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Separability of a mixture of Dicke states

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The structural relation between multipartite entanglement and symmetry is one of the central mysteries of quantum mechanics. In this paper, we study the separability of quantum states in the bosonic system. We show that a mixture of multiqubit Dicke states is separable if and only if its partial transpose is positive semidefinite, which confirms the hypothesis of Wolfe and Yelin [E. Wolfe and S. F. Yelin, Phys. Rev. Lett. **112**, 140402 (2014)]. We generalize this result to a class of bosonic states in the $d \otimes d$ system; and for general d, we determine its separability is NP-hard although verifiable conditions for separability are easily derived when $d = 3,4$.

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Introduction. Quantum entanglement has been regarded as a resource of cryptography and metrology [1,2]. Therefore, it is a fundamental problem to qualitatively test whether a given state is entangled or not. In multipartite systems, a quantum state is called *fully separable*, not entangled, if it can be written as a statistical mixture of product states. Although it is known to be NP-hard of testing separability [3], a considerable number of different separability criteria have been discovered (see the references in [4,5]), including the famous positive partial transpose (PPT) criterion [6]. One widely used tool of detecting entanglement is the entanglement witness [7,8]. Another key concept for entanglement detection is symmetry. The k-symmetric extension provides a hierarchy of separability criteria [9–13], which converges exactly to the set of separable states when k goes to infinity.

Due to the essential role of symmetry played in entanglement theory, it becomes of great interest to study the relation between multipartite entanglement and symmetry, more precisely, the entanglement of bosonic systems. For N-qubit bosonic systems, a natural basis is the N-qubit Dicke state (unnormalized),

$$|D_{N,n}\rangle := \binom{N}{n} P_{\text{sym}}(|0\rangle^{\otimes n} \otimes |1\rangle^{\otimes N-n}),$$

with P_{sym} being the projection onto the bosonic (fully symmetric) subspace, i.e., $P_{\text{sym}} = \frac{1}{N!}\sum_{\pi \in S_N} U_\pi$, the sum extending over all permutation operators U_π of the N-qubit systems. Dicke states are particularly suitable for cold atomic systems, where the particle number is usually thousands. Considerable effort has been devoted to the study of entanglement of Dicke states, both theoretically [14–20] and experimentally [21–26]. The separability of bosonic states, especially the role of PPT in the separability of bosonic systems, has attracted a lot of attention. Eckert *et al.* proved that there is no PPT entanglement in three-qubit bosonic system [14]. PPT entanglement was found in five- and six-qubit bosonic systems in [27]. The existence of four-qubit bosonic PPT entanglement is demonstrated in Ref. [28]. Particularly, analytical criteria of the separability of the mixture of Dicke states have been pursued extensively [27,29–32]. For instance, in Ref. [30], Quesada *et al.* provided the analytical expression for the best separable approximation of a mixture of Dicke states by using the idea introduced by Lewenstein *et al.* in [31]. In Ref. [32], Wolfe and Yelin proposed the hypothesis that a mixture of Dicke states is separable if and only if it is PPT, according to their ideas on generating sufficient separability criteria numerically.

In this paper, we confirm the validity of the hypothesis that PPT indicates separability of a mixture of Dicke states by giving a two-step proof. At the first step, we show that a mixture of Dicke states is separable if and only if two Hankel matrices [33] generated by its eigenvalues is positive semidefinite. Second, by using the relation between Dicke states, we demonstrate that PPT of a mixture of Dicke states implies the positive semidefinite of the two Hankel matrices. These two parts directly lead us to the conclusion that PPT is a sufficient and necessary condition of the separability of a mixture of Dicke states. Notably, in order to ensure the positive semidefinite of the two Hankel matrices, we only need to take the partial transpose in the half cut of the subsystems. Our main tool in the proof of the first step is to give a *complete* characterization of the *general* entanglement witness of the class of Dicke states by studying the non-negative polynomials. This idea is generalized to prove that the separability of a mixture of bipartite high-dimensional Dicke states is NP-complete, although a very simple criterion is given when the local dimension is three or four.

Main results. In the N-qudit system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ with d being the dimension of each Hilbert space $\mathcal{H}_i$, the bosonic space is a subspace of pure states which are invariant under the swap of any two subsystems among all N subsystems, i.e., for a swap operator $F_{i,j}$ exchanging a two-qudit system,

$$S :\equiv \{|\psi\rangle : |\psi\rangle = F_{i,j}|\psi\rangle, \text{ for all } i,j\}.$$

In the $d = 2$ case, the symmetric space is spanned by Dicke states $|D_{N,0}\rangle, \ldots, |D_{N,N}\rangle$. A mixed state ρ is called bosonic if its support is a subspace of bosonic space where the support of ρ is the subspace spanned by the eigenvectors corresponding

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to its nonzero eigenvalues. In other words, $\rho = F_{i,j}\rho = \rho F_{i,j}$ holds for $1 \leqslant i,j \leqslant N$.

An N-qubit bosonic state ρ is called *fully* separable, if ρ can be written as $\rho = \sum_j p_j \otimes_{k=1}^N |\alpha_{j_k}\rangle\langle\alpha_{j_k}|$,

$$\otimes_{k=1}^N|\alpha_{j_k}\rangle \in S \Rightarrow \exists|\alpha_j\rangle, \quad \otimes_{k=1}^N|\alpha_{j_k}\rangle = |\alpha_j\rangle^{\otimes N}.$$

That is, $\rho = \sum_j p_j {\alpha_j}^{\otimes N}$ with $\alpha_j = |\alpha_j\rangle\langle\alpha_j|$.

Now, we study *general* entanglement witnesses for a *bosonic* system as a useful tool which was introduced in [27]. For the N-qudit system, a Hermitian W is called a general entanglement witness of the bosonic system if $W = P_S W P_S$ with P_S being the projection onto the bosonic space S and

$$\mathrm{tr}(W\alpha^{\otimes N}) \geqslant 0, \quad \text{for all } \alpha = |\alpha\rangle\langle\alpha|.$$

By invoking the hyperplane separation theorem, we make the followng proposition:

Proposition 1. A bosonic state ρ is separable if and only if $\mathrm{tr}(W\rho) \geqslant 0$ holds for any general entanglement witness W of the bosonic system.

Proof. The "only if" part follows directly. For the "if" part, assume there is some bosonic entangled state ρ such that $\mathrm{tr}(W\rho) \geqslant 0$ holds for any general entanglement witness W. Separable states of the bosonic system is convex and compact. By the hyperplane separation theorem, there exists H such that $\mathrm{tr}(H\rho) < 0$ and $\mathrm{tr}(H\alpha^{\otimes N}) \geqslant 0$ holds for any α. Then $W = P_S H P_S$ is a general entanglement witness. On the other hand, $\mathrm{tr}(W\rho) = \mathrm{tr}(P_S H P_S \rho) = \mathrm{tr}(H\rho) < 0$, which contradicts the assumption. ■

In the following, we mainly focus on the N-*qubit* bosonic system, that is, $d = 2$. Naturally, the set of general entanglement witnesses forms a convex cone, i.e., a positive combination of two general entanglement witnesses is still a general entanglement witness.

One way to test the separability of bosonic states is to parametrize the set of general entanglement witnesses, at least its boundary. Notice that any Hermitian $W = P_S W P_S$ corresponds to a Hermitian matrix $M := (m_{i,j})_{(N+1)\times(N+1)}$ as follows:

$$W := \sum_{i,j=0}^{N} m_{i,j}|\widetilde{D_{N,i}}\rangle\langle\widetilde{D_{N,j}}|,$$

where $|\widetilde{D_{N,n}}\rangle := \binom{N}{n}^{-1}|D_{N,n}\rangle$, i.e., $\langle D_{N,m}|\widetilde{D_{N,n}}\rangle = \delta_{m,n}$, the Dirac delta function.

The condition of W being a general entanglement witness $\mathrm{tr}(W\alpha^{\otimes N}) \geqslant 0$ holds for all one-qubit $|\alpha\rangle$. That is, for any $z \in \mathbb{C}$,

$$\mathrm{tr}(W|0\rangle\langle 0|^{\otimes N}) \geqslant 0 \Leftrightarrow m_{N,N} \geqslant 0, \tag{1}$$

$$\mathrm{tr}\{W[(|1\rangle + z|0\rangle)(\langle 1| + z^*\langle 0|)]^{\otimes N}\} \geqslant 0, \tag{2}$$

$$\Leftrightarrow \vec{z}^\dagger M \vec{z} \geqslant 0 \quad \text{with } \vec{z} = (1,z,z^2,\dots,z^N)^T \in \mathbb{C}^{N+1}. \tag{3}$$

The second condition indicates the first as $|z| \to \infty$. The last equivalence condition is derived from the fact that $(|1\rangle + z|0\rangle)^{\otimes N} = \sum_{j=0}^N z^j|D_{N,j}\rangle$.

By employing this condition, we provide a necessary and sufficient analytical condition for N-qubit separability of the mixture of Dicke states, which was called diagonal symmetric states in previous literature [27–30,32],

$$\rho = \sum_{n=0}^{N} \chi_n|D_{N,n}\rangle\langle D_{N,n}|.$$

Such ρ enjoys the property that for all diagonal qubit unitary $U_\theta = \mathrm{diag}\{1,e^{i\theta}\}$,

$$\rho = U_\theta^{\otimes N}\rho U_\theta^{\dagger\otimes N}.$$

Thus, for any general entanglement witness W, we construct a "diagonal" general entanglement witness W_0 with $\mathrm{tr}(W_0\rho) = \mathrm{tr}(W\rho)$ by

$$W_0 = \frac{1}{2\pi}\int_0^{2\pi} U_\theta^{\dagger\otimes N} W U_\theta^{\otimes N} d\theta = \sum_{k=0}^{N} m_{k,k}|\widetilde{D_{N,k}}\rangle\langle\widetilde{D_{N,k}}|.$$

$$\mathrm{tr}(W\rho) = \mathrm{tr}\big(WU_\theta^{\otimes N}\rho U_\theta^{\dagger\otimes N}\big) = \mathrm{tr}\big(U_\theta^{\dagger\otimes N} W U_\theta^{\otimes N}\rho\big) = \mathrm{tr}(W_0\rho).$$

Proposition 1 indicates the following:

Proposition 2. The N-qubit mixture of Dicke states ρ is separable if and only if $\mathrm{tr}(W_0\rho) \geqslant 0$ for any diagonal general entanglement witness W_0.

For any $W_0 = \sum_{k=0}^N m_{k,k}|\widetilde{D_{N,k}}\rangle\langle\widetilde{D_{N,k}}|$, we define its corresponding real coefficient polynomial

$$g(x) := \sum_{k=0}^{N} m_{k,k}x^k.$$

Then, invoking Eq. (1), W_0 is a diagonal general entanglement witness if and only if $\sum_{k=0}^N m_{k,k}|z|^{2k}$ is always non-negative for all $z \in \mathbb{C}$. That is,

$$g(r) \geqslant 0 \; \forall\, r \geqslant 0.$$

The characterization of such polynomials is accomplished by the following proposition:

Proposition 3. A real coefficient polynomial $g(x)$ satisfies that $g(r) \geqslant 0$ for all $r \geqslant 0$, if and only if there exists a real coefficient polynomial $P_i(x), Q_i(x)$ such that

$$g(x) = \sum_i xP_i^2(x) + \sum_i Q_i^2(x).$$

Proof. The "if" part is direct. For the "only if" part, we use the fundamental theorem of algebra,

$$g(x) = a_0\prod(x - z_k)^{l_k}.$$

For non-real root z_k, we know that for all real r,

$$(r - z_k)(r - \bar{z_k}) = [r - \mathrm{Re}(z_k)]^2 + \mathrm{Im}^2(z_k) \geqslant 0.$$

For nonpositive z_k, we know that for all $r \geqslant 0$,

$$r - z_k = r + (-z_k) \geqslant 0.$$

For positive z_k, its power l_k must be even according to $g(r) \geqslant 0$ for all $r \geqslant 0$. Then, expanding $g(x) = a_0\prod(x - z_k)^{l_k}$, we know that $g(x)$ has the wanted form. ■

Invoking the relation between the diagonal general entanglement witness W_0 and $g(x)$, one can deduce the following:

Proposition 4. Any diagonal general entanglement witness for an N-qubit mixture of Dicke states can be written as a convex combination of the following two types of general entanglement witnesses:

$$R = \sum_{0\leqslant i,j\leqslant \frac{N}{2}} a_i a_j |\widetilde{D_{N,i+j}}\rangle\langle\widetilde{D_{N,i+j}}|,$$

$$T = \sum_{0\leqslant i,j\leqslant \frac{N-1}{2}} b_i b_j |\widetilde{D_{N,i+j+1}}\rangle\langle\widetilde{D_{N,i+j+1}}|,$$

with $a_k,b_k \in \mathbb{R}$. Here we correspond R and T to $Q(x)^2$ and $xP(x)^2$, respectively, with $Q(x) = \sum_{k=0}^{\frac{N}{2}} a_i x^i$ and $P(x) = \sum_{k=0}^{\frac{N-1}{2}} b_i x^i$. Now we are ready to show our main result:

Theorem 1. The mixture of Dicke states $\rho = \sum_{n=0}^{N} \chi_n |D_{N,n}\rangle\langle D_{N,n}|$ is separable if and only if the following two Hankel matrices [33] M_0,M_1 are positive semidefinite, i.e.,

$$M_0 := \begin{pmatrix} \chi_0 & \cdots & \chi_{m_0} \\ \cdots & \cdots & \cdots \\ \chi_{m_0} & \cdots & \chi_{2m_0} \end{pmatrix} \geqslant 0, \tag{4}$$

$$M_1 := \begin{pmatrix} \chi_1 & \cdots & \chi_{m_1} \\ \cdots & \cdots & \cdots \\ \chi_{m_1} & \cdots & \chi_{2m_1-1} \end{pmatrix} \geqslant 0, \tag{5}$$

where $m_0 := [\frac{N}{2}]$ and $m_1 := [\frac{N+1}{2}]$.

Proof. According to Propositions 2 and 4, ρ is separable if and only if $\mathrm{tr}(W_0\rho) \geqslant 0$ holds for any extreme point of diagonal general entanglement witness, that is, R and T types of general entanglement witnesses. Equivalently, for all $\vec{a} = (a_0,\ldots,a_{m_0})^T \in \mathbb{R}^{m_0+1}$ and $\vec{b} = (b_1,\ldots,b_{m_1})^T \in \mathbb{R}^{m_1}$ the following quadratic forms are non-negative:

$$\mathrm{tr}(R\rho) = \sum_{0\leqslant i,j\leqslant m_0} \chi_{i+j} a_i a_j = \vec{a}^T M_0 \vec{a} \geqslant 0,$$

$$\mathrm{tr}(T\rho) = \sum_{1\leqslant i,j\leqslant m_1} \chi_{i+j-1} b_i b_j = \vec{b}^T M_1 \vec{b} \geqslant 0.$$

Notice that M_0,M_1 are real matrices. The above condition is equivalent to $M_0,M_1 \geqslant 0$. ■

In addition, we know the following:

Theorem 2. An N-qubit mixture of Dicke states $\rho = \sum_{n=0}^{N} \chi_n |D_{N,n}\rangle\langle D_{N,n}|$ is separable if and only if it is PPT. More precisely, ρ is separable if and only if it is PPT under the partial transpose of $m_0 = [\frac{N}{2}]$ subsystems.

Remark: A positive semidefinite matrix M acting on a bipartite system is called PPT if $M^\Gamma \geqslant 0$ holds, where Γ means the partial transpose, i.e., $(|ij\rangle\langle kl|)^{\Gamma_A} = |kj\rangle\langle il|$.

Proof. Assume ρ is positive under the partial transpose of the first $m_0 = [\frac{N}{2}]$ subsystems, according to Theorem 1, we only need to show $M_0,M_1 \geqslant 0$ of Eqs. (1) and (2).

Write ρ^Γ in basis $|D_{m_0,j}\rangle|D_{m_1,k}\rangle$ by verifying the following relation between Dicke states:

$$|D_{N,n}\rangle = \sum_{j=\max\{0,n-m_1\}}^{\min\{n,m_0\}} |D_{m_0,j}\rangle|D_{m_1,n-j}\rangle.$$

Since $\rho^\Gamma \geqslant 0$, then the restriction of ρ^Γ on the subspace spanned by $\{|D_{m_0,j}\rangle|D_{m_1,j}\rangle, 0 \leqslant j \leqslant m_0\}$ is non-negative. Direct calculation indicates $M_0 \geqslant 0$.

On the other hand, the restriction of ρ^Γ on the subspace spanned by $\{|D_{m_0,j-1}\rangle|D_{m_1,j}\rangle, 1 \leqslant j \leqslant m_1\}$ is non-negative. That leads us to $M_1 \geqslant 0$. Therefore, ρ^Γ implies the separability of ρ. ■

If bosonic state ρ is separable, then the mixture of Dicke states $\sigma = \int U^{\otimes N}\rho U^{\dagger\otimes N} dU$ is separable, where dU ranges over the diagonal qubit unitaries. Then, Theorem 2 indeed provides a necessary condition on the separability of the general N-qubit bosonic state.

These techniques to study multiqubit Dicke states can be generalized to study the mixture of higher dimensional bipartite Dicke states,

$$\rho = \sum_{i,j=1}^{d} \chi_{i,j} |\psi_{i,j}\rangle\langle\psi_{i,j}|,$$

with $|\psi_{i,j}\rangle := \{ \begin{smallmatrix} |ii\rangle & \text{if } i=j, \\ |ij\rangle+|ji\rangle & \text{otherwise.} \end{smallmatrix}$ being some basis of the $d \otimes d$ symmetric subspace.

By using that $\rho = (U\otimes U)\rho(U\otimes U)^\dagger$ holds for all diagonal qudit unitary U, we know the following:

$$\begin{aligned} \rho &= \sum_k \alpha_k^{\otimes 2} \\ &= \sum_k \int (U\otimes U)\alpha_k^{\otimes 2}(U\otimes U)^\dagger dU \\ &= \sum_k |x_{k,i}|^2 |x_{k,j}|^2 |\psi_{i,j}\rangle\langle\psi_{i,j}|, \\ \Leftrightarrow \chi &:= (\chi_{ij})_{d\times d} = \sum_k \vec{x}_k \vec{x}_k^{\,T}, \end{aligned}$$

where dU ranges over all diagonal unitaries, $|\alpha_k\rangle = \sum_{j=1}^{d} x_{k,j}|j\rangle$, $\alpha_k = |\alpha_k\rangle\langle\alpha_k|$, and $\vec{x}_k = (|x_{k,1}|^2,\ldots,|x_{k,d}|^2)^T \in \mathbb{R}_+^d$, with $\mathbb{R}_+^d$ standing for the d-dimensional vector space whose entries are non-negative.

The above argument indicates that ρ is separable if and only if χ is a completely positive matrix, where the cone of completely positive matrices [34] is defined as

$$\mathcal{C} = \left\{ \sum_i \vec{y}_k \vec{y}_k^{\,T} : \vec{y}_k \in \mathbb{R}_+^d \right\}.$$

Recalling the known hardness result on testing the membership of completely positive matrices in Refs. [35,36], we have the following:

Theorem 3. It is NP-hard to decide whether $\rho = \sum_{i,j=1}^{d} \chi_{i,j}|\psi_{i,j}\rangle\langle\psi_{i,j}|$ is separable. On the other hand, for $d = 3,4$, it is separable if and only if $\chi = (\chi_{ij})_{d\times d}$ is semi-definite positive.

Conclusion. In this paper, we study the separability of bosonic state. We prove the validity of the hypothesis of Ref. [32] by demonstrating an analytical condition for the separability of a mixture of N-qubit Dicke states. These techniques are also applied to the mixture of $d\otimes d$ Dicke states, and the hardness result is shown. We hope that our techniques for certifying entanglement witnesses and positive polynomials may prove useful in furthering the understanding of entanglement. One interesting goal is to provide complete criteria for the separability of general N-qubit bosonic states.

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[1] C. H. Bennett and G. Brassard, in *International Conference on Computers, Systems, & Signal Processing: Proceedings, Dec. 9–12, 1984, Bangalore, India* (IEEE, New York, 1984).
[2] C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, Nature **464**, 1165 (2010).
[3] L. Gurvits, J. Comput. Syst. Sci. **69**, 448 (2004).
[4] L. M. Ioannou, Quantum Inf. Comput. **7**, 335 (2007).
[5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[6] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996); **283**, 1 (2001).
[8] B. M. Terhal, Phys. Lett. A **271**, 319 (2000).
[9] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002).
[10] M. Christandl, R. Konig, G. Mitchison, and R. Renner, Commun. Math. Phys. **273**, 473 (2007).
[11] G. Chiribella, in *Lecture Notes in Computer Science*, Vol. 6519 (Springer, New York, 2011), p. 9.
[12] A. W. Harrow, arXiv:1308.6595.
[13] F. G. S. L. Brãndao, and M. Christandl, Phys. Rev. Lett. **109**, 160502 (2012).
[14] K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein, Ann. Phys. (NY) **299**, 88 (2002).
[15] N. Yu, Phys. Rev. A **87**, 052310 (2013); N. Yu, E. Chitambar, C. Guo, and R. Duan, *ibid.* **81**, 014301 (2010); N. Yu, C. Guo, and R. Duan, Phys. Rev. Lett **112**, 160401 (2014).
[16] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[17] R. Hübener, M. Kleinmann, T.-C. Wei, C. González-Guillén, and O. Gühne, Phys. Rev. A **80**, 032324 (2009).
[18] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. **103**, 070503 (2009); T. Bastin, C. Thiel, J. von Zanthier, L. Lamata, E. Solano, and G. S. Agarwal, *ibid.* **102**, 053601 (2009).
[19] W. Wieczorek, N. Kiesel, C. Schmid, and H. Weinfurter, Phys. Rev. A **79**, 022311 (2009).
[20] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Phys. Rev. A **67**, 022112 (2003).
[21] M. Cramer, A. Bernard, N. Fabbri, L. Fallani, C. Fort, S. Rosi, F. Caruso, M. Inguscio, and M. B. Plenio, Nat. Commun. **4**, 2161 (2013).
[22] R. McConnell, H. Zhang, J. Hu, S. Ćuk, and V. Vuletić, Nature **519**, 439 (2015).
[23] C. Lu, Xiao-Qi Zhou, O. Gühne, W. Gao, J. Zhang, Z. Yuan, A. Goebel, T. Yang, and J. Pan, Nat. Phys. **3**, 91 (2007).
[24] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, Phys. Rev. Lett. **103**, 020504 (2009).
[25] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **103**, 020503 (2009).
[26] C. D. Hamley, C. S. Gerving, T. M. Hoang, E. M. Bookjans, and M. S. Chapman, Nat. Phys. **8**, 305 (2012).
[27] G. Tóth and O. Gühne, Appl. Phys. B **98**, 617 (2009); Phys. Rev. Lett. **102**, 170503 (2009); O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).
[28] J. Tura, R. Augusiak, P. Hyllus, M. Kus, J. Samsonowicz, and M. Lewenstein, Phys. Rev. A **85**, 060302(R) (2012).
[29] L. Novo, and T. Moroder, and O. Gühne, Phys. Rev. A **88**, 012305 (2013).
[30] R. Quesada and A. Sanpera, Phys. Rev. A **89**, 052319 (2014).
[31] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. **80**, 2261 (1998).
[32] E. Wolfe and S. F. Yelin, Phys. Rev. Lett. **112**, 140402 (2014); E. Wolfe, arXiv:1409.2517; E. Wolfe and S. F. Yelin, arXiv:1405.5288.
[33] J. R. Partington, *An Introduction to Hankel Operators*, London Mathematical Society Student Texts 13 (Cambridge University, New York, 1988).
[34] L. J. Gray and D. G. Wilson, Lin. Algebra Appl. **31**, 119 (1980).
[35] P. J. C. Dickinson, and L. Gijben, Comput. Optim. Appl. **57**, 403 (2014).
[36] H. Dianada, Math. Proc. Camb. Phil. Soc. **58**, 17 (1961).