Comparison of incoherent operations and measures of coherence

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(Received 23 May 2016; published 30 November 2016)

A resource theory of quantum coherence attempts to characterize the quantum coherence that exists in a given quantum system. Many different approaches to a resource theory of coherence have recently been proposed, with their differences lying primarily in the identification of "free" or "incoherent" operations. In this article, we compare a number of these operational classes. In particular, the recently introduced class of dephasing-covariant operations is analyzed, and we characterize the Kraus operators of such maps. A number of coherence measures are introduced based on relative Rényi entropies, and we study incoherent state transformations under different operational classes. In particular, we show that the incoherent Schmidt rank can be increased arbitrarily large by certain noncoherence generating operations. The distinction between asymmetry-based versus basis-dependent notions of coherence theory is clarified, and we further develop the resource theory of N asymmetry, where N is the group of all diagonal incoherent unitaries.

DOI: 10.1103/PhysRevA.94.052336

In quantum systems, the notion of coherence is ubiquitous. For instance, the state $|+\rangle = \sqrt{1/2}(|0\rangle + |1\rangle)$ can be seen as a coherent superposition of the states $|0\rangle$ and $|1\rangle$, while the state $|0\rangle$ can itself be seen as a coherent superposition of $|+\rangle$ and $|-\rangle = \sqrt{1/2}(|0\rangle - |1\rangle)$. Thus, without further qualification, it is completely ambiguous to say that one state has coherence while another does not. One way to make such a statement meaningful involves first identifying a fixed reference basis, and then defining coherence with respect to this basis. More precisely, a basis for the system's state space is specified (called the incoherent basis), and then a given state is deemed incoherent if it is diagonal in this basis.

Recently, researchers have used this distinction between coherent and incoherent states to construct resource theories of quantum coherence [1–10]. A general quantum resource theory consists of a class of "free" states along with a class of "free" or allowable operations [11]. The essential resource-theoretic condition is that the set of free states is closed under the set of free operations. Hence, any state that is not free is a resource since it cannot be obtained using the allowable operations. For quantum coherence, the free states are the incoherent states $\mathcal{I}$. As for the free or "incoherent" operations, many different approaches have been proposed, motivated by various degrees of physical and mathematical considerations.

The largest class of incoherent operations are the so-called "maximal" incoherent operations (MIO) [1,12], and these consist of all completely positive (CP) maps that act invariantly on $\mathcal{I}$. A smaller set of operations was introduced by Baumgratz *et al.* and simply goes by the name "incoherent operations" (IO) [3]. Two other proposed classes of operations are the strictly incoherent operations (SIO) [7,9] and the dephasing-covariant incoherent operations (DIO) [13,14]. Each of these operational classes is defined to reflect different measurement scenarios. However, from a resource-theoretic perspective, they all lack physical consistency in terms of their implementation: in order to perform a general MIO/IO/DIO/SIO map, coherence needs to be consumed on some ancilla system [13]. The class of operations that maintain implementation consistency was introduced in Ref. [13] under the name of physical incoherent operations (PIO). The relationship between these various incoherent operations is depicted in Fig. 1.

In conjunction with each of the operational classes, one can define different measures of coherence. From a resource-theoretic perspective, the crucial property of these measures is that they are monotonic under the specified class of operations. To give the measures physical meaning, one seeks to find some operational interpretation of the measure, thereby enabling the measure to quantify some particular physical property or process. A number of coherence measures have been proposed in the literature such as the relative entropy of coherence and the ℓ_1 norm of coherence [3], entanglement-induced measures of entanglement [15], distillable coherence and coherence of formation [6,7], and the robustness of coherence [16]. For a nice summary of different coherence measures, see the recent review article [17]. In this paper, we introduce a general prescription for generating a number of "distance-based" measures of coherence.

Close parallels can be drawn between the resource theory of coherence and the resource theories of asymmetry [18–20]. In the latter, one identifies a particular unitary group G, and the free states are those that are invariant under the G-twirling operation $\rho \to \int_G dg\, U(g)\rho U(g)^\dagger$. The free operations are those that commute with the unitary action of the group and are called G covariant. In physical systems, it is natural to choose G as the group of unitaries that commute with the time-translation operator e^{itH}, where H is the Hamiltonian of the system. In this way, one can speak of "coherences" between the eigenspaces of H, and a state is incoherent if it commutes with e^{itH} for all time; states that do not commute possess asymmetry with respect to the unitary group. Thus, one obtains a type of coherence resource theory based on this

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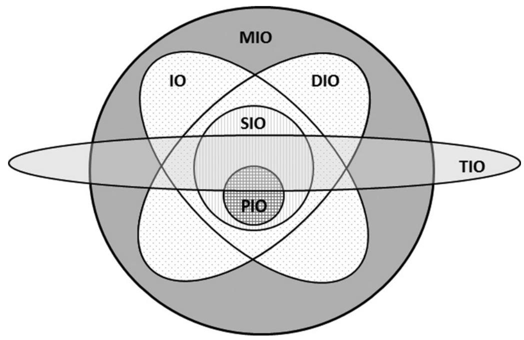

FIG. 1. A heuristic comparison between the five incoherence operations MIO/DIO/IO/SIO/PIO and TIO. The first five are classes of CPTP maps defined on a system with a specified incoherent basis. Maps from each of these classes act invariantly on the set of states diagonal in this basis. TIO is fundamentally different since the allowed operations in this class depend on invariances of the generator H which may or may not fix a single incoherent basis. In general, PIO will not be a subset of TIO since the former includes all permutations while the latter may not. On the other hand, TIO $\not\subset$ MIO because TIO allows for decoherence-free subspaces.

notion of asymmetry. The G-covariant operations are free in this resource theory, and in Ref. [21] these operations were also called translationally invariant operations (TIO).

Note that an asymmetry-based resource theory of coherence is categorically distinct from PIO/SIO/IO/DIO/MIO resource theories since in general the set of G-invariant states will not coincide with $\mathcal{I}$. The resource theories of PIO/SIO/IO/DIO/MIO are based on a *basis-dependent definition of coherence* which consists of first specifying an incoherent basis and then defining incoherent states as being diagonal in this basis. In contrast, the resource theory of TIO consists of first specifying a symmetry and then defining incoherent states as those possessing this symmetry. Consequently, the asymmetry approach can lead to decoherence-free subspaces when extending the symmetry to multiple systems, as we discuss in Sec. IV of this paper. A detailed discussion on the distinction between TIO and the family of operations PIO/SIO/IO/DIO/MIO can also be found in [14].

The purpose of this article is to provide a comparative investigation into the resource theories of coherence under different operational classes. This is an accompanying paper to Ref. [13] and covers the detailed proofs omitted from the latter. A summary of results and an outline of the paper is as follows.

(i) In Sec. I, we give a quantitative overview of the operational classes PIO/DIO/IO/DIO/ MIO. Aside from characterizing the structure of maps belonging to these classes, we focus on the ability of these maps to transform states. In particular, we consider the question of pure-state transformations using incoherent operations. We show that the so-called majorization condition decides transformation feasibility for the classes SIO and a special subclass of IO that we denote by sIO. However, whether or not the majorization condition also holds for IO remains an open problem and we point out mistakes in recent proofs claiming it does [22,23]. By constructing an explicit family of transformations, we show that the majorization condition can be violated by MIO, even stronger the Schmidt rank can be increased by MIO (Theorem 14).

(ii) For a general single-party state ρ, one can associate a bipartite maximally correlated state $\rho^{(mc)}$ with respect to a fixed incoherent basis according to

$$\rho = \sum_{xy} c_{xy}|x\rangle\langle y| \quad \Leftrightarrow \quad \rho^{(mc)} = \sum_{xy} c_{cy}|xx\rangle(5\langle yy|. \tag{1}$$

The question is then whether a transformation $\rho \to \sigma$ using one of the incoherent operational classes implies that the corresponding transformation $\rho^{(mc)} \to \sigma^{(mc)}$ is possible using local operations and classical communication (LOCC). We show that transforming states using PIO/SIO/sIO indeed implies the ability to transform the corresponding maximally correlated states using zero-communication LOCC/one-way LOCC/ two-way LOCC, respectively.

(iii) In Sec. II, we introduce a number of incoherent monotones and measures for the various operational classes based. All of these measures are unified within a very general framework for constructing incoherent measures. Two classes of measures included in this framework are the relative Rényi α entropies of incoherence and the quantum relative Rényi α entropies of incoherence. Within this class are the robustness of coherence and the Δ robustness of coherence.

(iv) In Sec. III, we provide a comprehensive overview of coherence in qubit systems. Necessary and sufficient conditions are proven for the transformation of qubit mixed states using SIO/IO/DIO/MIO, a result first reported in [13]. We show that all measures of coherence for qubits can be expressed in terms of the robustness of coherence and the Δ robustness, and we provide such expressions for the relative entropy of coherence, and the ℓ-1 norm of coherence.

(v) In Sec. IV, we discuss in greater detail the relationship between coherence resource theories based on asymmetry and those using a basis-dependent definition of coherence. We develop the resource theories of G asymmetry and N asymmetry, where G is the group of all incoherent unitaries and N is the group of all diagonal incoherent unitaries.

(vi) Finally, Sec. V describes a number of open problems related to the coherence measures and incoherent state transformations studied in this paper.

Throughout the paper we assume that an incoherent basis has been fixed and is taken as the computational basis. We consider d-dimensional quantum systems, and for bipartite systems the dimensions of the subsystems will be denoted by d_A and d_B. The map which completely dephases in the computational basis will be denoted by Δ, and its action is given by

$$\rho \mapsto \Delta(\rho) = \sum_{i=1}^{d} |i\rangle\langle i|\rho|i\rangle\langle i|. \tag{2}$$

I. FIVE TYPES OF INCOHERENT OPERATIONS

A. Physical incoherent operations (PIO)

The class of PIO is defined as the collection of operations so obtained via actions on a primary and an ancillary system that are noncoherence generating on both systems [13]. Denoting the primary system by A and the ancilla by B, a general PIO operation corresponds to performing an incoherent unitary U_{AB} on the input state ρ_A and some fixed incoherent state $\hat{\rho}_B$, and then performing a general incoherent projective measurement on system B. The Kraus operators for a general completely positive trace-preserving map (CPTP) belonging to PIO can be characterized by the following.

Proposition 1 ([13]). A CPTP map $\hat{\mathcal{E}}$ is a physically incoherent operation if and only if it can be expressed as a convex combination of maps each having Kraus operators $\{K_j\}_{j=1}^r$ of the form

$$K_j = U_j P_j = \sum_x e^{i\theta_x} |\pi_j(x)\rangle\langle x| P_j, \tag{3}$$

where the P_j form an orthogonal and complete set of incoherent projectors on system A and π_j are permutations.

State transformations

Proposition 1 shows that there is very little freedom in the allowable Kraus operators for a PIO map. The following lemma completely characterizes pure-state transformations by PIO.

Proposition 2. For any two states $|\psi\rangle$ and $|\phi\rangle$, the transformation $|\psi\rangle \to |\phi\rangle$ is possible by PIO if and only if

$$|\psi\rangle = \sum_{i=1}^k \sqrt{p_i} U_i |\phi\rangle, \tag{4}$$

where the U_i are incoherent isometries such that $P_i U_i |\phi\rangle = U_i |\phi\rangle$ for an orthogonal and complete set of incoherent projectors $\{P_i\}_i$.

Proof. Necessity of this condition follows from the form of K_j as given in Eq. (3). Since $K_j|\psi\rangle \propto |\phi\rangle$ for every j, we must have $\frac{1}{\sqrt{p_j}} U_j P_j |\psi\rangle = |\phi\rangle$. Thus,

$$\frac{1}{\sqrt{p_j}} P_j |\psi\rangle = U_j^\dagger |\phi\rangle = P_j U_j^\dagger |\phi\rangle.$$

Sufficiency of Eq. (4) can likewise be seen. Given the form of Eq. (4), one performs the incoherent projection $\{P_i\}_i$ on $|\psi\rangle$. Since $P_j U_i |\phi\rangle = 0$ for $i \neq j$, outcome P_j renders the post-measurement state $U_j|\phi\rangle$. The transformation is complete by applying $U_j^\dagger$. ■

A generic state $|\psi\rangle$ will not have a decomposition given by Eq. (4) for $k > 1$. Thus, most pure states cannot be transformed into any other outside of their respective incoherent unitary equivalence class. This situation is highly reminiscent of multipartite entanglement in which most pure states cannot be transformed to any another other outside their respective LU equivalence class.

In the asymptotic setting of many copies, the power of PIO is greatly improved. The following proposition shows that PIO is just as powerful as maximally incoherent operations (MIO) in terms of distilling maximally coherent bits $|+\rangle = \sqrt{1/2}(|0\rangle + |1\rangle)$ from many copies of a pure state. The optimal distillation rate under MIO is given by $S[\Delta(\psi)]$, where $S[\rho] = -\mathrm{tr}[\rho \log_2 \rho]$ is the von Neumann entropy [7].

Proposition 3 ([7]). For any $\epsilon > 0$ and n sufficiently large, the transformation $|\psi\rangle^{\otimes n} \to \overset{\epsilon}{\approx} |+\rangle^{\otimes \lfloor nR \rfloor}$ is possible by PIO whenever $R < S[\Delta(\psi)]$.

Proof. The proof for this is presented in Theorem 3 of Ref. [7] where the authors consider distillation using more general incoherent operations (IO). However, their protocol consists of incoherent unitaries and projections, and therefore it can be accomplished using PIO. ■

Rather surprisingly, the reverse transformation $|+\rangle^{\otimes m} \to \overset{\epsilon}{\approx} |\psi\rangle^{\otimes n}$ is not possible for any coherent state $|\psi\rangle$ that is not maximally coherent, i.e., if $|\psi\rangle$ is not of the form $\frac{1}{\sqrt{d}} \sum_{x=1}^d e^{i\theta_x}|x\rangle$. As described in the main text, a proof of this fact follows from communication complexity results in LOCC entanglement transformations. The key idea is that a PIO transformation $\rho \to \sum_j p_j \rho_j \otimes |j\rangle\langle j|$ can be converted into a bipartite LOCC transformation $\rho^{(mc)} \to \sum_j p_j \rho_j^{(mc)} \otimes |jj\rangle\langle jj|$ with no communication, where the correspondence between ρ_j and $\rho_j^{(mc)}$ is given by Eq. (1). Specifically, if $\{U_j P_j\}$ is the PIO measurement, then the corresponding LOCC protocol consists of Alice locally measuring $\{U_j P_j\}$, Bob learning the outcome of this measurement through the projective measurement $\{P_j\}$, and then him applying the corresponding U_j. Therefore, if $|+\rangle\langle +|^{\otimes m} \to \sum_j p_j |\psi_j\rangle\langle\psi_j| \otimes |j\rangle\langle j|$ by PIO with $\sum_j p_j |\psi_j\rangle\langle\psi_j| \overset{\epsilon}{\approx} |\psi\rangle\langle\psi|^{\otimes n}$ for arbitrarily small ϵ and m is sufficiently large, then it is possible to transform sufficiently large copies of an EPR state arbitrarily close to $|\psi^{(mc)}\rangle^{\otimes n}$ by local operations and no communication. However, as proven in Refs. [24,25], for any fixed n, there exists an ϵ-dependent lower bound on the communication needed to perform such an entanglement dilution, provided $|\psi^{(mc)}\rangle$ is not maximally entangled or a product state.

From this result we see that maximally coherent states are the weakest among all pure states, in terms of their ability to transform into other states. Under asymptotic PIO, the entire hierarchy of coherent states gets turned upside down.

B. Strictly incoherent operations (SIO)

The class of SIO is defined as the collection of operations so obtained via actions on a primary and an ancillary system that are noncoherence generating on just the primary system. Note the difference in description between SIO and PIO as stated above. A precise definition of SIO is given in terms of Kraus operator representations as follows.

Definition 1 ([7,9]). Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is said to be a strictly incoherent operation (SIO) if it can be represented by Kraus operators $\{M_j\}$ such that

$$\Delta(M_j \rho M_j^\dagger) = M_j \Delta(\rho) M_j^\dagger \quad \forall\, j, \forall\, \rho. \tag{5}$$

The following lemma characterizes the form of Kraus operators belonging to an SIO CPTP map.

Lemma 4. Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is SIO if and only if it can be represented by Kraus operators $\{M_j\}$ of the form

$$M_j = \sum_{x=1}^{d_A} c_{jx} |\pi_j(x)\rangle\langle x|. \tag{6}$$

Proof. Sufficiency is obvious to check. Suppose now that $\mathcal{E}^{A\to B}$ is SIO. Following the same arguments of Lemma 12, there must exist Kraus operators $\{M_j\}$ with the properties that

$$\Delta(M_j|x\rangle\langle x|M_j^\dagger) = M_j|x\rangle\langle x|M_j^\dagger \quad \text{and} \tag{7}$$

$$\Delta(M_j|x\rangle\langle x'|M_j^\dagger) = 0 \tag{8}$$

for all $x',x \in \{1,\dots,d_A\}$ with $x' \neq x$. Equation (7) implies that

$$M_j = \sum_{x=1}^{d_A} c_{j,x}|f_j(x)\rangle\langle x|, \tag{9}$$

where $f_j : \{1,\dots,d_A\} \to \{1,\dots,d_A\}$. Equation (8) implies that

$$\langle y|M_j|x\rangle\langle x'|M_j^\dagger|y\rangle = 0 \quad \forall\, x,x',y, \tag{10}$$

which is equivalent to the condition that f_j is one to one. Thus, f_j is a permutation π_j and M_j takes the form of Eq. (6). ■

1. Relating SIO to maximally correlated LOCC

The discussion after Proposition 3 describes how every PIO operation can be translated into a zero-communication LOCC protocol. A similar relationship holds for SIO and one-way LOCC.

Proposition 5. Using the notation of Eq. (1), if $\rho \to \sigma$ by SIO, then there exists a bipartite LOCC transformation $\rho^{(mc)} \to \sigma^{(mc)}$.

Proof. Let $\{M_j\}$ be a set of SIO Kraus operators so that for state $\rho = \sum_{xy} d_{xy}|x\rangle\langle y|$ the QC post-measurement state is

$$\begin{aligned}\sigma &= \sum_j M_j\rho M_j^\dagger \otimes |j\rangle\langle j| \\ &= \sum_{x,y} c_{jx}c_{jy}^* d_{xy}|\pi_j(x)\rangle\langle\pi_j(y)| \otimes |j\rangle\langle j|,\end{aligned} \tag{11}$$

where we have used Eq. (6). Then, the transformation $\rho^{(mc)} \to \sigma^{(mc)}$ can be accomplished by Alice performing the measurement $\{M_j\}$, announcing her result "j" to Bob, and then Bob performing the local permutation Π_j:$|x\rangle\to|\pi_j(x)\rangle$. ■

2. State transformations

Using Proposition 5, we can completely classify pure state transformations under SIO. The following is an analog to Nielsen's theorem for entanglement transformations of bipartite pure states [26]. Consider two states

$$|\psi\rangle = \sum_{i=1}^m \sqrt{\psi_i^\downarrow}|i\rangle, \qquad |\phi\rangle = \sum_{i=1}^n \sqrt{\phi_i^\downarrow}|i\rangle$$

where we have assumed without loss of generality that the $\psi_i^\downarrow$ are non-negative and ordered such that $\psi_i^\downarrow \geqslant \psi_{i+1}^\downarrow$, and likewise for the $\phi_i^\downarrow$. We say that $|\phi\rangle$ majorizes $|\psi\rangle$ [denoted by $\vec{\tau}(\psi) \prec \vec{\tau}(\phi)$] if $\sum_{i=1}^k \psi_i^\downarrow \leqslant \sum_{i=1}^k \phi_i^\downarrow$ for all $k = 1,\dots,\max\{m,n\}$, where a sufficient number of zeros are padded to the vector of shorter length so that both summations can be taken over $\max\{m,n\}$ elements.

Lemma 6. The state transformation $|\psi\rangle \to |\phi\rangle$ is possible by SIO iff $\vec{\tau}(\psi) \prec \vec{\tau}(\phi)$.

Proof. Sufficiency: Suppose that $\vec{\tau}(\psi) \prec \vec{\tau}(\phi)$. Then, there exists a doubly stochastic matrix D such that $\vec{\tau}(\psi) = D\vec{\tau}(\phi)$ [27]. Birkhoff's theorem assures that $D = \sum_\alpha p_\alpha \Pi_\alpha$, where the p_α form a probability distribution and the Π_α are permutation matrices. Then, define the operators $M_\alpha := \sqrt{p_\alpha}\Pi_\alpha^\dagger \bullet S$, where the elements of S are given by $[[S]]_{ij} = \sqrt{\phi_i}/\sqrt{\psi_j}$ and "$\bullet$" denotes the Hadamard product. Recall that the Hadamard product of two matrices A and B is the matrix $A \bullet B$ with elements $[[A \bullet B]]_{ij} = [[A]]_{ij}[[B]]_{ij}$. Note that each M_α has the form of Eq. (6). By construction $M_\alpha|\psi\rangle \propto |\phi\rangle$ for every α, and the relation $\vec{\tau}(\psi) = \sum_\alpha p_\alpha \Pi_\alpha \vec{\tau}(\phi)$ readily implies that $\sum_\alpha M_\alpha^\dagger M_\alpha = \mathbb{I}$.

Necessity: Now, suppose that $|\psi\rangle \to |\phi\rangle$ by SIO. By Proposition 5, this means that $|\psi^{(mc)}\rangle \to |\phi^{(mc)}\rangle$ by bipartite LOCC. However, a necessary condition for this is that $\vec{\tau}(\psi) \prec \vec{\tau}(\phi)$ [26]. ■

By the same arguments, additional statements about SIO pure-state transformations can be made that are analogous to statements in bipartite LOCC. The following are the coherence versions of the results presented in [28] and [29], respectively.

Proposition 7. The multioutcome transformation $|\psi\rangle \to \{|\phi_i\rangle, p_i\}$ is possible by SIO iff $\vec{\tau}(\psi) \prec \sum_i p_i\vec{\tau}(\phi_i)$.

Proposition 8. The maximum probability of converting $|\psi\rangle \to |\phi\rangle$ is given by

$$\min_{k\in\{1,\dots,\max\{m,n\}\}} \frac{\sum_{i=k}^n \psi_i^\downarrow}{\sum_{i=k}^n \phi_I^\downarrow}. \tag{12}$$

With Lemma 6, the asymptotic transformation of pure states becomes reversible under SIO. Indeed, the dilution protocol described in Ref. [7] relies on being able to perform any pure-state transformation provided the majorization condition is satisfied. We thus have the following.

Corollary 9 ([7]). For any $\epsilon > 0$ and n sufficiently large, the transformation $|\psi\rangle^{\otimes\lfloor nR\rfloor} \to\overset{\epsilon}{\approx} |\varphi\rangle^{\otimes n}$ is possible whenever $R < S[\Delta(\psi)]/S[\Delta(\varphi)]$.

C. Incoherent operations (IO)

The incoherent operations of Baumgratz *et al.* have received a considerable amount of attention in the resource-theoretic development of quantum coherence. Physically, these can be seen as generalized measurements performed on a quantum system that are coherence nongenerating for each measurement outcome; however, their physical implementation may require performing a coherence-generating unitary across the primary system and the ancillary system. Formally, their definition is given by the following.

Definition 2 ([3]). Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is said to be an incoherent operation (IO) if it can be represented by Kraus operators $\{M_\alpha\}$ such that

$$\Delta(M_\alpha|x\rangle\langle x|M_\alpha^\dagger) = M_\alpha|x\rangle\langle x|M_\alpha^\dagger \quad \forall\, x. \tag{13}$$

From this definition, it is easy to see that an arbitrary incoherent measurement has Kraus operators $\{M_\alpha\}_\alpha$ of the form

$$M_\alpha = \sum_{i=1}^d c_{\alpha,i}|f_\alpha(i)\rangle\langle i|, \tag{14}$$

where $f_\alpha : \{1,\dots,d\} \to \{1,\dots,d\}$ and the completion identity demands

$$\sum_{\substack{\alpha \text{ such that}\\ f_\alpha(i)=f_\alpha(j)}} c^*_{\alpha,i} c_{\alpha,j} = \delta_{ij}. \tag{15}$$

Note that we could further decompose the sum as

$$\sum_{\substack{\alpha \text{ such that}\\ f_\alpha(i)=f_\alpha(j)}} c^*_{\alpha,i} c_{\alpha,j} = \sum_{k=1}^{d} \sum_{\substack{\alpha \text{ such that}\\ i,j\in f_\alpha^{-1}(k)}} c^*_{\alpha,i} c_{\alpha,j} = \delta_{ij}. \tag{16}$$

Let us comment on the problem of transforming pure states using incoherent operations. It has been reported that the majorization condition characterizes pure-state transformations under IO, i.e., that Lemma 6 can be extended to IO [22,23]. However, as we now discuss, the proofs given in these references are not correct. It is still an open question whether $\vec{\tau}(\psi) \prec \vec{\tau}(\phi)$ is necessary for an IO transformation $|\psi\rangle \to |\phi\rangle$.

1. Mistakes in the majorization proofs

To explain the mistake made in Ref. [22], let us begin by studying the action of an incoherent operator M_α on a state $|\psi\rangle = \sum_i \psi_i |i\rangle$. We have

$$M_\alpha|\psi\rangle = \sum_{k=1}^{d} \left(\sum_{i\in f^{-1}(k)} c_{\alpha,i}\psi_i \right) |k\rangle. \tag{17}$$

What interests us are the diagonal elements of $\Delta(\sum_\alpha M_\alpha|\psi\rangle\langle\psi|M_\alpha^\dagger)$. They have undergone the transformation

$$\begin{aligned}(|\psi_k|^2)_k &\to \left[\sum_\alpha \left(\sum_{i\in f_\alpha^{-1}(k)} c_{\alpha,i}\psi_i \right) \left(\sum_{j\in f_\alpha^{-1}(k)} c^*_{\alpha,j}\psi^*_j \right) \right]_k \\ &= \left[\sum_\alpha \left(\sum_{i,j\in f_\alpha^{-1}(k)} \psi_i\psi^*_j c_{\alpha,i}c^*_{\alpha,j} \right) \right]_k \\ &= \left(\sum_{i,j} \psi_i\psi^*_j \sum_{\substack{\alpha \text{ such that}\\ i,j\in f_\alpha^{-1}(k)}} c_{\alpha,i}c^*_{\alpha,j} \right)_k. \end{aligned} \tag{18}$$

In Ref. [22], the authors assume that for each value of k, the cross terms vanish. In other words, the assumption is that

$$\sum_{\substack{\alpha \text{ such that}\\ i,j\in f_\alpha^{-1}(k)}} c_{\alpha,i}c^*_{\alpha,j} = \delta_{ij}$$

when, in fact, the full condition is given by Eq. (16).

To bring this out more explicitly, we adopt the notation used in [22]. From the completion identity, Eq. (18) of [22] gives

$$\sum_n (\delta_{1,i(2)}\delta_{1,i(3)} + \delta_{2,i(2)}\delta_{2,i(3)})\overline{k_2^{(n)}}k_3^{(n)} = 0. \tag{19}$$

Note here the authors are assuming that the $\delta_{j,i(l)}$ do depend on n, which is not true in general. Nevertheless, let us momentarily continue with the argument with $\delta_{j,i(l)}$ being independent of n. Because the measurement is incoherent, we have that

$$\begin{aligned} \delta_{1,i(2)}\delta_{1,i(3)} \neq 0 \quad &\Rightarrow \quad \delta_{2,i(2)}\delta_{2,i(3)} = 0, \\ \delta_{2,i(2)}\delta_{2,i(3)} \neq 0 \quad &\Rightarrow \quad \delta_{1,i(2)}\delta_{1,i(3)} = 0. \end{aligned} \tag{20}$$

This means that Eq. (19) implies

$$\sum_n \delta_{1,i(2)}\delta_{1,i(3)}\overline{k_2^{(n)}}k_3^{(n)} = \sum_n \delta_{2,i(2)}\delta_{2,i(3)}\overline{k_2^{(n)}}k_3^{(n)} = 0. \tag{21}$$

Therefore, when computing $\sum_n |\dots|^2$ in their Eq. (21), the left-hand side of the second equation becomes

$$\begin{aligned} &\sum_n \left|\delta_{2,i(2)}k_2^{(n)}\psi_2 + \delta_{2,i(3)}k_3^{(n)}\psi_3\right|^2 \\ &\quad= \delta_{2,i(2)}\psi_2^2 + \delta_{2,i(3)}\psi_3^2 \\ &\qquad+\psi_2\psi_3 \sum_n \delta_{2,i(2)}\delta_{2,i(3)}\left(\overline{k_2^{(n)}}k_3^{(n)} + \overline{k_3^{(n)}}k_2^{(n)}\right) \\ &\quad= \delta_{2,i(2)}\psi_2^2 + \delta_{2,i(3)}\psi_3^2, \end{aligned} \tag{22}$$

where we use Eq. (21). But, now let us consider the most general IO measurement by allowing $\delta_{j,i(l)}$ to depend on n. That is, we make the replacement $\delta_{j,i(j)} \to \delta^{(n)}_{j,i(j)}$. Then, Eq. (19) becomes

$$\sum_n \left(\delta^{(n)}_{1,i(2)}\delta^{(n)}_{1,i(3)} + \delta^{(n)}_{2,i(2)}\delta^{(n)}_{2,i(3)}\right)\overline{k_2^{(n)}}k_3^{(n)} = 0. \tag{23}$$

However, we no longer have Eq. (21) because of the dependence on n. In other words, in general $\sum_n \delta_{2,i(2)}\delta_{2,i(3)}\overline{k_2^{(n)}}k_3^{(n)} \neq 0$. Therefore,

$$\begin{aligned} &\sum_n \left|\delta^{(n)}_{2,i(2)}k_2^{(n)}\psi_2 + \delta^{(n)}_{2,i(3)}k_3^{(n)}\psi_3\right|^2 \\ &\quad= \sum_n \delta^{(n)}_{2,i(2)}\left|k_2^{(n)}\right|^2\psi_2^2 + \sum_n \delta^{(n)}_{2,i(3)}\left|k_3^{(n)}\right|^2\psi_3^2 \\ &\qquad+ \psi_2\psi_3 \sum_n \delta^{(n)}_{2,i(2)}\delta^{(n)}_{2,i(3)}\left(\overline{k_2^{(n)}}k_3^{(n)} + \overline{k_3^{(n)}}k_2^{(n)}\right). \end{aligned} \tag{24}$$

The cross term no longer vanishes.

An alternative proof for the majorization condition was presented in Ref. [23]. The proof technique used is similar to the proof of Lemma 6 in which the incoherent transformation is mapped to a bipartite LOCC pure-state transformation. However, the LOCC measurement described in that paper is not trace preserving, and it is not clear how this can be remedied [30].

2. Majorization for a special subclass of IO

At the present, it remains unknown whether or not the majorization criterion dictates the feasibility of pure-state transformations by IO. However, we can introduce yet another class of operations more general than SIO for which majorization precisely captures pure-state convertibility.

Definition 3. Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is said to be a special incoherent operation (sIO) if it can be represented by Kraus operators $\{M_\alpha\}$ each having the form

$$M_\alpha = \sum_x c_{\alpha x}\Pi_\alpha|f(x)\rangle\langle x|, \tag{25}$$

where $f : \{1,\dots,d\} \to \{1,\dots,d\}$ and Π_α is a permutation. Note that SIO $\subset$ sIO $\subset$ IO.

We first show that the statement of Proposition 5 can be extended to sIO operations. However, the corresponding LOCC transformation now uses two-way classical communication.

Proposition 10. If $\rho \to \sigma$ by sIO, then there exists a bipartite two-way LOCC transformation $\rho^{(mc)} \to \sigma^{(mc)}$.

Proof. Suppose that $\rho \to \sigma = \sum_\alpha M_\alpha \rho M_\alpha^\dagger \otimes |\alpha\rangle\langle\alpha|$ for sIO Kraus operators $\{M_\alpha\}$ given by Eq. (25). Let $S \subset \{1,\ldots,d\}$ denote the range of f, $\kappa_s = |f^{-1}(s)|$ for $s \in S$, and $\kappa = \prod_{s\in S}\kappa_s$. For each $s \in S$, let $\{|s,j_s\rangle : j_s = 0,\ldots,|f^{-1}(s)|-1\}$ be a relabeling of the kets $|x\rangle$ with $x \in f^{-1}(s)$. Next, we want to define a generalized Hadamard basis with respect to the $|s,j_s\rangle$:

$$\left\{ |\widetilde{s,k_s}\rangle := \sum_{j_s=0}^{\kappa_s-1} e^{i2\pi j_s k_s/\kappa_s} |s,j_s\rangle \right\}_{k_s=0,\ldots,\kappa_s-1}.$$

Finally, for every sequence $\vec{k} = (k_1,k_2,\ldots,k_{|S|})$ with $k_s \in \{0,\ldots,\kappa_s-1\}$, define the operator

$$N_{\vec{k}} = \frac{1}{\sqrt{\kappa}} \sum_{s=1}^{|S|} |s\rangle\langle\widetilde{s,k_s}|. \tag{26}$$

It can be seen that $\sum_{\vec{k}} N_{\vec{k}}^\dagger N_{\vec{k}} = \mathbb{I}$. The LOCC protocol then consists of Bob first performing the measurement $\{N_{\vec{k}}\}_{\vec{k}}$. The state transformation corresponding to outcome $\vec{k} = (k_s)_{s=1}^{|S|}$ is

$$\begin{aligned}
\rho^{(mc)} &= \sum_{xy} d_{xy} |xx\rangle\langle yy| \\
&= \sum_{ss'} \sum_{j_s,j_{s'}} d_{sj_s,s'j_{s'}} |s,j_s\rangle\langle s',j_{s'}|_A \otimes |s,j_s\rangle\langle s',j_{s'}|_B \\
&\to \propto \sum_{ss'} \sum_{j_s,j_{s'}} d_{sj_s,s'j_{s'}} e^{i2\pi(j_s-j_{s'})k_s/\kappa_s} \\
&\quad \times |s,j_s\rangle\langle s',j_{s'}|_A \otimes |s\rangle\langle s'|_B.
\end{aligned} \tag{27}$$

Bob then announces his outcome $\vec{k} = (k_s)_{s=1}^{|S|}$ to Alice who subsequently performs the unitary

$$U_{\vec{k}} = \sum_s \sum_{j_s} e^{-i2\pi j_s k_s/\kappa_s} |s,j_s\rangle\langle s,j_s|. \tag{28}$$

At this stage, Alice and Bob share the state

$$\hat{\rho}^{(mc)} = \sum_{xy} d_{x,y} |x\rangle\langle y|_A \otimes |f(x)\rangle\langle f(y)|_B, \tag{29}$$

regardless of Bob's outcome $\vec{k}$. Alice now locally performs the sIO measurement $\{M_\alpha\}$. She announces her result to Bob who then performs the conditional permutation Π_α on his system. Thus, the resulting QC state is

$$\begin{aligned}
\sigma^{(mc)} &= \sum_{xy} d_{x,y} c_{\alpha,x} c_{\alpha,y}^* \\
&\quad \times (\Pi_\alpha \otimes \Pi_\alpha)|f(x)f(x)\rangle\langle f(y)f(y)|_{A_1B_1}(\Pi_\alpha \otimes \Pi_\alpha) \\
&\quad \otimes |\alpha\alpha\rangle\langle\alpha\alpha|_{A_2B_2}.
\end{aligned} \tag{30}$$

■

Corollary 11. The state transformation $|\psi\rangle \to |\phi\rangle$ is possible by sIO iff $\vec{\tau}(\psi) \prec \vec{\tau}(\phi)$.

D. Dephasing-covariant incoherent operations (DIO)

We next introduce a class of operations that generalizes SIO. Notice that SIO is defined in terms of the Kraus operators of a generalized measurement and their covariance with the completely dephasing channel. But, what if one looks more generally at CPTP maps and not just specific Kraus operator representations? DIO represents the class of all CPTP maps that possess covariance with the completely dephasing channel.

Definition 4. Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is said to be a dephasing-covariant incoherent operation (DIO) if

$$[\Delta, \mathcal{E}^{A\to B}] = 0 \tag{31}$$

which is equivalent to

$$\Delta(\mathcal{E}^{A\to B}(\rho)) = \mathcal{E}^{A\to B}(\Delta(\rho)) \ \ \forall\ \rho. \tag{32}$$

The following provides an alternative characterization of DIO maps that is computationally convenient.

Lemma 12. Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is DIO if and only if for all $x',x \in \{1,\ldots,d_A\}$ with $x' \neq x$:

$$\mathcal{E}^{A\to B}(|x\rangle\langle x|) \in \mathcal{I} \quad \text{and} \tag{33}$$

$$\Delta(\mathcal{E}^{A\to B}(|x\rangle\langle x'|)) = 0. \tag{34}$$

Proof. The first condition in the equation above ensures that $\mathcal{E}^{A\to B}$ is a MIO. Therefore, this is a necessary condition. The second condition is also necessary since

$$\begin{aligned}
&\Delta(\mathcal{E}^{A\to B}(|x\rangle\langle x'|)) \\
&\quad = \mathcal{E}^{A\to B}(\Delta(|x\rangle\langle x'|)) = \mathcal{E}^{A\to B}(0) = 0.
\end{aligned}$$

Now, to see that these two conditions are sufficient, note that any density matrix ρ acting on $\mathcal{H}^A$ can be decomposed as

$$\rho = \Delta(\rho) + Z, \tag{35}$$

where Z is a Hermitian matrix with zeros on the diagonal. We therefore have

$$\begin{aligned}
\Delta(\mathcal{E}^{A\to B}(\rho)) &= \Delta(\mathcal{E}^{A\to B}(\Delta(\rho))) + \Delta(\mathcal{E}^{A\to B}(Z)) \\
&= \mathcal{E}^{A\to B}(\Delta(\rho)) + \Delta(\mathcal{E}^{A\to B}(Z)) \\
&= \mathcal{E}^{A\to B}(\Delta(\rho)),
\end{aligned} \tag{36}$$

where the second equality follows from (33), and the third equality follows from (34). Hence, $\mathcal{E}^{A\to B}$ is DIO iff (33) and (34) hold. ■

Note that if we denote by

$$\mathbf{v}_{y|x} \equiv \begin{pmatrix} \langle y|M_1|x\rangle \\ \langle y|M_2|x\rangle \\ \vdots \\ \langle y|M_m|x\rangle \end{pmatrix} \in \mathbb{C}^m, \tag{37}$$

we get the following corollary:

Corollary 13. Using the notation of (37), a CPTP map $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ is a DIO if and only if there exist conditional probabilities $r_{y|x}$ such that

$$\mathbf{v}^\dagger_{y'|x}\mathbf{v}_{y|x} = r_{y|x}\delta_{yy'}, \tag{38}$$

$$\mathbf{v}^\dagger_{y|x}\mathbf{v}_{y|x'} = r_{y|x}\delta_{xx'}. \tag{39}$$

Consider now the equation $\sigma = \mathcal{E}(\rho)$ where $\mathcal{E}$ is DIO. We therefore have

$$\sigma_{yy'} = \sum_{x,x'} \rho_{xx'}\langle y|\mathcal{E}(|x\rangle\langle x'|)|y'\rangle. \tag{40}$$

In the notations above, this is equivalent to

$$\sigma_{yy'} = \sum_{x,x'} \rho_{xx'}\mathbf{v}^\dagger_{y'|x'}\mathbf{v}_{y|x}. \tag{41}$$

The diagonal terms have the form

$$\sigma_{yy} = \sum_x r_{y|x}\rho_{xx}. \tag{42}$$

E. Maximal incoherent operations (MIO)

We reach the final class of operations in our overview. These are simply the class of CPTP maps that act invariantly on the set of incoherent states. It is not difficult to see that this can be equivalently defined as follows.

Definition 5 ([1,12]). Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map. Then, $\mathcal{E}^{A\to B}$ is a maximal incoherent operation (MIO) if

$$\Delta \circ \mathcal{E}^{A\to B} \circ \Delta = \mathcal{E}^{A\to B} \circ \Delta. \tag{43}$$

Let $\mathcal{E}^{A\to B} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^B)$ be a CPTP map with an operator sum representation $\{M_j\}_{j=1}^m$, and let $\mathcal{M}$ denote the set of MIOs. Then, from the definition above, $\mathcal{E}^{A\to B} \in \mathcal{M}$ if and only if

$$\sum_{j=1}^m \langle y|M_j|x\rangle\langle x|M_j^\dagger|y'\rangle = 0 \tag{44}$$

for all $x \in \{1,\ldots,d_A\}$ and $y \neq y'$ with $y,y' \in \{1,\ldots,d_B\}$. Using the notation of (37) we get that then $\mathcal{E}^{A\to B} \in \mathcal{M}$ if and only if there exist $d_A d_B$ vectors $\mathbf{v}_{y|x} \in \mathbb{C}^m$, and conditional probability distribution $r_{y|x}$ (i.e., $r_{y|x} \geqslant 0$ and $\sum_y r_{y|x} = 1$) such that

$$\mathbf{v}^\dagger_{y'|x}\mathbf{v}_{y|x} = r_{y|x}\delta_{yy'}, \tag{45}$$

$$\sum_{y=1}^{d_B} \mathbf{v}^\dagger_{y|x}\mathbf{v}_{y|x'} = \delta_{xx'}, \tag{46}$$

where the first equation follows from (44) and the second from $\sum_j M_j^\dagger M_j = I$.

State transformations

Consider a MIO CPTP map that converts $|\psi\rangle = \sum_x \sqrt{p_x}|x\rangle$ to $|\phi\rangle = \sum_y \sqrt{q_y}|y\rangle$. In this case, we have $|\phi\rangle\langle\phi| = \mathcal{E}(|\psi\rangle\langle\psi|)$, where $\mathcal{E}$ is MIO. Then, there must exist coefficients c_j such that $\sum_{j=1}^m |c_j|^2 = 1$ and $M_j|\psi\rangle = c_j|\phi\rangle$. Denoting $\mathbf{c} \equiv (c_j)_j \in \mathbb{C}^m$ gives

$$\sqrt{q_y}\mathbf{c} = \sum_x \sqrt{p_x}\mathbf{v}_{y|x} \quad \forall\ y. \tag{47}$$

Consider now the simpler case of $d_A = 2$. We will also assume that $q_y > 0$ and $d_B \geqslant 3$. The case $d_B = 2$ is a special case of the qubit mixed-state transformation to be discussed later. Denote by $r_y \equiv r_{y|0}$ and $t_y \equiv r_{y|1}$ the two probability distributions, and denote also $\mathbf{v}_{y|0} \equiv \mathbf{v}_y$ and $\mathbf{v}_{y|1} \equiv \mathbf{u}_y$. With these notations, conditions (45), (46), and (47) take the form

$$\begin{gathered}\mathbf{v}^\dagger_y\mathbf{v}_{y'} = r_y\delta_{yy'},\ \ \mathbf{u}^\dagger_y\mathbf{u}_{y'} = t_y\delta_{yy'},\\ \sum_{y=1}^{d_B}\mathbf{v}^\dagger_y\mathbf{u}_y = 0,\ \ \sqrt{q_y}\mathbf{c} = \sqrt{p_0}\mathbf{v}_y + \sqrt{p_1}\mathbf{u}_y.\end{gathered} \tag{48}$$

The last equation can be written as

$$\sqrt{p_1}\mathbf{u}_y = \sqrt{q_y}\mathbf{c} - \sqrt{p_0}\mathbf{v}_y. \tag{49}$$

Hence, we must have

$$\begin{aligned}p_1 t_y\delta_{yy'} =& p_1\mathbf{u}^\dagger_y\mathbf{u}_{y'} = \sqrt{q_y q_{y'}} + p_0 r_y\delta_{yy'}\\ &- \sqrt{p_0}(\sqrt{q_y}\mathbf{c}^\dagger\mathbf{v}_{y'} + \sqrt{q_{y'}}\mathbf{v}^\dagger_y\mathbf{c}),\end{aligned} \tag{50}$$

where we have used the normalization of $\mathbf{c}$ and the orthogonality of $\{\mathbf{v}_y\}$ and of $\{\mathbf{u}_y\}$. Therefore, after dividing both sides of the equation by $\sqrt{q_y q_{y'}}$ (which is nonzero) we get

$$1 = \sqrt{p_0}\left(\frac{\mathbf{c}^\dagger\mathbf{v}_{y'}}{\sqrt{q_{y'}}} + \frac{\mathbf{v}^\dagger_y\mathbf{c}}{\sqrt{q_y}}\right)\ \forall\ y \neq y' \tag{51}$$

and for $y = y'$

$$p_1 t_y = q_y + p_0 r_y - \sqrt{p_0 q_y}(\mathbf{c}^\dagger\mathbf{v}_y + \mathbf{v}^\dagger_y\mathbf{c}). \tag{52}$$

From (51) we get that

$$\sqrt{p_0}\frac{\mathbf{v}^\dagger_y\mathbf{c}}{\sqrt{q_y}} \equiv a, \tag{53}$$

where a is some complex number independent of y satisfying $a + \bar{a} = 1$. Substituting this into (52) we get

$$p_1 t_y = q_y + p_0 r_y - q_y. \tag{54}$$

This equation holds iff

$$p_0 = p_1 = \tfrac{1}{2} \quad \text{and} \quad t_y = r_y. \tag{55}$$

With these choices, the first equation of (48) gives

$$0 = \sum_{y=1}^{d_B}\mathbf{v}^\dagger_y\mathbf{u}_y = \sum_{y=1}^{d_B}\mathbf{v}^\dagger_y(\sqrt{2q_y}\mathbf{c} - \mathbf{v}_y), \tag{56}$$

which is equivalent to

$$1 = \sum_{y=1}^{d_B}\sqrt{2q_y}\mathbf{v}^\dagger_y\mathbf{c} = 2a. \tag{57}$$

We therefore conclude that

$$\mathbf{v}^\dagger_y\mathbf{c} = \sqrt{\frac{q_y}{2}}. \tag{58}$$

Since $q_y > 0$ we get that $\mathbf{v}_y \neq 0$ for all y and therefore $r_y > 0$ for all y. Together with the orthogonality relation of $\mathbf{v}_y$, this implies that the set of vectors $\{\frac{1}{\sqrt{r_y}}\mathbf{v}_y\}$ is orthonormal. Therefore, the number of Kraus operators m (which is the dimension of $\mathbf{v}_{y|x}$) must be at least d_B. Hence, the equation above gives

$$\sum_{y=1}^{d_B}\frac{q_y}{2r_y}=\sum_{y=1}^{d_B}\frac{\mathbf{c}^\dagger\mathbf{v}_y\mathbf{v}_y^\dagger\mathbf{c}}{r_y}\leqslant\mathbf{c}^\dagger\mathbf{c}=1.$$

A simple calculation shows that $\sum_y q_y/r_y$ obtains its minimum value when

$$r_y=\frac{\sqrt{q_y}}{\sum_{y'=1}^{d_B}\sqrt{q_{y'}}}. \tag{59}$$

Therefore, we get

$$1\geqslant\sum_{y=1}^{d_B}\frac{q_y}{2r_y}\geqslant\frac{1}{2}\left(\sum_{y=1}^{d_B}\sqrt{q_y}\right)^2. \tag{60}$$

We therefore arrive at the following theorem.

Theorem 14. Let $|\psi\rangle=\sqrt{p_0}|0\rangle+\sqrt{p_1}|1\rangle$ and $|\phi\rangle=\sum_{y=1}^{d_B}\sqrt{q_y}|y\rangle$, where $q_y>0$ and $d_B>2$. Then, $|\psi\rangle$ can be converted to $|\phi\rangle$ if and only if $p_0=p_1=\frac{1}{2}$ and

$$\sum_{y=1}^{d_B}\sqrt{q_y}\leqslant\sqrt{2}. \tag{61}$$

Proof. The necessity of this condition follows from the arguments above. To prove sufficiency, take $m=d_B+1$ and $\mathbf{v}_y=\sqrt{r_y}\mathbf{e}_y$, where $\{\mathbf{e}_y\}$ is the standard basis of $\mathbb{C}^m$, and r_y is given in (59). To be consistent with (58) we define for $j=1,\ldots,d_B$

$$c_j=\frac{\sqrt{q_j}}{\sqrt{2}\sum_{y=1}^{d_B}\sqrt{q_y}} \tag{62}$$

and for $j=d_B+1$ we define

$$c_{d_B+1}=\sqrt{1-\sum_{j=1}^{d_B+1}c_j^2}. \tag{63}$$

Note that the term inside the sum is positive due to (61). Finally, we define for $y=1,\ldots,d_B$

$$\mathbf{u}_y=\sqrt{2q_y}\mathbf{c}-\mathbf{v}_y. \tag{64}$$

With these choices, all the conditions in (48) are satisfied. This completes the proof. ■

Example 1. Consider the following two states:

$$|+\rangle=\sqrt{\tfrac{1}{2}}|0\rangle+\sqrt{\tfrac{1}{2}}|1\rangle \tag{65}$$

and

$$|\psi\rangle:=\sqrt{\frac{8}{9}}|0\rangle+\sqrt{\frac{1}{18}}|1\rangle+\sqrt{\frac{1}{18}}|2\rangle. \tag{66}$$

We show that the transformation $|+\rangle\to|\psi\rangle$ is achievable by maximally incoherent operations. Indeed, consider the following three Kraus operators:

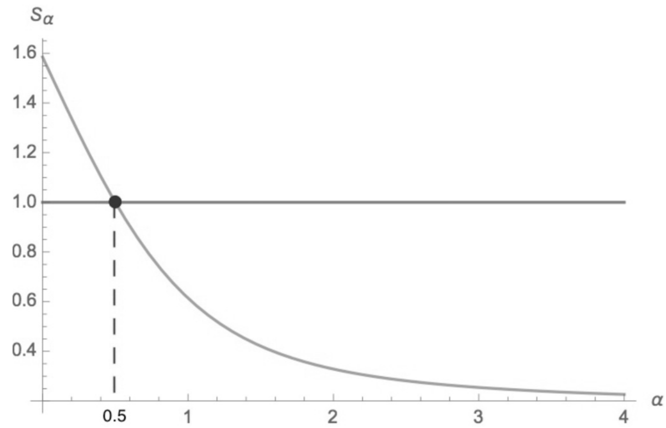

FIG. 2. Comparison of $S_\alpha(|+\rangle)=1$ (the horizontal line) and $S_\alpha(|\psi\rangle)$ (the curved line) as a function of α. For $0\leqslant\alpha<\frac{1}{2}$, $S_\alpha(|\psi\rangle)>S_\alpha(|+\rangle)$, and for $\alpha>\frac{1}{2}$, $S_\alpha(|\psi\rangle)<S_\alpha(|+\rangle)$.

$$M_1=\frac{\sqrt{2}}{3\sqrt{3}}\begin{pmatrix}3&1\\0&1\\0&1\end{pmatrix}, \tag{67}$$

$$M_2=\frac{1}{3\sqrt{6}}\begin{pmatrix}0&4\\3&-2\\0&1\end{pmatrix}, \tag{68}$$

$$M_2=\frac{1}{3\sqrt{6}}\begin{pmatrix}0&4\\0&1\\3&-2\end{pmatrix}. \tag{69}$$

It is straightforward to check that $\sum_{j=1}^3 M_j^\dagger M_j=I_2$ where I_2 is the 2×2 identity matrix. Furthermore, note that

$$M_j|+\rangle\propto 4|0\rangle+|1\rangle+|2\rangle\propto|\psi\rangle\quad\forall\ j=1,2,3. \tag{70}$$

To see that it is a maximal incoherent operation, note that

$$\sum_{j=1}^{3}M_j|0\rangle\langle 0|M_j^\dagger=\sum_{j=1}^{3}M_j|1\rangle\langle 1|M_j^\dagger=\frac{1}{6}\begin{pmatrix}4&0&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{71}$$

In Fig. 2, we plot the Rényi entropies of these two states. From the graph it is clear that $S_\alpha(|\psi\rangle)>S_\alpha(|+\rangle)=1$ for $\alpha\in[0,\frac{1}{2})$. Therefore, this example also demonstrates that all the Rényi entropies with $\alpha\in[0,\frac{1}{2})$ are *not* monotones and therefore are not measures of coherence. Furthermore, it provides an independent proof that the Rényi divergences D_α and $D_\alpha^{(q)}$ do *not* satisfy the data processing inequality in the α ranges $(2,\infty]$ and $[0,\frac{1}{2})$, respectively.

II. FAMILY OF MONOTONES

In this section, we provide a general framework for constructing distance-based coherence monotones and discuss specific examples. Our main distinction will be functions that behave monotonically under MIO and those that behave monotonically under DIO.

Theorem 15. Let $D(\rho\parallel\sigma)$ be a contractive function, i.e., $D[\mathcal{E}(\rho)\parallel\mathcal{E}(\sigma)]\leqslant D(\rho\parallel\sigma)$ if $\mathcal{E}$ is a CPTP map. Let A_ρ be a set of density matrices acting on $\mathbb{C}^d$. Note that the set A_ρ can depend of the state ρ. If $\mathcal{E}(A_\rho) \subseteq A_{\mathcal{E}(\rho)}$ for all free operations $\mathcal{E}$, then the two functions

$$C_A^R(\rho) = \min_{\sigma \in A_\rho} D(\rho \parallel \sigma),$$
$$C_A^L(\rho) = \min_{\sigma \in A_\rho} D(\sigma \parallel \rho) \tag{72}$$

are monotonic under the set of free operations.

Proof.

$$\begin{aligned} C_A^R[\mathcal{E}(\rho)] &= \min_{\tau \in A_{\mathcal{E}(\rho)}} D[\mathcal{E}(\rho) \parallel \tau] \\ &\leqslant \min_{\tau \in \mathcal{E}(A_\rho)} D[\mathcal{E}(\rho) \parallel \tau] \\ &= \min_{\sigma \in A_\rho} D[\mathcal{E}(\rho) \parallel \mathcal{E}(\sigma)] \\ &\leqslant \min_{\sigma \in A_\rho} D(\rho \parallel \sigma) = C_A^R(\rho). \end{aligned} \tag{73}$$

Similar arguments prove that C_A^L is also a monotone. ■

A. MIO monotones

As a simple application of Theorem 15 take $A_\rho = \mathcal{I}$, the set of incoherent diagonal states. In this case, A_ρ is independent of ρ so we get trivially that

$$\mathcal{E}(A_\rho) = \mathcal{E}(\mathcal{I}) \subseteq \mathcal{I} = A_{\mathcal{E}(\rho)} \tag{74}$$

for any DIO (or MIO) $\mathcal{E}$. Moreover, in this case,

$$C_A^R(\rho) = \min_{\sigma \in \mathcal{I}} D(\rho \parallel \sigma), \tag{75}$$

which reduces to the the well-known relatively entropy of coherence [3] when take $D(\rho \parallel \sigma)$ to be the relative entropy. However, note that under PIO, SIO, IO, DIO, or MIO

$$C_A^L(\rho) = \min_{\sigma \in \mathcal{I}} D(\sigma \parallel \rho) \tag{76}$$

is also a monotone.

1. *Relative Rényi α monotones*

Beyond the relative Shannon entropy, one can consider the more general relative Rényi entropies. For $\alpha \in [0,\infty]$ the relative Rényi entropy is defined by

$$D_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log_2 \mathrm{tr}(\rho^\alpha \sigma^{1-\alpha}). \tag{77}$$

This quantity is contractive (or equivalently satisfies the data processing inequality) for all $\alpha \in [0,2]$. We will therefore be interested here only in this range of α. Define the α-coherence monotone by $(0 \leqslant \alpha \leqslant 2)$

$$C_\alpha(\rho) := \min_{\sigma \in \mathcal{I}} D_\alpha(\rho \parallel \sigma). \tag{78}$$

We can compute this monotone explicitly, and part of the following work overlaps with independent work conducted by Rastegin in Ref. [31]. Let $\sigma = \sum_x q_x |x\rangle\langle x|$ be some free state. Then,

$$C_\alpha(\rho) := \min_{\{q_x\}} \frac{1}{\alpha - 1} \log_2 \sum_x q_x^{1-\alpha} \langle x|\rho^\alpha|x\rangle. \tag{79}$$

Denote

$$r_x \equiv \frac{(\langle x|\rho^\alpha|x\rangle)^{1/\alpha}}{r} \quad \text{where} \quad r \equiv \sum_x (\langle x|\rho^\alpha|x\rangle)^{1/\alpha}. \tag{80}$$

By definition, $\sum_x r_x = 1$ and $r_x \geqslant 0$. Therefore,

$$\begin{aligned} C_\alpha(\rho) &= \frac{\alpha}{\alpha - 1} \log_2 r + \min_{\{q_x\}} \frac{1}{\alpha - 1} \log_2 \sum_x q_x^{1-\alpha} r_x^\alpha \\ &= \frac{\alpha}{\alpha - 1} \log_2 r + \min_{\{q_x\}} D_\alpha(\{r_x\} \parallel \{q_x\}) \\ &= \frac{\alpha}{\alpha - 1} \log_2 r, \end{aligned} \tag{81}$$

where $D_\alpha(\{r_x\} \| \{q_x\})$ is the classical Rényi divergence. We therefore conclude that for $\alpha \in [0,2]$, the quantities

$$C_\alpha(\rho) = \frac{\alpha}{\alpha - 1} \log_2 \sum_x (\langle x|\rho^\alpha|x\rangle)^{1/\alpha} \tag{82}$$

are coherence monotones. Note that in the limit $\alpha \to 1$ we get $C_\alpha(\rho) \to C_{rel}(\rho)$. Furthermore, in terms of the completely dephasing map $\Delta(\rho) := \sum_x \langle x|\rho|x\rangle\ |x\rangle\langle x|$, we have

$$\begin{aligned} C_\alpha(\rho) &= \frac{\alpha}{\alpha - 1} \log_2 \mathrm{tr}[(\Delta(\rho^\alpha))^{1/\alpha}] \\ &= \frac{1}{\alpha - 1} \log_2 \mathrm{tr}[\|\Delta(\rho^\alpha)\|_{1/\alpha}]. \end{aligned} \tag{83}$$

$C_\alpha(\rho)$ can also be written in terms of the eigenvalues of ρ as follows. Suppose the spectrum decomposition of ρ is given by

$$\rho = \sum_{y=1}^n \lambda_y |v_y\rangle\langle v_y|, \tag{84}$$

where λ_y are the eigenvalues of ρ, with corresponding eigenvectors $|v_y\rangle$. Denote by D the $n \times n$ doubly stochastic matrix whose elements are $D_{xy} \equiv |\langle x|v_y\rangle|^2$. Then, Eq. (83) takes the form

$$C_\alpha(\rho) = \frac{\alpha}{\alpha - 1} \log_2 \sum_x \left(\sum_y D_{xy} \lambda_y^\alpha \right)^{1/\alpha}. \tag{85}$$

Note that for a pure state $\rho = |\psi\rangle\langle\psi|$ we have

$$C_\alpha(\psi) = \frac{\alpha}{\alpha - 1} \log_2 \sum_j p_j^{1/\alpha} = S_{1/\alpha}(p), \tag{86}$$

where $S_{1/\alpha}$ is the Rényi entropy with parameter $1/\alpha \in [1/2,\infty]$.

Example 2. Consider $\alpha = 2$ in (83). Then, this monotone has a particular simple expression. Denoting by ρ_{xy} the components of ρ we get

$$\begin{aligned} C_{\alpha=2}(\rho) &= 2 \log_2 \sum_x \sqrt{\langle x|\rho^2|x\rangle} \\ &= 2 \log_2 \sum_x \left(\sum_y |\rho_{xy}|^2 \right)^{1/2}. \end{aligned} \tag{87}$$

We now apply this to the qubit case where

$$\rho = \begin{pmatrix} p & r \\ r & 1 - p \end{pmatrix}. \tag{88}$$

Then,

$$C_{\alpha=2}(\rho) = 2\log_2[\sqrt{p^2+r^2} + \sqrt{(1-p)^2+r^2}]. \quad (89)$$

2. Quantum relative Rényi α monotones

For $\alpha \in [\frac{1}{2},\infty]$, the quantum relative Rényi entropy is given by

$$D_\alpha^{(q)}(\rho \parallel \sigma) := \frac{1}{\alpha-1}\log_2 \mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]. \quad (90)$$

Define the quantum α-coherence monotone by

$$C_\alpha^{(q)}(\rho) := \min_{\sigma\in\mathcal{I}} D_\alpha^{(q)}(\rho \parallel \sigma). \quad (91)$$

The minimization in this case is harder to perform. However, for a pure state $\rho = |\psi\rangle\langle\psi|$ we have

$$D_\alpha^{(q)}(\rho \parallel \sigma) = \frac{1}{\alpha-1}\log_2 \mathrm{tr}\big[\big(\sigma^{\frac{1-\alpha}{2\alpha}}|\psi\rangle\langle\psi|\sigma^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]$$
$$= \frac{\alpha}{\alpha-1}\log_2\langle\psi|\sigma^{\frac{1-\alpha}{\alpha}}|\psi\rangle,$$

which is very similar to the expression we get for the relative Rényi entropy. We therefore conclude that for pure states

$$C_\alpha^{(q)}(\psi) = \frac{2\alpha-1}{\alpha-1}\log_2\left(\sum_j p_j^{\frac{\alpha}{2\alpha-1}}\right). \quad (92)$$

Denoting $\gamma \equiv \frac{\alpha}{2\alpha-1}$ we can rewrite the expression above as

$$C_\alpha^{(q)}(\psi) = \frac{1}{1-\gamma}\log_2\left(\sum_j p_j^\gamma\right) \equiv S_\gamma(\mathbf{p}). \quad (93)$$

Note that the range of γ is also $[\frac{1}{2},\infty]$. Also, the other two parameter quantum divergences introduced in [32] lead to the same Rényi entropies for pure states. Therefore, one may be tempted to conjecture that the transformation

$$|\psi\rangle \to |\phi\rangle \quad (94)$$

is possible by MIO if and only if

$$S_\alpha(\mathbf{p}) \geqslant S_\alpha(\mathbf{q}) \ \ \forall\, \alpha \in \left[\tfrac{1}{2},\infty\right], \quad (95)$$

where the probability vectors $\mathbf{p}$ and $\mathbf{q}$ correspond to $|\psi\rangle$ and $|\phi\rangle$, respectively. However, note that the requirements $p_0 = p_1 = \frac{1}{2}$ in Theorem 14 show that this conjecture is false. That is, the above equation is necessary but not sufficient for the existence of a MIO from $|\psi\rangle \to |\phi\rangle$.

Example 3. Consider the case $\alpha = \infty$ in (90). In this case, $D_\alpha^{(q)}$ is known to be equal to the max relative entropy given by

$$D_\infty^{(q)}(\rho \parallel \sigma) = \log_2 \min\{\lambda \ : \ \rho \leqslant \lambda\sigma\}. \quad (96)$$

The corresponding monotone is therefore

$$C_\infty^{(q)}(\rho) = \log_2 \min\left\{\mathrm{tr}(\sigma) \ : \ \rho \leqslant \sigma \ ; \ \frac{\sigma}{\mathrm{tr}(\sigma)} \in \mathcal{I}\right\}. \quad (97)$$

To calculate this expression, observe that it can be rewritten as

$$C_\infty^{(q)}(\rho) = \log_2 \min\{\mathrm{tr}(\sigma) \ : \ \rho \leqslant \Delta(\sigma) \ ; \ \sigma \geqslant 0\}. \quad (98)$$

Next, we recall the dual formulation in linear programming (see, e.g. Renes' paper on subrelative majorization [23], as well as recent work by Piani *et al.* [16]). Consider the following setting of linear programming. Let V_1 and V_2 be two (inner product) vector spaces with two cones $K_1 \subset V_1$ and $K_2 \subset V_2$. Consider two vectors $v_1 \in V_1$ and $v_2 \in V_2$, and a linear map $\mathcal{T} : V_1 \to V_2$. Then, the primal form

$$\max_{\substack{x\,\in\, K_1 \\ v_2 - \mathcal{T}(x)\,\in\, K_2}} \langle v_1, x\rangle_1. \quad (99)$$

The dual form involves $\mathcal{T}^* : V_2 \to V_1$:

$$\min_{\substack{y\,\in\, K_2 \\ \mathcal{T}^*(y) - v_1\,\in\, K_1}} \langle v_2, y\rangle_2. \quad (100)$$

Applying this to our formulation, take $V_1 = V_2 = H_n$ the vector space of $n\times n$ Hermitian matrices. Take $K_1 = K_2 = H_{n,+}$ as the cone of positive-semidefinite matrices in H_n. Take $\mathcal{T} = \Delta$ which is self-adjoint. Finally, take $v_2 = I$, $v_1 = \rho$, $y = \sigma$, $x = \tau$. With these choices the dual is our original expression for C_∞ and the primal is the following expression:

$$C_\infty^{(q)}(\rho) = \log_2 \max\{\mathrm{tr}(\rho\tau) \ : \ \Delta(\tau) \leqslant I \ ; \ \tau \geqslant 0\} \quad (101)$$
$$= \log_2 \max\{\mathrm{tr}(\rho\tau) \ : \ \Delta(\tau) = I \ ; \ \tau \geqslant 0\}. \quad (102)$$

Note that for $j \neq k$, $|\tau_{jk}| \leqslant 1$. Otherwise, if $|\tau_{jk}| > 1$, one can find $\theta \in [0,2\pi]$ such that for $|\psi\rangle = |j\rangle + e^{i\theta}|k\rangle$, the expectation value $\langle\psi|\tau|\psi\rangle < 0$. We therefore conclude that

$$\mathrm{tr}(\rho\tau) = 1 + \sum_{j\neq k}\rho_{jk}\tau_{kj} \leqslant 1 + \sum_{j\neq k}|\rho_{jk}|$$
$$= 1 + C_{\ell_1}(\rho), \quad (103)$$

where

$$C_{\ell_1}(\rho) = \sum_{j\neq k}|\rho_{jk}| \quad (104)$$

is the so-called ℓ_1 coherence measure [3]. This bound can be saturated in the case where ρ is real with non-negative off-diagonal terms, in which case we take $\tau = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_x |x\rangle$.

Note the relation between $C_\infty^{(q)}$ and the robustness of coherence C_R, which is defined as

$$C_R(\rho) = \min_{t\geqslant 0}\left\{t \ \middle| \ \frac{\rho + t\sigma}{1+t} \in \mathcal{I},\, \sigma \geqslant 0\right\}. \quad (105)$$

Letting $\hat{\sigma} = \rho + t\sigma$ so that $t = \mathrm{tr}[\hat{\sigma}] - 1$, we can rewrite this as

$$C_R(\rho) = \min_{\hat{\sigma}}\left\{\mathrm{tr}[\hat{\sigma}] - 1 \ \middle| \ \frac{\hat{\sigma}}{\mathrm{tr}[\hat{\sigma}]} \in \mathcal{I},\, \hat{\sigma} \geqslant \rho\right\}. \quad (106)$$

Putting everything together, we obtain the following:

Proposition 16.

$$C_\infty^{(q)}(\rho) = \log_2[1 + C_R(\rho)]. \quad (107)$$

Moreover, $C_R(\rho) = C_{\ell_1}(\rho)$ for pure states, qubit mixed states, and any state ρ with non-negative real matrix elements when expressed in the incoherent basis.

It is still an open problem whether C_{ℓ_1} is a MIO monotone in general, although it is a known monotone under IO [3].

B. DIO monotones

Next, we turn to DIO operations and consider DIO operations derived from Theorem 15. Take $A_\rho = \{\Delta(\rho)\}$ which contains only a single state. Note that under DIO $\mathcal{E}$ we have

$$\mathcal{E}(A_\rho) = \{\mathcal{E}(\Delta(\rho))\} = \{\Delta(\mathcal{E}(\rho))\} = A_{\mathcal{E}(\rho)}. \tag{108}$$

Therefore, both the functions

$$C_A^R(\rho) = D(\rho \parallel \Delta(\rho)), \quad C_A^L(\rho) = D(\Delta(\rho) \parallel \rho) \tag{109}$$

are monotones. If we take $D(\rho,\sigma) = \|\rho - \sigma\|$, where $\|\ldots\|$ is the trace norm, we get

$$C_A^R(\rho) = C_A^L(\rho) = \|\rho - \Delta(\rho)\| , \tag{110}$$

which is a function only of the off-diagonal terms.

If we choose D as in (77), then we get the following monotones:

$$\begin{aligned} C_\alpha^R(\rho) &= \frac{1}{\alpha-1}\log_2 \mathrm{tr}[\rho^\alpha(\Delta(\rho))^{1-\alpha}], \\ C_\alpha^L(\rho) &= \frac{1}{\alpha-1}\log_2 \mathrm{tr}[(\Delta(\rho))^\alpha \rho^{1-\alpha}]. \end{aligned} \tag{111}$$

For a pure state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_x \sqrt{p_x}|x\rangle$ we have

$$\begin{aligned} C_\alpha^R(\rho) &= \frac{1}{\alpha-1}\log_2\langle\psi|(\Delta(\rho))^{1-\alpha}|\psi\rangle \\ &= \frac{1}{\alpha-1}\log_2\sum_x p_x^{2-\alpha} \equiv \frac{1}{1-\gamma}\log_2\sum_x p_x^\gamma = S_\gamma(\rho), \end{aligned} \tag{112}$$

where we denoted $\gamma \equiv 2-\alpha$. Since the C_α^R is a DIO monotone for $\alpha \in [0,2]$, together with the fact that DIO $\subset$ MIO, we have that all Rényi entropies are DIO monotones. This is in contrast with the set of MIO for which S_γ is a monotone only for $\gamma \geqslant \frac{1}{2}$.

Robustness of coherence

To obtain another DIO monotone, take

$$A_\rho = \left\{ \frac{(1+t)\Delta(\rho)-\rho}{t} \;\middle|\; t>0\,;\;\; (1+t)\Delta(\rho)-\rho \geqslant 0 \right\}. \tag{113}$$

In this case, it is straightforward to check that $\mathcal{E}(A_\rho) \subseteq A_{\mathcal{E}(\rho)}$ for all $\mathcal{E} \in$ DIO. We consider the quantum Rényi relative entropy $C^{(q)}_{\Delta,\alpha}(\rho) := \min_{\sigma\in A_\rho} D^{(q)}(\rho \parallel \sigma)$. Then, in the limit $\alpha \to \infty$, we obtain analogs to Eqs. (96) and (98):

$$\begin{aligned} &C^{(q)}_{\Delta,\infty}(\rho) \\ &= \log_2 \min\left\{\mathrm{tr}(\sigma) \;\middle|\; \rho \leqslant \sigma;\;\; \frac{\sigma}{\mathrm{tr}(\sigma)} \in A_\rho\right\} \\ &= \min_{t,\lambda>0}\left\{\lambda \;\middle|\; \rho \leqslant \lambda\frac{(1+t)\Delta(\rho)-\rho}{t};\;\; (1+t)\Delta(\rho) \geqslant \rho\right\} \\ &= \min_{t,\lambda>0}\left\{\lambda \;\middle|\; \frac{t+\lambda}{\lambda}\rho \leqslant (1+t)\Delta(\rho);\;\; (1+t)\Delta(\rho) \geqslant \rho\right\} \\ &= \min_{t,\lambda>0}\left\{\lambda \;\middle|\; \frac{t+\lambda}{\lambda}\rho \leqslant (1+t)\Delta(\rho)\right\}, \end{aligned} \tag{114}$$

where the last equality follows from the fact that $\frac{t+\lambda}{\lambda} \geqslant 1$. Note that $0 \leqslant \lambda\frac{(1+t)\Delta(\rho)-\rho}{t} - \rho$, which means that $0 \leqslant (\lambda - 1)\Delta(\rho)$; thus, $\lambda \geqslant 1$. Then, the minimum above can be written as

$$\min_{t,\lambda>0}\left\{\lambda \;:\; \frac{t+\lambda}{1+t}\rho \leqslant \lambda\Delta(\rho)\right\}. \tag{115}$$

But, since $\frac{t+\lambda}{1+t} > 1$ we must have $\rho \leqslant \lambda\Delta(\rho)$. On the other hand, taking the limit $t \to \infty$ in the above minimum gives $\rho \leqslant \lambda\Delta(\rho)$. We therefore conclude that the above minimum is equal to

$$\min_{\lambda>0}\{\lambda \;:\; \rho \leqslant \lambda\Delta(\rho)\} \tag{116}$$

or equivalently

$$1 + \min_{t>0}\{t \;:\; \rho \leqslant (1+t)\Delta(\rho)\}. \tag{117}$$

Finally, note that $t \geqslant 0$ satisfies $\rho \leqslant (1+t)\Delta(\rho)$ iff there exists a matrix σ such that (i) $\frac{\rho+t\sigma}{1+t} \in \mathcal{I}$, (ii) $\sigma \geqslant 0$, and (iii) $\Delta(\sigma) = \Delta(\rho)$. Therefore, we have the Δ analog of Proposition 16:

$$C^{(q)}_{\Delta,\infty}(\rho) = \log_2[1 + C_{\Delta,R}(\rho)], \tag{118}$$

where $C_{\Delta,R}(\rho)$ is a quantity we shall call the Δ robustness of coherence:

$$\begin{aligned} &C_{\Delta,R}(\rho) \\ &:= \min\left\{t \geqslant 0 \;\middle|\; \frac{\rho+t\sigma}{1+t} \in \mathcal{I}\;,\; \sigma \geqslant 0\;,\; \Delta(\sigma) = \Delta(\rho)\right\}. \end{aligned} \tag{119}$$

By construction, $C_{\Delta,R}$ is a DIO monotone.

Example 4. Consider the qubit state

$$\rho = \begin{pmatrix} p & r \\ r & 1-p \end{pmatrix}. \tag{120}$$

Then, the matrix σ must have the form

$$\sigma = \begin{pmatrix} p & -\frac{r}{t} \\ -\frac{r}{t} & 1-p \end{pmatrix} \tag{121}$$

to ensure that $\rho + t\sigma$ is diagonal and $\Delta(\sigma) = \Delta(\rho)$. Now, the condition $\sigma \geqslant 0$ gives a lower bound on t. We therefore conclude that for $0 < p < 1$,

$$C_R(\rho) = \frac{r}{\sqrt{p(1-p)}} \tag{122}$$

and, otherwise, for $p = 0$ or $p = 1$, $C_R(\rho) = 0$.

The form of σ above can be generalized to any dimension. That is, for $\rho = \Delta(\rho) + Z$, σ must have the form

$$\sigma = \Delta(\rho) - \frac{1}{t}Z. \tag{123}$$

Hence, $C_R(\rho)$ equals the minimum values of $t \geqslant 0$ such that σ above is positive semidefinite. Note that the positivity of σ is equivalent to the positivity of

$$t\Delta(\rho) - Z = t\Delta(\rho) - [\rho - \Delta(\rho)] = (1+t)\Delta(\rho) - \rho. \tag{124}$$

We therefore arrive at the following expression for C_R:

$$\begin{aligned} C_{\Delta,R}(\rho) &= \min\{t \geqslant 0 \mid (1+t)\Delta(\rho) - \rho \geqslant 0\} \\ &= \max\left\{\frac{\langle\phi|\rho|\phi\rangle}{\langle\phi|\Delta(\rho)|\phi\rangle} \;\middle|\; |\phi\rangle \in \mathbb{C}^d \;,\; \langle\phi|\phi\rangle = 1\right\}. \end{aligned} \tag{125}$$

Theorem 17. Consider the linear map

$$\Phi_t(\rho) \equiv (1+t)\Delta(\rho) - \rho. \tag{126}$$

The following are equivalent:
(1) $\Phi_t(\rho)$ is positive,
(2) $\Phi_t(\rho)$ is completely positive,
(3) the parmeter $t \geqslant d-1$.
Proof. The Choi matrix

$$\begin{aligned} I &\otimes \Phi_t(|\psi^+\rangle\langle\psi^+|) \\ &= \sum_{j,k} |j\rangle\langle k| \otimes \Phi_t(|j\rangle\langle k|) \\ &= \sum_{j} |j\rangle\langle j| \otimes \Phi_t(|j\rangle\langle j|) + \sum_{j\neq k} |j\rangle\langle k| \otimes \Phi_t(|j\rangle\langle k|) \\ &= t\sum_{j} |j\rangle\langle j| \otimes |j\rangle\langle j| - \sum_{j\neq k} |j\rangle\langle k| \otimes |j\rangle\langle k| \\ &= (1+t)\sum_{j} |j\rangle\langle j| \otimes |j\rangle\langle j| - |\psi^+\rangle\langle\psi^+|. \end{aligned}$$

Finally, note that the last term is positive if and only if $1+t \geqslant d$. This completes the proof that (2) and (3) are equivalent. It is therefore left to show that (1) implies (3). To see it, note that

$$\Phi_t(|+\rangle\langle+|) = \frac{1+t}{d}I - |+\rangle\langle+|, \tag{127}$$

where $|+\rangle \equiv \frac{1}{\sqrt{d}}\sum_j |j\rangle$. Since we assume that Φ_t is positive, it follows that $1+t \geqslant d$. ■

Corollary 18. The function

$$R_D(\rho) := \log_2[1 + C_R(\rho)] \tag{128}$$

which we call *logarithmic robustness of dephasing* is a faithful measure of coherence [i.e., $R_D(\rho) = 0$ iff $\Delta(\rho) = \rho$] satisfying

$$0 \leqslant R_D(\rho) \leqslant \log_2 d. \tag{129}$$

Conjecture 19. R_D is additive. It is true for pure states (see below), unknown for mixed states.

Lemma 20. For a pure state $|\psi\rangle = \sum_{x=1}^n \sqrt{p_x}|x\rangle$, with $n \leqslant d$ and $p_x > 0$,

$$C_R(|\psi\rangle) = n - 1. \tag{130}$$

Proof. Let $|\phi\rangle = \sum_{x=1}^n \sqrt{q_x}e^{i\theta_x}|x\rangle$, then

$$\begin{aligned} \frac{\langle\psi|\rho|\psi\rangle}{\langle\psi|\Delta(\rho)|\psi\rangle} &= \frac{\sum_{x\neq x'}\sqrt{p_xq_xp_{x'}q_{x'}}e^{i(\theta_x-\theta_{x'})}}{\sum_x p_xq_x} \\ &\leqslant \frac{\sum_{x\neq x'}\sqrt{p_xq_xp_{x'}q_{x'}}}{\sum_x p_xq_x} \\ &= \mathbf{u}^\dagger A\mathbf{u}, \end{aligned} \tag{131}$$

where $\mathbf{u}$ is a unit vector in $\mathbf{C}^n$ with components

$$u_x \equiv \frac{\sqrt{p_xq_x}}{\sqrt{\sum_{x'=1}^n p_{x'}q_{x'}}} \tag{132}$$

and A is the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}. \tag{133}$$

Hence, by taking $\theta_x = 0$ and

$$q_x = \frac{1}{p_x}\bigg/\sum_{x'=1}^n \frac{1}{p_{x'}} \tag{134}$$

we get that $\mathbf{u} = \frac{1}{\sqrt{n}}(1,\ldots,1)^T$ corresponds to the maximal eigenvalue of A; i.e. for this choice $\mathbf{u}^\dagger A\mathbf{u} = n-1$. This completes the proof. ■

III. QUBIT COHERENCE

In this section we focus exclusively on maps whose input and output space consists of single-qubit density matrices. We will say that a qubit state ρ is in *standard form* when expressed as

$$\rho = \begin{pmatrix} p & r \\ r & 1-p \end{pmatrix}, \quad p \geqslant \frac{1}{2}, r \geqslant 0 \tag{135}$$

in the incoherent basis. Any state ρ can always be transformed into standard form by an incoherent unitary transformation, and thus each state can be uniquely parametrized by the tuple (p,r) with $p \geqslant \frac{1}{2}$, $r \geqslant 0$.

A. Channels: IO-MIO equivalence

The main result we prove here is that every MIO channel $\mathcal{E}$ has a Kraus operator implementation that belongs to IO.

Theorem 21. IO=MIO for CPTP maps $\mathcal{E} : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$.

Proof. Consider an arbitrary MIO CPTP map $\mathcal{E}$ with Kraus operator representation $\{M_j\}_{j=0}^t$. We want to prove that $\mathcal{E}$ has another Kraus operator representation with each operator having one of the forms given in Eq. (141). Since $\mathcal{E}$ is MIO, we have

$$\sum_{j=0}^{m-1}\langle y|M_j|x\rangle\langle x|M_j^\dagger|y\oplus 1\rangle = 0, \qquad \forall\, x,y \in \{0,1\}. \tag{136}$$

Our goal is to find another Kraus operator representation $\{\tilde{M}_j\}_{j=0}^{\tilde{t}}$ of the channel $\mathcal{E}$ such that

$$\langle y|\tilde{M}_j|x\rangle\langle x|\tilde{M}_j^\dagger|y\oplus 1\rangle = 0, \qquad \forall\, x,y \in \{0,1\}, \forall\, j. \tag{137}$$

We describe iteratively how this can always be done. In the following, recall that Kraus operators $\{\tilde{M}_j\}_{j=0}^{t}$ generate the same channel $\mathcal{E}$ iff $\tilde{M}_j = \sum_{k=0}^{m-1} u_{jk}M_k$ for some unitary matrix u_{jk}.

(1) Take $x=0$. Find two distinct values (j,j') such that $\langle 0|M_j|x\rangle\langle x|M_j^\dagger|1\rangle \neq 0$ and $\langle 0|M_{j'}|x\rangle\langle x|M_{j'}^\dagger|1\rangle \neq 0$; relabel

and denote these by $(j,j') = (0,1)$. If two distinct values cannot be found, then by Eq. (136) we must have that $\langle 0|M_j|x\rangle\langle x|M_j^\dagger|1\rangle = 0$ for all j, and in which case set $\tilde{M}_j = M_j$ for all j and proceed to step 4. Otherwise, proceed to step 2.

(2) Consider an $m \times m$ unitary matrix whose only nontrivial action consists of a 2×2 block $\begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$. Then, a different Kraus operator representation for $\mathcal{E}$ is realized by the elements $\tilde{M}_i = u_{i0}M_0 + u_{i1}M_1$ for $i = 0,1$ and $\tilde{M}_i = M_i$ for $i = 2,\dots,m-1$. The unitary matrix is chosen such that (u_{00},u_{01}) is the normalized vector of $(-\langle 0|M_1|x\rangle, \langle 0|M_0|x\rangle)$. With this choice, we have

$$\langle 0|\tilde{M}_0|x\rangle = u_{00}\langle 0|M_0|x\rangle + u_{01}\langle 0|M_1|x\rangle = 0. \tag{138}$$

(3) Repeat step 1 with the updated set of Kraus operators $\{\tilde{M}_0,\tilde{M}_1,\tilde{M}_i\}_{i=2}^{m-1}$.

(4) At this step in the procedure, we have a Kraus representation $\{\tilde{M}_j\}_{j=0}^{m-1}$ for $\mathcal{E}$ such that either $\langle 0|\tilde{M}_j|x\rangle = 0$ or $\langle 1|\tilde{M}_j|x\rangle = 0$ for all j.

(5) Repeat the previous steps except with choosing $x = 1$. In the end, we obtain an ensemble satisfying Eq. (137). This completes the procedure. ■

B. Transformations: SIO-DIO-IO-MIO equivalence

We now proceed to show that in terms of a single incoherent transformation $\rho \to \sigma$, MIO is just as powerful as SIO. Since SIO is both a subset of IO and DIO it follows that SIO=OI=DIO=MIO on qubits. As demonstrated above, the robustness of coherence and the Δ robustness of coherence for qubits can be computed explicitly:

$$\begin{aligned} C_R(\rho) &= 2r, \\ C_{\Delta,R}(\rho) &= \frac{r}{\sqrt{p(1-p)}}. \end{aligned} \tag{139}$$

In general, C_R is a MIO monotone while $C_{\Delta,R}$ is DIO monotone. However, we will now show that $C_{\Delta,R}$ is also a MIO monotone for qubits.

Theorem 22. $C_{\Delta,R}$ is monotonic under MIO channels $\mathcal{E} : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$.

Proof. By Theorem 21, it suffices to prove that $C_{\Delta,R}$ is an IO monotone. For qubits, any CP map $\mathcal{E}$ that belongs to IO can always be expressed as

$$\begin{aligned} \sigma = \mathcal{E}(\rho) = &\sum_\alpha J_\alpha \rho J_\alpha^\dagger + \sum_\beta K_\beta \rho K_\beta^\dagger \\ &+ \sum_\gamma L_\gamma \rho L_\gamma^\dagger + \sum_\delta M_\delta \rho M_\delta^\dagger, \end{aligned} \tag{140}$$

where the Kraus operators $\{J_\alpha,K_\beta,L_\gamma,M_\delta\}_{\alpha,\beta,\gamma,\delta}$ have the general form

$$\begin{aligned} J_\alpha &= j_{\alpha 0}|0\rangle\langle 0| + j_{\alpha 1}|1\rangle\langle 1|, \\ K_\beta &= k_{\beta 0}|1\rangle\langle 0| + k_{\beta 1}|0\rangle\langle 1|, \\ L_\gamma &= l_{\gamma 0}|0\rangle\langle 0| + l_{\gamma 1}|0\rangle\langle 1|, \\ M_\delta &= m_{\delta 0}|1\rangle\langle 0| + m_{\delta 1}|1\rangle\langle 1|. \end{aligned} \tag{141}$$

Crucially, these operators share the following relationships with Δ:

$$\begin{aligned} \Delta(J_\alpha \rho J_\alpha^\dagger) &= J_\alpha \Delta(\rho) J_\alpha^\dagger, \\ \Delta(K_\beta \rho K_\beta^\dagger) &= K_\beta \Delta(\rho) J_\beta^\dagger, \\ \Delta(L_\gamma \rho L_\gamma^\dagger) &= L_\gamma \rho L_\gamma^\dagger, \\ \Delta(M_\delta \rho M_\delta^\dagger) &= M_\delta \rho M_\delta^\dagger \end{aligned} \tag{142}$$

for all ρ. Suppose now that $t \geqslant 0$ satisfies $(1+t)\Delta(\rho) - \rho \geqslant 0$. Then, for an IO channel $\mathcal{E}$ we have

$$\begin{aligned} (1+t)\Delta[\mathcal{E}(\rho)] - \mathcal{E}(\rho) = &t\omega + \sum_\alpha J_\alpha[(1+t)\Delta(\rho) - \rho]J_\alpha^\dagger \\ &+ \sum_\beta K_\beta[(1+t)\Delta(\rho) - \rho]K_\beta^\dagger, \end{aligned}$$

where

$$\omega = t\left(\sum_\gamma L_\gamma \rho L_\gamma^\dagger + \sum_\delta M_\delta \rho M_\delta^\dagger\right) \geqslant 0.$$

By the assumption $(1+t)\Delta(\rho) - \rho \geqslant 0$ we likewise have $(1+t)\Delta[\mathcal{E}(\rho)] - \mathcal{E}(\rho) \geqslant 0$. From the definition of $C_{\Delta,R}$, it therefore follows that

$$C_{\Delta,R}(\rho) \geqslant C_{\Delta,R}[\mathcal{E}(\rho)]. \tag{143}$$

■

Next, we prove that monotonicity of $C_{\Delta,R}(\rho)$ is also sufficient for an SIO (and therefore also MIO) transformation.

Lemma 23. Let ρ and σ have standard-form parametrizations (p,r) and (q,t), respectively. Then, ρ can be transformed into σ by SIO if and only if

$$C_R(\rho) \geqslant C_R(\sigma) \quad \text{and} \quad C_{\Delta,R}(\rho) \geqslant C_{\Delta,R}(\sigma). \tag{144}$$

Proof. We will describe a channel $\mathcal{E}$ consisting exclusively of Kraus operators having the form J_α and K_β as given in Eq. (137). The transformation will consist of two steps $\rho \to \sigma_{\max} \to \sigma$, where $\sigma_{\max}$ has parameters $[q,t_{\max}(q)]$ with

$$t_{\max}(q) = \begin{cases} r & \text{if } p \geqslant q, \\ r\sqrt{\frac{q(1-q)}{p(1-p)}} & \text{if } q \geqslant p. \end{cases} \tag{145}$$

The channel attaining $t_{\max}$ is given by $\rho \mapsto \sigma_{\max} = J\rho J^\dagger + K\rho K^\dagger$, where

$$\begin{aligned} j_0^2 &= \begin{cases} \frac{p+q-1}{2p-1} & \text{if } p \geqslant q, \\ \frac{q}{p}\frac{p+q-1}{2q-1} & \text{if } q \geqslant p, \end{cases} \\ j_1^2 &= \begin{cases} \frac{p-q}{2p-1} & \text{if } p \geqslant q, \\ \frac{1-q}{1-p}\frac{p+q-1}{2q-1} & \text{if } q \geqslant p, \end{cases} \\ k_0^2 &= 1 - j_0^2, \\ k_1^2 &= 1 - j_1^2. \end{aligned} \tag{146}$$

Finally, the transformation $\sigma_{\max} \to \sigma$ can be seen as SIO feasible by noting that any $t < t_{\max}(q)$ can be reached for a fixed value of q by applying a dephasing channel $\rho = J_1\rho J_1^\dagger + J_2\rho J_2^\dagger$ where $J_1 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$ and $J_2 = \begin{pmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{pmatrix}$, for some appropriately chosen θ. ■

Combining Theorem 22 with Lemma 23, we therefore obtain the main result:

Theorem 24. For qubit states ρ and σ, the transformation $\rho \to \sigma$ is possible by either DIO, IO, or MIO if and only if both $C_R(\rho) \geqslant C_R(\sigma)$ and $C_{\Delta,R}(\rho) \geqslant C_{\Delta,R}(\sigma)$.

C. Coherence measures

For qubit states, a number of coherence measures have been proposed and evaluated, in direct analogy to entanglement measures in two-qubit systems. For instance, the so-called coherence of formation and concurrence of coherence [6,7] have been proposed, and both can be shown as being equivalent to the ℓ_1 norm: $C_{\ell_1}(\rho) = 2r$ [6,33]. Distinct from these is the relative entropy of coherence, which was known before under the name G asymmetry (see [34] and references therein), which takes the form

$$C_{\rm rel}(\rho) = S[\Delta(\rho)] - S(\rho). \tag{147}$$

All measures in qubit systems can be seen as arising from the two robustness measures C_R and $C_{\Delta,R}$ according to

$$C_{\ell_1}(\rho) = C_R(\rho),$$
$$C_{\rm rel}(\rho) = f\left[\frac{C_R(\rho)}{C_{\Delta,R}(\rho)}\right] - f\left[\frac{C_R(\rho)}{C_{\Delta,R}(\rho)}\sqrt{1 - C_{\Delta,R}(\rho)^2}\right], \tag{148}$$

where $f(x) = h(\frac{1}{2}[1 - \sqrt{1-x^2}])$ and $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

IV. COHERENCE THEORIES BASED ON ASYMMETRY

A. Translation-invariant operations (TIO)

Let us now comment further on asymmetry-based resource theories of coherence. For a general compact group G', a G'-asymmetry resource theory identifies its free states as those that are invariant under G' twirling

$$\mathcal{G}(\rho) = \int_{G'} dg\, U(g)\rho U(g)^\dagger,$$

where $U : G' \to \mathcal{H}$ is the representation of G' on the Hilbert space $\mathcal{H}$ and dg the Haar measure. The free operations are G' covariant:

$$\mathcal{E}[U(g)\rho U(g)^\dagger] = U(t)[\mathcal{E}(\rho)]U(g)^\dagger$$

for all $g \in G'$ and all ρ. A coherence resource theory based on asymmetry then identifies incoherent states (resp. operations) with the free states (resp. operations) defined with respect to the particular symmetry. For instance, if H is some observable, say the Hamiltonian, one can consider the unitary group of translations $\{e^{-itH} : t \in \mathbb{R}\}$. A state ρ is said to be incoherent if it commutes with every element of the group, i.e., $e^{-itH}\rho e^{itH} = \rho$ for all t. The class of translation-invariant operations (TIO) consists of all CPTP maps $\mathcal{E}$ that commute with the unitary action of the group, i.e.,

$$\mathcal{E}[e^{-itH}(\rho)e^{itH}] = e^{-itH}[\mathcal{E}(\rho)]e^{itH}$$

for all t and all ρ. The class TIO was first introduced and studied in Ref. [21]. When H is proportional to the number operator $\hat{N}$, then the unitary group of translations provides a representation for $U(1)$ [18].

Notice that the approach to defining coherence in the asymmetry picture is different than the approach used in the PIO/SIO/IO/DIO/MIO theories. The latter adopts a basis-dependent definition of coherence in which a state is incoherent if and only if it is diagonal in some specified basis $\mathcal{I}$, called the incoherent basis. In order that a G'-asymmetry theory likewise identifies $\mathcal{I}$ as the free states, one needs that G' and its representation U are such that

$$\mathcal{G}(\rho) = \Delta(\rho).$$

In the case of TIO, the condition that $\mathcal{G}(\rho) \in \mathcal{I}$ amounts to the generator H having a nondegenerate spectrum. But, in general, degeneracies will exist and the resulting resource theory will look very different than the basis-dependent theories of PIO/SIO/IO/DIO/MIO.

As an example of how TIO can define a resource theory fundamentally different than PIO/SIO/IO/DIO/MIO, consider a pair of bosons such as the electrons of a helium atom. Due to the exchange symmetry, a natural incoherent basis to consider for this system is $\{|b_0\rangle = \sqrt{1/2}(|01\rangle + |10\rangle), |b_1\rangle = \sqrt{1/2}(|01\rangle - |10\rangle), |b_2\rangle = |00\rangle, |b_3\rangle = |11\rangle\}$. In the basis-dependent theories of PIO/SIO/IO/DIO/MIO, a state of this system is incoherent if and only if it is diagonal in this basis. However, in a coherence resource theory based on $U(1)$ asymmetry of the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$, $|b_0\rangle$ and $|b_1\rangle$ are still identified as incoherent states, but so is the superposition state $|\psi\rangle = \sqrt{1/2}(|b_0\rangle + |b_1\rangle)$ as well as the mixture $\rho = 1/2(|b_0\rangle\langle b_0| + |b_1\rangle\langle b_1|)$. Typically, $|\psi\rangle$ is called a coherent superposition whereas ρ is an incoherent superposition. This example shows how the notion of coherence in a TIO resource theory depends crucially on the particular representation of the symmetry group. Therefore, one cannot make a general comparison between PIO/SIO/IO/DIO/MIO and TIO since their relationship will depend on the representation.

Conceptuallly, a TIO-based resource theory can be interpreted as defining coherence with respect to just individual degrees of freedom for a system, whereas a basis-dependent definition of coherence considers *all* degrees of freedom. In this sense, a basis-dependent theory of coherence may be seen as capturing a more complete notion of coherence for a system. In terms of the generator H, TIO theory characterizes coherence between different *eigenspaces* of H rather than among a specific set of *eigenstates*. In certain settings, it may be desirable to think of coherence in this way [21]. See also Ref. [14] for a complementary exposition of the different approaches to defining coherence.

In the following, we introduce two resource theories of asymmetry with the property that $\mathcal{G}(\rho) = \Delta(\rho)$. In fact, we identify the largest group with this property (Proposition 28).

B. *G*-asymmetry and *N*-asymmetry resource theories

The set of all incoherent unitary matrices forms a group which we denote by G. The group G consists of all $d \times d$ unitaries of the form πu, where π is a permutation matrix and u is a diagonal unitary matrix (with phases on the diagonal). We denote by N the group of $d \times d$ diagonal unitary matrices and by Π the group of permutation matrices. Note that N is a normal subgroup of G, and $G = N \rtimes \Pi$ is the semidirect product of N and Π. Clearly, the group G is compact and the twirlings over N and G are given by

$$\int_N dg\ \mathcal{T}_g(\rho) = \Delta(\rho) \ \text{and} \ \int_G dg\ \mathcal{T}_g(\rho) = \frac{1}{d}I, \quad (149)$$

where $\mathcal{T}_g(\rho) := g\rho g^\dagger$, and the integration is with respect to the Haar measure dg.

1. G-covariant maps

We would like to characterize the set of all G-covariant quantum channels. That is, we would like to characterize all CPTP maps that satisfy

$$[\mathcal{E},\mathcal{T}_g] = 0,\ \ \forall\ g \in G. \quad (150)$$

Consider the following three CPTP maps that are all G covariant:

$$\mathcal{E}^{(1)}(\rho) = \rho,$$
$$\mathcal{E}^{(2)}(\rho) = \frac{1}{d-1}[I - \Delta(\rho)], \quad (151)$$
$$\mathcal{E}^{(3)}(\rho) = \frac{1}{d-1}[d\Delta(\rho) - \rho].$$

Remark. (1) The map $\mathcal{E}^{(1)}$ is the trivial map and it is covariant under *all* groups (with unitary representations), whereas the last two maps are nontrivial as they are not covariant with respect to all groups. (2) The two convex combinations of $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, and $\mathcal{E}^{(3)}$,

$$\frac{1}{d^2}\mathcal{E}^{(1)}(\rho) + \frac{d-1}{d}\mathcal{E}^{(2)}(\rho) + \frac{d-1}{d^2}\mathcal{E}^{(3)}(\rho) = \frac{1}{d}I,$$
$$\frac{d}{d+1}\mathcal{E}^{(2)}(\rho) + \frac{1}{d+1}\mathcal{E}^{(3)}(\rho) = \frac{1}{d^2-1}(dI - \rho)$$

are also covariant under all groups (note that the coefficient d in front of I in the right-hand side of the second equation is necessary since otherwise the map is not completely positive). (3) The map $\mathcal{E}^{(3)}$ is completely positive (see Theorem 17) and the coefficient d in front of $\Delta(\rho)$ is necessary since otherwise the map is not positive. (4) The dephasing map is the following convex combination of $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(3)}$:

$$\Delta(\rho) = \frac{1}{d}\mathcal{E}^{(1)}(\rho) + \frac{d-1}{d}\mathcal{E}^{(3)}(\rho). \quad (152)$$

The following theorem shows that up to convex combinations, these three CPTP maps are all the G-covariant maps.

Theorem 25. (a) Let G be as above, U be a unitary matrix, and $\mathcal{U}(\rho) := U\rho U^\dagger$. Then,

$$[\mathcal{U},\Delta] = 0 \iff U \in G. \quad (153)$$

(b) A CPTP map $\mathcal{E}$ is G covariant if and only if $\mathcal{E}$ is a convex combination of the three CPTP maps defined above. Explicitly, $\mathcal{E}$ is G covariant if and only if

$$\mathcal{E}(\rho) = q_1\rho + \frac{q_2}{d-1}[I - \Delta(\rho)] + \frac{q_3}{d-1}[d\Delta(\rho) - \rho] \quad (154)$$

for some $q_i \geqslant 0$ with $\sum_{i=1}^3 q_i = 1$.

Proof. (a) A direct calculation shows that Δ is a G-covariant map [it also follows from part (b)]. Conversely, suppose $[\Delta,\mathcal{U}] = 0$. Note that that for a given fixed x,

$$\Delta[\mathcal{U}(|x\rangle\langle x|)] = \sum_{x'} |\langle x'|U|x\rangle|^2 |x'\rangle\langle x'|,$$
$$\mathcal{U}[\Delta(|x\rangle\langle x|)] = U|x\rangle\langle x|U^\dagger. \quad (155)$$

Comparing the two expressions gives $\langle x'|U|x\rangle = 0$ except for one value of x'. Hence, $U \in G$. ■

Before we prove part (b) of the theorem, we first prove the following lemma:

Lemma 26. Let $\mathcal{E}$ be an N-covariant CPTP map; that is,

$$[\mathcal{E},\mathcal{T}_g] = 0,\ \ \forall\ g \in N. \quad (156)$$

Then, $\mathcal{E}$ has the following Kraus decomposition:

$$\mathcal{E}(\rho) = \sum_j M_j\rho M_j^\dagger + \sum_{x\neq x'} J_{xx'}\rho J_{xx'}^\dagger, \quad (157)$$

where all $M_j = \sum_x a_{jx}|x\rangle\langle x|$ are diagonal matrices and $J_{xx'} = b_{xx'}|x\rangle\langle x'|$.

Proof. We will apply Lemma 1 of [18] to the characterization of N-invariant operations. Note first that the irreducible representations of $N \cong U(1)^d$ are labeled by d integers $\mathbf{k} = (k_1,\ldots,k_d)$, and are all one dimensional. The $\mathbf{k}$th irreducible representation $u_{\mathbf{k}} : N \to \mathbb{C}$ has the form

$$u_{\mathbf{k}}(\vec{\theta}) = e^{i\vec{\theta}\cdot\mathbf{k}}, \quad (158)$$

where $\vec{\theta} = (\theta_1,\ldots,\theta_d) \in U(1)^d$. It follows from Lemma 1 of [18] that the Kraus operators $K_{\mathbf{k},\alpha}$ of an N-invariant operation can be labeled by the irrep $\mathbf{k}$ and a multiplicity index α, and satisfy

$$g_{\vec{\theta}}\ K_{\mathbf{k},\alpha}\ g_{\vec{\theta}}^\dagger = e^{i\vec{\theta}\cdot\mathbf{k}}K_{\mathbf{k},\alpha},\ \forall\ \vec{\theta} \in U(1)^d, \quad (159)$$

where $g_{\vec{\theta}}$ is the diagonal matrix with components $e^{i\theta_1},\ldots,e^{i\theta_d}$ on the diagonal.

Note that by virtue of the fact that the irreps are one dimensional, the Kraus operators do not get mixed with one another under the action of N (this provides a significant simplification relative to non-Abelian groups). The most general expression for $K_{\mathbf{k},\alpha}$ is

$$K_{\mathbf{k},\alpha} = \sum_{x,x'} c_{xx'}^{\mathbf{k},\alpha}|x\rangle\langle x'|, \quad (160)$$

with some coefficients $c_{xx'}^{\mathbf{k},\alpha}$. Plugging this into (159) yields the constraint

$$c_{xx'}^{\mathbf{k},\alpha}(e^{i(\theta_x-\theta_{x'})} - e^{i\vec{\theta}\cdot\mathbf{k}}) = 0,\ \forall\ \vec{\theta} \in U(1)^d. \quad (161)$$

Hence, $c_{xx'}^{\mathbf{k},\alpha}$ must be zero unless $\mathbf{k} = 0$ and $x = x'$, or the x and x' components of $\mathbf{k}$ are 1 and -1, respectively, and all other components are zero. This completes the proof of the lemma. ■

Note that the lemma above provides the form of the Kraus operators in the resource theory of symmetric operations under the group N. This can be viewed as a physical resource theory of coherence. However, as discussed in the paper, resource theories of asymmetry cannot be used for coherence due to decoherence subspaces. Moreover, as we can see from the above form of the Kraus operators, in the resource theory of N-asymmetry permutations are not free! We are now ready to prove Theorem 25.

Proof. In addition to the form in (157), $\mathcal{E}$ also has to commute with all permutations:

$$[\mathcal{E},\mathcal{T}_\pi] = 0, \ \ \forall\, \pi \in \Pi. \tag{162}$$

In particular, we get

$$\mathcal{T}_\pi[\mathcal{E}(\rho)] = \sum_{j,x,x'} a_{jx}\bar{a}_{jx'}\rho_{xx'}|\pi(x)\rangle\langle\pi(x')| + \sum_{x'\neq x}|b_{xx'}|^2\rho_{x'x'}|\pi(x)\rangle\langle\pi(x)|, \tag{163}$$

whereas

$$\mathcal{E}[\mathcal{T}_\pi(\rho)] = \sum_{j,x,x'} a_{j\pi(x)}\bar{a}_{j\pi(x')}\rho_{xx'}|\pi(x)\rangle\langle\pi(x')| + \sum_{x'\neq x}|b_{\pi(x)\pi(x')}|^2\rho_{x'x'}|\pi(x)\rangle\langle\pi(x)|. \tag{164}$$

Hence, comparing the off-diagonal terms of $\mathcal{E}[\mathcal{T}_\pi(\rho)] = \mathcal{T}_\pi[\mathcal{E}(\rho)]$ give

$$\sum_j a_{jx}\bar{a}_{jx'} = \sum_j a_{j\pi(x)}\bar{a}_{j\pi(x')} \equiv c \tag{165}$$

since $\mathcal{E}[\mathcal{T}_\pi(\rho)] = \mathcal{T}_\pi[\mathcal{E}(\rho)]$ holds for all ρ and for all permutations $\pi \in \Pi$. The constant $c \in \mathbb{R}$ and is independent of x and x' Comparing the diagonal terms of $\mathcal{E}[\mathcal{T}_\pi(\rho)] = \mathcal{T}_\pi[\mathcal{E}(\rho)]$ gives

$$\sum_j |a_{jx}|^2\rho_{xx} + \sum_{\{x':x'\neq x\}}|b_{xx'}|^2\rho_{x'x'} = \sum_j |a_{j\pi(x)}|^2\rho_{xx} + \sum_{\{x':x'\neq x\}}|b_{\pi(x)\pi(x')}|^2\rho_{x'x'} \ \ \forall\,\rho. \tag{166}$$

Since the equation above holds for all ρ we must have

$$\sum_j |a_{jx}|^2 = \sum_j |a_{j\pi(x)}|^2 \equiv a \tag{167}$$

and

$$|b_{xx'}|^2 = |b_{\pi(x)\pi(x')}|^2 \equiv b, \tag{168}$$

where a and b are non-negative real numbers independent of x and x'. We therefore get that

$$\begin{aligned}\mathcal{E}(\rho) &= \sum_x a\rho_{xx}|x\rangle\langle x| + \sum_{x\neq x'} c\rho_{xx'}|x\rangle\langle x'| + \sum_{x'\neq x} b\rho_{xx}|x'\rangle\langle x'| \\ &= a\Delta(\rho) + c[\rho - \Delta(\rho)] + b\sum_x \rho_{xx}(I - |x\rangle\langle x|) \\ &= a\Delta(\rho) + c[\rho - \Delta(\rho)] + b[I - \Delta(\rho)].\end{aligned} \tag{169}$$

Note that the condition $\sum_j M_j^\dagger M_j + \sum_{x\neq x'} J_{xx'}^\dagger J_{xx'} = I$ gives

$$a + b(d-1) = 1. \tag{170}$$

We therefore conclude

$$\mathcal{E}(\rho) = a\Delta(\rho) + c[\rho - \Delta(\rho)] + \frac{1-a}{d-1}[I - \Delta(\rho)], \tag{171}$$

where $0 \leqslant a \leqslant 1$. We now argue that

$$-\frac{a}{d-1} \leqslant c \leqslant a. \tag{172}$$

Indeed,

$$|c| \leqslant \sum_j |a_{jx}\bar{a}_{jx'}| \leqslant \sum_j \frac{1}{2}(|a_{jx}|^2 + |a_{jx'}|^2) = a \tag{173}$$

and we also have

$$\begin{aligned}0 &\leqslant \sum_j \left(\sum_x a_{jx}\right)\left(\sum_{x'} \bar{a}_{jx'}\right) \\ &= \sum_x\sum_j |a_{jx}|^2 + \sum_{x\neq x'}\sum_j a_{jx}\bar{a}_{jx'} = da + d(d-1)c,\end{aligned}$$

which is equivalent to $c \geqslant -a/(d-1)$. Finally, we note that (171) can be expressed as

$$\mathcal{E}(\rho) = \frac{a + c(d-1)}{d}\mathcal{E}^{(1)}(\rho) + (1-a)\mathcal{E}^{(2)}(\rho) + \frac{(a-c)(d-1)}{d}\mathcal{E}^{(3)}(\rho). \tag{174}$$

The constraints on c in (172) ensure that the above equation is a convex combination of $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, and $\mathcal{E}^{(3)}$. This completes the proof of the theorem. ∎

2. *N*-covariant maps

The N-covariant operations given in Lemma 26 are very similar to the "cooling operations" given in [35]. The only difference is that $J_{xx'}$ is zero unless $x < x'$ (in the context of thermodynamics, the x index corresponds to energy levels, and cooling operations can not increase the energy). Therefore, N-covariant operations are a bit more powerful than cooling operations, as can be seen from the following theorem, when compared with Theorem 1 in [35].

Theorem 27. Let ρ,σ be two density matrices of the same dimensions, with all the off-diagonal terms of ρ being nonzero. Define the matrix $Q = (q_{xx'})$ as follows:

$$q_{xx'} := \begin{cases}\min\left\{\frac{\sigma_{xx}}{\rho_{xx}},\, 1\right\} & \text{if } x = x', \\ \frac{\sigma_{xx'}}{\rho_{xx'}} & \text{if } x \neq x'.\end{cases} \tag{175}$$

Then, $\sigma = \mathcal{E}(\rho)$ where $\mathcal{E}$ is N-invariant operation if and only if $Q \geqslant 0$.

Proof. Let $\mathbf{a}_x \equiv (a_{jx})_j$ where a_{jx} are the coefficients of M_j as in Eq. (157). Denote also $h_{xx'} \equiv \mathbf{a}_x^\dagger\mathbf{a}_{x'}$, and

$$r_{x'|x} \equiv \begin{cases}h_{xx} & \text{if } \ x = x', \\ |b_{xx'}|^2 & \text{if } \ x \neq x',\end{cases} \tag{176}$$

where $b_{xx'}$ are the coefficients associated with the operator $J_{xx'}$ in Eq. (157). Since $\mathcal{E}$ is trace preserving, $\sum_{x'} r_{x'|x} = 1$. Note that the matrix $H = (h_{xx'})$ is Gramian and therefore positive semidefinite. Recall also that the components of any positive-semidefinite matrix can be written as $\mathbf{a}_x^\dagger\mathbf{a}_{x'}$ for some vectors $\mathbf{a}_x$. Hence, from (157) it follows that there exists N-covariant map $\mathcal{E}$ such that $\sigma = \mathcal{E}(\rho)$ iff there exists $H \geqslant 0$ and a column stochastic matrix $R = (r_{x|x'})$ with diagonal elements

$r_{x|x} = h_{xx}$ such that

$$\sigma_{xx'} \equiv \begin{cases} \sum_y r_{x|y}\rho_{yy} & \text{if} \quad x = x', \\ h_{xx'}\rho_{xx'} & \text{if} \quad x \neq x'. \end{cases} \tag{177}$$

From the relation above we get

$$h_{xx'} = \frac{\sigma_{xx'}}{\rho_{xx'}} \equiv q_{xx'} \quad \text{for} \quad x \neq x',$$
$$h_{xx} = r_{x|x} \leqslant \min\left\{\frac{\sigma_{xx}}{\rho_{xx}}, 1\right\} \equiv q_{xx}. \tag{178}$$

Suppose now that $\sigma = \mathcal{E}(\rho)$. Then, there exists $H \geqslant 0$ that satisfies the above relations. Since Q and H are only different in the diagonal elements we can write $Q = H + D$ where D is some diagonal matrix. The equation above shows that $D \geqslant 0$. Therefore, $Q \geqslant 0$. Conversely, suppose $Q \geqslant 0$. We need to show that there exists $H \geqslant 0$ and column stochastic matrix R (with the same diagonal as H) that satisfy Eq. (177). We take $H = Q$ and show that there exists R with the desired properties. For simplicity of the exposition here, suppose that $\rho_{xx} \leqslant \sigma_{xx}$ for $x = 1,\ldots,k$ and $\rho_{xx} > \sigma_{xx}$ for $x = k+1,\ldots,d$. We take the column stochastic matrix R to have the following form:

$$R = \begin{pmatrix} I_k & CD' \\ \mathbf{0} & D \end{pmatrix}, \tag{179}$$

where I_k is the $k \times k$ identity matrix, $\mathbf{0}$ is the $(d-k) \times k$ zero matrix, D is the $(d-k) \times (d-k)$ diagonal matrix with diagonal elements $\{\sigma_{xx}/\rho_{xx}\}$ with $x = k+1,\ldots,d$, the matrix C is a $k \times (d-k)$ column stochastic matrix, and D' is a $(d-k) \times (d-k)$ diagonal matrix with diagonal elements $\{1 - \sigma_{xx}/\rho_{xx}\}$ with $x = k+1,\ldots,d$. Hence, R is column stochastic as long as C is column stochastic. With this form of R, the condition $\sigma_{xx} = \sum_y r_{x|y}\rho_{yy}$ is equivalent to

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \\ \sigma_{kk} \end{pmatrix} = \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \vdots \\ \rho_{kk} \end{pmatrix} + C \begin{pmatrix} \rho_{(k+1)(k+1)} - \sigma_{k+1)(k+1)} \\ \rho_{(k+2)(k+2)} - \sigma_{(k+2)(k+2)} \\ \vdots \\ \rho_{dd} - \sigma_{dd} \end{pmatrix}. \tag{180}$$

Define $\mathbf{r}$ to be the k-dimensional vector whose components are $\sigma_{xx} - \rho_{xx}$ for $x = 1,\ldots,k$, and $\mathbf{t}$ the $(d-k)$-dimensional vector whose components are $\rho_{xx} - \sigma_{xx}$ for $x = k+1,\ldots,d$. By definition, both vectors have non-negative components, and note also that the sum of the components of $\mathbf{r}$ is the same as the sum of the components of $\mathbf{t}$. Hence, there exists a column stochastic matrix C that satisfies $\mathbf{r} = C\mathbf{t}$. This completes the proof. ■

In the next proposition we show that the group N is the largest group possible with the property that its twirling is the dephasing map Δ.

Proposition 28. Let G' be any group with unitary representation $U(g)$ for $g \in G'$ such that

$$\int_{G'} dg\, U(g)\rho U(g)^\dagger = \Delta(\rho). \tag{181}$$

Then, the set $\{U(g)\}_{g\in G'}$ is a subgroup of N.

Proof. If $\int_{G'} dg\, U(g)\rho U(g)^\dagger = \Delta(\rho)$, then

$$\int_{G'} dg\, U(g)|x\rangle\langle x|U(g)^\dagger = |x\rangle\langle x|, \quad \forall\, x = 1,\ldots,d \tag{182}$$

which gives

$$U(g)|x\rangle = e^{i\theta_x(g)}|x\rangle, \tag{183}$$

where $\{\theta_x\}_{x=1}^d$ are one-dimensional representations of G'. The equation above clearly indicates that $U(g) \in N$ so that $U(G')$ must be a subgroup of N. In this sense, N is the largest group with the property that $\mathcal{G}(\rho) = \Delta(\rho)$.

The requirement $\mathcal{G}(|x\rangle\langle x'|) = 0$ for $x \neq x'$ gives in addition

$$\int_{G'} dg\, e^{i[\theta_x(g) - \theta_{x'}(g)]} = \delta_{xx'}. \tag{184}$$

Taking $dg = \frac{d\alpha}{2\pi}$ and $\theta_x(g) = x\alpha$ with $\alpha \in [0,2\pi]$ reproduce the $U(1)$ twirling. Of course, the equation above is also satisfied for $\theta_x(g) = x^2\alpha$, but still the group $G' = U(1)$. ■

V. OPEN PROBLEMS

We conclude with a few open questions.

A. State transformations

Pure-state transformations under SIO (both asymptotic and single-copy cases) have been completely characterized in this paper via the one-to-one correspondence with LOCC. Consequently, among all coherence models discussed here, the SIO model is the most similar to the theory of pure bipartite entanglement. Particularly, in the single-copy regime, pure-state transformations are determined by the majorization criterion (similar to Nielsen theorem in entanglement theory). A key open question is whether or not this criterion can be extended to the IO and DIO models.

Since majorization is both a necessary and sufficient condition for an SIO pure-state transformation $|\psi\rangle \to |\phi\rangle$, it follows that it is sufficient for both IO and DIO (recall SIO is a subset of both IO and DIO). In IO it is also known to be necessary if both pure states have a full Schmidt rank since here the transformation is actually accomplished by sIO. But, as we discussed in this paper, it is not clear if it is still the case when the Schmidt rank of the target state $|\phi\rangle$ is strictly smaller than the Schmidt rank of $|\psi\rangle$.

As for DIO, we have shown that all the Rényi entropies of the Schmidt components of a pure state are monotones under DIO. In [36] it was shown that if $S_\alpha(\psi) \geqslant S_\alpha(\phi)$ for all α then there exists a catalyst $|C\rangle$ such that the Schmidt components of $|\psi\rangle|C\rangle$ are majorized by the Schmidt components of $|\phi\rangle|C\rangle$. Therefore, the existence of a catalyst provides a sufficient condition for the transformation $|\psi\rangle \to |\phi\rangle$ under DIO. This means that necessary *and* sufficient condition for pure-state transformation under DIO are somewhere between majorization and catalytic majorization.

Majorization also provides sufficient condition for $|\psi\rangle \to |\phi\rangle$ under MIO, but here we also know that it is not necessary. In fact, MIO can increase the Schmidt rank as demonstrated in Theorem 14. However, Theorem 14 only involves a transformation from pure qubit to pure qudit. It is left open to extend it to higher dimensions.

Necessary and sufficient conditions for mixed-state transformations have only been found for the qubit case, and a special type of asymmetry-based theory with symmetry groups G and N. However, in higher dimensions, necessary and sufficient conditions for mixed-state transformations for SIO/IO/DIO/MIO are not known. In the asymptotic limit of many copies of a mixed state we know that IO is not a reversible model, and distillation and formation rates have been calculated in [7]. MIO, on the other hand, is a reversible quantum resource theory (QRT) in the asymptotic limit of many copies, due to a general QRT theorem proved in [11]. However, the asymptotic distillation and formation rates are not known for SIO and DIO.

Finally, another area of open inquiry pertains to determining the precise relationship between SIO, IO, and DIO. To our knowledge, no operational gap in terms of state transformation is known between these classes, despite the fact that they represent distinct collections of CP maps. More precisely, for every transformation $\rho \to \sigma$ feasible by IO (resp. DIO), is it also feasible by DIO (resp. IO) as well as SIO? We suspect that such examples can be found, but perhaps not when ρ is pure.

B. Monotones

There are few open problems regarding coherence monotones. In [3] a measure of coherence under IO was introduced. This measure was defined by

$$C_{\ell_1}(\rho) = \sum_{x \neq y} \rho_{xy}, \tag{185}$$

where ρ_{xy} are components of ρ in the incoherent basis. We have shown that the robustness of coherence as defined in (105) equals C_{ℓ_1} for pure states and mixed states with non-negative real off-diagonal terms. While the robustness of coherence is a monotone under MIO, it is not known if C_{ℓ_1} is also a monotone under MIO.

In the Appendix we have also introduced many monotones under DIO. These sets of monotones are closely related to monotones under thermal operations. In the resource theory of quantum thermodynamics, the free (or "thermal") operations take the form $\rho_A \to \mathrm{tr}_B[U(\rho_A \otimes \gamma_B^{(T)})U^\dagger]$, where U is any unitary that commutes with the joint Hamiltonian, and $\gamma_B^{(T)}$ is the Gibbs state at temperature T [37,38]. It was also observed in [39] that thermal operations are time-translation symmetric, and in particular belong to DIO when the incoherent basis is taken to be the energy eigenstates, assuming no degeneracy in the energy eigenstates. Therefore, all the DIO monotones introduced in this appendix are also monotones under thermal operations. In the case of degeneracy in the energy eigenstates, it is left open how to apply the DIO monotones to thermodynamics.

C. Relating coherence with maximally correlated entanglement

Propositions 5 and 10 show that every transformation $\rho \to \sigma$ by either SIO or sIO corresponds to an LOCC transformation between the corresponding maximally correlated states $\rho^{(mc)} \to \sigma^{(mc)}$. One obtains the maximally correlated state $\rho^{(mc)}$ from the single-system state ρ via the "coherent channel" $|x\rangle \to |xx\rangle$. In and of itself, such a channel appears in the theory of coherent communication where the tasks of coherent superdense coding and coherent teleportation are fully dual to one another (see Chap. 7 of [40]). We have been interested in using this channel to map the theory of SIO/sIO into one-way/two-way LOCC. A natural question is whether or not such a connection can also be established between IO and LOCC. Such a relationship has been conjectured in Ref. [7], and a probabilistic version of it was proven in Ref. [41]. Specifically, it was shown that for every IO transformation $\rho \to \sigma$, the transformation $\rho^{(mc)} \to \sigma^{(mc)}$ can always be accomplished with some nonzero probability. It is unknown whether a deterministic LOCC implementation is always possible, and whether such a result also holds for transformations $\rho \to \sigma$ that are feasible using DIO.

Lastly, Theorem 14 shows that $\rho \to \sigma$ by MIO fails to imply $\rho^{(mc)} \to \sigma^{(mc)}$ by LOCC. Unlike LOCC, MIO is able to increase the Schmidt rank under pure-state transformations. An interesting open question is whether, analogous to MIO, the Schmidt rank can be increased by some nonentangling operation.

Note added. Recently, we became aware of independent work by Marvian and Spekkens [14], where the physical meaning of incoherent operations is analyzed and the class of dephasing-covariant incoherent operations is presented. Also recently, Bu and Xiong have demonstrated a state transformation that can be performed by DIO but not IO [42]. Their example also shows that ℓ-1 norm is not a monotone under MIO, thus resolving one of the open problems listed above.

ACKNOWLEDGMENTS

We would like to thank I. Marvian and R. Spekkens for constructive discussions during the preparation of this manuscript. We also thank B. Yadin and J. De Vicente for helpful comments on an earlier draft of this work. E.C. is supported by the National Science Foundation (NSF) Early CAREER Award No. 1352326. G.G. research is supported by NSERC.

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