Distillability of non-positive-partial-transpose bipartite quantum states of rank four

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We show that a bipartite quantum state of rank four is distillable if its partial transpose has at least one negative eigenvalue; that is, the state is a non-positive-partial transpose (NPT). For this purpose we prove that if the partial transpose of a two-qutrit NPT state has at least two non-positive eigenvalues, then the state is distillable. We further construct a parametrized two-qutrit NPT state of rank five which is not 1-distillable and show that it is not n-distillable for any given n when the parameter is sufficiently small. This state has the smallest rank among all 1-undistillable NPT states. We conjecture that the state is not distillable.

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I. INTRODUCTION

Many quantum-information tasks require entangled pure states as a necessary resource. In entanglement theory, a long-standing open problem is whether bipartite non-positive-partial transpose (NPT) states can be asymptotically converted into pure entangled states under local operations and classical communication (LOCC). This is the well-known distillability problem [1,2] which is related to the superactivation of bound entanglement and zero-capacity quantum channels.

In spite of much effort being devoted to it in recent years, the distillability problem is still far from a solution due to the rapidly increasing number of parameters in the density matrices of states. A method of avoiding this difficulty is to convert, by LOCC, the states into the Werner states containing only one parameter [1]. Thus to solve the distillability problem, it suffices to distill Werner states. In recent years there were several attempts to do that [1,3–8]. On the other hand, progress towards distilling entangled states of given dimensions or deficient rank has been made steadily. In contrast to the Werner states, in the case of states of deficient rank, one can make use of more fruitful features of density matrices such as the existence of product vectors in the kernel or the range. Entangled states of ranks two and three [9–11], $2 \times N$ NPT states [2], and $M \times N$ entangled states of rank at most $\max(M,N)$ [9,12] have been proven to be distillable.

In this paper we show that if the partial transpose of an NPT two-qutrit state has at least two non-positive eigenvalues, counting multiplicities, then the state is distillable. This is presented in Theorem 1. In part (a) of Theorem 2 we show that if the kernel of a two-qutrit NPT state contains a product vector, then the state is 1-distillable. In part (b) of the same theorem we further show that, in any bipartite system, any NPT state of rank four is distillable. Since entangled states of rank of at most three are distillable [9,10], any NPT state of rank at most four is distillable. So the distillable entanglement measure is positive for these states [13]. Based on these facts, we give in Corollary 1 some necessary conditions for a bipartite state not to be 1-distillable. We construct a two-qutrit NPT state ρ of rank five [see Eq. (7)] depending on a parameter $\epsilon > 0$, which is not 1-distillable. This is achieved by using the positive-partial-transpose (PPT) entangled edge states constructed in [14]. Since any NPT state of rank at most four is 1-distillable, ρ has the smallest rank among all 1-undistillable NPT states. We further show in Lemma 4 that for any given integer n and sufficiently small $\epsilon > 0$, ρ is n-undistillable. This is based on some estimates for many-copy separable Werner states, presented in Lemma 3. We conjecture that ρ is not distillable for small $\epsilon > 0$.

In the literature many results for two-qutrit states of rank four have been found by using the PPT states constructed from the unextendible product bases (UPB) [15]. Next, any such states were proved to be constructed by UPB up to stochastic LOCC (SLOCC) [16]. Further, a necessary and sufficient condition for the separability of states of rank four has been proposed in [12]. All these results are about the PPT states. In contrast, our results show the distillability of two-qutrit NPT states of rank four.

The rest of the paper is organized as follows. In Sec. II we introduce the mathematical formulation of the distillability problem and the notation used in the paper. In Sec. III we present the conditions for 1-distillable and 1-undistillable two-qutrit NPT states. In Theorem 2 we prove our main result that, in any bipartite system, all NPT states of rank four are 1-distillable. We also construct 1-undistillable two-qutrit NPT states of rank five. In Sec. IV we construct n-undistillable two-qutrit NPT states of rank five for any integer n. We give our conclusions in Sec. V.

II. PRELIMINARIES

In this section we introduce the notation and recall facts of the distillability problem which will be used throughout the paper. Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the bipartite Hilbert space with $\dim \mathcal{H}_A = M$ and $\dim \mathcal{H}_A = N$. We shall work with bipartite quantum states ρ on $\mathcal{H}$. We shall write I_k for the identity $k \times k$ matrix. We denote by $\mathcal{R}(\rho)$ and $\ker \rho$ the range and kernel of a linear map ρ, respectively. From now on, unless stated otherwise, the states will not be normalized. We shall denote by $\{|i\rangle_A : i = 0, \ldots, M-1\}$ and $\{|j\rangle_B : j = 0, \ldots, N-1\}$ orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The partial transpose of ρ with respect to the system A is defined as

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$\rho^\Gamma := \sum_{i,j} |j\rangle\langle i| \otimes \langle i|\rho|j\rangle$. We say that ρ is PPT if $\rho^\Gamma \geqslant 0$. Otherwise, ρ is NPT; that is, ρ^Γ has at least one negative eigenvalue. The NPT states are always entangled due to the Peres-Horodecki criterion for separable states [17].

The distillability problem requires many-copy states, so we introduce the notion of a composite system. Let $\rho_{A_iB_i}$ be an $M_i \times N_i$ state of rank r_i acting on the Hilbert space $\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$, $i = 1,2$. Suppose ρ of systems A_1,A_2 and B_1,B_2 is a state acting on the Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$, such that $\mathrm{Tr}_{A_1B_1}\,\rho = \rho_{A_2B_2}$ and $\mathrm{Tr}_{A_2B_2}\,\rho = \rho_{A_1B_1}$. By switching the two middle factors, we can consider ρ a *composite* bipartite state acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $\mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$. In that case we shall write $\rho = \rho_{A_1A_2:B_1B_2}$. So ρ is an $M_1M_2 \times N_1N_2$ state of rank not larger than r_1r_2. In particular for the *tensor product* $\rho = \rho_{A_1B_1} \otimes \rho_{A_2B_2}$, it is easy to see that ρ is an $M_1M_2 \times N_1N_2$ state of rank r_1r_2.

The above definition can be easily generalized to the tensor product of N states $\rho_{A_iB_i}, i = 1,\ldots,N$. They form a bipartite state on the Hilbert space $\mathcal{H}_{A_1,\ldots,A_N} \otimes \mathcal{H}_{B_1,\ldots,B_N}$. It is written as $\mathcal{H}^{\otimes n}$ when $\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i} = \mathcal{H}$. Using this terminology, we introduce the definition of distillable states [1].

Definition 1. A bipartite state ρ is *n-distillable* under LOCC if there exists a Schmidt-rank-two state $|\psi\rangle \in \mathcal{H}^{\otimes n}$ such that $\langle\psi|(\rho^{\otimes n})^\Gamma|\psi\rangle < 0$. Otherwise, we say that ρ is n-undistillable. We say that ρ is *distillable* if it is n-distillable for some $n \geqslant 1$. If an entangled state ρ is not distillable, then we say that it is *bound entangled*.

It is immediately clear from this definition that no PPT state is distillable. Hence PPT entangled states are bound entangled states. The distillability problem asks whether a bound entangled state can be NPT.

It is also immediately clear from the same definition that the set of k-distillable states is open in the set of all states; that is, if ρ is a k-distillable state, then there exists $\epsilon > 0$ such that every state ρ' satisfying $\|\rho' - \rho\| < \epsilon$ is k-distillable.

Let us recall some basic methods for proving the separability, distillability, and PPT properties of bipartite states. We say that two bipartite states ρ and σ are equivalent under SLOCC if there exists an invertible local operator (ILO) $A \otimes B$ such that $\rho = (A^\dagger \otimes B^\dagger)\sigma(A \otimes B)$ [18]. It is easy to see that any ILO transforms distillable, PPT, entangled, or separable states into the same kind of states. We often use ILOs to simplify the density matrices of states. A subspace which contains no product state is referred to as a completely entangled subspace (CES).

III. 1-DISTILLABILITY OF NPT STATES

For convenience, we denote by $\mathrm{sr}(x)$ the Schmidt rank of a state $|x\rangle \in \mathcal{H}$. If $|x\rangle = \sum_{i,j} \xi_{ij}|i,j\rangle$, $0 \leqslant i < M$, $0 \leqslant j < N$, we say that $[\xi_{ij}]$ is the matrix of $|x\rangle$. By definition, $\mathrm{sr}(x)$ is the ordinary rank of $[\xi_{ij}]$. The following lemma gives a necessary condition for the 1-distillability of quantum states.

Lemma 1. If ρ is a bipartite state and $\langle\psi|\rho^\Gamma|\psi\rangle < 0$ for some vector $|\psi\rangle$, then $|\psi\rangle$ is entangled. ■

Proof. Assume that $|\psi\rangle = |x,y\rangle$. Then $\langle\psi|\rho^\Gamma|\psi\rangle = \langle x^*,y|\rho|x^*,y\rangle \geqslant 0$ gives a contradiction.

The following result generalizes [12, Lemma 4], and their proofs are also similar.

Lemma 2. Let ρ be a bipartite state such that ρ^Γ has a principal 2×2 submatrix of negative determinant. Then ρ is distillable.

Proof. Let $\rho = \sum_{i,j} |i\rangle\langle j| \otimes \sigma_{ij}$ and the 2×2 submatrix be $[\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}]$. We have $\rho^\Gamma = \sum_{i,j} |i\rangle\langle j| \otimes \sigma_{ji}$. Since $\sigma_{ii} \geqslant 0$, the diagonal entries a and c must belong to different diagonal blocks, say, σ_{kk} and σ_{ll}. Let P be the orthogonal projector onto the two-dimensional subspace of $\mathcal{H}_A$ spanned by $|k\rangle$ and $|l\rangle$. Then the projected state $(P \otimes I_B)\rho(P \otimes I_B)$ is an NPT state on the $2 \times N$ system. Hence it is distillable, and ρ is also distillable. ■

We can now prove the main results of this section.

Theorem 1. If ρ is a two-qutrit NPT state and ρ^Γ has at least two non-positive eigenvalues counting multiplicities, then ρ is 1-distillable.

Proof. By the hypothesis, there exist two eigenvectors of ρ^Γ, say, $|\alpha\rangle$ and $|\beta\rangle$, with matrices A and B, such that $\rho^\Gamma|\alpha\rangle = \lambda|\alpha\rangle$, $\lambda < 0$, $\rho^\Gamma|\beta\rangle = \mu|\beta\rangle$, $\mu \leqslant 0$, and $\langle\alpha|\beta\rangle = 0$. If A is not invertible, then its rank is 2, and so ρ is 1-distillable. From now on we assume that A is invertible and also that B is invertible if $\mu < 0$.

If $N := A^{-1}B$ is not nilpotent, then $\det(I_3 + tN)$ is a nonconstant polynomial in t, and we can choose t so that this determinant is zero. Thus $A + tB$ is singular, and $|\phi\rangle := |\alpha\rangle + t|\beta\rangle$ satisfies $\langle\phi|\rho^\Gamma|\phi\rangle = \lambda\|\alpha\|^2 + \mu|t|^2\|\beta\|^2 < 0$. Hence ρ is 1-distillable.

If N is nilpotent, we must have $\mu = 0$. We can choose $|\alpha'\rangle$, with matrix A', close to $|\alpha\rangle$ such that $\langle\alpha'|\rho^\Gamma|\alpha'\rangle < 0$ and $N' := (A')^{-1}B$ is not nilpotent. We choose t so that $A' + tB$ is singular. For $|\phi'\rangle := |\alpha'\rangle + t|\beta\rangle$ we have $\langle\phi'|\rho^\Gamma|\phi'\rangle = \langle\alpha'|\rho^\Gamma|\alpha'\rangle < 0$ and $\mathrm{sr}(\phi') = 2$. Hence ρ is 1-distillable. ■

Using this result, we show that the existence of product vectors in the kernel is related to 1-distillability.

Theorem 2. (a) If the kernel of a two-qutrit NPT state ρ contains a product vector, then ρ is 1-distillable. (b) Any bipartite NPT state ρ of rank four is 1-distillable.

Proof. (a) We can assume that $|0,0\rangle \in \ker\rho$. Consequently, the first diagonal entry of ρ is zero, and the same is true for ρ^Γ. If the first column of ρ^Γ is not zero, then ρ is 1-distillable by [12, Lemma 4]. Otherwise, $|0,0\rangle \in \ker\rho^\Gamma$, and ρ is 1-distillable by Theorem 1. We have proved the first assertion.

(b) To prove the second assertion, we recall that $M \times N$ NPT states of rank four are 1-distillable when $\max(M,N) \geqslant 4$ [9,12] or $\min(M,N) = 2$. It remains to consider the case $M = N = 3$. Since $\dim\ker\rho = 5$, $\ker\rho$ contains a product vector. Hence the first assertion implies the second in this case.

Our results show that if we can convert, by LOCC, an entangled state ρ into an NPT state of rank four, then ρ is distillable. This provides a new way of attacking the distillability problem. Let us mention that the result proved in [10] is the special case of this theorem when $r = 3$ and that [12, Theorem 28] follows from the case $r = 4$. It is immediately clear from the definition of bad states in [19] that the kernel of such a state contains at least one product vector. Consequently, all bad two-qutrit NPT states are 1-distillable.

Given any bipartite NPT state σ of rank four, we can further construct 1-distillable states ρ of any rank $r > 4$. This follows from the fact that σ itself is 1-distillable and the observation made earlier that the set of 1-distillable states is open.

In spite of these results, there exist 1-undistillable two-qutrit NPT states of rank five.

Corollary 1. If ρ is a 1-undistillable two-qutrit NPT state, then $\ker\rho$ is a CES, and ρ^Γ has exactly one negative and eight positive eigenvalues. Consequently, $\operatorname{rank}\rho > 4$ and $\det\rho^\Gamma \neq 0$.

One can verify that the 1-undistillable two-qutrit NPT Werner states satisfy the conditions of this corollary. They have full rank, nine. Here we analytically construct an example of such ρ of rank five. By Corollary 1, this is the minimum rank for 1-undistillable NPT states.

We use the fact [14] that there exists a two-parameter family of two-qutrit edge-entangled states σ^Γ with $\operatorname{rank}\sigma = 5$ and $\operatorname{rank}\sigma^\Gamma = 8$. Explicitly, they are given by

$$\sigma := \frac{1}{3(2\cos\theta + b + 1/b)}\begin{bmatrix} 2\cos\theta & 0 & 0 & 0 & -\cos\theta & 0 & 0 & 0 & -\cos\theta \\ 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & -e^{i\theta} & 0 & 0 \\ 0 & -e^{i\theta} & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -\cos\theta & 0 & 0 & 0 & 2\cos\theta & 0 & 0 & 0 & -\cos\theta \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 \\ 0 & 0 & -e^{-i\theta} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{i\theta} & 0 & b & 0 \\ -\cos\theta & 0 & 0 & 0 & -\cos\theta & 0 & 0 & 0 & 2\cos\theta \end{bmatrix}, \tag{1}$$

and thus

$$\sigma^\Gamma = \frac{1}{3(2\cos\theta + b + 1/b)}\begin{bmatrix} 2\cos\theta & 0 & 0 & 0 & -e^{i\theta} & 0 & 0 & 0 & -e^{-i\theta} \\ 0 & \frac{1}{b} & 0 & -\cos\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & -\cos\theta & 0 & 0 \\ 0 & -\cos\theta & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -e^{-i\theta} & 0 & 0 & 0 & 2\cos\theta & 0 & 0 & 0 & -e^{i\theta} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & -\cos\theta & 0 \\ 0 & 0 & -\cos\theta & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\cos\theta & 0 & b & 0 \\ -e^{i\theta} & 0 & 0 & 0 & -e^{-i\theta} & 0 & 0 & 0 & 2\cos\theta \end{bmatrix}, \tag{2}$$

where the two parameters b and θ are subject to the constraints $b > 0$ and $0 < |\theta| < \pi/3$. We have the spectral decomposition

$$\sigma^\Gamma = \sum_{i=1}^{8} p_i |\varphi_i\rangle\langle\varphi_i|, \tag{3}$$

where p_i are the positive eigenvalues of σ^Γ. Let the eigenvalues be in the ascending order, i.e., $p_j \leqslant p_{j+1}$. By a computation, we find that

$$p_1 = \frac{1}{3(2\cos\theta + b + 1/b)} \min\left\{3\cos\theta - \sqrt{3}|\sin\theta|,\right.$$
$$\left.\frac{1 + b^2 - \sqrt{1 + b^4 + 2b^2\cos 2\theta}}{2b}\right\}. \tag{4}$$

Equation (2) implies that $\ker\sigma^\Gamma$ is spanned by the two-qutrit maximally entangled state

$$|\Psi\rangle := \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{5}$$

$\mathcal{R}(\sigma)$ contains the normalized product state

$$|f,g\rangle := \frac{1}{b^{\frac{1}{2}} + b^{-\frac{1}{2}}}(|0\rangle + b^{\frac{1}{2}} e^{\frac{i}{2}\theta}|1\rangle)(|0\rangle - b^{-\frac{1}{2}} e^{-\frac{i}{2}\theta}|1\rangle). \tag{6}$$

One can verify that $|f^*,g\rangle \notin \mathcal{R}(\sigma^\Gamma)$. For $\epsilon = p_1/3$, the matrix

$$\rho = \sigma - \epsilon|f,g\rangle\langle f,g| \tag{7}$$

is positive semidefinite and has rank five. So ρ is a non-normalized quantum state. Since $|f^*,g\rangle \notin \mathcal{R}(\sigma^\Gamma)$, the partial transpose $\rho^\Gamma = \sigma^\Gamma - \epsilon|f^*,g\rangle\langle f^*,g|$ is not positive semidefinite. So ρ is NPT.

Let $|\psi\rangle$ be a normalized pure state of Schmidt rank at most two. Since $\sigma^\Gamma \geqslant 0$ and its kernel is spanned by $|\Psi\rangle$, we have $\langle\psi|\sigma^\Gamma|\psi\rangle > 0$. By [1, Lemma 3] we have

$$\max_\psi |\langle\Psi|\psi\rangle|^2 = \tfrac{2}{3}, \tag{8}$$

and Eq. (3) implies that

$$\langle\psi|\sigma^\Gamma|\psi\rangle = \sum_i p_i\langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle \geqslant p_1\sum_i\langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle$$
$$= p_1\langle\psi|(I_9 - |\Psi\rangle\langle\Psi|)|\psi\rangle \geqslant \frac{p_1}{3}. \tag{9}$$

As $|\langle f^*,g|\psi\rangle| < 1$, we have $\langle\psi|\rho^\Gamma|\psi\rangle = \langle\psi|(\sigma^\Gamma - \epsilon|f^*,g\rangle\langle f^*,g|)|\psi\rangle > p_1/3 - \epsilon = 0$. Hence ρ is 1-undistillable.

IV. THE *n*-DISTILLABILITY OF NPT STATES

In this section we investigate the n-distillability of NPT states. In Lemma 4, we will construct a parametrized family of two-qutrit NPT states of rank five. These states are not n-distillable when the parameter is small enough. This is similar to the n-undistillable Werner states presented in [1–3], while our states have a smaller rank. For this purpose we present a preliminary lemma. We denote by sr_k the set of normalized bipartite pure states of Schmidt rank k.

Lemma 3. Let $\rho_s = \frac{1}{8}(I_9 - |\Psi\rangle\langle\Psi|)$ and σ be the state (1), with p_1 and p_8 being, respectively, the smallest and the largest positive eigenvalues of σ^Γ. We have

$$8p_8\rho_s \geqslant \sigma^\Gamma \geqslant 8p_1\rho_s, \tag{10}$$

$$\max_{\psi\in \mathrm{sr}_1}\langle\psi|(\rho_s)^{\otimes n}|\psi\rangle = \max_{\psi\in \mathrm{sr}_2}\langle\psi|(\rho_s)^{\otimes n}|\psi\rangle = \frac{1}{8^n}, \tag{11}$$

$$\frac{1}{2}\frac{1}{12^n} \geqslant \min_{\psi\in \mathrm{sr}_1\cup \mathrm{sr}_2}\langle\psi|(\rho_s)^{\otimes n}|\psi\rangle \geqslant \frac{1}{24^n}, \tag{12}$$

$$\min_{\psi\in \mathrm{sr}_1}\langle\psi|(\rho_s)^{\otimes n}|\psi\rangle = \frac{1}{12^n}. \tag{13}$$

Proof. Equation (3) implies (10). Next, (11) follows from the fact that $\rho_s^{\otimes n}$ is a normalized projector of rank 8^n, and the maximum is achieved when $|\psi\rangle = |0\cdots0\rangle_{A_1\cdots A_n}\otimes|1\cdots1\rangle_{B_1\cdots B_n}$ or $|\psi\rangle = |0\cdots0\rangle_{A_1\cdots A_{n-1}}\otimes|1\cdots1\rangle_{B_1\cdots B_{n-1}}(\frac{|01\rangle+|10\rangle}{\sqrt{2}})_{A_nB_n}$. Third, the first equality of (12) is achieved when $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)_{A_1B_1}\otimes(\otimes_{j=2}^n|00\rangle_{A_jB_j})$. So we have proved the first inequality of (12). Below we prove the second inequality of (12) by induction. Equation (8) implies that the inequality holds for $n=1$. Suppose it holds for $n-1$, namely, $\min_\psi\langle\psi|(\rho_s)^{\otimes(n-1)}|\psi\rangle \geqslant \frac{1}{24^{n-1}}$. It is known that ρ_s is the partial transpose of a two-qutrit separable Werner state [20], and we note that rank $\rho_s = 8$. Let $\rho_s = \sum_i q_i|a_i,b_i\rangle\langle a_i,b_i|$, where $q_i>0$, $\sum_i q_i = 1$, and any $|a_i,b_i\rangle$ is a unit vector. We have

$$\begin{aligned}\langle\psi|(\rho_s)^{\otimes n}|\psi\rangle &= \langle\psi|(\rho_s)^{\otimes(n-1)}\otimes\left(\sum_i q_i|a_i,b_i\rangle\langle a_i,b_i|\right)_{A_nB_n}|\psi\rangle\\ &= \sum_i q_i\|\langle a_i,b_i|\psi\rangle\|^2\langle\psi_i|(\rho_s)^{\otimes(n-1)}|\psi_i\rangle\\ &\geqslant \sum_i q_i\|\langle a_i,b_i|\psi\rangle\|^2\frac{1}{24^{n-1}}\\ &= \langle\psi|(\rho_s)_{A_nB_n}|\psi\rangle\frac{1}{24^{n-1}} \geqslant \frac{1}{24^n}.\end{aligned} \tag{14}$$

Here $|\psi_i\rangle = \frac{\langle a_i,b_i|\psi\rangle}{\|\langle a_i,b_i|\psi\rangle\|} \in \mathcal{H}_{A_1\cdots A_{n-1}}\otimes\mathcal{H}_{B_1\cdots B_{n-1}}$ is a normalized bipartite state of Schmidt rank at most two. The first inequality in (14) follows from the induction assumption. To prove the last inequality in (14) we assume that $|\psi\rangle = \sqrt{\lambda}|w,x\rangle + \sqrt{1-\lambda}|y,z\rangle$ is the Schmidt decomposition, where

$$|w\rangle = \sum_j\sqrt{\alpha_j}|j,\epsilon_j\rangle_{A_1\cdots A_{n-1},A_n}, \tag{15}$$

$$|x\rangle = \sum_k\sqrt{\beta_k}|k,\zeta_k\rangle_{B_1\cdots B_{n-1},B_n}, \tag{16}$$

$$|y\rangle = \sum_j\sqrt{\gamma_j}|j,\iota_j\rangle_{A_1\cdots A_{n-1},A_n}, \tag{17}$$

$$|z\rangle = \sum_k\sqrt{\delta_k}|k,\kappa_k\rangle_{B_1\cdots B_{n-1},B_n}, \tag{18}$$

and $\langle w|y\rangle = \langle x|z\rangle = 0$. For $|\varphi_{jk}\rangle := \sqrt{\lambda\alpha_j\beta_k}|\epsilon_j,\zeta_k\rangle + \sqrt{(1-\lambda)\gamma_j\delta_k}|\iota_j,\kappa_k\rangle$ and its normalization $|\varphi'_{jk}\rangle$, we have

$$\begin{aligned}\langle\psi|(\rho_s)_{A_nB_n}|\psi\rangle &= \sum_{j,k}\langle\varphi_{jk}|\rho_s|\varphi_{jk}\rangle\\ &= \sum_{j,k}\|\varphi_{jk}\|^2\langle\varphi'_{jk}|\rho_s|\varphi'_{jk}\rangle\\ &\geqslant \sum_{j,k}\|\varphi_{jk}\|^2\min_\mu\langle\mu|\rho_s|\mu\rangle\\ &= \min_\mu\langle\mu|\rho_s|\mu\rangle = \frac{1}{24},\end{aligned} \tag{19}$$

where $|\mu\rangle$ is an arbitrary bipartite pure state of Schmidt rank at most two. So the last inequality in (14) holds, and we have proved the second inequality in (12). One can similarly prove (13), where the minimum is attained at $|\psi\rangle = |0\cdots0\rangle_{A_1\cdots A_n}\otimes|1\cdots1\rangle_{B_1\cdots B_n}$. This completes the proof. ■

Using Lemma 3 we have the following.

Lemma 4. For any integer n and sufficiently small $\epsilon = \epsilon(n) > 0$, the two-qutrit NPT state ρ in (7) is n-undistillable.

Proof. For any pure state $|\psi\rangle$ of Schmidt rank two, we have

$$\begin{aligned}\langle\psi|(\rho^\Gamma)^{\otimes n}|\psi\rangle &= \langle\psi|(\sigma^\Gamma - \epsilon|f^*,g\rangle\langle f^*,g|)^{\otimes n}|\psi\rangle\\ &:= \langle\psi|(\sigma^\Gamma)^{\otimes n}|\psi\rangle + \sum_{k=1}^n c_k\epsilon^k,\end{aligned} \tag{20}$$

where c_k are real numbers. The definition of σ implies that $\sigma^\Gamma \geqslant p_1(I_9 - |\Psi\rangle\langle\Psi|) \geqslant 0$, where p_1 is the smallest positive eigenvalue of σ^Γ. Hence

$$\langle\psi|(\rho^\Gamma)^{\otimes n}|\psi\rangle \geqslant p_1^n\langle\psi|(I_9 - |\Psi\rangle\langle\Psi|)^{\otimes n}|\psi\rangle + \sum_{k=1}^n c_k\epsilon^k. \tag{21}$$

By Lemma 3, the first summand is positive. Since it is independent of ϵ, the claim follows. ■

V. CONCLUSION

We have proposed methods to detect the 1-distillability of two-qutrit NPT states under LOCC. By using them, we have proved that bipartite NPT states of rank four are 1-distillable. (It is known that this is also true when the rank is less than four.) So they are a useful resource for quantum information tasks. The bound four is sharp; that is, there exist bipartite NPT states of rank five which are not 1-distillable. We give concrete examples of such states. We conjecture that these states are not distillable and so are related to the distillability problem. The next step is to investigate these states and study their 2-distillability.

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