

Vector nematicons: Coupled spatial solitons in nematic liquid crystalsTheodoros P. Horikis¹ and Dimitrios J. Frantzeskakis²¹*Department of Mathematics, University of Ioannina, Ioannina 45110, Greece*²*Department of Physics, University of Athens, Panepistimiopolis, Zografos, Athens 15784, Greece*

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Families of soliton pairs, namely vector solitons, are found within the context of a coupled nonlocal nonlinear Schrödinger system of equations, as appropriate for modeling beam propagation in nematic liquid crystals. In the focusing case, bright soliton pairs have been found to exist provided their amplitudes satisfy a specific condition. In our analytical approach, focused on the defocusing regime, we rely on a multiscale expansion methods, which reveals the existence of dark-dark and antidark-antidark solitons, obeying an effective Korteweg-de Vries equation, as well as dark-bright solitons, obeying an effective Mel'nikov system. These pairs are discriminated by the sign of a constant that links all physical parameters of the system to the amplitude of the stable continuous wave solutions, and, much like the focusing case, the solitons' amplitudes are linked, leading to mutual guiding.

DOI: [10.1103/PhysRevA.94.053805](https://doi.org/10.1103/PhysRevA.94.053805)**I. INTRODUCTION**

Solitons, namely robust localized waveforms propagating in nonlinear dispersive (or diffractive) media, have been studied extensively in various physical contexts [1]. In nonlinear optics, solitons appear either as pulses localized in time (temporal solitons) or as bounded self-guided beams in space (spatial solitons) [2]. These structures are usually described by the two main variants of the nonlinear Schrödinger (NLS) equation, with a local Kerr (cubic) nonlinearity and depend on the relative sign of dispersion or diffraction and nonlinearity: the focusing, where dispersion or diffraction and nonlinearity share the same signs and bright solitons are exhibited and defocusing where the two effects have opposite signs and the NLS supports dark solitons. In the case where more than one-component scalar fields are involved (as in the case of fields of different frequencies or different polarizations), their nonlinear interaction leads to vector NLS models, which support *vector solitons* of various types, e.g., bright-bright, dark-dark, dark-bright, and so on, depending again on the relative signs of dispersion or diffraction, as well as inter- and intracomponent nonlinearity coefficients [2,3]. Note that a similar picture, regarding vector NLS models with local cubic nonlinearities and the types of vector solitons they support, appear in other physical systems, such as atomic Bose-Einstein condensates [4,5].

On the other hand, there has been an increased interest in physical systems (and their corresponding mathematical models) featuring a spatially nonlocal nonlinear response, where beam dynamics and solitons are relevant. Pertinent examples include partially ionized plasmas [6,7], atomic vapors [8], lead glasses featuring strong thermal nonlinearity [9], as well as media with a long-range interparticle interaction. The latter include dipolar bosonic quantum gases [10], and nematic liquid crystals with long-range molecular reorientational interactions [11]. Nematic liquid crystals are known to support spatial solitons [12], which are usually called *nematicons* [13–15]. These structures, are described by nonlocal NLS equations which, in general, do not possess exact analytical solutions with the freedom of various parameters describing the soliton's properties (amplitude, velocity, etc.). Thus, variational techniques are usually employed for the study of either

bright [13,16–18] or dark [19–23] nematicons. More recently, in the self-defocusing setting, multiscale expansion methods were used to study dark nematicons in one-dimensional (1D) [24] and higher-dimensional [25,26] geometries; these studies, apart from investigating the dynamics of dark solitons, also predicted the existence of *antidark* solitons, namely humps on top of a continuous-wave (cw) background. These solutions are discriminated from dark solitons by the sign of a specific parameter, which associates the degree of nonlocality with the amplitude of the cw wave on top of which these solutions are formed.

Two-color nematicons, i.e., vectorial nematicon structures excited at different wavelengths, have also been experimentally realized and studied theoretically as well [17,27–29]. In the focusing 1D setting, the existence of exact bright-bright soliton solutions, provided that their amplitudes satisfy a specific condition, was recently reported [30]. In the same 1D setting, but in the defocusing regime, nonlocal dark-dark [31] and dark-bright [32] solitons were studied by means of variational methods, similar to those used in the one-component problem. Notice that the defocusing regime is also accessible in the context of nematic liquid crystals: indeed, as shown in Ref. [20] where dark nematicons were observed for the first time, azo-doped nematic liquid crystals exhibit a self-defocusing response for extraordinary waves. Generally, instead of exploiting a thermo-optic response, self-defocusing in this setting can be obtained by introducing dopants [33].

In this work, our aim is to present families of vector nematicons in the 1D, self-defocusing setting. Our main findings, as well as the outline of the paper, are as follows. First, in Sec. II, we present the model, as well as review its cw solution, its stability, and the derivation of the appropriate condition for bright solitons to exist. Note that the complete analysis for this case was presented in Ref. [30], we briefly summarize these findings for completeness. Then, in Sec. III, seeking solutions that feature nontrivial boundary conditions at infinity, we develop a multiscale expansion method that reduces the nonlocal system to a single Korteweg-de Vries (KdV) equation; we also obtain an additional equation that links the amplitudes of the two modes, and thus derive dark-dark and antidark-antidark soliton pairs. In Sec. IV,

assuming that one mode decays to zero at infinity, we develop another multiscale expansion method to reduce the nonlocal system to the Mel'nikov system [34,35]; this system, which is completely integrable by means of the inverse scattering transform [36], allows for the derivation of dark-bright soliton solutions in the original nonlocal system. In all cases, our analytical findings are corroborated by direct numerical simulations. Finally, in Sec. V, we summarize our findings and suggest further generalizations.

II. THE GOVERNING EQUATIONS

We consider the equations that describe two polarized, coherent light beams, of two different wavelengths, propagating through a cell filled with a nematic liquid crystal. These equations are expressed in dimensionless form as follows [27,37]:

$$i \frac{\partial E_1}{\partial z} + \frac{d_1}{2} \frac{\partial^2 E_1}{\partial x^2} + 2g_1 \theta E_1 = 0, \quad (1a)$$

$$i \frac{\partial E_2}{\partial z} + \frac{d_2}{2} \frac{\partial^2 E_2}{\partial x^2} + 2g_2 \theta E_2 = 0, \quad (1b)$$

$$v \frac{\partial^2 \theta}{\partial x^2} - 2q\theta = -2(g_1 |E_1|^2 + g_2 |E_2|^2). \quad (1c)$$

The variables E_1 and E_2 are the complex valued, slowly varying envelopes of the electric fields, and θ is the optically induced deviation of the director angle. Diffraction is characterized by the coefficients d_1, d_2 , while nonlinearity by g_1, g_2 . The nonlocality parameter v measures the strength of the response of the nematic in space, with a highly nonlocal response corresponding to v large. The parameter q is related to the square of the applied static field which pretilts the nematic dielectric [13,15,18]. Note that the above system corresponds to the nonlocal regime with v large, where the optically induced rotation θ is small [13]; in particular, d_1, g_1, d_2, g_2, q are $O(1)$, while v is $O(10^2)$ [17,37]. Depending on the relative signs between diffraction and nonlinearity the relative system is deemed focusing ($d_1 g_1, d_2 g_2 > 0$) or defocusing ($d_1 g_1, d_2 g_2 < 0$). These equations assume an incoherent interaction between the beams and that they only interact through the nematic. That is, there are no coupling terms between E_1 and E_2 .

The simplest solution of this system is a pair of cw's of the form

$$E_1(z) = u_0 e^{2ig_1 \theta_0 z}, \quad E_2(z) = v_0 e^{2ig_2 \theta_0 z}, \quad \theta_0 = \frac{g_1 u_0^2 + g_2 v_0^2}{q},$$

where u_0 and v_0 are real constants. By considering small perturbations to these solutions in Ref. [30], the dispersion relation

$$p_1(k)\omega^4 + p_2(k)\omega^2 + p_3(k) = 0$$

was derived, where

$$\begin{aligned} p_1(k) &= 16(k^2 v + 2q), \\ p_2(k) &= -4v(d_1^2 + d_2^2)k^6 - 8q(d_1^2 + d_2^2)k^4 \\ &\quad + 64(d_1 g_1^2 u_0^2 + d_2 g_2^2 v_0^2)k^2, \end{aligned}$$

$$\begin{aligned} p_3(k) &= d_1^2 d_2^2 v k^{10} + 2d_1^2 d_2^2 q k^8 \\ &\quad - 16d_1 d_2 (d_2 g_1^2 u_0^2 + d_1 g_2^2 v_0^2) k^6. \end{aligned}$$

This dispersion relation was shown to have real roots, i.e., the system would be modulationally stable, provided the diffraction and nonlinearity signs are opposite, i.e., the fully defocusing case. Hereafter, we fix this sign difference into the nonlocal system and we write

$$i \frac{\partial E_1}{\partial z} + \frac{d_1}{2} \frac{\partial^2 E_1}{\partial x^2} - 2g_1 \theta E_1 = 0, \quad (2a)$$

$$i \frac{\partial E_2}{\partial z} + \frac{d_2}{2} \frac{\partial^2 E_2}{\partial x^2} - 2g_2 \theta E_2 = 0, \quad (2b)$$

$$v \frac{\partial^2 \theta}{\partial x^2} - 2q\theta = -2(g_1 |E_1|^2 + g_2 |E_2|^2), \quad (2c)$$

where now d_1, g_1, d_2, g_2, v, q are all positive. Bright soliton pairs of Eqs. (1) have already been discussed in Refs. [30,38] and will not be considered here where the focus is turned on the defocusing case.

III. DARK AND ANTIDARK SOLITON PAIRS

Our analysis is now focused on soliton pairs that rely on the existence of a stable cw background and hence on the defocusing system where $d_1 g_1, d_2 g_2 < 0$. As such, we only consider Eqs. (2). Write the solutions of this system in the form

$$E_1 = u_b(z)u(z, x), \quad (3a)$$

$$E_2 = v_b(z)v(z, x), \quad (3b)$$

$$\theta = \theta_b w(z, x), \quad (3c)$$

where the functions $u_b(z)$ and $v_b(z)$ correspond to the relative cw backgrounds so that

$$\begin{cases} i u'_b - 2g_1 \theta_b u_b = 0 \\ i v'_b - 2g_2 \theta_b v_b = 0 \end{cases} \Rightarrow \begin{cases} u_b(z) = u_0 e^{-2ig_1 \theta_b z + ic_1} \\ v_b(z) = v_0 e^{-2ig_2 \theta_b z + ic_2} \end{cases}$$

where $u_0, v_0, c_1, c_2 \in \mathbb{R}$ and $\theta_b = \frac{1}{q}(g_1 u_0^2 + g_2 v_0^2)$. Substituting back to Eqs. (2) gives

$$i \frac{\partial u}{\partial z} + \frac{d_1}{2} \frac{\partial^2 u}{\partial x^2} - 2g_1 \theta_b (w - 1)u = 0, \quad (4a)$$

$$i \frac{\partial v}{\partial z} + \frac{d_2}{2} \frac{\partial^2 v}{\partial x^2} - 2g_2 \theta_b (w - 1)v = 0, \quad (4b)$$

$$v \frac{\partial^2 w}{\partial x^2} - 2qw = -\frac{2}{\theta_b} (g_1 u_0^2 |E_1|^2 + g_2 v_0^2 |E_2|^2). \quad (4c)$$

It is trivial to check that these are also satisfied at the boundaries where $u = v = w = 1$, and any evolution of the boundary conditions has been absorbed by the background functions. This way, the resulting equations have now fixed boundary conditions. Next, we employ the Madelung transformation:

$$u(x, z) = \rho_1(x, z) \exp[i\phi_1(x, z)], \quad (5a)$$

$$v(x, z) = \rho_2(x, z) \exp[i\phi_2(x, z)], \quad (5b)$$

so that

$$d_j \frac{\partial^2 \rho_j}{\partial x^2} - 2\rho_j \frac{\partial \phi_j}{\partial z} - d_j \rho_j \left(\frac{\partial \phi_j}{\partial x} \right)^2 - 4g_j \theta_b \rho_j (w - 1) = 0, \quad (6a)$$

$$\frac{\partial \rho_j}{\partial z} + \frac{1}{2} d_j \rho_j \frac{\partial^2 \phi_j}{\partial x^2} + d_j \frac{\partial \rho_j}{\partial x} \frac{\partial \phi_j}{\partial x} = 0, \quad (6b)$$

$$v \frac{\partial^2 w}{\partial x^2} - 2qw = -\frac{2}{\theta_b} (g_1 u_0^2 \rho_1^2 + g_2 v_0^2 \rho_2^2), \quad (6c)$$

where $j = 1, 2$, and recall that $w(z, x) \in \mathbb{R}$.

To analytically study system Eq. (6), and determine the unknown functions ρ_j , ϕ_j , and w , we now employ the reductive perturbation method [39]. We thus introduce the stretched variables:

$$Z = \varepsilon^3 z, \quad X = \varepsilon(x - Cz), \quad (7)$$

where C is the speed of sound (to be determined later in the analysis), namely the velocity of small-amplitude and long-wavelength waves propagating along the background. Additionally, we expand amplitudes and phases in powers of ε as follows:

$$\rho_j = \rho_{j0} + \varepsilon^2 \rho_{j2} + \varepsilon^4 \rho_{j4} + \dots, \quad (8a)$$

$$\phi_j = \varepsilon \phi_{j1} + \varepsilon^3 \phi_{j3} + \varepsilon^5 \phi_{j5} + \dots, \quad (8b)$$

$$w = 1 + \varepsilon^2 w_2 + \varepsilon^4 w_4 + \dots, \quad (8c)$$

where $\rho_{j0} = 1$ and the rest of the unknown fields depend on the stretched variables Eq. (7). These values for ρ_{j0} is not only a result obtained from the perturbation analysis but is also anticipated from Eqs. (3) and (4). Recall that the background has been removed, absorbed by the functions u_b and v_b , which, in general, are not equal.

Substituting back to Eqs. (6) we obtain the following results (see details in Appendix). First, in the linear limit, i.e., at the lowest-order approximation in ε , we derive equations connecting the unknown fields, namely:

$$w_2 = \frac{2}{q\theta_b} (g_1 u_0^2 \rho_{21} + g_2 v_0^2 \rho_{22}), \quad \phi_{21} = \frac{g_2}{g_1} \phi_{11}, \quad (9a)$$

$$\rho_{22} = \frac{d_2 g_2}{d_1 g_1} \rho_{21}, \quad \frac{d_j}{2} \frac{\partial \phi_{j2}}{\partial X} = C \rho_{j2}, \quad (9b)$$

as well as the speed of sound,

$$C^2 = \frac{2}{q} (d_1 g_1 u_0^2 + d_2 g_2 v_0^2). \quad (10)$$

Obviously, Eqs. (9) suggest that only one equation for one of these fields will suffice to determine the rest of the unknown fields ρ_{j2} , ϕ_{j1} , and w_2 . This equation is derived to the next order of approximation and turns out to be the following nonlinear equation for the field ρ_{12} :

$$\frac{\partial \rho_{12}}{\partial Z} + A_1 \frac{\partial^3 \rho_{12}}{\partial X^3} + 6A_2 \rho_{12} \frac{\partial \rho_{12}}{\partial X} = 0, \quad (11)$$

where coefficients A_1 and A_2 are given by

$$A_1 = \frac{vC^4 - (d_1^3 g_1^2 u_0^2 + d_2^3 g_2^2 v_0^2)}{4C^2 q},$$

$$A_2 = \frac{d_1^2 g_1^3 u_0^2 + d_2^2 g_2^3 v_0^2}{C d_1 g_1 q}.$$

Equation (11) is the renowned KdV equation, which is completely integrable by means of the IST [40], and finds numerous applications in a variety of physical contexts [1,41]. More recently, a KdV equation was derived from the single-component version of Eqs. (2) and used to describe small-amplitude nematicons [24]; notice that the KdV model derived in Ref. [24] is identical with Eq. (11) when the coupling constants are set to zero. Notably, the same procedure can result in other integrable forms of the KdV in higher dimensions, such as the Kadomtsev-Petviashvili (KP) equation, Johnson's equation, and others [25,26].

These asymptotic reductions provide information on the type of the soliton solutions the original system may exhibit up to (and including) $O(\varepsilon^2)$. Indeed, first we note that the soliton solution of Eq. (11) takes the form (e.g., Ref. [41])

$$\rho_{12}(Z, X) = \frac{2A_1}{A_2} \eta^2 \text{sech}^2(\eta X - 4\eta^3 A_1 Z + X_0), \quad (12)$$

where η and X_0 are free parameters, setting the amplitude and width and initial position of the soliton, respectively. Then, it is straightforward to retrieve the pertinent phase,

$$\phi_{11} = -\frac{4A_1 C}{A_2 d_1} \eta \tanh(\eta X - 4\eta^3 A_1 Z + X_0), \quad (13)$$

so that, finally, the solutions for the two components may be written as

$$E_1(z, x) \approx u_b(z) (1 + \varepsilon^2 \rho_{12}) \exp(i\varepsilon \phi_{12}), \quad (14)$$

$$E_2(z, x) \approx v_b(z) \left(1 + \varepsilon^2 \frac{d_2 g_2}{d_1 g_1} \rho_{12} \right) \exp\left(i\varepsilon \frac{g_2}{g_1} \phi_{12} \right). \quad (15)$$

It is now important to notice that the type of the soliton Eqs. (14) and (15) depends crucially on the sign of the ratio A_1/A_2 ; this quantity changes sign according to the critical value v_c , given by

$$v_c = \frac{q^2 (d_1^3 g_1^2 u_0^2 + d_2^3 g_2^2 v_0^2)}{4(d_1 g_1 u_0^2 + d_2 g_2 v_0^2)^2}. \quad (16)$$

Indeed, if the nonlocality parameter v is such that $v < v_c$ (i.e., $A_1/A_2 > 0$), the solitons are dark, namely are intensity dips off of the cw background. On the other hand, if $v > v_c$ (i.e., $A_1/A_2 < 0$) the solitons are *antidark*, namely intensity elevations on top of the cw background. Notice that Eqs. (A7) suggest that the relative signs between the modes are the same and, as such, the only allowed pairs are solitons of the same kind. It should also be mentioned that if $A_1 = 0$, modification of the asymptotic analysis and inclusion of higher-order terms is needed. This has been addressed, to a certain extent, in Ref. [42], where, it was found that higher-order dispersive terms can lead to resonant interactions with radiation, as expected, for the higher (fifth) order KdV equation.

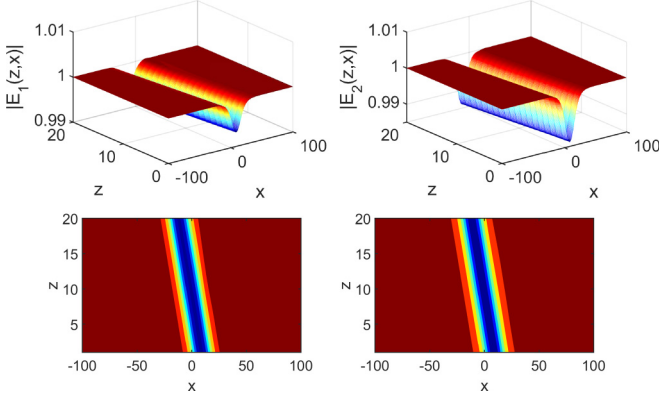


FIG. 1. The evolution of a typical dark soliton pair. Left and right columns depict the two components, while top and bottom panels show three-dimensional plots, and spatio-temporal contour plots, respectively.

To demonstrate the validity of our analysis, we perform direct numerical simulations, and we thus integrate Eqs. (2), employing a high-accuracy spectral integrator, and use initial conditions (at $z = 0$) taken from Eqs. (14) and (15), for both the dark and the antidark soliton pairs. The results are shown in Fig. 1, where a typical evolution of a dark soliton pair is depicted. Here, we choose parameter values $d_1 = d_2/1.5 = g_1 = g_2 = 1$, $u_0 = v_0 = 1$, and $q/5 = v = 1$. Similarly, in Fig. 2, we show a typical evolution of an antidark soliton pair; all parameters remain the same except $q = 1$. In both cases, it is clear that the solitons not only exist but also propagate undistorted on top of the cw background. It is also observed that the solitons propagate with constant speed, with the antidark soliton pair traveling faster than the dark one, as expected from Eq. (10).

IV. DARK-BRIGHT SOLITON PAIRS

Apart from soliton pairs of the same type, it is also possible to derive vector soliton solutions composed by different types of solitons. This can be done upon seeking solutions of the system of Eqs. (2), such that one of the components decays to zero at infinity, while the other tends to a constant, as before.

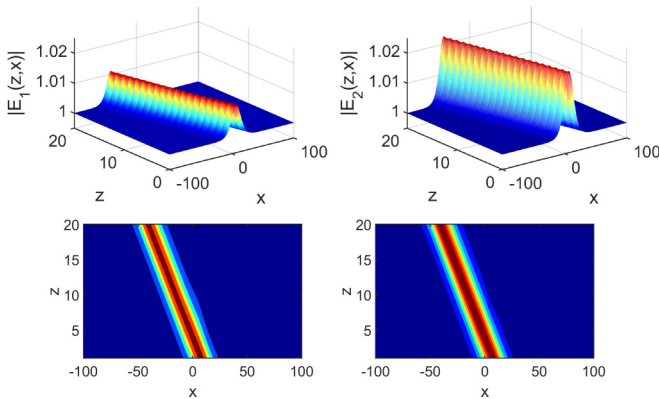


FIG. 2. Similar to Fig. 1, but now for a typical antidark soliton pair.

In such a case, solutions of Eqs. (2) are again taken to be of the form of Eqs. (3), but now we assume that the background functions are given by

$$u_b(x, z) = \exp[ikx - i(\omega - \varepsilon^2\Omega)z], \quad (17a)$$

$$\omega = \frac{1}{2q} (d_1 k^2 q + 4g_1 g_2 v_0^2), \quad (17b)$$

$$v_b(z) = v_0 \exp(-2i g_2 \theta_b z + i \psi_1), \quad \theta_b = \frac{g_2 v_0^2}{q}. \quad (17c)$$

Then, the system Eq. (2) is reduced to the form

$$i u_z + \frac{d_1}{2} u_{xx} - 2g_1 \theta_b (w - 1) u - i d_1 k u_x = 0, \quad (18)$$

$$i v_z + \frac{d_2}{2} v_{xx} - 2g_2 \theta_b (w - 1) v = 0, \quad (19)$$

$$v w_{xx} - 2q w = -\frac{2}{\theta_b} (g_1 |u|^2 + g_2 v_0^2 |v|^2). \quad (20)$$

Then, using the stretched variables Eq. (7) and the asymptotic expansions Eq. (8), and following the procedure of the previous section, we obtain the following results. First, at the leading order, $O(1)$, we get $\rho_{10} = 0$ and $\rho_{20} = w_0 = 1$, while in the linear limit, i.e., at the orders $O(\varepsilon^2)$ and $O(\varepsilon^3)$, we derive equations connecting the unknown fields, namely,

$$w_2 = 2\rho_{22}, \quad C \frac{\partial \phi_{21}}{\partial X} = \frac{4g_2^2 v_0^2}{q} \rho_{22}, \quad (21a)$$

$$\frac{d_2}{2} \frac{\partial^2 \phi_{21}}{\partial X^2} = C \frac{\partial \rho_{22}}{\partial X}, \quad k = \frac{C}{d_1}. \quad (21b)$$

The above equations suggest that, now, the speed of sound is given by

$$C^2 = \frac{2g_2^2 v_0^2 d_2}{q}. \quad (22)$$

Next, in the nonlinear regime, namely at $O(\varepsilon^4)$ and $O(\varepsilon^5)$, we obtain the following system for the fields ρ_{j2} :

$$\begin{aligned} \frac{8g_2^2 v_0^2}{Cq} \frac{\partial \rho_{22}}{\partial Z} - \frac{(d_2 q^2 - 4g_2^2 v_0^2 v)}{2q^2} \frac{\partial^3 \rho_{22}}{\partial X^3} + \frac{24g_2^2 v_0^2}{q} \rho_{22} \frac{\partial \rho_{22}}{\partial X} \\ + \frac{2g_1 g_2}{q} \frac{\partial}{\partial X} (\rho_{12}^2) = 0, \end{aligned} \quad (23a)$$

$$\frac{d_1}{2} \frac{\partial^2 \rho_{12}}{\partial X^2} - \frac{4g_1 g_2 v_0^2}{q} \rho_{12} \rho_{22} = \Omega \rho_{12}, \quad (23b)$$

as well as equations connecting fields that can be determined at a higher-order approximation. The system of Eqs. (23) is the so-called Mel'nikov system [34–36] and is apparently composed of a KdV equation with a self-consistent source, which satisfies a stationary Schrödinger equation. This system has been derived in earlier works to describe dark-bright solitons in nonlinear optical systems [43] and in Bose-Einstein condensates [44,45]. The Mel'nikov system is completely integrable by the inverse scattering transform and possesses

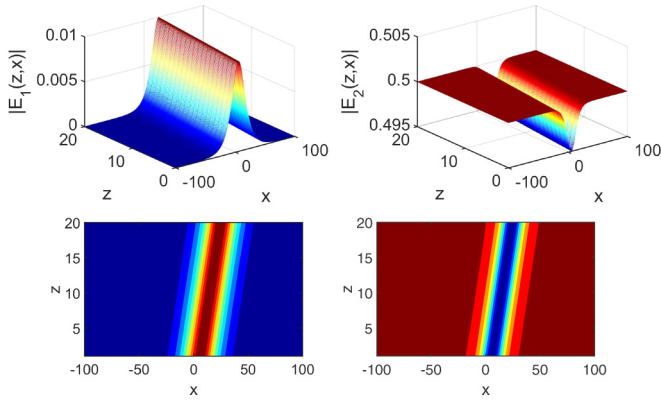


FIG. 3. Similar to Fig. 1, but now for a typical dark-bright soliton pair.

a soliton solution of the form [35]

$$\rho_{22}(Z, X) = -\frac{d_1 q}{4g_1 g_2 v_0^2} \eta^2 \operatorname{sech}^2(\eta X + bZ + X_0), \quad (24)$$

$$\rho_{12}(Z, X) = A \operatorname{sech}(\eta X + bZ + X_0), \quad (25)$$

where $\Omega = (1/2)\eta^2 d_1$, while parameters η , A , and b are connected through the following equation:

$$C d_1 (4v g_2^2 v_0^2 - d_2 q^2) \eta^4 + 4q d_1 g_2^2 v_0^2 b \eta - 4C g_1^2 g_2^2 v_0^2 A^2 = 0. \quad (26)$$

Using the above expressions, we can now express the relevant approximate [valid up to $O(\varepsilon^2)$] solutions of the original system for the two components $E_{1,2}$ as follows:

$$E_1(z, x) \approx \varepsilon^2 u_b(z) \rho_{12} \exp(i\varepsilon \phi_{12}) \quad (27)$$

$$E_2(z, x) \approx v_b(z) (1 + \varepsilon^2 \rho_{22}) \exp(i\varepsilon \phi_{22}). \quad (28)$$

It is clear that the above solution represents a dark-bright soliton pair, for the components E_2 and E_1 , respectively.

As in the case of the dark and antidark soliton pairs, we numerically integrate Eqs. (2), using initial conditions (at $z = 0$) taken from Eqs. (27) and (28). The results are shown in Fig. 3, where a typical evolution of a dark-bright soliton pair is depicted. Here we choose all parameters equal to unity, except $v_0 = 1/2$. In this case, too, the dark-bright soliton not only exist but also propagates undistorted with constant velocity, in excellent agreement with our analytical predictions.

V. CONCLUSIONS

Concluding, in this work we have completed the analysis started in Ref. [30] for the coupled focusing nonlocal NLS system. As such we studied vector nematicons in the defocusing regime using multiscale expansion methods to derive various types of such vector solitons.

In particular, first we have found dark-dark and antidark-antidark solitons. These structures, which have the form of propagating dips or humps on top of a stable vectorial continuous-wave background, respectively, were found to obey an effective KdV equation. The existence of the dark or the antidark soliton pair was connected with the magnitude

of the nonlocality parameter: it was found that below (above) a certain critical value of this parameter—which depends on the parameters of the system, as well as the background amplitudes—the soliton pair is dark (antidark), much like the single nematicon system [24]. In addition, we have found dark-bright soliton pairs, namely a dark soliton in one component, coupled with a bright soliton in the other component. It was shown that this soliton pair obeys another completely integrable effective model, namely the so-called Mel’nikov system. In all cases, we have numerically integrated the original nonlocal system and verified the existence and robustness of the vector nematicons, in excellent agreement with our analytical approach.

It would be interesting to extend our considerations in higher-dimensional settings and investigate existence, stability, and dynamics of such vector solitons, as well as other localized structures, such as vortices, and combinations thereof. However, the extension of the nematicon equations from $(1+1)$ to $(2+1)$ dimensions is a nontrivial extension. The stability will depend on the value of v with the solitary waves or vortices becoming unstable if v is small enough. This is because the nematicon equations become the $(2+1)$ -dimensional NLS equation as $v \rightarrow 0$.

APPENDIX: DETAILS ON THE PERTURBATION METHOD

Substituting Eqs. (7) and (8) into Eqs. (6), we obtain at $O(\varepsilon^2)$ the following equations:

$$C \frac{\partial \phi_{11}}{\partial X} = \frac{4g_1}{q} (g_1 u_0^2 \rho_{21} + g_2 v_0^2 \rho_{22}), \quad (A1)$$

$$C \frac{\partial \phi_{21}}{\partial X} = \frac{4g_2}{q} (g_1 u_0^2 \rho_{21} + g_2 v_0^2 \rho_{22}), \quad (A2)$$

$$w_2 = \frac{2}{q\theta_b} (g_1 u_0^2 \rho_{21} + g_2 v_0^2 \rho_{22}), \quad (A3)$$

which clearly suggest that

$$g_2 \frac{\partial \phi_{11}}{\partial X} = g_1 \frac{\partial \phi_{21}}{\partial X} \Rightarrow \phi_{21} = \frac{g_2}{g_1} \phi_{11}. \quad (A4)$$

Notice that any integrating constants are set to zero in order for the boundary conditions to be satisfied; recall in this formulation the boundary conditions are fixed at infinity. In addition, at $O(\varepsilon^2)$ we obtain

$$\frac{d_1}{2} \frac{\partial^2 \phi_{11}}{\partial X^2} = C \frac{\partial \rho_{12}}{\partial X}, \quad (A5)$$

$$\frac{d_2}{2} \frac{\partial^2 \phi_{21}}{\partial X^2} = C \frac{\partial \rho_{22}}{\partial X}, \quad (A6)$$

which also suggest that

$$d_2 g_2 \frac{\partial \rho_{21}}{\partial X} = d_1 g_1 \frac{\partial \rho_{22}}{\partial X} \Rightarrow \rho_{22} = \frac{d_2 g_2}{d_1 g_1} \rho_{21}, \quad (A7)$$

where again integrating constants have been ignored in order for the boundary conditions to be satisfied. Obviously, the compatibility condition of the equations yields the speed of sound Eq. (10). The same procedure follows for the higher-order equations. When applying the above, and since

ρ_{21} and ρ_{11} are related, one expects to find a single equation for one of the two. Hence, at $O(\varepsilon^4)$ we derive the equations

$$(d_1^2 g_1^2 q^2 \theta_b) w_4 = (d_1^2 g_1^3 q u_0^2 + d_2^2 g_2^3 q v_0^2) \rho_{12}^2 + 2d_1^2 g_1^3 q u_0^2 \rho_{14} + 2d_1^2 g_1^2 g_2 q v_0^2 \rho_{24} + d_1 g_1 v (d_1 g_1^2 u_0^2 + d_2 g_2^2 v_0^2) \frac{\partial^2 \rho_{12}}{\partial X^2}, \quad (\text{A8})$$

$$-2g_1 \theta_b (w_4 + w_2 \rho_{12} - \rho_{14} + \rho_{14}) + C \rho_{12} \frac{\partial \phi_{11}}{\partial X} - \frac{1}{2} d_1 \left(\frac{\partial \phi_{11}}{\partial X} \right)^2 + C \frac{\partial \phi_{13}}{\partial X} + \frac{1}{2} d_1 \frac{\partial^2 \rho_{12}}{\partial X^2} - \frac{\partial \phi_{11}}{\partial Z} = 0, \quad (\text{A9})$$

and

$$\frac{\partial^2 \phi_{13}}{\partial X^2} = -\frac{2}{d_1} \left(3C \rho_{12} \frac{\partial \rho_{12}}{\partial X} - C \frac{\partial \rho_{14}}{\partial X} + \frac{\partial \rho_{12}}{\partial Z} \right), \quad (\text{A10})$$

while at $O(\varepsilon^5)$ we obtain:

$$(8d_1^2 g_1^2 g_2^2 q v_0^2) \rho_{24} = -8d_1^2 g_1^3 g_2 q u_0^2 \rho_{14} - 4d_1 g_1 g_2 q (d_1 g_1^2 u_0^2 + 2d_2 g_1 g_2 u_0^2 + 3d_1 g_2^3 v_0^2) \rho_{12}^2 - d_1 d_2 g_1 g_2 (4d_1 g_1^2 v u_0^2 + 4g_2^2 v v_0^2 - d_2 q^2) \frac{\partial^2 \rho_{12}}{\partial X^2} + 2C d_1^2 g_1^2 q^2 \frac{\partial \phi_{23}}{\partial X} - 2d_1^2 g_1 g_2 q^2 \frac{\partial \phi_{11}}{\partial Z}, \quad (\text{A11})$$

$$\frac{2d_1^2 g_1^2 q (C^2 q - 2d_2 g_2^2 v_0^2)}{g_2} \frac{\partial^2 \phi_{23}}{\partial X^2} = 8C d_1^2 g_1^3 q u_0^2 \frac{\partial \rho_{14}}{\partial X} + 8C g_1 q (d_1^2 g_1^2 u_0^2 + 2d_1 d_2 g_1 g_2 u_0^2 + 6d_2^2 g_2^3 v_0^2) \rho_{12} \frac{\partial \rho_{12}}{\partial X} + C d_1 g_1 (-d_2^2 q^2 + 4d_1 g_1^2 u_0^2 v + 4d_2 g_2^2 v_0^2 v) \frac{\partial^3 \rho_{12}}{\partial X^3} + 4d_1 g_1 q (C^2 q + 2d_2 g_2^2 v_0^2) \frac{\partial \rho_{12}}{\partial Z}. \quad (\text{A12})$$

To this end, after a tedious but straightforward calculation, we eliminate all phase terms from these systems and derive the KdV Eq. (11).

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