mode j. The B function for $\widehat{I}_M$ is

$$\{\widehat{I}_M\}_\mathrm{p} = \exp\left(\sum_j e_j e_j^*\right). \tag{23}$$

Combining together Eqs. (21), (22), and (20), we find the formula for the expected value of normally ordered observable $:o(\widehat{\boldsymbol{a}}^\dagger,\widehat{\boldsymbol{a}}):$,

$$\begin{aligned}\mathrm{Tr}[:o(\widehat{\boldsymbol{a}}^\dagger,\widehat{\boldsymbol{a}}):\widehat{\rho}] &= \int de_1^* \dots de_M^* de_M \dots de_1 \\ &\quad\times \exp\left(\sum_j e_j e_j^*\right) o(-\boldsymbol{e}^*,\boldsymbol{e})\rho_\mathrm{h}(\boldsymbol{e},\boldsymbol{e}^*).\end{aligned} \tag{24}$$

Reiterating the same arguments for antinormally ordered observables $\{o(\widehat{\boldsymbol{a}},\widehat{\boldsymbol{a}}^{\dagger})\}$ and for $\widehat{I}_M$ in the hole picture, we find the result

$$\mathrm{Tr}[\{o(\widehat{\boldsymbol{a}},\widehat{\boldsymbol{a}}^{\dagger})\}\widehat{\rho}] = \int de_1^* \dots de_M^* de_M \dots de_1 \times \exp\left(\sum_j e_j e_j^*\right) o(-\boldsymbol{e}^*,\boldsymbol{e})\rho_{\mathrm{p}}(\boldsymbol{e},\boldsymbol{e}^*). \tag{25}$$

Analogously to the bosonic Drummond-Gardiner positive-P representation [15], in the following it will be convenient to treat $\boldsymbol{e}$ and $\boldsymbol{e}^*$ as completely independent nonconjugate complex g numbers [11]. Therefore, we double the dimension of our Grassmann algebra by introducing additional basis elements e'_j, $j = 1,\dots,M$. Then, in all the relations starting from Eq. (6), we replace all the occurrences of $\boldsymbol{e}^*$ by $\boldsymbol{e}'^*$, and all the occurrences of e_j^* by $e_j'^*$. This operation does not change the validity and scope of the presented results.

B. Equations of motion

In this section, we review how the master equations for B functions are derived, starting from the equations of motion for the density operator.

Let us consider a quantum system with Hamiltonian

$$\widehat{H} = \widehat{a}_p^{\dagger} T_{pq} \widehat{a}_q - \tfrac{1}{4} \widehat{a}_p^{\dagger} \widehat{a}_q^{\dagger} V_{pqrs} \widehat{a}_r \widehat{a}_s. \tag{26}$$

From now on, summation over repeated indices is implied. Real-time evolution of the density operator is governed by the von Neumann equation,

$$i\partial_t \widehat{\rho} = [\widehat{H},\widehat{\rho}]. \tag{27}$$

We can use the properties of coherent states (5) in order to find a master equation for the corresponding B representation. In particular, by Grassmann integration by parts, it can be shown that [11]

$$\{\widehat{a}_j \widehat{\rho}\}_{\mathrm{h}} = e_j \rho_{\mathrm{h}},\ \{\widehat{a}_j^{\dagger} \widehat{\rho}\}_{\mathrm{h}} = \overrightarrow{\partial}_j \rho_{\mathrm{h}}, \tag{28}$$

$$\{\widehat{\rho}\widehat{a}_j\}_{\mathrm{h}} = \rho_{\mathrm{h}} \overleftarrow{\partial}'_{j^*},\ \{\widehat{\rho}\widehat{a}_j^{\dagger}\}_{\mathrm{h}} = \rho_{\mathrm{h}} e_j'^*. \tag{29}$$

Here, $\overleftarrow{\partial}'_{j^*}$ is the right Grassmann derivative with respect to element $e_j'^*$; we omit the arguments $(\boldsymbol{e},\boldsymbol{e}'^*)$ of B function ρ_{h}. We can apply these rules for the von Neumann equation (27) and find [11]

$$\partial_t \rho_{\mathrm{h}} = -\overrightarrow{\partial}_p (i T_{pq} e_q)\rho_{\mathrm{h}} - \rho_{\mathrm{h}}[\overrightarrow{\partial}'_p (i T_{pq} e'_q)]^* + \frac{1}{2}\overrightarrow{\partial}_p \overrightarrow{\partial}_q \left(\frac{i}{2} V_{pqrs} e_r e_s\right)\rho_{\mathrm{h}} + \frac{1}{2}\rho_{\mathrm{h}}\left[\overrightarrow{\partial}'_p \overrightarrow{\partial}'_q \left(\frac{i}{2} V_{pqrs} e'_r e'_s\right)\right]^*. \tag{30}$$

Now, if we compare this equation with the Fokker-Planck equation for a classical c-number probability distribution P expressed in terms of complex variables [18],

$$\partial_t P = -\partial_p A_p P - \partial_p^* A_p^* P + \tfrac{1}{2}\partial_p \partial_q B_{pl} B_{ql} P + \partial_p \partial_q^* B_{pl} B_{ql}^* P + \tfrac{1}{2}\partial_p^* \partial_q^* B_{pl}^* B_{ql}^* P, \tag{31}$$

we observe that B-function master equation (30) looks like the anticommuting analog of Fokker-Planck equation (31). This analogy encourages us to find a certain Grassmann analog of stochastic process which has $\rho_{\mathrm{p}}(\boldsymbol{e},\boldsymbol{e}'^*;t)$ as its "probability" density. It fact, it has been done in [10,11], but without considering the emerging classical probability distributions. In the following sections, we will do it by explicitly introducing a c-number probability distribution into the B representation (14).

III. GRASSMANN CALCULUS REVISITED

We want to construct a classical stochastic interpretation of the B-function master equation (30). Before we do so, we need to carry out some preparatory work. The classical stochastic process is defined through infinitesimal increments of the process variables. The appearance of the term "infinitesimal" means we need to discuss how to introduce the norm of an arbitrary g-number g. Moreover, the behavior under infinitesimal variations is described in terms of derivatives. However, the conventional Grassmann derivative operators $\overrightarrow{\partial}_j$ and $\overleftarrow{\partial}_j$ are defined as formal algebraic manipulations on the basis elements e_j. Therefore, we need to find another notion of Grassmann derivatives which is connected with infinitesimal variations. Next, in order to introduce probability distributions on Grassmann algebra, we need to discuss the notion of function of arbitrary g number and how to integrate it.

A. Norm of Grassmann number

In [[11], p. 49], it is argued that g numbers do not have notions of size and magnitude, and thus there is no notion of proximity for them. Nevertheless, we believe that such a concept can be defined consistently.

Since "analytic" Grassmann numbers are defined according to Eq. (A2) of Appendix A, we see that each g number is equivalent to a hierarchy of n-point functions $G(i_1,\dots,i_n)$. Physically, we can interpret the g number as a quantum many-body state, and the functions $G(i_1,\dots,i_n)$ can be interpreted as its n-particle amplitudes. Due to anticommutation between the basis elements, $G(i_1,\dots,i_n)$ are not unique: we can represent the n-point function as a sum,

$$G(i_1,\dots,i_n) = G_A(i_1,\dots,i_n) + Z(i_1,\dots,i_n), \tag{32}$$

where $G_A(i_1,\dots,i_n)$ is completely antisymmetric, and $Z(i_1,\dots,i_n)$ is arbitrary but which has the symmetry of any Young tableau except complete antisymmetry. In the following, we always choose $Z(i_1,\dots,i_n) = 0$. We can introduce the norm of the g number as the sum of norms of its n-particle amplitudes,

$$\|g\|^2 \equiv |G_A(0)|^2 + \|G_A(1)\|^2 + \dots + \|G_A(M)\|^2, \tag{33}$$

where $\|G_A(i)\|^2$ is a certain (yet to be defined) n-point-function norm. Then, the distance between two g-numbers g and h is defined as $\|g-h\|^2$. Such a definition is appealing from the physical point of view, since the two quantum states should be regarded as similar if all their n-point functions (correlations) are similar. If we choose the n-point-function norm as the

Hilbert-Schmidt norm,

$$\|G_A(n)\|^2 = \sum_{i_1<\dots<i_n} |G_A(i_1,\dots,i_n)|^2, \tag{34}$$

then our g-number norm satisfies all the expected and reasonable inequalities,

$$\|g+h\| \leqslant \|g\| + \|h\|, \quad \|gh\| \leqslant \|g\|\|h\|, \tag{35}$$

and if

$$\|g-h\| = 0, \quad \text{then} \quad g = h. \tag{36}$$

From a physical point of view, the norm (34) has the meaning of the (unnormalized) probability of observing any n-point configuration, and $\|g\|^2$ is its normalization factor.

B. Functions of Grassmann numbers

1. Algebraic functions

The major objects of our theory, i.e., the B function $\rho_{\rm h}(\boldsymbol{e},\boldsymbol{e}'^*)$, a coherent state dyadic $|\boldsymbol{e}\rangle\langle\boldsymbol{e}'^*|$, and master equation (30), are formulated as depending on the basis elements e_j and e'^*_j. This means that in a stochastic interpretation, e_j and e'^*_j should be replaced with stochastic process variables g_j and g'^*_j, which in turn should be considered as arbitrary g numbers with random n-point functions $G(i_1,\dots,i_n)$. Therefore, we need to consider functions of arbitrary Grassmann numbers, e.g., $|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|$. The general analytic function f of arbitrary g-numbers g_j is a sum of monomials,

$$(g^+_{i_1})^{p_1},\dots,(g^+_{i_n})^{p_n} g^-_{j_1},\dots,g^-_{j_m}, \tag{37}$$

where p_k are non-negative integer powers since, in general, $(g^+_k)^2 \neq 0$; however, the indices $j_1,\dots,j_m$ should all be different since $(g^-_k)^2 = 0$. We call such functions algebraic since they can be expressed in terms of algebraic operations: multiplication, addition, and taking even and odd parts.

2. Nonalgebraic functions

In order to define the c-number stochastic process, we also need to introduce classical probabilities on Grassmann numbers. Apparently they cannot be expressed as a sum or series of monomials (37). This is because classical probability should take real positive values for all g, whereas algebraic functions of the form (37) will, in general, take g-number values. However, such nonalgebraic functions of Grassmann numbers as classical probabilities naturally depend on n-point functions. For a given g-number g, we will denote its n-point function by the corresponding capital letter, $G(\boldsymbol{i}_n)$, where $\boldsymbol{i}_n = (i_1,\dots,i_n)$; the set of all $G(\boldsymbol{i}_n)$ of a given order n, for all values of $\boldsymbol{i}_n$, will be denoted by $G(n)$; and the full hierarchy $[G(0),\dots,G(M)]$ will be designated by G. Therefore, a classical probability P depending on g will be denoted as $P(G_A)$. Observe that we take the antisymmetric part of G.

C. Metric Grassmann derivatives

Now we have the notion of proximity and magnitude. We can introduce the novel Grassmann derivatives which are based on infinitesimal variations of arguments. In order to distinguish them from the ordinary formal Grassmann derivatives, we call them "metric Grassmann derivatives." According to standard calculus, the derivative of algebraic function f is defined through its local behavior,

$$f(g+\delta) - f(g) = \sum_{\boldsymbol{i}_n} \Delta_A(\boldsymbol{i}_n)\partial_{G_A(\boldsymbol{i}_n)} f(g) + O(\|\delta\|^2), \tag{38}$$

where $\Delta_A(\boldsymbol{i}_n)$ is the antisymmetric part of the n-point function $\Delta(\boldsymbol{i}_n)$ of δ; and $\partial_{G_A(\boldsymbol{i}_n)}$ is the usual c-number derivative with respect to n-point function $G_A(\boldsymbol{i}_n)$. This definition is precise. However, it is insufficient since it ignores the algebraic structure and the commutation properties of δ. This is because we can write

$$\delta = \delta^+ + \delta^-, \tag{39}$$

and substitute it into $f(g+\delta)$. Since $f(g+\delta^+ + \delta^-)$ is a polynomial (or a series), we expand it and move all δ^+ and δ^- to the left (or to the right) respecting their commutation properties. Keeping only the first-order terms in δ^+ and δ^-, we arrive at the following representations of local behavior:

$$f(g+\delta) - f(g) = \delta^+ \overrightarrow{\partial}^{+}_{g} f(g) + \delta^- \overrightarrow{\partial}^{-}_{g} f(g) + O(\|\delta\|^2) \tag{40}$$

and

$$f(g+\delta) - f(g) = f(g)\overleftarrow{\partial}^{+}_{g}\delta^+ + f(g)\overleftarrow{\partial}^{-}_{g}\delta^- + O(\|\delta\|^2), \tag{41}$$

where we have introduced the novel odd-left $\overrightarrow{\partial}^{-}_{g}$, odd-right $\overleftarrow{\partial}^{-}_{g}$, even-left $\overrightarrow{\partial}^{+}_{g}$, and even-right $\overleftarrow{\partial}^{+}_{g}$ "metric Grassmann derivatives." We define these derivatives operationally as coefficients of monomials $\delta^\pm$ after all δ^+ and δ^- where commuted to the left (right). From this definition, we deduce their properties. The even-left metric Grassmann derivative $\overrightarrow{\partial}^{+}_{g_i}$ has the following properties:

$$\overrightarrow{\partial}^{+}_{g_i} c = 0, \quad \overrightarrow{\partial}^{+}_{g_i} g^+_j = \delta_{ij}, \quad \overrightarrow{\partial}^{+}_{g_i} g^-_j = 0, \tag{42}$$

where c is a g-number constant. More complex objects are differentiated according to linearity,

$$\overrightarrow{\partial}^{+}_{g}\{c_1 f_1(g) + c_2 f_2(g)\} = c_1 \overrightarrow{\partial}^{+}_{g} f_1(g) + c_2 \overrightarrow{\partial}^{+}_{g} f_2(g), \tag{43}$$

and by employing the following commutation relation:

$$\overrightarrow{\partial}^{+}_{g} f = (\overrightarrow{\partial}^{+}_{g} f) + f\,\overrightarrow{\partial}^{+}_{g}. \tag{44}$$

The odd-left metric Grassmann derivative $\overrightarrow{\partial}^{-}_{g_i}$ has the following properties:

$$\overrightarrow{\partial}^{-}_{g_i} c = 0, \quad \overrightarrow{\partial}^{-}_{g_i} g^-_j = \delta_{ij}, \quad \overrightarrow{\partial}^{-}_{g_i} g^+_j = 0. \tag{45}$$

Compound objects are differentiated according to the antilinearity,

$$\overrightarrow{\partial}^{-}_{g}\{c_1 f_1(g) + c_2 f_2(g)\} = \overline{c}_1 \overrightarrow{\partial}^{-}_{g} f_1(g) + \overline{c}_2 \overrightarrow{\partial}^{-}_{g} f_2(g), \tag{46}$$

and the anticommutation relation

$$\overrightarrow{\partial}^{-}_{g} f = (\overrightarrow{\partial}^{-}_{g} f) + \overline{f}\,\overrightarrow{\partial}^{-}_{g}. \tag{47}$$

Here, for each Grassmann number g, we have introduced its involution $\overline{g}$ as the negation of its odd part:

$$\overline{g} = g^+ - g^-. \tag{48}$$

The properties of the right metric derivatives are obtained through complex conjugation, according to the following relations:

$$[\overrightarrow{\partial}_g^{\pm}]^* = \overleftarrow{\partial}_{g^*}^{\pm}, \quad [\overleftarrow{\partial}_g^{\pm}]^* = \overrightarrow{\partial}_{g^*}^{\pm}. \tag{49}$$

Different derivatives have the following commutation relations:

$$\overrightarrow{\partial}_{g_i}^{+}\overrightarrow{\partial}_{g_j}^{\pm} = \overrightarrow{\partial}_{g_j}^{\pm}\overrightarrow{\partial}_{g_i}^{+}, \quad \overrightarrow{\partial}_{g_i}^{-}\overrightarrow{\partial}_{g_j}^{\pm} = (\pm 1)\overrightarrow{\partial}_{g_j}^{\pm}\overrightarrow{\partial}_{g_i}^{-}. \tag{50}$$

Left and right derivatives are related as

$$\overrightarrow{\partial}_g^{+} f(g) = f(g)\overleftarrow{\partial}_g^{+}, \quad \overrightarrow{\partial}_g^{-} f(g) = -\overline{f}(g)\overleftarrow{\partial}_g^{-}. \tag{51}$$

We observe that the properties of $\overrightarrow{\partial}_g^{-}$ are similar to the conventional (formal) Grassmann derivatives, except that they act upon variable g instead of fixed basis element e_j. However, the even derivative $\overrightarrow{\partial}_g^{+}$ is a novel object. There is a relation between the ordinary c-number calculus derivatives and the metric Grassmann derivatives:

$$\begin{aligned}\sum_{\boldsymbol{i}_n} \Delta_A(\boldsymbol{i}_n)\partial_{G_A(\boldsymbol{i}_n)} f(g) &= \{\delta^+\overrightarrow{\partial}_g^{+} + \delta^-\overrightarrow{\partial}_g^{-}\} f(g) \\ &= f(g)\{\overleftarrow{\partial}_g^{+}\delta^+ + \overleftarrow{\partial}_g^{-}\delta^-\}.\end{aligned} \tag{52}$$

D. Integration over Grassmann algebra

In order to work with classical probability, we need to integrate it over Grassmann numbers. Therefore, we introduce integration in the space of n-point functions,

$$\int dG_A P(G_A) := \prod_{n=1}^{M}\prod_{\boldsymbol{i}_n}\int_{\mathbb{C}} dG_A(\boldsymbol{i}_n) dG_A^*(\boldsymbol{i}_n) P(G_A), \tag{53}$$

where $\prod_{\boldsymbol{i}_n}$ means the product over all the ordered sequences $i_1 < \cdots < i_n$; $P(G_A)$ is an arbitrary c-number function of n-point functions $G_A(\boldsymbol{i}_n)$. Using our definitions, it can be shown that there is the following integration-by-parts formula:

$$\begin{aligned}&\int dG_A f(g)\left\{\sum_{\boldsymbol{i}_n}\partial_{G_A(\boldsymbol{i}_n)} H(\boldsymbol{i}_n)\right\} P(G_A) \\ &= -\int dG_A P(G_A)\{h^+\overrightarrow{\partial}_g^{+} + h^-\overrightarrow{\partial}_g^{-}\} f(g) \\ &= -\int dG_A f(g)\{\overleftarrow{\partial}_g^{+}h^+ + \overleftarrow{\partial}_g^{-}h^-\} P(G_A),\end{aligned} \tag{54}$$

where $f(g)$ is arbitrary algebraic g-number function. From now on, we assume that n-point functions are always antisymmetric, and the subscript A will be omitted.

E. Chain rule

Suppose that we have algebraic g-number function $f(g)$ such that its expression contains basis elements e_i only indirectly through the variable g. Let us evaluate $\overrightarrow{\partial}_i f(g)$, where $\overrightarrow{\partial}_i$ is the conventional (formal) Grassmann derivative with respect to basis element e_i. The derivative $\overrightarrow{\partial}_i f(g)$ means that we go through all the occurrences of e_i sequentially and commute them to the left, one at a time. We can do it in two stages: first we commute to the left all the occurrences of g (one at a time), then we compute the derivative of g. Therefore, we obtain the following chain formulas:

$$\overrightarrow{\partial}_i f(g) = \{(\overrightarrow{\partial}_i g^+)\overrightarrow{\partial}_g^{+} + (\overrightarrow{\partial}_i g^-)\overrightarrow{\partial}_g^{-}\} f(g). \tag{55}$$

Arguing similarly, we get the chain rule for the right derivative:

$$f(g)\overleftarrow{\partial}_i = f(g)\{\overleftarrow{\partial}_g^{+}(g^+\overleftarrow{\partial}_i) + \overleftarrow{\partial}_g^{-}(g^-\overleftarrow{\partial}_i)\}. \tag{56}$$

IV. FIRST PROBABILISTIC INTERPRETATION: FIXED HOLE EXCITATION IN A STOCHASTIC FERMI SEA

Now we are ready to introduce the stochastic interpretation of the formal Grassmann B-function master equation (30). The idea is that we introduce random g-number vectors $\boldsymbol{g} = (g_1,\ldots,g_M)$ and $\boldsymbol{g}'$. However, the dependence on $\boldsymbol{g}$ and $\boldsymbol{g}'$ can be introduced in two ways. The first way is to consider the coherent state dyadic as a function of these vectors, $|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|$. The second way is to consider the B function as depending on these vectors, $\rho_{\mathrm{h}}(\boldsymbol{g},\boldsymbol{g}'^*)$.

In this section, we study the first way which may be called the "stochastic Fermi-sea" representation, as will become clear. The second way will be discussed in the next section.

A. Definition of representation

The coherent state dyadic as a function of g-number variables, $|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|$, has the following properties:

$$\widehat{a}_i^\dagger|\boldsymbol{g}\rangle = -\overrightarrow{\partial}_{g_i}^{\pm}|\boldsymbol{g}\rangle = (\mp 1)|\boldsymbol{g}\rangle\overleftarrow{\partial}_{g_i}^{\pm}, \tag{57}$$

$$\widehat{a}_i|\boldsymbol{g}\rangle = \widehat{a}_i(1 - g_p\widehat{a}_p^\dagger)|0\rangle = -\overline{g}_p\widehat{a}_i\widehat{a}_p^\dagger|0\rangle = -\overline{g}_i|0\rangle. \tag{58}$$

The last equation is problematic: its form is not suitable for construction of a phase-space representation. However, if $\boldsymbol{g}$ is odd, so that $g_p = g_p^-$, then we obtain

$$\widehat{a}_i|\boldsymbol{g}^-\rangle = g_i^-|0\rangle = g_i^-(1 - g_p^-\widehat{a}_p^\dagger)|0\rangle = g_i^-|\boldsymbol{g}^-\rangle. \tag{59}$$

We see that suitable differential correspondences are realized only when $\boldsymbol{g}$ belongs to the odd sector. Therefore, from now on, we impose this restriction on $\boldsymbol{g}$ and $\boldsymbol{g}'$. The conjugated relations are

$$\langle(\boldsymbol{g}^-)^*|\widehat{a}_i = -\langle(\boldsymbol{g}^-)^*|\overleftarrow{\partial}_{g_i^*}^{-} = \overrightarrow{\partial}_{g_i^*}^{-}\langle(\boldsymbol{g}^-)^*|, \tag{60}$$

$$\langle(\boldsymbol{g}^-)^*|\widehat{a}_i^\dagger = \langle(\boldsymbol{g}^-)^*|(g_i^-)^*. \tag{61}$$

At the time moment $t = 0$, the random vectors $\boldsymbol{g}$ and $\boldsymbol{g}'$ should coincide with the vectors of basis elements, $\boldsymbol{g} = \boldsymbol{e}$ and $\boldsymbol{g}' = \boldsymbol{e}'$. However, at later time, they begin to diffuse. We express this fact by inserting integration over the probability distribution into the Grassmann B representation (6):

$$\begin{aligned}\widehat{\rho}(t) = &\int_{\mathrm{odd}} d\boldsymbol{G}d\boldsymbol{G}'^* P(\boldsymbol{G},\boldsymbol{G}'^*;t) \\ &\times \int de_1'^*\ldots de_M'^* de_M\ldots de_1\rho_{\mathrm{h}}(\boldsymbol{e},\boldsymbol{e}'^*)|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|,\end{aligned} \tag{62}$$

with the initial condition

$$P(\boldsymbol{G},\boldsymbol{G}'^*;0) = \delta(\boldsymbol{G}-\boldsymbol{E})\delta(\boldsymbol{G}'^*-\boldsymbol{E}'^*). \tag{63}$$

Here, bold capital letters designate vectors $\boldsymbol{G} = (G_0,\ldots,G_M)$, $\boldsymbol{E} = (E_0,\ldots,E_M)$, etc.; the symbol G_j means the full hierarchy of n-point functions for g_j. In fact, our Grassmann representation is equivalent to ordinary c-number phase-space representation,

$$\widehat{\rho}(t) = \int_{\text{odd}} d\boldsymbol{G}d\boldsymbol{G}'^* P(\boldsymbol{G},\boldsymbol{G}'^*;t)\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*), \tag{64}$$

with the overcomplete operator basis

$$\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*) = \int de_1'^*\ldots de_M'^* de_M\ldots de_1 \rho_{\text{h}}(\boldsymbol{e},\boldsymbol{e}'^*)|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|. \tag{65}$$

Let us discuss the physical content of this phase-space picture. Observe that in the operator basis (65), the B-function $\rho_{\text{h}}(\boldsymbol{e},\boldsymbol{e}'^*)$ is a parameter: for different $\rho_{\text{h}}(\boldsymbol{e},\boldsymbol{e}'^*)$, we obtain different operator bases $\widehat{\Lambda}$ and different phase-space pictures (64). We can interpret the structure of $\widehat{\Lambda}$ as a "quantum kinematics." Analogously to the kinematics of classical mechanics, we (i) postulate the existence of space which is spanned by modes $j = 1,\ldots,M$, then (ii) select the reference frame in this space by the choice of filled sea of states $g_1,\ldots,g_M$, and, finally, (iii) define the state of the "body" by specifying the hole excitation field $\rho_{\text{h}}(\boldsymbol{e},\boldsymbol{e}'^*)$. Therefore, such a description corresponds to a fixed hole-excitation field in a moving background Fermi sea. We call this the "stochastic Fermi-sea representation."

B. Equations of motion

We symbolically denote the relation (64) as

$$P(\boldsymbol{G},\boldsymbol{G}'^*;t) = \{\widehat{\rho}(t)\}_{\text{sea}}(\boldsymbol{G},\boldsymbol{G}'^*). \tag{66}$$

In order to construct a master equation for the stochastic Fermi-sea representation, we proceed analogously to Sec. II B: we find expressions for $\{\widehat{a}_i^\dagger\widehat{a}_j\widehat{\rho}(t)\}_{\text{sea}}$, etc. Note that since integration-by-parts formula (54) contains only the combinations $h^-\overrightarrow{\partial}_g^-$ and $\overleftarrow{\partial}_g^- h^-$, there are no rules for nonconserving terms such as $\{\widehat{a}_j\widehat{\rho}(t)\}_{\text{sea}}$. Using the coherent state properties (57) and (59)–(61), we find

$$\widehat{a}_i^\dagger\widehat{a}_j|\boldsymbol{g}\rangle = g_j\overrightarrow{\partial}_{g_i}^-|\boldsymbol{g}\rangle. \tag{67}$$

Using this relation and its conjugated variant in the stochastic Grassmann B representation (62), and integrating by parts according to (54), we find

$$\{\widehat{a}_i^\dagger\widehat{a}_j\widehat{\rho}(t)\}_{\text{sea}} = -\sum_{\boldsymbol{i}_n}\partial_{G_i(\boldsymbol{i}_n)}G_j(\boldsymbol{i}_n)\{\widehat{\rho}(t)\}_{\text{sea}}, \tag{68}$$

$$\{\widehat{\rho}(t)\widehat{a}_i^\dagger\widehat{a}_j\}_{\text{sea}} = -\sum_{\boldsymbol{i}_n}\partial_{G_j'^*(\boldsymbol{i}_n)}G_i'^*(\boldsymbol{i}_n)\{\widehat{\rho}(t)\}_{\text{sea}}. \tag{69}$$

Representation for quartic terms such as $\{\widehat{a}_i^\dagger\widehat{a}_j^\dagger\widehat{a}_k\widehat{a}_l\widehat{\rho}(t)\}_{\text{sea}}$ can be found by repeated application of Eqs. (68) and (69), and by using the anticommutation relation

$$\overrightarrow{\partial}_{g_p}^- g_s = \delta_{ps} - g_s\overrightarrow{\partial}_{g_p}^-. \tag{70}$$

In the stochastic Fermi-sea representation, the von Neumann equation (27) assumes the form

$$\begin{aligned}\partial_t\{\widehat{\rho}(t)\}_{\text{sea}} = \Bigg(&\partial_{G_p(\boldsymbol{i}_n)}\left[iT_{pq}G_q(\boldsymbol{i}_n) - \frac{i}{4}V_{lpql}G_q(\boldsymbol{i}_n)\right] \\ &- \frac{i}{4}\partial_{G_p(\boldsymbol{i}_m)}G_r(\boldsymbol{i}_m)\partial_{G_q(\boldsymbol{i}_n)}G_s(\boldsymbol{i}_n)V_{pqrs} \\ &+ \Bigg\{\partial_{G_p'(\boldsymbol{i}_n)}\left[iT_{pq}G_q'(\boldsymbol{i}_n) - \frac{i}{4}V_{lpql}G_q'(\boldsymbol{i}_n)\right] \\ &- \frac{i}{4}\partial_{G_p'(\boldsymbol{i}_m)}G_r'(\boldsymbol{i}_m)\partial_{G_q'(\boldsymbol{i}_n)}G_s'(\boldsymbol{i}_n)V_{pqrs}\Bigg\}^*\Bigg) \\ &\times\{\widehat{\rho}(t)\}_{\text{sea}}.\end{aligned} \tag{71}$$

We see that the evolution equation for the distribution $\{\widehat{\rho}(t)\}_{\text{sea}}$ has the form of the Fokker-Planck equation in Stratonovich form [19], except that it is lacking a number of complex conjugated terms of the form (see Appendix B of Ref. [18])

$$\partial_{G_p^*(\boldsymbol{i}_n)}\{\ldots\} + \partial_{G_p'(\boldsymbol{i}_n)}\{\ldots\}. \tag{72}$$

However, since the Grassmann coherent state dyadic is analytic,

$$\partial_{G_p^*(\boldsymbol{i}_n)}|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*| = 0,\quad \partial_{G_p'(\boldsymbol{i}_n)}|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*| = 0, \tag{73}$$

we can add the required terms to the right-hand side of Eq. (71) (see Appendix B of Ref. [18]). After performing this addition, we conclude that $\{\widehat{\rho}(t)\}_{\text{sea}}$ is a joint probability distribution for the stochastic process (in a Stratonovich sense),

$$\begin{aligned}dG_p(\boldsymbol{i}_n) = &-i\sum_q T_{pq}G_q(\boldsymbol{i}_n)dt + \frac{i}{4}\sum_{lq}V_{lpql}G_q(\boldsymbol{i}_n)dt \\ &+\sqrt{\frac{\omega_\gamma}{2i}}\sum_{\gamma q}O_{pq}^{(\gamma)}G_q(\boldsymbol{i}_n)dX_\gamma,\end{aligned} \tag{74}$$

$$\begin{aligned}dG_p'(\boldsymbol{i}_n) = &-i\sum_q T_{pq}G_q'(\boldsymbol{i}_n)dt + \frac{i}{4}\sum_{lq}V_{lpql}G_q'(\boldsymbol{i}_n)dt \\ &+\sqrt{\frac{\omega_\gamma}{2i}}\sum_{\gamma q}O_{pq}^{(\gamma)}G_q'(\boldsymbol{i}_n)dY_\gamma.\end{aligned} \tag{75}$$

Here we have decomposed the pair potential as [18,20,21]

$$V_{pqrs} = \sum_\gamma \omega_\gamma O_{pr}^{(\gamma)}O_{qs}^{(\gamma)}. \tag{76}$$

The real Wiener increments dX_γ and dY_γ obey the standard statistics,

$$E[dX_\gamma] = E[dY_\gamma] = E[dX_\gamma dY_\mu] = 0, \tag{77}$$

$$E[dX_\gamma dX_\mu] = E[dY_\gamma dY_\mu] = dt\delta_{\gamma\mu}. \tag{78}$$

We note that Eqs. (74) and (75) actually form a set of equations for each of the n-point functions $G_p(i_1,\ldots,i_n)$ and $G_p'(i_1,\ldots,i_n)$, which are uncoupled for different n and for different values of $i_1,\ldots,i_n$. We can multiply each equation for $G_p(i_1,\ldots,i_n)$ and $G_p'(i_1,\ldots,i_n)$ by $e_{i_1},\ldots,e_{i_n}$ and $e_{i_1}',\ldots,e_{i_n}'$ correspondingly, then sum them up over $i_1,\ldots,i_n$ and over n, and obtain a system of coupled stochastic equations for odd

Grassmann numbers $g_1,\ldots,g_M$ and $g'_1,\ldots,g'_M$:

$$dg_p = -i\sum_q T_{pq}g_q dt + \frac{i}{4}\sum_{lq} V_{lpql}g_q dt + \sum_\gamma \sqrt{\frac{\omega_\gamma}{2i}}\sum_q O^{(\gamma)}_{pq} g_q dX_\gamma, \quad (79)$$

$$dg'_p = -i\sum_q T_{pq}g'_q dt + \frac{i}{4}\sum_{lq} V_{lpql}g'_q dt + \sum_\gamma \sqrt{\frac{\omega_\gamma}{2i}}\sum_q O^{(\gamma)}_{pq} g'_q dY_\gamma. \quad (80)$$

In fact, these equations are the same as those obtained in [11], except for the notational difference for Hamiltonian terms (26) and the fact that our equations are in Stratonovich form, whereas equations in Ref. [11] are in Ito form. However, for numerical calculations, we always have to interpret these equations in the n-point picture [Eqs. (74) and (75)].

C. Simulation procedure

Here we outline the possible scheme of a numerical simulation. A specific form of $\rho_h(\boldsymbol{e},\boldsymbol{e}'^*)$ is selected depending on the problem. Then, the initial density operator is defined by $P(\boldsymbol{G},\boldsymbol{G}'^*;t=0)$. A Monte Carlo sampling of $\boldsymbol{g}(0)$ and $\boldsymbol{g}'^*(0)$ is performed from the initial distribution $P(\boldsymbol{G},\boldsymbol{G}'^*;t=0)$. For each realization of $\boldsymbol{g}(0)$ and $\boldsymbol{g}'^*(0)$, the stochastic equations (79) and (80) are solved. The expected values of observables are computed according to the formula

$$\langle : o(\widehat{\boldsymbol{a}}^\dagger,\widehat{\boldsymbol{a}}) :\rangle(t) = E\left\{\int de_1'^*\ldots de_M'^* de_M\ldots de_1 \times \exp\left(\sum_j e_j e_j'^*\right) o[-\boldsymbol{g}'^*(t),\boldsymbol{g}(t)]\rho_h(\boldsymbol{e},\boldsymbol{e}'^*)\right\}, \quad (81)$$

where for given $\rho_h(\boldsymbol{e},\boldsymbol{e}'^*)$ the Grassmann integral should be computed analytically.

V. SECOND PROBABILISTIC INTERPRETATION: STOCHASTIC HOLE EXCITATIONS OF A FIXED FERMI SEA

A. Definition of representation

In this section, we explore the second opportunity: when the B function depends on stochastic variables,

$$\widehat{\rho}(t) = \int_{\text{odd}} d\boldsymbol{G}d\boldsymbol{G}'^* P(\boldsymbol{G},\boldsymbol{G}'^*;t) \times \int de_1'^*\ldots de_M'^* de_M\ldots de_1 \rho_h(\boldsymbol{g},\boldsymbol{g}'^*)|\boldsymbol{e}\rangle\langle\boldsymbol{e}'^*|, \quad (82)$$

with the initial condition

$$P\big(\boldsymbol{G},\boldsymbol{G}'^*;0\big) = \delta(\boldsymbol{G}-\boldsymbol{E})\delta(\boldsymbol{G}'^*-\boldsymbol{E}'^*). \quad (83)$$

Our Grassmann representation is equivalent to ordinary c-number phase-space representation,

$$\widehat{\rho}(t) = \int_{\text{odd}} d\boldsymbol{G}d\boldsymbol{G}'^* P(\boldsymbol{G},\boldsymbol{G}'^*;t)\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*), \quad (84)$$

with the overcomplete operator basis

$$\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*) = \int de_1'^*\ldots de_M'^* de_M\ldots de_1 \rho_h(\boldsymbol{g},\boldsymbol{g}'^*)|\boldsymbol{e}\rangle\langle\boldsymbol{e}'^*|. \quad (85)$$

Physically this corresponds to a description when the background Fermi sea is fixed, and the hole-excitation field is stochastically evolving. Therefore, we call this the "stochastic hole-excitation field representation."

B. Equations of motion

We denote symbolically the relation (84) as

$$P(\boldsymbol{G},\boldsymbol{G}'^*;t) = \{\widehat{\rho}(t)\}_{\text{exc}}(\boldsymbol{G},\boldsymbol{G}'^*). \quad (86)$$

In Appendix B, it is shown that proceeding analogously to Sec. IV B, we obtain the following Stratonovich differential stochastic equations:

$$dg_k = i\sum_{pq}\left(T_{pq} - \frac{3}{4}V_{pllq}\right)e_q \overrightarrow{\partial}_p g_k dt + \sum_\gamma \sqrt{\frac{\omega_\gamma}{2i}}\sum_{ip} O^{(\gamma)}_{ip} e_p \overrightarrow{\partial}_i g_k dX_\gamma, \quad (87)$$

$$dg'_k = i\sum_{pq}\left(T_{pq} - \frac{3}{4}V_{pllq}\right)e'_q \overrightarrow{\partial}'_p g'_k dt + \sum_\gamma \sqrt{\frac{\omega_\gamma}{2i}}\sum_{ip} O^{(\gamma)}_{ip} e'_p \overrightarrow{\partial}'_i g'_k dY_\gamma. \quad (88)$$

C. Simulation procedure

A specific form of $\rho_h(\boldsymbol{g},\boldsymbol{g}'^*)$ is selected. Then, the initial density operator is defined by $P(\boldsymbol{G},\boldsymbol{G}'^*;t=0)$. A Monte Carlo sampling of $\boldsymbol{g}(0)$ and $\boldsymbol{g}'^*(0)$ is performed from the initial distribution $P(\boldsymbol{G},\boldsymbol{G}'^*;t=0)$. For each realization of $\boldsymbol{g}(0)$ and $\boldsymbol{g}'^*(0)$, the stochastic equations (87) and (88) are solved. The expected values of observables are computed according to the formula

$$\langle : o(\widehat{\boldsymbol{a}}^\dagger,\widehat{\boldsymbol{a}}) :\rangle(t) = E\left\{\int de_1'^*\ldots de_M'^* de_M\ldots de_1 \times \exp\left(\sum_j e_j e_j'^*\right) o(-\boldsymbol{e}'^*,\boldsymbol{e})\rho_h[\boldsymbol{g},(t)\boldsymbol{g}'^*(t)]\right\}. \quad (89)$$

For a given form of $\rho_h(\boldsymbol{g},\boldsymbol{g}'^*)$, the Grassmann integral in the last two lines should be evaluated analytically. An example of actual simulation of this kind can be found in work [10].

VI. MARKOVIAN DYNAMICS OF OPEN QUANTUM SYSTEM

In this section, we investigate how to employ the Grassmann B function in order to describe the dynamics of a system which is coupled to a reservoir.

Consider a fermionic degree of freedom which is coupled to a Markovian environment in the ground state. The master equation for the density operator is given by

$$\partial_t\widehat{\rho} = -i\omega\widehat{a}^\dagger\widehat{a}\widehat{\rho} + i\omega\widehat{\rho}\widehat{a}^\dagger\widehat{a} + \gamma\left(\widehat{a}\widehat{\rho}\widehat{a}^\dagger - \tfrac{1}{2}\widehat{a}^\dagger\widehat{a}\widehat{\rho} - \tfrac{1}{2}\widehat{\rho}\widehat{a}^\dagger\widehat{a}\right). \tag{90}$$

First of all, we observe that the term $\widehat{a}\widehat{\rho}\widehat{a}^\dagger$ is nonconserving. In Grassmann B representation, this term is represented as $e_1^* e_1'^* \rho_{\rm h}(e_1,e_1'^*)$, and we cannot transform it into a c-number linear operator for the classical probability $\{\widehat{\rho}(t)\}_{\rm sea}$ or $\{\widehat{\rho}(t)\}_{\rm exc}$. Here we propose the two routes to overcome this problem.

A. Mutually inverted vacuums

The term $\widehat{a}\widehat{\rho}\widehat{a}^\dagger$ is nonconserving because it annihilates particles in ket and in bra states. Now suppose that we invert the vacuum for the bra state and expand the density operator over the dyadics $|e_1\rangle\langle e_1'^*|'$,

$$\widehat{\rho} = \int de_1'^* de_1 \rho_{\text{h-p}}(e_1,e_1'^*)|e_1\rangle\langle e_1'^*|', \tag{91}$$

where the superscript h-p denotes the mixed hole-particle representation. In this picture, the term $\widehat{a}\widehat{\rho}\widehat{a}^\dagger$ annihilates a particle in ket and create a hole in bra; therefore, the total number of particles and holes is conserved in the bra and ket states, and we can represent it probabilistically. Indeed, using the properties of particle [Eqs. (5)] and hole [Eq. (13)] coherent states, we obtain the following rules:

$$\widehat{a}|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|'\widehat{a}^\dagger = -g\overrightarrow{\partial}'_*|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|', \tag{92}$$

$$\widehat{a}\widehat{a}^\dagger|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|' = -g\overrightarrow{\partial}|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|', \tag{93}$$

$$|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|'\widehat{a}\widehat{a}^\dagger = \overrightarrow{\partial}'_* g'^*|\boldsymbol{g}\rangle\langle\boldsymbol{g}'^*|'. \tag{94}$$

Employing these rules for the master equation (90), then introducing c-number probability as in the stochastic Fermi-sea approach in Sec. IV, we obtain

$$\begin{aligned}\partial_t\{\widehat{\rho}(t)\}_{\rm sea} = &-\left(-i\omega-\frac{\gamma}{2}\right)\partial_{G(\boldsymbol{i}_n)}G(\boldsymbol{i}_n)\{\widehat{\rho}(t)\}_{\rm sea}\\ &+\left(i\omega-\frac{\gamma}{2}\right)\partial_{G'^*(\boldsymbol{i}_n)}G'^*(\boldsymbol{i}_n)\{\widehat{\rho}(t)\}_{\rm sea}\\ &+\left(i\omega-\frac{\gamma}{2}\right)\{\widehat{\rho}(t)\}_{\rm sea} - \gamma\,\partial_{G'^*(\boldsymbol{i}_n)}G(\boldsymbol{i}_n)\{\widehat{\rho}(t)\}_{\rm sea}.\end{aligned} \tag{95}$$

In the derivation of this equation we have taken into account that $\rho_{\text{h-p}}(e_1,e_1'^*)$ is odd. In the last line, we see a potential term $(i\omega-\frac{\gamma}{2})\{\widehat{\rho}(t)\}_{\rm sea}$. We absorb it by making the B-function $\rho_{\text{h-p}}(e_1,e_1'^*)$ time dependent,

$$\rho_{\text{h-p}}(e_1,e_1'^*;t) = e^{(i\omega-\frac{\gamma}{2})t}\rho^{(0)}_{\text{h-p}}(e_1,e_1'^*). \tag{96}$$

We get the following Grassmann equations of motion:

$$\partial_t g_1 = -\left(i\omega+\frac{\gamma}{2}\right)g_1, \tag{97}$$

$$\partial_t g_1'^* = \left(-i\omega+\frac{\gamma}{2}\right)g_1'^* + \gamma g_1, \tag{98}$$

with the initial conditions $g_1(0)=e_1$, $g_1'^*(0)=e_1'^*$. Their solution is

$$g_1(t) = e_1 e^{-(i\omega+\frac{\gamma}{2})t}, \tag{99}$$

$$g_1'^*(t) = e_1'^* e^{(-i\omega+\frac{\gamma}{2})t} + 2e_1 e^{-i\omega t}\sinh\left(\frac{\gamma}{2}t\right). \tag{100}$$

Now suppose that initially the fermionic degree of freedom is in the excited state,

$$\rho^{(0)}_{\text{h-p}}(e_1,e_1'^*) = e_1'^*. \tag{101}$$

Let us compute the dynamics of the average population,

$$\begin{aligned}\langle\widehat{a}^\dagger\widehat{a}\rangle(t) &= \int de_1'^* de_1 e^{(i\omega-\frac{\gamma}{2})t}e_1'^*{\rm Tr}[\widehat{a}^\dagger\widehat{a}|g_1(t)\rangle\langle g_1'^*(t)|']\\ &= e^{-\gamma t},\end{aligned} \tag{102}$$

i.e., the population of excited state decays with rate γ as expected. It can also be checked that the normalization is conserved,

$$\langle 1\rangle(t) = \int de_1'^* de_1 e^{(i\omega-\frac{\gamma}{2})t}e_1'^*{\rm Tr}|g_1(t)\rangle\langle g_1'^*(t)|' = 1. \tag{103}$$

The approach presented in this section is a rather exotic trick. We present it here only to demonstrate the various possibilities offered by B representation.

In the next section, we present a second approach, which we believe is more "standard" and more suited to treat general problems.

B. Gaussian *B* function

We observe that the B function is actually a part of overcomplete basis $\widehat{\Lambda}$, given by Eqs. (65) and (85). Therefore, we can introduce parameters into it and use them to modify the phase-space correspondences for the master equation. For example, consider the stochastic Fermi-sea representation (62). Let us take the following ansatz:

$$\rho_{\rm h}(e_1,e_1'^*;\lambda) = e^{e_1\lambda e_1'^*}\rho^{(0)}_{\rm h}(e_1,e_1'^*), \tag{104}$$

where λ is the additional parameter of basis $\widehat{\Lambda}$. Then, the terms $\widehat{a}^\dagger\widehat{a}\widehat{\rho}$ and $\widehat{\rho}\widehat{a}^\dagger\widehat{a}$ are represented as earlier, given by Eqs. (68) and (69). In order to map the nonconserving term $\widehat{a}\widehat{\rho}\widehat{a}^\dagger$, we proceed as follows:

$$\widehat{a}\widehat{\Lambda}(G,G'^*;\lambda)\widehat{a}^\dagger = \int de_1'^* de_1 e^{e_1\lambda e_1'^*}\rho^{(0)}_{\rm h}(e_1,e_1'^*)g_1 g_1'^*|g_1\rangle\langle g_1'^*|'. \tag{105}$$

Suppose that $g_1 = Ge_1$ and $g_1'^* = G'^* e_1'^*$. Then, we obtain

$$e^{e_1\lambda e_1'^*}g_1 g_1'^* = GG'^* e_1 e_1'^* e^{e_1\lambda e_1'^*} = GG'^*\partial_\lambda e^{e_1\lambda e_1'^*}. \tag{106}$$

We finish Eq. (105),

$$\widehat{a}\widehat{\Lambda}(G,G'^*;\lambda)\widehat{a}^\dagger = GG'^*\partial_\lambda\widehat{\Lambda}(G,G'^*;\lambda), \tag{107}$$

and after integration by parts, we obtain the rule for the nonconserving term,

$$\{\widehat{a}\widehat{\rho}(t)\widehat{a}^{\dagger}\}_{\text{sea}} = -GG'^{*}\partial_{\lambda}\{\widehat{\rho}(t)\}_{\text{sea}}. \tag{108}$$

In the resulting Gaussian B-function representation,

$$\widehat{\rho}(t) = \int dGdG'^{*}\{\widehat{\rho}\}_{\text{sea}}(G,G'^{*};\lambda;t)\widehat{\Lambda}(G,G'^{*};\lambda), \tag{109}$$

the classical probability $\{\widehat{\rho}\}_{\text{sea}}$ depends on parameter λ. The master equation (90) is represented in this picture as

$$\partial_t\{\widehat{\rho}\}_{\text{sea}} = \left(i\omega+\frac{\gamma}{2}\right)\partial_G G\{\widehat{\rho}(t)\}_{\text{sea}} + \left(-i\omega+\frac{\gamma}{2}\right) \times \partial_{G'^{*}}G'^{*}\{\widehat{\rho}(t)\}_{\text{sea}} - \gamma GG'^{*}\partial_{\lambda}\{\widehat{\rho}(t)\}_{\text{sea}} \tag{110}$$

The corresponding g-number drift process is

$$\partial_t g_1 = \left(-i\omega-\frac{\gamma}{2}\right)g_1,\ \ \partial_t g_1'^{*} = \left(i\omega-\frac{\gamma}{2}\right)g_1'^{*}, \tag{111}$$

$$\partial_t\lambda = \gamma GG'^{*}, \tag{112}$$

with initial conditions $g_1(0) = e_1$, $g_1'^{*}(0) = e'^{*}$, and $\lambda(0) = 0$. The solution of these equations is

$$g_1 = e_1 e^{-i\omega t}e^{-\frac{\gamma}{2}t},\ \ g_1'^{*} = e_1'^{*}e^{i\omega t}e^{-\frac{\gamma}{2}t}, \tag{113}$$

$$\lambda = 1 - e^{-\gamma t}. \tag{114}$$

Now we evaluate the observables for the initial excited state for which $\rho_{\text{h}}^{(0)} = 1$ and we get the same correct results as in the previous section.

VII. DISCUSSION

The Grassmann numbers are objects of high computational complexity, but they are not as abstract as they are usually considered to be. We can interpret the g number as a physical many-body state, with a hierarchy of correlations. This leads to natural notions of size and proximity between them. With the help of these notions, we were able to develop a c-number stochastic calculus on Grassmann algebra.

The Grassmann B representation describes the state of the quantum system as a hole excitation of a certain Fermi sea. This hole excitation is represented by the B function, and the Fermi-sea states are generated by coherent state dyadics. The Grassmann B representation can be converted into c-number phase-space representation by introducing into it a probability distribution on Grassmann algebra. There are two possible ways of doing so. We can consider either that (a) the coherent state dyadics depends on random variables or (b) the B function depends on random variables. Physically this corresponds to the two possible pictures of how the state of a quantum system evolves: either (a) the fixed hole excitation on top of a randomly moving Fermi sea or (b) the dual picture where on a fixed Fermi sea a random hole excitation is acting.

In these stochastic representations, the B function becomes a part of the overcomplete operator basis. Therefore, we can choose a certain parametrized ansatz for the B function in order to make the phase-space representation more flexible. This idea is illustrated on the problem of the phase-space description of Markovian dynamics of fermionic degree of freedom which is coupled to a reservoir. We choose a Gaussian ansatz for the B function and this allows us to find a phase-space representation for the nonconserving Lindblad term. We believe this technique will be useful in the derivation and in c-number stochastic unraveling of master equations for open quantum systems in fermionic environments [22,23].

Other g-number phase-space representations can be written in the general form

$$\widehat{\rho} = \int de_1^{*}\dots de_M^{*}de_M\dots de_1\rho(\boldsymbol{e},\boldsymbol{e}^{*})\widehat{\Lambda}(\boldsymbol{e},\boldsymbol{e}^{*}), \tag{115}$$

where $\rho(\boldsymbol{e},\boldsymbol{e}^{*})$ is a g-number quasiprobability distribution, and $\widehat{\Lambda}(\boldsymbol{e},\boldsymbol{e}^{*})$ is a certain Grassmann operator basis. From this relation, it is seen that the technique described in this work can be applied for other Grassmann representations. There are also the two ways of inserting c-number probability: by making either $\widehat{\Lambda}(\boldsymbol{e},\boldsymbol{e}^{*})$ or $\rho(\boldsymbol{e},\boldsymbol{e}^{*})$ depending on random variables. However, the concrete physical interpretation will depend on specific details of $\widehat{\Lambda}(\boldsymbol{e},\boldsymbol{e}^{*})$. In this way, the other g-number methods can be made accessible to computations.

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APPENDIX A: BASIC GRASSMANN CALCULUS

We consider g-numbers e_j as linearly independent basis elements, with anticommuting multiplication law

$$e_i e_j = -e_j e_i, \tag{A1}$$

and which generate algebra of arbitrary g numbers

$$g = G(0) + \sum_i G(i)e_i + \sum_{i_1<i_2} G(i_1 i_2)e_{i_1}e_{i_2}\dots + \sum_{i_1<\dots<i_M} G(i_1,\dots,i_M)e_{i_1},\dots,e_{i_M}. \tag{A2}$$

Every Grassmann number g can be unambiguously decomposed into even and odd parts,

$$g = g^{+} + g^{-}, \tag{A3}$$

where even part g^{+} consists of even powers of e_i in the representation (A2), and odd part g^{-} consists of odd powers of e_i, respectively. We introduce the Grassmann complex conjugated basis elements e_j^{*}, which are defined to be linearly independent of e_j. The elements e_j^{*} generate arbitrary conjugated g-numbers g^{*}, whose general form is given by a conjugated variant of (A2). Although the most general g number contains both elements e_i and their conjugates e_j^{*}, we do not encounter such g numbers in our problem, and thus we assume that all g numbers contain either e_j or e_j^{*}. To put it another way, we are dealing only with "analytic" g numbers. We assume that what we call complex conjugation has the property that for any g-numbers α, β, γ, we have

$$(\alpha\beta\gamma)^{*} = \gamma^{*}\beta^{*}\alpha^{*}. \tag{A4}$$

Due to this rule, Grassmann complex conjugation can be interpreted in a way which is consistent with Hermitian conjugation; for example,

$$(\alpha\widehat{a}_j\gamma)^\dagger = \gamma^*\widehat{a}_j^\dagger\alpha^*. \quad \text{(A5)}$$

By $\overrightarrow{\partial}_j$ and $\overleftarrow{\partial}_j$, we denote the usual left and right Grassmann derivative operators with respect to basis element e_j. Left and right derivatives with respect to complex conjugate elements e_j^* are denoted as $\overrightarrow{\partial}_{j^*}$ and $\overleftarrow{\partial}_{j^*}$. In order to maintain consistency with the properties (A4) and (A5), the complex conjugation of derivatives is defined as

$$[\overrightarrow{\partial}_j]^* = \overleftarrow{\partial}_{j^*}, \quad [\overleftarrow{\partial}_j]^* = \overrightarrow{\partial}_{j^*}. \quad \text{(A6)}$$

For example,

$$(\alpha\overrightarrow{\partial}_j\beta\gamma)^* = \gamma^*\beta^*\overleftarrow{\partial}_{j^*}\alpha^* \quad \text{(A7)}$$

and

$$(\alpha\widehat{a}_j\gamma\overrightarrow{\partial}_j)^\dagger = \overleftarrow{\partial}_{j^*}\gamma^*\widehat{a}_j^\dagger\alpha^*. \quad \text{(A8)}$$

APPENDIX B: DERIVATION OF MASTER EQUATIONS FOR THE STOCHASTIC HOLE-EXCITATION REPRESENTATION

In order to find the master equation for $\{\widehat{\rho}(t)\}_{\text{exc}}$, let us see how the number-conserving product $\widehat{a}_i^\dagger\widehat{a}_j$ acts on $\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*)$ given by Eq. (85):

$$\widehat{a}_i^\dagger\widehat{a}_j\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*) = \int de_1'^*\dots de_M'^* de_M\dots de_1 \times \rho_{\text{h}}(\boldsymbol{g},\boldsymbol{g}'^*)e_j\partial_i|\boldsymbol{e}\rangle\langle\boldsymbol{e}'^*|. \quad \text{(B1)}$$

Next the Grassmann integration by parts is performed,

$$\widehat{a}_i^\dagger\widehat{a}_j\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*) = \int de_1'^*\dots de_M'^* de_M\dots de_1 \times (\delta_{ij} - e_j\overrightarrow{\partial}_i)\rho_{\text{h}}(\boldsymbol{g},\boldsymbol{g}'^*)|\boldsymbol{e}\rangle\langle\boldsymbol{e}'^*|. \quad \text{(B2)}$$

Now, since ρ_{h} depends on basis elements e_j only through the variables g_k, we apply the chain rule (55),

$$\begin{aligned} e_j\overrightarrow{\partial}_i\rho_{\text{h}}(\boldsymbol{g},\boldsymbol{g}'^*) &= e_j(\overrightarrow{\partial}_i g_k)\overrightarrow{\partial}_{g_k}\rho_{\text{h}}(\boldsymbol{g},\boldsymbol{g}'^*) \\ &= [\![e_j\overrightarrow{\partial}_i g_k]\!](\boldsymbol{i}_n)\partial_{G_k(\boldsymbol{i}_n)}\rho_{\text{h}}(\boldsymbol{g},\boldsymbol{g}'^*). \end{aligned} \quad \text{(B3)}$$

Here, by $[\![\cdot]\!](\boldsymbol{i}_n)$, we denote the n-point function of the g-number expression which is inside the brackets. Therefore, we have

$$\widehat{a}_i^\dagger\widehat{a}_j\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*) = (\delta_{ij} - [\![e_j\overrightarrow{\partial}_i g_k]\!](\boldsymbol{i}_n)\partial_{G_k(\boldsymbol{i}_n)})\widehat{\Lambda}(\boldsymbol{G},\boldsymbol{G}'^*). \quad \text{(B4)}$$

We substitute this relation into the stochastic hole-excitation representation (84), integrate by parts, and find the correspondence rule,

$$\{\widehat{a}_i^\dagger\widehat{a}_j\widehat{\rho}\}_{\text{exc}} = (\delta_{ij} + \partial_{G_k(\boldsymbol{i}_n)}[\![e_j\overrightarrow{\partial}_i g_k]\!](\boldsymbol{i}_n))\{\widehat{\rho}\}_{\text{exc}}. \quad \text{(B5)}$$

In the same way, we obtain the conjugated rule,

$$\{\widehat{\rho}\widehat{a}_i^\dagger\widehat{a}_j\}_{\text{exc}} = (\delta_{ij} + \partial_{G_k'^*(\boldsymbol{i}_n)}[\![e_i'^*\overrightarrow{\partial}'_{j^*}g_k'^*]\!](\boldsymbol{i}_n))\{\widehat{\rho}\}_{\text{exc}}. \quad \text{(B6)}$$

Using these rules, the von Neumann equation (27) assumes the following form:

$$\begin{aligned} \partial_t\{\widehat{\rho}(t)\}_{\text{exc}} &= \Bigg\{ -\partial_{G_k(\boldsymbol{i}_n)}i\left(T_{pq} - \frac{3}{4}V_{pllq}\right)[\![e_q\overrightarrow{\partial}_p g_k]\!](\boldsymbol{i}_n) \\ &\quad - \frac{i}{4}\partial_{G_p(\boldsymbol{i}'_m)}[\![e_k\overrightarrow{\partial}_i g_p]\!](\boldsymbol{i}'_m)\partial_{G_q(\boldsymbol{i}_n)}[\![e_l\overrightarrow{\partial}_j g_q]\!](\boldsymbol{i}_n)V_{ijkl} \\ &\quad + \Bigg[-\partial_{G_k'(\boldsymbol{i}_n)}i\left(T_{pq} - \frac{3}{4}V_{pllq}\right)[\![e_q'\overrightarrow{\partial}'_p g_k']\!](\boldsymbol{i}_n) \\ &\quad - \frac{i}{4}\partial_{G_p'(\boldsymbol{i}'_m)}[\![e_k'\overrightarrow{\partial}'_i g_p']\!](\boldsymbol{i}'_m)\partial_{G_q'(\boldsymbol{i}_n)} \\ &\quad \times [\![e_l'\overrightarrow{\partial}'_j g_q']\!](\boldsymbol{i}_n)V_{ijkl}\Bigg]^*\Bigg\}\{\widehat{\rho}(t)\}_{\text{exc}}. \end{aligned} \quad \text{(B7)}$$

In the same way as in Sec. IV B, using the analyticity of $\rho_{\text{h}}(\boldsymbol{g},\boldsymbol{g}'^*)$, we can interpret this master equation as corresponding to the following stochastic process in the Stratonovich form:

$$\begin{aligned} dG_k(\boldsymbol{i}_n) &= i\sum_{pq}\left(T_{pq} - \frac{3}{4}V_{pllq}\right)[\![e_q\overrightarrow{\partial}_p g_k]\!](\boldsymbol{i}_n)dt \\ &\quad + \sum_\gamma\sqrt{\frac{\omega_\gamma}{2i}}\sum_{ip}O_{ip}^{(\gamma)}[\![e_p\overrightarrow{\partial}_i g_k]\!](\boldsymbol{i}_n)dX_\gamma, \end{aligned} \quad \text{(B8)}$$

$$\begin{aligned} dG_k'(\boldsymbol{i}_n) &= i\sum_{pq}\left(T_{pq} - \frac{3}{4}V_{pllq}\right)[\![e_q'\overrightarrow{\partial}'_p g_k']\!](\boldsymbol{i}_n)dt \\ &\quad + \sum_\gamma\sqrt{\frac{\omega_\gamma}{2i}}\sum_{ip}O_{ip}^{(\gamma)}[\![e_p'\overrightarrow{\partial}'_i g_k']\!](\boldsymbol{i}_n)dY_\gamma. \end{aligned} \quad \text{(B9)}$$

Now, if we multiply these equations by $e_{i_1},\dots,e_{i_n}$ and $e'_{i_1},\dots,e'_{i_n}$ correspondingly, then sum them up over $i_1,\dots,i_n$ and over n, we obtain the stochastic equations for g-numbers g_k and g_k' given by Eqs. (87) and (88).

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