

# Determining stationary-state quantum properties directly from system-environment interactions

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Considering stationary states of continuous-variable systems undergoing an open dynamics, we unveil the connection between properties and symmetries of the latter and the dynamical parameters. In particular, we explore the relation between the Lyapunov equation for dynamical systems and the steady-state solutions of a time-independent Lindblad master equation for bosonic modes. Exploiting bona fide relations that characterize some genuine quantum properties (entanglement, classicality, and steerability), we obtain conditions on the dynamical parameters for which the system is driven to a steady state possessing such properties. We also develop a method to capture the symmetries of a steady state based on symmetries of the Lyapunov equation. All the results and examples can be useful for steady-state engineering processes.

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The manipulation of the environment affecting the dynamics of a quantum system, with the aim of driving the latter towards a specific state, embodies a valuable tool for quantum state engineering. Depending on assumptions about the couplings, the open dynamics can lead to either an equilibrium state or to a dynamical steady state. On the other hand, in this scenario, it is critical to ensure that the desired state is achieved regardless the fluctuations in the initial state of the system. Protocols of this sort are known as reservoir engineering, stabilization, and design [1–5].

A standard approach to the modeling of the evolution of an open system is the Lindblad master equation (LME) for the density operator  $\hat{\rho}$  [3,6,7]:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{1}{2\hbar} \sum_{m=1}^M (\{\hat{L}_m^\dagger \hat{L}_m, \hat{\rho}\} - 2\hat{L}_m \hat{\rho} \hat{L}_m^\dagger), \quad (1)$$

which, besides the unitary dynamics ruled by the Hamiltonian operator  $\hat{H}$ , accounts for a nonunitary dynamics as resulting from the weak coupling (*via* the operators  $\hat{L}_k$ ) to uncontrollable environmental degrees of freedom. The LME is the most general type of Markovian and time-homogeneous master equation guaranteeing trace preservation and complete positivity. Despite the fundamental and very restrictive Markovianity assumption, the LME is crucial for the description of an ample set of dynamics in quantum optics and information, mesoscopic physics, and quantum chemistry [3,7,8].

In this work we investigate properties and symmetries of continuous-variable states driven to equilibrium by a linear evolution governed by the time-independent Lindblad dynamics. Gaussian states, which play a preponderant role in quantum information science and are the natural candidates for the implementation of quantum computation with continuous variables [9], belong to such set of states.

From the mathematical point of view, the problem of whether a linear LME has a stable steady state is equivalent to the solution of a Lyapunov equation for the covariance matrix of the quantum state. The methodology used to solve Lyapunov equations, known as Lyapunov stability theory [10],

was developed in Ref. [11] in the context of dynamical systems. This formalism was explored in Ref. [12] to determine conditions for a state to be pure in the stationary regime.

In our work, we make use of the connection between the LME and the Lyapunov theory to determine several properties of continuous-variable steady states, such as classicality [13], separability [14] (or bound entanglement [15]), and steerability [16]. Furthermore, we also explore the steady-state symmetries induced by the dynamical symmetries of the Lyapunov equation. This is particularly interesting, because it is in general hard to characterize the symmetries of the states working directly on the LME (1). This task becomes instead fully manageable when dealing with finite matrices. Our results are applicable to systems with a generic number of degrees of freedom and their analyticity brings in turn robustness for numerical examinations of the mentioned properties.

The remaining of this paper is organized as follows: In Sec. I, we set the notation and describe the linear dynamics, discussing the connection between the LME and the Lyapunov theory. The mathematical results concerning the Lyapunov equation are developed in Sec. II, which will be extensively applied to find general properties of stationary solutions in Sec. III. Symmetries of the system are analyzed in Sec. IV, while examples are given in Sec. V. A method for engineering steady states is presented in Sec. VI. Section VII presents our conclusions, while in the Appendixes we further discuss some technical aspects of the mathematical approach, including a brief summary of the notation.

## I. LINEAR DYNAMICS AND STATIONARY CONDITIONS

For a system of  $n$  continuous degrees of freedom (DF), the generalized coordinates together with the canonical conjugated momenta are collected in a  $2n$  column vector:

$$\hat{x} := (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)^\dagger. \quad (2)$$

In this notation, the canonical commutation relation (CCR) is written compactly as  $[\hat{x}_j, \hat{x}_k] = i\hbar \mathbf{J}_{jk}$  with  $\mathbf{J}_{jk}$  given by the elements of the symplectic matrix

$$\mathbf{J} := \begin{pmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0}_n \end{pmatrix}. \quad (3)$$

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We consider the evolution of a quantum state governed by the LME with a quadratic Hamiltonian and linear Lindblad operators, viz.,

$$\begin{aligned} \hat{H} &= \frac{1}{2}\hat{x} \cdot \mathbf{H}\hat{x} + \xi \cdot \mathbf{J}\hat{x} + H_0, \\ \hat{L}_m &= \lambda_m \cdot \mathbf{J}\hat{x} + \mu_m, \end{aligned} \quad (4)$$

where  $\xi \in \mathbb{R}^{2n}$  and  $\lambda_m \in \mathbb{C}^{2n}$  are column vectors,  $H_0 \in \mathbb{R}$  and  $\mu_m \in \mathbb{C}$  are constants, and  $m = 1, \dots, M$ . The Hessian of the Hamiltonian is symmetric by definition:  $\mathbf{H} = \mathbf{H}^\top \in \text{Mat}(2n, \mathbb{R})$ . Under such conditions, the evolution of the mean value vector  $\langle \hat{x} \rangle_t := \text{Tr}[\hat{x}\hat{\rho}(t)] \in \mathbb{R}^{2n}$  can be retrieved from Eq. (1) by using only the CCR [3,17]:

$$\frac{d\langle \hat{x} \rangle_t}{dt} = \xi - \eta + \mathbf{\Gamma}\langle \hat{x} \rangle_t, \quad (5)$$

where we have introduced  $\eta := \sum_{m=1}^M \text{Im}(\mu_m^* \lambda_m) \in \mathbb{C}^{2n}$ ,

$$\mathbf{\Gamma} := \mathbf{J}\mathbf{H} - \text{Im}\mathbf{\Upsilon}\mathbf{J} \in \text{Mat}(2n, \mathbb{R}), \quad (6)$$

and

$$\mathbf{\Upsilon} := \sum_{m=1}^M \lambda_m \lambda_m^\dagger \in \text{Mat}(2n, \mathbb{C}). \quad (7)$$

The natural question that arises at this point is whether a solution of Eq. (5) attains a finite asymptotic value when  $t \rightarrow \infty$ . The answer is provided in the context of the Lyapunov stability theory [10]. All solutions will be driven to an asymptotic point, if the matrix  $\mathbf{\Gamma}$  is *asymptotically stable* (AS), i.e., if its spectrum has negative real part.

Interestingly enough, from the Lindblad dynamics (1) with the operators in Eq. (4), a Lyapunov equation (LE) emerges naturally for the stationary value of the covariance matrix (CM) of the system, as we shall see. Defining the CM of the state  $\hat{\rho}$  as  $\mathbf{V} = \mathbf{V}^\top \in \text{Mat}(2n, \mathbb{R})$  with elements [18]

$$\mathbf{V}_{jk} = \frac{1}{\hbar} \text{Tr}[\{\hat{x}_j - \langle \hat{x}_j \rangle_t, \hat{x}_k - \langle \hat{x}_k \rangle_t\} \hat{\rho}(t)], \quad (8)$$

and calculating its evolution [3,17], the (possible) stationary value of the CM will be the solution of the LE

$$\mathbf{V}\mathbf{\Gamma}^\top + \mathbf{\Gamma}\mathbf{V} + \mathbf{D} = 0, \quad (9)$$

with  $\mathbf{\Gamma}$  in Eq. (6) and

$$\mathbf{D} := 2\text{Re}\mathbf{\Upsilon} = \mathbf{D}^\top \in \text{Mat}(2n, \mathbb{R}), \quad (10)$$

which is positive semidefinite,  $\mathbf{D} \geq 0$ , by the definition of  $\mathbf{\Upsilon}$ . The Lyapunov theorem and its extensions [10,11,19] guarantee that, for an AS matrix  $\mathbf{\Gamma}$ , the solution of Eq. (9) exists and is unique. Furthermore, those theorems also relate the stability nature of the matrix  $\mathbf{\Gamma}$  to the existence of matrices (in our case  $\mathbf{V}$  and  $\mathbf{D}$ ) satisfying the LE (9).

We stress that, in order to deduce Eqs. (5) and (9), we do not need any assumptions about the initial state of the system. The derivation of such equations only uses the LME, the particular structure of Eqs. (4), and the CCR. Meanwhile, the LME with the operators (4) will always preserve the Gaussian character of an initial Gaussian state [17]. Once the CM of a steady state

of the system is a solution of Eq. (9), which is unique and does not depend on the initial state, any initial state will end in a Gaussian steady state.

## II. LYAPUNOV EQUATIONS

In this section we show and develop results concerning the generic Lyapunov equation

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\dagger + \mathbf{Q} = 0, \quad (11)$$

and its solution. Since our objective is to understand properties of stationary solutions of the LME, it is convenient to assume that (i)  $\mathbf{A} \in \text{Mat}(m, \mathbb{C})$  is AS, (ii)  $\mathbf{P} = \mathbf{P}^\dagger \in \text{Mat}(m, \mathbb{C})$ , and (iii)  $\mathbf{Q} = \mathbf{Q}^\dagger \in \text{Mat}(m, \mathbb{C})$ . From now on, the LE (11) will be represented by the triple  $[\mathbf{P}, \mathbf{A}, \mathbf{Q}]$ .

The first of those assumptions ( $\mathbf{A}$  is AS) is enough to prove [10,19] that the *unique* solution for the LE in (11) is written as

$$\mathbf{P}(\mathbf{Q}, \mathbf{A}) = \int_0^\infty dt e^{\mathbf{A}t} \mathbf{Q} e^{\mathbf{A}^\dagger t}. \quad (12)$$

Furthermore, for any  $\mathbf{A} \in \text{GL}(m, \mathbb{C})$ , it is true that

$$\text{In}(e^{\mathbf{A}t} \mathbf{Q} e^{\mathbf{A}^\dagger t}) = \text{In}(\mathbf{Q}), \quad (13)$$

because the expression inside the parenthesis on the LHS is a congruence transformation of  $\mathbf{Q}$ . The symbol “In” refers to the *inertia index* of a matrix, as defined in Appendix A. On the other hand, once  $\mathbf{A}$  is AS, then  $\lim_{t \rightarrow \infty} e^{\mathbf{A}t} = 0$ , which guarantees the convergence of the integral in Eq. (12). These last arguments about the structure of Eq. (12) are used in the proof of the following result [10,19]:

*Proposition 1.* Consider the solution (12) for the LE in Eq. (11). If  $\mathbf{Q} \geq 0$  ( $\mathbf{Q} > 0$ ), then  $\mathbf{P} \geq 0$  ( $\mathbf{P} > 0$ ).

Note that Proposition 1 does not exclude the statement  $\mathbf{Q} \geq 0 \Rightarrow \mathbf{P} > 0$ , since the set of matrices  $\mathbf{P}$  such that  $\mathbf{P} > 0$  is a subset of  $\mathbf{P} \geq 0$ ; cf. Appendix A. This is the case provided the pair  $(\mathbf{Q}, \mathbf{A})$  is observable [10,19]. For our purposes, this statement is not necessary; however, it is for the results in Ref. [12].

Now, let us go a bit further with the results in Proposition 1, specializing the properties of the AS matrix  $\mathbf{A}$ :

*Proposition 2.* Consider the LE  $[\mathbf{P}, \mathbf{A}, \mathbf{Q}]$  with  $\mathbf{A} = \mathbf{A}^\dagger$ , then  $\mathbf{P} \geq 0$  ( $\mathbf{P} > 0$ ) if and only if  $\mathbf{Q} \geq 0$  ( $\mathbf{Q} > 0$ ).

*Proof.* Since  $\mathbf{A}$  is self-adjoint and negative definite (AS), it is possible to write  $\mathbf{A} = -\sqrt{-\mathbf{A}}\sqrt{-\mathbf{A}}$ , where  $\sqrt{-\mathbf{A}}$  is the unique self-adjoint positive-definite square root of  $-\mathbf{A}$ . From the LE (11),

$$\begin{aligned} \text{Spec}_\mathbb{R}(\mathbf{Q}) &= -\text{Spec}_\mathbb{R}(\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}) \\ &= -\text{Spec}_\mathbb{R}[(\mathbf{P} + \mathbf{A}\mathbf{P}\mathbf{A}^{-1})\mathbf{A}] \\ &= \text{Spec}_\mathbb{R}[\sqrt{-\mathbf{A}}(\mathbf{P} + \mathbf{A}\mathbf{P}\mathbf{A}^{-1})\sqrt{-\mathbf{A}}]. \end{aligned} \quad (14)$$

Since the sum of positive-semidefinite (positive-definite) matrices is positive semidefinite (positive definite), and since a congruence transformation does not change the signs of the eigenvalues (or the inertia of the matrix), it follows that  $\mathbf{Q} \geq 0$  ( $\mathbf{Q} > 0$ ) if  $\mathbf{P} \geq 0$  ( $\mathbf{P} > 0$ ), which proves the necessary condition. The sufficiency is in Proposition 1. ■

Note that, once one statement in Proposition 2 is  $\mathbf{Q} > 0 \Leftrightarrow \mathbf{P} > 0$ , then the statement  $\mathbf{Q} \geq 0 \Leftrightarrow \mathbf{P} \geq 0$  necessarily means

that if  $\mathbf{Q}$  has one null eigenvalue, then  $\mathbf{P}$  will also have one null eigenvalue, and vice versa.

In the direction of the main task of this work, we must develop some results concerning matrices of the form  $\mathbf{P} + \mathbf{\Xi}$ , where  $\mathbf{P}$  is the solution in Eq. (12) of  $[\mathbf{P}, \mathbf{A}, \mathbf{Q}]$  and  $\mathbf{\Xi} = \mathbf{\Xi}^\dagger \in \text{Mat}(m, \mathbb{C})$ .

*Corollary 1.*  $\mathbf{P} + \mathbf{\Xi} \geq 0$  if  $\mathbf{Q}_{[\mathbf{\Xi}]} := \mathbf{Q} - \mathbf{\Xi} \mathbf{A}^\dagger - \mathbf{A} \mathbf{\Xi} \geq 0$ .

*Proof.* It is easy to see that the LE  $[\mathbf{P} + \mathbf{\Xi}, \mathbf{A}, \mathbf{Q}_{[\mathbf{\Xi}]}]$  is equivalent to the LE in Eq. (11). Thus, the proof follows from Proposition 1, since  $\mathbf{Q}_{[\mathbf{\Xi}]} = \mathbf{Q}_{[\mathbf{\Xi}]}^\dagger$ . ■

The converse statement of Corollary 1 is not true in general. By using the restriction for the matrices  $\mathbf{A}$ , as in Proposition 2, we obtain the following corollary giving a necessary and sufficient condition:

*Corollary 2.* Consider  $\mathbf{A} = \mathbf{A}^\dagger$  ( $\mathbf{A}$  is AS), then  $\mathbf{P} + \mathbf{\Xi} \geq 0$  if and only if  $\tilde{\mathbf{Q}}_{[\mathbf{\Xi}]} := \mathbf{Q} - \{\mathbf{\Xi}, \mathbf{A}\}_+ \geq 0$ .

*Proof.* As before, we construct the equivalent Lyapunov equation  $[\mathbf{P} + \mathbf{\Xi}, \mathbf{A}, \tilde{\mathbf{Q}}_{[\mathbf{\Xi}]}]$ , and use Proposition 2 since  $\tilde{\mathbf{Q}}_{[\mathbf{\Xi}]} = \tilde{\mathbf{Q}}_{[\mathbf{\Xi}]}^\dagger$ . ■

Note that, contrary to Propositions 1 and 2, we did not mention the strictly positive cases ( $\mathbf{Q}_{[\mathbf{\Xi}]} > 0$ ,  $\tilde{\mathbf{Q}}_{[\mathbf{\Xi}]} > 0$ ) in Corollaries 1 and 2. Actually, these cases follow the same prescription, but they are not necessary for our next results.

In what follows, properties of the steady states, driven to equilibrium under the linear evolution generated by Eq. (4), will be considered from the perspective of the results developed in this section.

### III. BONA FIDE RELATIONS AND STEADY STATES

Through the temporal evolution of a state by the LME conditioned to an AS dynamics, the dependence on the initial condition is progressively erased by the environmental action. Therefore, the steady-state properties must be completely determined only by the environment. An usual way to describe properties of continuous-variable states is given by bona fide relations involving the CM of the states. From now on, we will assume that the CM of a quantum state evolves with  $\mathbf{\Gamma}$  AS and  $\mathbf{D} \geq 0$  and attains an asymptotic value described by the LE (9) with solution  $\mathbf{V} := \mathbf{P}(\mathbf{D}, \mathbf{\Gamma})$  in Eq. (12).

It is convenient to recall the definitions of the auxiliary matrices defined in Corollaries 1 and 2, but now for the LE in question. For any matrix  $\mathbf{\Xi} = \mathbf{\Xi}^\dagger \in \text{Mat}(2n, \mathbb{C})$ , we have

$$\mathbf{D}_{[\mathbf{\Xi}]} := \mathbf{D} - \mathbf{\Xi} \mathbf{\Gamma}^\top - \mathbf{\Gamma} \mathbf{\Xi}, \quad \tilde{\mathbf{D}}_{[\mathbf{\Xi}]} := \mathbf{D} - \{\mathbf{\Xi}, \mathbf{\Gamma}\}_+. \quad (15)$$

#### A. Uncertainty principle

Any quantum state is subjected to constraints imposed by the uncertainty principle, which is only a consequence of the CCR. For the continuous-variable case, this principle takes into account only the CM (8). A genuine physical state has a CM such that [20]

$$\mathbf{V} + i\mathbf{J}_{2n} \geq 0. \quad (16)$$

Given a Hamiltonian and a collection of Lindblad operators as in Eq. (4), what can our corollaries say about the genuineness of the steady state generated by the LME? Invoking Corollary 1, the matrix  $\mathbf{V}$  of a steady state is a bona fide CM if  $\mathbf{D}_{[iJ]} = \mathbf{D} - i\mathbf{\Gamma} \mathbf{J} - i\mathbf{J} \mathbf{\Gamma}^\top \geq 0$ . However, by using Eqs. (6)

and (7), it is not difficult to show that  $\mathbf{D}_{[iJ]} = 2\mathbf{\Upsilon}^*$ , which is always positive semidefinite, according to the definition of  $\mathbf{\Upsilon}$  in Eq. (7). Tautologically, this says that all linear LMEs with  $\mathbf{\Gamma}$  AS and  $\mathbf{D} \geq 0$  will drive the system to a steady state obeying the relation (16), i.e., a genuine physical state.

On the other hand, Eq. (9) is a consequence of the CCR, as mentioned before. Accordingly, the LME guarantees that the uncertainty principle holds for all times, including the steady-state limit, whereof the condition in Corollary 1 is necessary and sufficient regardless of whether  $\mathbf{\Gamma}$  is symmetric. Before going to the next bona fide relation, it is important to remark that this extension of Corollary 1 to an “if and only if” condition is only true for the relation in Eq. (16). For all the other relations which will appear in what follows, the differences between Corollaries 1 and 2 should be considered.

#### B. Classical states

Classical states are defined as having a positive Glauber–Sudarshan distribution function, they are also called  $P$ -representable states. This definition relies on the possibility to express a desired state as a classical mixture of coherent states. The necessary and sufficient condition for  $P$ -representability of a Gaussian state is written in terms of its CM as the bona fide relation (see Appendix B)

$$\mathbf{V} - \mathbf{I}_{2n} \geq 0. \quad (17)$$

The evolution that drives the system to a classical steady state is subjected to the sufficient condition given by Corollary 1:

$$\mathbf{D}_{[-1]} = \mathbf{\Gamma} + \mathbf{\Gamma}^\top + \mathbf{D} \geq 0 \Rightarrow \mathbf{V} - \mathbf{I}_{2n} \geq 0. \quad (18)$$

The contrapositive of the above statement says that, if a given LME is such that  $\mathbf{D}_{[-1]}$  has at least one negative eigenvalue, it will lead the system to a nonclassical stationary state. The matrix  $\mathbf{V}$  is, by hypothesis, the CM of a steady state of the LME. Once the converse statement of Eq. (18) is not true, one can conclude that there are classical steady states which cannot be generated by a LME with dynamical matrices such that  $\mathbf{D}_{[-1]} \geq 0$ .

If we consider only steady states generated by a LME with  $\mathbf{\Gamma}$  symmetric, by Corollary 2 the possible classical states will obey the necessary and sufficient condition

$$\tilde{\mathbf{D}}_{[-1]} = 2\mathbf{\Gamma} + \mathbf{D} \geq 0 \Leftrightarrow \mathbf{V} - \mathbf{I}_{2n} \geq 0. \quad (19)$$

This means that all states generated by a LME with  $\mathbf{\Gamma} = \mathbf{\Gamma}^\top$  and  $\tilde{\mathbf{D}}_{[-1]} \geq 0$  are classical states. Conversely, all classical states with CM  $\mathbf{V}$  which are solutions of a LE with  $\mathbf{\Gamma} = \mathbf{\Gamma}^\top$  are steady states of a LME satisfying  $\tilde{\mathbf{D}}_{[-1]} \geq 0$ .

Since classicality is related to mixtures of coherent states, one instructive example is given by the CM  $\mathbf{V} = \alpha \mathbf{I}_{2n}$ —i.e., the CM of any  $n$ -mode (or  $n$ -DF) coherent state. The simplicity of this case enables us to derive a useful necessary and sufficient condition besides the relation in Eq. (19). Actually, we will be concerned with the slightly more general situation: the tensor product of  $n$  Gibbs states with the same occupation number. These states have the global CM written as  $\mathbf{V} = \alpha \mathbf{I}_{2n}$ , which is a CM of a classical state if  $\alpha \geq 1$ .

*Corollary 3.* For any  $\alpha > 0$  and any  $\mathbf{\Gamma}$  such that  $(\mathbf{\Gamma} + \mathbf{\Gamma}^\top) \leq 0$ , the matrix  $\mathbf{V} = \alpha \mathbf{I}_{2n}$  is a solution of the LE  $[\mathbf{V}, \mathbf{\Gamma}, \mathbf{D}]$

if and only if  $\alpha(\mathbf{\Gamma} + \mathbf{\Gamma}^\top) + \mathbf{D} = 0$ . Furthermore,  $\alpha = \frac{1}{2}\text{Tr}\mathbf{D}/\text{Tr}(\text{Im}\mathbf{\Upsilon}\mathbf{J})$ .

*Proof.* The sufficient condition is trivially obtained by constructing the LE  $[\alpha\mathbf{I}_{2n}, \mathbf{\Gamma}, \mathbf{D}]$  from Eq. (9). To prove the necessary condition, one defines

$$\mathcal{I} := \int_0^\infty dt e^{\mathbf{\Gamma}t} e^{\mathbf{\Gamma}^\top t}$$

and integrates it by parts to show that  $\mathcal{I}\mathbf{\Gamma}^\top + \mathbf{\Gamma}\mathcal{I} = -\mathbf{I}_{2n}$ . The solution (12) with  $\mathbf{Q} = \mathbf{D} = -\alpha(\mathbf{\Gamma} + \mathbf{\Gamma}^\top)$  and  $\mathbf{A} = \mathbf{\Gamma}$  shows that  $\mathbf{V} = -\alpha(\mathbf{\Gamma}\mathcal{I} + \mathcal{I}\mathbf{\Gamma}^\top) = \alpha\mathbf{I}$ . Since  $\mathbf{\Gamma} \in \text{Mat}(2n, \mathbb{R})$ , its complex eigenvalues will occur in conjugate pairs, then  $\text{Tr}\mathbf{\Gamma} \in \mathbb{R}$ . Since it is also AS,  $\text{Tr}\mathbf{\Gamma} \neq 0$ , then the value of  $\alpha$  holds if one considers the definition of  $\mathbf{\Gamma}$  in Eq. (6) and the fact that  $\text{Tr}(\mathbf{J}\mathbf{H}) = 0$ , since  $\mathbf{H}$  is symmetric. ■

The state studied in Corollary 3 is an example of the multiplicity of the steady state with respect to different matrices  $\mathbf{\Gamma}$  and  $\mathbf{D}$ . There is an infinite number of matrices satisfying the relation  $\alpha(\mathbf{\Gamma} + \mathbf{\Gamma}^\top) + \mathbf{D} = 0$  and giving rise to the same steady state. However, for a given pair of matrices  $\mathbf{\Gamma}$  and  $\mathbf{D}$ , the solution  $\mathbf{V}(\mathbf{\Gamma}, \mathbf{D})$  is uniquely given in Eq. (12).

### C. Separable states

A necessary and sufficient condition for an  $n$ -mode Gaussian state to be *separable* with respect to one of the modes, say the  $k$ th mode, is defined in terms of its CM as  $\mathbf{V}^{\top k} + i\mathbf{J} \geq 0$ , where  $\mathbf{V}^{\top k} := \mathbf{T}_k \mathbf{V} \mathbf{T}_k$ . The transformation  $\mathbf{T}_k = \mathbf{T}_k^{-1} = \mathbf{T}_k^\top$  is a local time inversion on the  $\hat{x}$  operator, viz.,

$$\mathbf{T}_k \hat{x} = (\hat{q}_1, \dots, \hat{q}_k, \dots, q_n, \hat{p}_1, \dots, -\hat{p}_k, \dots, \hat{p}_n)^\top, \quad (20)$$

and, of course, cannot be implemented unitarily. Since  $\mathbf{T}_k$  is orthogonal, we can express the separability condition equivalently as [14,15]

$$\mathbf{V} + i\mathbf{J}^{\top k} \geq 0. \quad (21)$$

The statements in Eqs. (18) and (19) can be readily modified to the present case:

$$\text{(Corollary 1)} \quad \mathbf{D}_{[i\mathbf{J}^{\top k}]} \geq 0 \Rightarrow \mathbf{V} + i\mathbf{J}^{\top k} \geq 0, \quad (22a)$$

$$\text{(Corollary 2)} \quad \tilde{\mathbf{D}}_{[i\mathbf{J}^{\top k}]} \geq 0 \Leftrightarrow \mathbf{V} + i\mathbf{J}^{\top k} \geq 0. \quad (22b)$$

The interpretations of these conditions are also readily adapted from those in the previous section, it is just a question of changing the dichotomy ‘‘classical and/or nonclassical’’ to ‘‘separable and/or entangled.’’

All classical states (not only the Gaussian ones) are separable, since they are written as a convex sum of coherent states which are separable [see Eq. (B1)]. As a consequence, a hierarchy of the dynamics of LMEs can be established: the set of matrices such that  $\mathbf{D}_{[-]} \geq 0$  in Eq. (19) is a subset of those satisfying  $\tilde{\mathbf{D}}_{[i\mathbf{J}^{\top k}]} \geq 0$  in Eq. (22b). However, this is not true for the matrices in Eqs. (18) and (22a), because both only give a sufficient condition.

Now consider the following partition of the number of DF of a state:  $n = n_1 + n_2$ , where  $n_i$  is the number of DF of each partition. Define also the local time inversion operation as

$$\mathbf{T}_{n_2} \hat{x} = (\hat{q}_1, \dots, \hat{q}_{n_1}, \hat{p}_1, \dots, \hat{p}_{n_1}, -\hat{p}_{n_1+1}, \dots, -\hat{p}_{n_1+n_2})^\top. \quad (23)$$

The separability criteria already exposed is a necessary and sufficient condition only if  $n_2 = 1$ . For all other cases, entangled states with  $\mathbf{V} + i\mathbf{T}_{n_2} \mathbf{J} \mathbf{T}_{n_2} \geq 0$  are bound entangled, i.e., they have nondistillable entanglement [15]. One can also relate the reservoir properties with this bona fide relation through the replacement  $\mathbf{J}^{\top k} \rightarrow \mathbf{T}_{n_2} \mathbf{J} \mathbf{T}_{n_2}$  in Eqs. (22). Note that the bona fide relation in question does not say whether the state is separable or bound-entangled.

### D. Gaussian steerability

Quantum steering is a form of correlation related to the ability of one part of a system to modify the state of a companion system when only local measurements are performed on the former. More precisely, if through local measurements and classical communication one part of the system is able to convince the other part that they share an entangled state, the state is said to be steerable with respect to the first part [3].

As in the previous section, considering the partition of the DF as  $n = n_1 + n_2$ , a state is *non-Gaussian-steerable* with respect to the first part (with  $n_1$  DF) if and only if [16]

$$\mathbf{V} + i\mathbf{\Pi}_2 \geq 0, \quad \mathbf{\Pi}_2 := \frac{1}{2}(\mathbf{J} + \mathbf{T}_{n_2} \mathbf{J} \mathbf{T}_{n_2}), \quad (24)$$

with  $\mathbf{T}_{n_2}$  defined in Eq. (23). In other words, it is not possible to steer the state of part 1, making local Gaussian measurements on part 2, if the last condition holds. The steering relation with respect to the second part is obtained by changing the roles of  $n_1$  and  $n_2$ .

As before, this concept can be related to the dynamical matrices, and one can derive similar formulas by just replacing  $\mathbf{J}^{\top k} \rightarrow \mathbf{\Pi}_2$  in Eq. (22). All Gaussian steerable states are entangled [3], thus the set of matrices such that  $\tilde{\mathbf{D}}_{[i\mathbf{\Pi}_2]} \geq 0$  is a subset of those satisfying  $\tilde{\mathbf{D}}_{[i\mathbf{J}^{\top k}]} \geq 0$ .

This concludes our analysis of the bona fide relations used across this paper.

## IV. SYMMETRIES OF STEADY STATES

In principle, symmetries of the steady states can be associated with the symmetries of the dynamics governed by the LME [5,8]. In the perspective developed in this work, the relation of the steady-state symmetries and the symmetries of the dynamical matrices  $\mathbf{\Gamma}$  and  $\mathbf{D}$  will be investigated.

Two LEs,  $[\mathbf{V}, \mathbf{\Gamma}, \mathbf{D}]$  and  $[\mathbf{V}', \mathbf{\Gamma}', \mathbf{D}']$ , are said to be *covariant* when their matrices are related by

$$\mathbf{V}' = \mathbf{W}\mathbf{V}\mathbf{W}^\top, \quad \mathbf{\Gamma}' = \mathbf{W}\mathbf{\Gamma}\mathbf{W}^{-1}, \quad \mathbf{D}' = \mathbf{W}\mathbf{D}\mathbf{W}^\top, \quad (25)$$

for  $\mathbf{W} \in \text{GL}(2n, \mathbb{R})$ . Note that, for an orthogonal  $\mathbf{W}$ , all above matrices are subjected to the same transformation. This covariance is helpful to determine the invariance properties of a steady state, working directly at the level of the CM:

*Proposition 3.* If  $\mathbf{\Gamma}$  and  $\mathbf{D}$  are invariant under  $\mathbf{W} \in \text{GL}(2n, \mathbb{R})$  (i.e.,  $\mathbf{W}\mathbf{\Gamma}\mathbf{W}^{-1} = \mathbf{\Gamma}$  and  $\mathbf{W}\mathbf{D}\mathbf{W}^\top = \mathbf{D}$ ), then  $\mathbf{V}$  is invariant as well (i.e.,  $\mathbf{W}\mathbf{V}\mathbf{W}^\top = \mathbf{V}$ ). In this case, we say that  $\mathbf{W}$  is a symmetry transformation of the Lyapunov equation.

*Proof.* Writing the solution  $\mathbf{V}(\mathbf{\Gamma}, \mathbf{D})$  as in Eq. (12) for the LE  $[\mathbf{V}, \mathbf{\Gamma}, \mathbf{D}]$ , and since  $\det \mathbf{W} \neq 0$ , it is easy to see that  $\mathbf{V}(\mathbf{\Gamma}, \mathbf{D}) = \mathbf{V}(\mathbf{W}\mathbf{\Gamma}\mathbf{W}^{-1}, \mathbf{W}\mathbf{D}\mathbf{W}^\top)$ , i.e.,  $\mathbf{V}$  remains unchanged. ■



Let us give some particular but useful examples. Consider the transformation

$$\mathbf{W} = \hat{\sigma}_z \otimes I_2 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix} \in \text{GL}(4, \mathbb{R}). \quad (26)$$

If  $\mathbf{\Gamma}$  and  $\mathbf{D}$  are invariant under this transformation, they are necessarily written as  $\mathbf{\Gamma} = \mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2$  and  $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ , where  $\mathbf{\Gamma}_j, \mathbf{D}_j \in \text{Mat}(2, \mathbb{R})$ . As a consequence of Proposition 3,  $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$ , i.e., it will be the CM of a state without position-momentum correlations. This invariance can be retrieved directly from the solution (12): if  $\mathbf{A}$  and  $\mathbf{Q}$  are block diagonals then  $\mathbf{P}$  will also be block diagonal.

Another example is the transformation

$$\mathbf{W} = \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix} \in \text{GL}(4, \mathbb{R}). \quad (27)$$

If  $\mathbf{\Gamma}$  and  $\mathbf{D}$  are invariant under this, the CM will have momentum correlations equal to position correlations:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_{12} \\ \mathbf{V}_{12} & \mathbf{V}_1 \end{pmatrix}. \quad (28)$$

Focusing on the symplectic group, i.e., choosing  $\mathbf{W} \in \text{Sp}(2n, \mathbb{R}) \subset \text{GL}(2n, \mathbb{R})$ , we use the definition of  $\mathbf{\Gamma}$  in Eq. (6) and the covariance relation (25) to write

$$\mathbf{V}' = \mathbf{W}\mathbf{V}\mathbf{W}^\top, \quad \mathbf{H}' = \mathbf{W}^{-\top}\mathbf{H}\mathbf{W}^{-1}, \quad \lambda' = \mathbf{W}\lambda. \quad (29)$$

This symplectic covariance, by the Stone–von Neumann theorem [21], is nothing but the representation of a unitary transformation of the LME:

$$\hat{\rho} \rightarrow \hat{U}\hat{\rho}\hat{U}^\dagger, \quad \hat{H} \rightarrow \hat{U}\hat{H}\hat{U}^\dagger, \quad \hat{L}_i \rightarrow \hat{U}\hat{L}_i\hat{U}^\dagger, \quad (30)$$

where the unitary operator  $\hat{U}$  is the metaplectic operator associated with  $\mathbf{W} \in \text{Sp}(2n, \mathbb{R})$  [17,21].

If we rearrange the elements of Eq. (8) consistently with the reordering  $\hat{x} \mapsto \check{x} := (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)$  of Eq. (2), then the invariance under the (nonreordered) transformation in Eq. (26) implies that the steady state of the system, with CM  $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$ , is the product state  $\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2$ . In the reordered basis, the transformation (26) is symplectic. Similarly, the (nonreordered) matrix in Eq. (27) is also symplectic in the reordered basis, and it realizes the exchange of the subsystems. Consequently, the matrix in Eq. (28) is the CM of states with same local purity (symmetric states).

As a last example, consider a symplectic rotation  $\mathbf{R} \in \mathbf{K}(n) := \text{Sp}(2n, \mathbb{R}) \cap \text{O}(2n)$ . As a consequence of its symplecticity and orthogonality, it is written as [20]

$$\mathbf{R} = \begin{pmatrix} \mathbf{Y} & \mathbf{Z} \\ -\mathbf{Z} & \mathbf{Y} \end{pmatrix}, \quad (31)$$

with  $\mathbf{Y}, \mathbf{Z} \in \text{Mat}(n, \mathbb{R})$  satisfying the following conditions:

$$\mathbf{Y}\mathbf{Y}^\top + \mathbf{Z}\mathbf{Z}^\top = I_n, \quad \mathbf{Y}\mathbf{Z}^\top - \mathbf{Z}\mathbf{Y}^\top = 0_n. \quad (32)$$

Any matrix written as  $\mathbf{M} := m_1 I_{2n} + m_2 \mathbf{J} \in \text{Mat}(2n, \mathbb{R})$ , with  $m_1, m_2 \in \mathbb{R}$  is invariant under the whole group  $\mathbf{K}(n)$ . Note that, if  $\mathbf{M} = \mathbf{M}^\top$ , then  $m_2 = 0$ . If we consider

$$\mathbf{\Gamma} = -\gamma_1 I_{2n} + \gamma_2 \mathbf{J}, \quad \mathbf{D} = \delta I_{2n}, \quad (33)$$

i.e., both invariant under  $\mathbf{K}(n)$ , then Proposition 3 implies that  $\mathbf{V} = \nu I_{2n}$ . By the other side, Corollary 3 is a necessary and

sufficient condition for this CM, thus  $\nu = \frac{\gamma}{2\delta}$ . It is important to mention that the matrices  $\mathbf{\Gamma}$  and  $\mathbf{D}$  on that corollary need not be invariant.

The subgroup of local rotations in  $\mathbf{K}(n)$  is described as the set of matrices (31) with

$$\mathbf{Y} = \text{Diag}(y_1, y_2, \dots, y_n), \quad \mathbf{Z} = \text{Diag}(z_1, z_2, \dots, z_n), \quad (34)$$

which corresponds to a rotation

$$\mathbf{R}_i := \begin{pmatrix} y_i & z_i \\ -z_i & y_i \end{pmatrix} \in \mathbf{K}(1) \quad (35)$$

in each respective canonical pair  $(\hat{q}_i, \hat{p}_i)$ . The matrix  $\mathbf{\Gamma} \in \text{Mat}(2n, \mathbb{R})$  is invariant under the local rotation subgroup if it is of the following form:

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 \\ -\mathbf{\Gamma}_2 & \mathbf{\Gamma}_1 \end{pmatrix}, \quad (36)$$

with  $\mathbf{\Gamma}_i := \text{Diag}(\gamma_{i1}, \dots, \gamma_{in}) \in \text{Mat}(n, \mathbb{R})$ . Since  $\mathbf{D}$  is symmetric, it will be invariant under the same subgroup if it is written as

$$\mathbf{D} = \text{Diag}(d_1, \dots, d_n, d_1, \dots, d_n) \in \text{Mat}(2n, \mathbb{R}). \quad (37)$$

Assuming that  $\mathbf{\Gamma}$  and  $\mathbf{D}$  have this invariant structure, Proposition 3 guaranties that the CM of the steady state will be

$$\mathbf{V} = \text{Diag}(v_1, \dots, v_n, v_1, \dots, v_n), \quad v_i \in \mathbb{R} \quad \forall i, \quad (38)$$

which is the CM of  $n$ -mode thermal state.

## V. EXAMPLES

Let us now present some examples to show the usefulness of the results presented in this work.

### A. Two oscillators interacting with thermal baths

Consider two coupled harmonic oscillators, each one interacting with its own thermal bath. The frequency of the oscillators are  $\omega_1$  and  $\omega_2$  and the spring constant is  $\kappa$ .

The Hamiltonian of the system is given by Eq. (4) with  $\xi = 0$ ,  $H_0 = 0$  and

$$\mathbf{H} = \begin{bmatrix} \omega_1 + \frac{\kappa}{2} & -\frac{\kappa}{2} \\ -\frac{\kappa}{2} & \omega_2 + \frac{\kappa}{2} \end{bmatrix} \oplus \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}. \quad (39)$$

The coupling between a given oscillator and the respective reservoir is described by the Lindblad operators [3]

$$\hat{L}_k = \sqrt{\hbar\zeta_k(\bar{N}_k + 1)}\hat{a}_k, \quad \hat{L}'_k = \sqrt{\hbar\zeta_k\bar{N}_k}\hat{a}_k^\dagger, \quad k = 1, 2, \quad (40)$$

where  $\zeta_k \geq 0$  are the bath-oscillator couplings,  $\bar{N}_k \geq 0$  are thermal occupation numbers, and  $\hat{a}_k := (\hat{q}_k + i\hat{p}_k)/\sqrt{2\hbar}$  is the annihilation operator associated with mode  $k$ . This choice for

the reservoirs and Eq. (4) allow us to identify

$$\begin{aligned}\lambda_1 &= \sqrt{\frac{\zeta_1}{2}(\bar{N}_1 + 1)} (i, 0, -1, 0)^\top, \\ \lambda'_1 &= \sqrt{\frac{\zeta_1}{2}\bar{N}_1} (i, 0, -1, 0)^\dagger, \\ \lambda_2 &= \sqrt{\frac{\zeta_2}{2}(\bar{N}_2 + 1)} (0, i, 0, -1)^\top, \\ \lambda'_2 &= \sqrt{\frac{\zeta_2}{2}\bar{N}_2} (0, i, 0, -1)^\dagger.\end{aligned}\quad (41)$$

With the above vectors, and using Eqs. (6), (7), and (10), one finds

$$\mathbf{D} = \mathbf{D} \oplus \mathbf{D}, \quad \mathbf{D} := \text{Diag}[\zeta_1(2\bar{N}_1 + 1), \zeta_2(2\bar{N}_2 + 1)],$$

$$\text{Spec}_{\mathbb{R}}(\mathbf{D}_{[iJ^T_k]}) = \left\{ \zeta(\bar{N}_1 + \bar{N}_2 + 1) \pm \sqrt{\frac{\kappa^2}{2} + \zeta^2(\bar{N}_1 - \bar{N}_2)^2} + \zeta^2 \pm \sqrt{\frac{\kappa^4}{4} + \zeta^2\kappa^2 + 4\zeta^4(\bar{N}_1 - \bar{N}_2)^2} \right\}. \quad (44)$$

The steady state is separable if this spectrum is non-negative, or explicitly when

$$\frac{\zeta}{\kappa} \geq \mathcal{S}(\bar{N}_1, \bar{N}_2) := \sqrt{\frac{(2\bar{N}_1 + 1)(2\bar{N}_2 + 1)}{16\bar{N}_1\bar{N}_2(\bar{N}_1 + 1)(\bar{N}_2 + 1)}}. \quad (45)$$

In this example, states with  $0 \in \text{Spec}_{\mathbb{R}}(\mathbf{D}_{[iJ^T_k]})$  are all lying in the surface  $\zeta/\kappa = \mathcal{S}(\bar{N}_1, \bar{N}_2)$ . The classicality of the steady state will be determined by Eq. (18) with

$$\text{Spec}_{\mathbb{R}}(\mathbf{D}_{[-I]}) = \left\{ \zeta(\bar{N}_1 + \bar{N}_2) \pm \frac{\kappa}{2} \sqrt{\frac{\kappa^2}{4} + \zeta^2(\bar{N}_1 - \bar{N}_2)^2} \right\}, \quad (46)$$

and the steady state will be classical if

$$\frac{\zeta}{\kappa} \geq \mathcal{P}(\bar{N}_1, \bar{N}_2) := \frac{\bar{N}_1 + \bar{N}_2}{4\bar{N}_1\bar{N}_2}. \quad (47)$$

In Fig. 1 we show the functions in Eqs. (45) and (47). Since a classical state is always separable,  $\mathcal{S}(\bar{N}_1, \bar{N}_2) < \mathcal{P}(\bar{N}_1, \bar{N}_2)$ .

To understand the sufficiency of the results for the system in consideration, as a consequence of the fact that  $\mathbf{\Gamma} \neq \mathbf{\Gamma}^\top$ , we explore the separability and classicality criteria directly applying both to the CM of the steady state. Using Eqs. (42), we are able to obtain analytically the solution  $\mathbf{V} = \mathbf{P}(\mathbf{\Gamma}, \mathbf{D})$  using Eq. (12) or solving algebraically the LE  $[\mathbf{V}, \mathbf{\Gamma}, \mathbf{D}]$ . For simplicity and without loss of generality, we choose  $\bar{N}_1 = \bar{N}_2 = \bar{N}$  and the solution is

$$\begin{aligned}\mathbf{V} &= (2\bar{N} + 1)\mathbf{I}_4 \\ &+ \frac{(2\bar{N} + 1)\kappa}{\zeta^2 + 4\omega(\omega + \kappa)} \begin{bmatrix} \omega & \frac{1}{2}\zeta \\ \frac{1}{2}\zeta & \omega + \kappa \end{bmatrix} \otimes \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.\end{aligned}\quad (48)$$

The steady state is classical if and only if the above CM obeys Eq. (17), or working out its eigenvalues, if and only if

$$\frac{\zeta}{\kappa} \geq \mathcal{P}'(\bar{N}) := \sqrt{\frac{1}{4\bar{N}^2} - \left(\frac{2\omega}{\kappa} + 1\right)^2}. \quad (49)$$

$$\mathbf{\Gamma} = \begin{bmatrix} -\frac{\zeta_1}{2} & 0 & \omega_1 & 0 \\ 0 & -\frac{\zeta_2}{2} & 0 & \omega_2 \\ -\omega_1 - \frac{\kappa}{2} & \frac{\kappa}{2} & -\frac{\zeta_1}{2} & 0 \\ \frac{\kappa}{2} & -\omega_2 - \frac{\kappa}{2} & 0 & -\frac{\zeta_2}{2} \end{bmatrix}. \quad (42)$$

For simplicity, we will consider  $\zeta_1 = \zeta_2 = \zeta$ ,  $\omega_1 = \omega_2 = \omega$ , and the eigenvalues of  $\mathbf{\Gamma}$  become

$$\text{Spec}_{\mathbb{C}}(\mathbf{\Gamma}) = \left\{ -\frac{\zeta}{2} \pm i\omega, -\frac{\zeta}{2} \pm i\sqrt{\omega(\omega + \kappa)} \right\}. \quad (43)$$

As one can see,  $\mathbf{\Gamma}$  is AS since  $\text{Re}[\text{Spec}_{\mathbb{C}}(\mathbf{\Gamma})] < 0$  for all (positive) values of the parameters.

The separability of the steady state will be retrieved from Eqs. (22). Since  $\mathbf{\Gamma} \neq \mathbf{\Gamma}^\top$ , Eq. (22a) will be applied and, calculating the eigenvalues of  $\mathbf{D}_{[iJ^T_k]}$ , one finds

As for separability, the steady state will be separable if and only if the condition (21) is satisfied, which reads as

$$\frac{\zeta}{\kappa} \geq \mathcal{S}'(\bar{N}) := \sqrt{\frac{1}{16\bar{N}^2(\bar{N} + 1)^2} - \left(\frac{2\omega}{\kappa} + 1\right)^2}. \quad (50)$$

In Fig. 2, the two functions representing the necessary and sufficient conditions in Eqs. (49) and (50) are shown. We also compare them with the two sufficient conditions (45) and (47), already plotted in Fig. 1. It is clear that, for the system in question, even if the sufficient criteria become tighter for smaller values of  $\bar{N}$ , they are in general unable to determine whether a state is entangled or nonclassical.

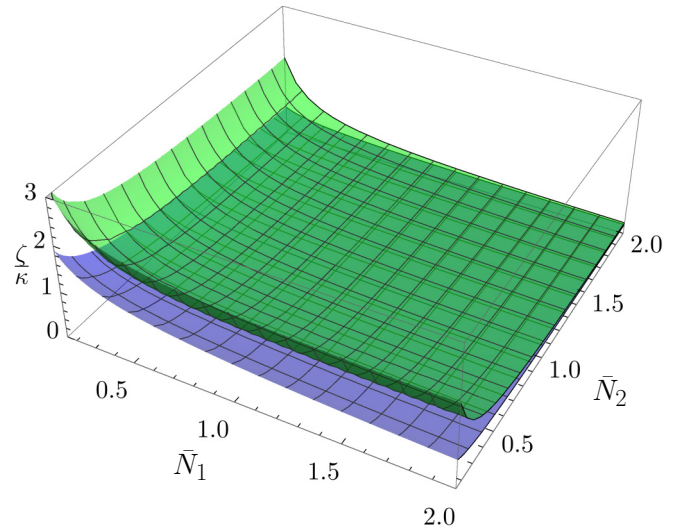


FIG. 1. Separability surface  $\mathcal{S}$  (bottom) and classicality surface  $\mathcal{P}$  (top). States either on or above  $\mathcal{S}$  are separable while states either on or above  $\mathcal{P}$  are classical. Between  $\mathcal{S}$  and  $\mathcal{P}$  there are classical and nonclassical separable states. Below  $\mathcal{S}$ , there are separable and entangled states. The divergence of both functions for  $\bar{N}_1 \rightarrow 0$  and for  $\bar{N}_2 \rightarrow 0$  are not shown in this plot.

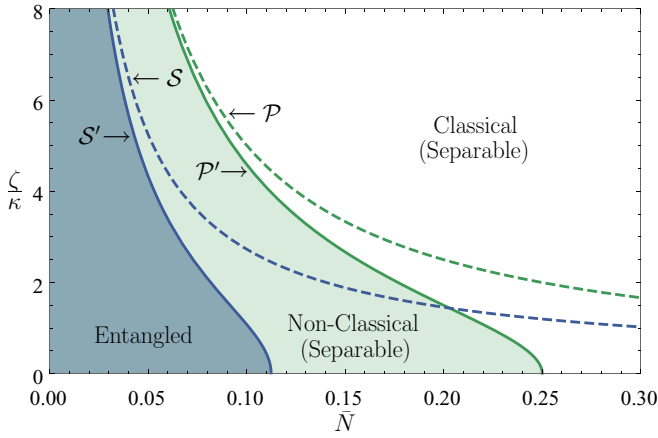


FIG. 2. Necessary and sufficient conditions for classicality and for separability of the steady state. We show the functions  $\mathcal{P}'(\bar{N})$  in Eq. (49) and  $\mathcal{S}'(\bar{N})$  in Eq. (50), as indicated in the graph. The regions limited by these two functions indicates the nature of the steady state. We also plot the functions  $\mathcal{S}(\bar{N}, \bar{N})$  in Eq. (45) and  $\mathcal{P}(\bar{N}, \bar{N})$  in Eq. (47) as dashed curves, see Fig. 1. In this plot  $\omega/\kappa = 0.5$ .

For the Gaussian-steering property, we also consider the case  $\bar{N}_1 = \bar{N}_2 = \bar{N}$  and the sufficient condition is determined by calculating the matrix  $\mathbf{D}_{[i\Pi_k]}$  [the matrix  $\Pi_k$  is defined in Eq. (24), and here  $n_1 = n_2 = 1$ ], which has the eigenvalues

$$\text{Spec}_{\mathbb{R}}(\mathbf{D}_{[i\Pi_1]}) = \text{Spec}_{\mathbb{R}}(\mathbf{D}_{[i\Pi_2]}) = \{(2\bar{N} + 1), \zeta(2\bar{N} + 1) \pm \sqrt{4\zeta^2 + \kappa^2}\}. \quad (51)$$

The steady state will be *non*-Gaussian steerable with respect to both partitions if  $\zeta/\kappa \geq [4(2\bar{N} + 1)^2 + 4]^{-1/2}$ .

### B. Two oscillators interacting with thermal baths in rotating wave approximation

Let us consider the same system as before, but with the Hamiltonian

$$\mathbf{H} = \begin{bmatrix} \varpi_1 & \Omega \\ \Omega & \varpi_2 \end{bmatrix} \oplus \begin{bmatrix} \varpi_1 & \Omega \\ \Omega & \varpi_2 \end{bmatrix}. \quad (52)$$

This Hamiltonian is derived from the one in Eq. (39) by applying a rotating wave approximation (RWA). The procedure and the validity of this result are carefully discussed in the appendix of Ref. [22], as well as the relation among the coupling constant  $\Omega$  in Eq. (52) with the parameters in Eq. (39); see the note [23] for details. Considering also the same structure for the reservoirs in (41), one finds

$$\mathbf{\Gamma} = \begin{bmatrix} -\frac{\zeta_1}{2} & 0 & \varpi_1 & \Omega \\ 0 & -\frac{\zeta_2}{2} & \Omega & \varpi_2 \\ -\varpi_1 & -\Omega & -\frac{\zeta_1}{2} & 0 \\ -\Omega & -\varpi_2 & 0 & -\frac{\zeta_2}{2} \end{bmatrix}, \quad (53)$$

which is AS with eigenvalues

$$\text{Spec}_{\mathbb{C}}(\mathbf{\Gamma}) = \left\{ -\frac{\zeta_1 + \zeta_2}{4} \pm \frac{1}{4} \sqrt{(\zeta_1 - \zeta_2)^2 - 4\Omega^2} \pm i\varpi \right\}, \quad (54)$$

where we used  $\varpi_1 = \varpi_2 =: \varpi$ . Note that  $\mathbf{\Gamma}$  in Eq. (53) and  $\mathbf{D}$  in Eq. (42) are invariant under a rotation by  $\mathbf{J} \in \mathbf{K}(2)$ . Following Proposition 3, the steady state CM for this system will also be, i.e.,  $\mathbf{V} = \mathbf{J}\mathbf{V}\mathbf{J}^T$ , which means that position-momentum correlations are antisymmetric and position-position correlations are equal to momentum-momentum correlations. This symmetry help us to solve algebraically the LE (9), obtaining [24]

$$\mathbf{V} = \begin{bmatrix} v_1 & 0 & 0 & v_{14} \\ 0 & v_2 & -v_{14} & 0 \\ 0 & -v_{14} & v_1 & 0 \\ v_{14} & 0 & 0 & v_2 \end{bmatrix}, \quad (55)$$

with

$$\begin{aligned} v_1 &= 2 \frac{\zeta_1 \bar{N}_1 + \zeta_2 \bar{N}_2}{\zeta_1 + \zeta_2} + \frac{2(\bar{N}_1 - \bar{N}_2)\zeta_1 \zeta_2^2}{(\zeta_1 + \zeta_2)(4\Omega^2 + \zeta_1 \zeta_2)} + 1, \\ v_2 &= 2 \frac{\zeta_1 \bar{N}_1 + \zeta_2 \bar{N}_2}{\zeta_1 + \zeta_2} + \frac{2(\bar{N}_2 - \bar{N}_1)\zeta_1^2 \zeta_2}{(\zeta_1 + \zeta_2)(4\Omega^2 + \zeta_1 \zeta_2)} + 1, \\ v_{14} &= \frac{4\zeta_1 \zeta_2 \Omega (\bar{N}_2 - \bar{N}_1)}{(\zeta_1 + \zeta_2)(\zeta_1 \zeta_2 + 4\Omega^2)}. \end{aligned} \quad (56)$$

Note that, if  $\zeta_2 = 0$ , then  $\mathbf{V} = (2\bar{N}_1 + 1)\mathbf{l}_4$ , which has a simple structure, but it cannot be simply retrieved by symmetries of  $\mathbf{\Gamma}$  and  $\mathbf{D}$ .

The simple form of Eq. (55) can be used to explicitly analyze the results in conditions in Eq. (18). Considering for simplicity  $\zeta_1 = \zeta_2 = \zeta$ , the (doubly degenerate) spectrum of Eq. (55) is

$$\text{Spec}_{\mathbb{R}}(\mathbf{V}) = \left\{ \bar{N}_1 + \bar{N}_2 + 1 \pm \frac{(\bar{N}_1 - \bar{N}_2)}{\sqrt{1 + 4\Omega^2/\zeta^2}} \right\}. \quad (57)$$

From this, it is easy to see that the state (55) is classical for any values of the parameters, since  $\mathbf{V} - \mathbf{l}_4 \geq 0$ . On the other hand, let us calculate

$$\text{Spec}_{\mathbb{R}}(\mathbf{D}_{[-1]}) = \{2\zeta \bar{N}_1, 2\zeta \bar{N}_2\}, \quad (58)$$

which is non-negative for any value of  $\bar{N}_1$  and  $\bar{N}_2$ . The statement in Eq. (18) thus tells us that all steady states of this system belong to the set of classical states, which was already found by means of Eq. (57).

### C. Cascaded optical parametric oscillator

Consider an optical parametric oscillator (OPO) coupled to the vacuum field [12]. The Hamiltonian is written as  $\hat{H} = i\hbar\epsilon(\hat{a}^{\dagger 2} - \hat{a}^2)/4$ , where  $\epsilon \geq 0$  denotes the effective pump intensity. The coupling with the vacuum is described by the operator  $\hat{L} = \sqrt{\hbar\kappa}\hat{a}$ , where  $\kappa > 0$  is the damping cavity rate.

With the help of Eqs. (4), (6), (7), and (10), one readily reaches

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{1}{2}(\epsilon - \kappa) & 0 \\ 0 & -\frac{1}{2}(\epsilon + \kappa) \end{bmatrix}, \quad \mathbf{D} = \kappa \mathbf{l}_2. \quad (59)$$

The matrix  $\mathbf{\Gamma} = \mathbf{\Gamma}^T$  will be AS in so far as  $\kappa > \epsilon$ . Also,  $\mathbf{D} > 0$ , once  $\kappa > 0$ . Following the statement in Eq. (19), since  $\mathbf{D}_{[i\mathbf{J}^T 2]} = \text{Diag}(\epsilon, -\epsilon)$ , the steady state will be nonclassical if  $\epsilon \neq 0$  and will be classical only if  $\epsilon = 0$ . The LE is trivially

solved to give the CM of the steady state:

$$\mathbf{V} = \text{Diag}[\kappa/(\kappa - \epsilon), \kappa/(\kappa + \epsilon)], \quad (60)$$

which is a squeezed thermal state and corresponds to the coherent-state solution in Ref. [12] if  $\epsilon = 0$ .

For a cascaded OPO [12], the system is described by the Hamiltonian  $\hat{H} = \hat{H}_1 + \hat{H}_2 + \frac{1}{2i}(\hat{L}_1^\dagger \hat{L}_2 - \hat{L}_2^\dagger \hat{L}_1)$  with  $\hat{H}_j = i\hbar\epsilon_j(\hat{a}_j^{\dagger 2} - \hat{a}_j^2)/4$  and  $\hat{L}_j = \sqrt{\hbar\kappa}\hat{a}_j$ . The coupling of the system with the intracavity vacuum is represented by  $\hat{L} = \hat{L}_1 + \hat{L}_2$ . Under these circumstances we write

$$\mathbf{D} = \begin{bmatrix} \kappa & \kappa \\ \kappa & \kappa \end{bmatrix} \oplus \begin{bmatrix} \kappa & \kappa \\ \kappa & \kappa \end{bmatrix}, \quad (61)$$

and

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{\epsilon_1 - \kappa}{2} & 0 \\ -\kappa & \frac{\epsilon_2 - \kappa}{2} \end{bmatrix} \oplus \begin{bmatrix} -\frac{\epsilon_1 + \kappa}{2} & 0 \\ -\kappa & -\frac{\epsilon_2 + \kappa}{2} \end{bmatrix}, \quad (62)$$

with spectrum given by

$$\text{Spec}(\mathbf{\Gamma}) = \left\{ -\frac{1}{2}(\kappa \pm \epsilon_1), -\frac{1}{2}(\kappa \pm \epsilon_2) \right\}. \quad (63)$$

Note that  $\mathbf{\Gamma} \neq \mathbf{\Gamma}^\top$  is AS if  $\kappa > \max\{|\epsilon_1|, |\epsilon_2|\}$ . Now the eigenvalues of  $\mathbf{D}_{[i, \mathbb{J}^{\top 2}]}$  in Eq. (22) can be calculated, yielding

$$\text{Spec}_{\mathbb{R}}(\mathbf{D}_{[i, \mathbb{J}^{\top 2}]}) = \{(1 \pm \sqrt{5})\kappa, 2\kappa, 0\}, \quad (64)$$

which shows that the state will be always entangled following Corollary 1. The same arguments can be applied to determine the Gaussian steerability. Calculating the matrix  $\mathbf{D}_{[i, \mathbf{\Pi}_k]}$  [the matrix  $\mathbf{\Pi}_k$  is defined in Eq. (24), and here  $n_1 = n_2 = 1$ ], which has the eigenvalues

$$\begin{aligned} \text{Spec}_{\mathbb{R}}(\mathbf{D}_{[i, \mathbf{\Pi}_1]}) &= \left\{ (1 \pm \sqrt{5})\frac{\kappa}{2}, (3 \pm \sqrt{5})\frac{\kappa}{2} \right\}, \\ \text{Spec}_{\mathbb{R}}(\mathbf{D}_{[i, \mathbf{\Pi}_2]}) &= \left\{ (3 \pm \sqrt{17})\frac{\kappa}{2}, k, 0 \right\}. \end{aligned} \quad (65)$$

From these, one concludes that the state will always be Gaussian steerable with respect to both modes since these spectra are nonpositive.

Following Proposition 3, the CM of the steady state of the cascaded OPO will have the symmetry induced by Eq. (26), and reads

$$\mathbf{V} = \begin{bmatrix} \frac{k}{k - \epsilon_1} & -\frac{2\kappa\epsilon_1}{g_-} \\ -\frac{2\kappa\epsilon_1}{g_-} & -\frac{\kappa h_+}{g_-} \end{bmatrix} \oplus \begin{bmatrix} \frac{k}{k + \epsilon_1} & \frac{2\kappa\epsilon_1}{g_+} \\ \frac{2\kappa\epsilon_1}{g_+} & \frac{\kappa h_-}{g_+} \end{bmatrix}, \quad (66)$$

where we have defined

$$\begin{aligned} g_{\pm} &= (\epsilon_1 + \epsilon_2 \pm 2\kappa)(\epsilon_1 \pm \kappa), \\ h_{\pm} &= (\epsilon_1^2 + \epsilon_1\epsilon_2 \pm \epsilon_1\kappa + 2\kappa^2 \mp \kappa\epsilon_2)(\epsilon_2 \mp \kappa)^{-1}. \end{aligned}$$

### D. Optical parametric oscillator and thermal baths

Consider the Hamiltonian dynamics of two particles described by

$$\hat{H} = \frac{\epsilon}{4}\{\hat{q}_1, \hat{p}_1\}_+ + \frac{\epsilon}{4}\{\hat{q}_2, \hat{p}_2\}_+ + \frac{\kappa}{2}(\hat{q}_2\hat{p}_1 + \hat{p}_2\hat{q}_1). \quad (67)$$

This Hamiltonian is similar to the one in the previous example, it is basically the Hamiltonian of the cascaded OPO with a

phase change [3]. If the particles are in contact with the thermal baths, as in Eq. (40), the dynamical matrices become

$$\mathbf{\Gamma} = \mathbf{\Gamma}^\top = \frac{1}{2} \begin{bmatrix} \epsilon - \zeta & k \\ k & \epsilon - \zeta \end{bmatrix} \oplus \frac{1}{2} \begin{bmatrix} -(\epsilon + \zeta) & k \\ k & -(\epsilon + \zeta) \end{bmatrix}, \quad (68)$$

$\mathbf{D}$  is as in Eq. (42), and now we are considering  $\bar{N}_1 = \bar{N}_2 = \bar{N}$  and  $\zeta_1 = \zeta_2 = \zeta$ . The eigenvalues of  $\mathbf{\Gamma}$  are

$$\text{Spec}(\mathbf{\Gamma}) = \left\{ -\frac{1}{2}\zeta \pm \frac{1}{2}(\epsilon + \kappa), -\frac{1}{2}\zeta \pm \frac{1}{2}(\epsilon - \kappa) \right\}, \quad (69)$$

and it will be AS if  $\zeta > \epsilon + \kappa$ . The eigenvalues of  $\mathbf{D}_{[-1]}$  in Eq. (19) are

$$\text{Spec}_{\mathbb{R}}(\mathbf{D}_{[-1]}) = \{2\zeta\bar{N} \pm (\epsilon + \kappa), 2\zeta\bar{N} \pm (\epsilon - \kappa)\}, \quad (70)$$

which shows that the state will be classical if and only if  $2\zeta\bar{N} \geq (\epsilon + \kappa)$ . The eigenvalues of  $\tilde{\mathbf{D}}_{[i, \mathbb{J}^{\top 2}]}$  in Eq. (22b) are

$$\text{Spec}_{\mathbb{R}}(\tilde{\mathbf{D}}_{[i, \mathbb{J}^{\top 2}]}) = \{2\zeta\bar{N} \pm \kappa, 2\zeta(\bar{N} + 1) \pm \kappa\}, \quad (71)$$

and the steady state will be entangled if and only if  $2\zeta\bar{N} < \kappa$ . In the interval  $(\epsilon + \kappa) > 2\zeta\bar{N} \geq \kappa$ , the state is nonclassical and separable. The steerability of the state is determined by

$$\begin{aligned} \text{Spec}_{\mathbb{R}}(\tilde{\mathbf{D}}_{[i, \mathbf{\Pi}_1]}) &= \text{Spec}_{\mathbb{R}}(\tilde{\mathbf{D}}_{[i, \mathbf{\Pi}_2]}) \\ &= \left\{ (2\bar{N} + \frac{1}{2})\zeta \pm \sqrt{\zeta^2 + \kappa^2}, (2\bar{N} + \frac{3}{2})\zeta \right. \\ &\quad \left. \pm \sqrt{\zeta^2 + \kappa^2} \right\}. \end{aligned} \quad (72)$$

As a consequence of the chosen parameters, the steady state is symmetric with respect to the steerings of both partitions. This state is steerable if and only if  $(2\bar{N} + \frac{1}{2})\zeta < (\zeta^2 + \kappa^2)^{1/2}$ .

## VI. ENGINEERING STEADY STATES

The unavoidable influence of uncontrollable degrees of freedom are usually responsible for losses of the quantumness of a system through the procedure called decoherence. However, a steady state with a desired quantum property can be produced by controlling the parameters of the system and of the environmental action. As the examples of the last section, systems of bosonic degrees of freedom have been extensively studied in what concerns entanglement generation [25,26], production of pure states [12,27], and engineering of graph states [12,26]. On the experimental side, realizations of these techniques in the context of atomic ensembles were already performed [28].

To develop a simple theoretical engineering-state method for bosonic degrees of freedom, we will use the results provided by the Williamson theorem [20,21,29]:

*Theorem 1 (Williamson).* Let  $\mathbf{M} \in \text{Mat}(2n, \mathbb{R})$  be a positive-definite matrix:  $\mathbf{M} = \mathbf{M}^\top > 0$ . This matrix can be diagonalized by a symplectic congruence, i.e., there exists  $\mathbf{S} \in \text{Sp}(2n, \mathbb{R})$  such that

$$\mathbf{SMS}^\top = \text{Diag}(\mu_1, \dots, \mu_n, \mu_1, \dots, \mu_n) =: \mathbf{\Lambda}, \quad (73)$$

where  $\mu_j \geq \mu_k > 0$  for  $j > k$ .

The double-paired ordered set (or the diagonal matrix)  $\mathbf{\Lambda}$  is called *symplectic spectrum* of  $\mathbf{M}$ , and  $\mu_k$  are its symplectic



eigenvalues (SEs). These can be found from the (Euclidean) eigenvalues of  $\mathbf{JM}$  [21], which turn out to be

$$\text{Spec}_{\mathbb{C}}(\mathbf{JM}) = \text{Diag}(i\mu_1, \dots, i\mu_n, -i\mu_1, \dots, -i\mu_n). \quad (74)$$

Suppose one wants to design a reservoir structure able to produce a steady state (with  $n$  degrees of freedom) described by a  $2n \times 2n$  CM  $\mathbf{V}'$ . If one identifies this CM with  $\mathbf{M}$  in Eq. (73), the first step is to find a suitable LE able to produce its corresponding symplectic spectrum  $\mathbf{\Lambda}$  as a solution, i.e., it is necessary to find matrices  $\mathbf{\Gamma}'$  and  $\mathbf{D}'$  satisfying the LE  $[\mathbf{\Lambda}, \mathbf{\Gamma}', \mathbf{D}']$ . Assuming that it is possible to design a system-reservoir structure with this LE, one applies the symplectic covariance (29) to it and finds

$$[\mathbf{V}', \mathbf{S}^{-1}\mathbf{\Gamma}'\mathbf{S}, \mathbf{S}^{-1}\mathbf{D}'\mathbf{S}^{-\top}], \quad (75)$$

which, by Eq. (73), is the LE with solution  $\mathbf{V}' = \mathbf{S}^{-1}\mathbf{\Lambda}\mathbf{S}^{-\top}$ . Reservoir engineering can then be realized by finding some convenient matrix  $\mathbf{\Lambda}$  and a system-reservoir structure suitable to the unitary transformations, as in Eq. (30). Following Eq. (29), the engineered Hamiltonian and Lindblad operators will be, respectively, such that  $\mathbf{H}' = \mathbf{S}^{\top}\mathbf{H}\mathbf{S}$  and  $\lambda' = \mathbf{S}^{-1}\lambda$ .

A peculiar example using Eq. (75) appears when one is able to produce a reservoir structure such that  $\mathbf{D}' = \beta\mathbf{\Lambda}$  and  $\mathbf{\Gamma}' = -\beta\mathbf{l}_{2n}$  with  $\beta > 0$ , thus the use of Eq. (73) gives that  $[\mathbf{V}', -\beta\mathbf{l}_{2n}, \beta\mathbf{V}']$ . This shows that a Lindblad equation with  $\mathbf{Y} = \frac{1}{2}\beta\mathbf{V}' + i\beta\mathbf{l}_{2n}$  [see Eq. (7)] will have a steady state automatically given by a Gaussian with CM  $\mathbf{V}'$ .

In fact, the matrix  $\mathbf{\Lambda}$  in Eq. (73) is like the CM in Eq. (38), thus any  $\mathbf{\Gamma}'$  as in Eq. (36) and  $\mathbf{D}'$  as in Eq. (37), which are invariants under the local rotations in Eq. (35), are appropriate to the first step of the method. It is noticeable that not all reservoir structures are suitable to produce a diagonal matrix  $\mathbf{\Lambda}$  as a CM of a steady state, since it cannot have the desired invariant structure for performing the first step. However, it is still possible to use the covariance relation (29) to design a specific steady state from a known simple steady state of some system, one of these cases (the OPO in Sec. VC) will be analyzed at the end of this section.

The special condition presented in Corollary 3 can be used to engineer a specific and important class of states. Explicitly, we will use the relation (29) to generate a state with CM  $\mathbf{V}' = \alpha\mathbf{S}\mathbf{S}^{\top}$ , where  $\mathbf{S} \in \text{Sp}(2n, \mathbb{R})$  and  $\alpha \geq 1$ . In other words, we want to know which matrices  $\mathbf{\Gamma}_p$  AS and  $\mathbf{D}_p \geq 0$  are necessary to construct a LE with the solution given by the desired CM. Note that the above-mentioned CM represents a pure state if and only if  $\alpha = 1$  [12,20,21].

As stated in Corollary 3, the  $n$ -mode Gibbs state with  $\mathbf{V} = \alpha\mathbf{l}_{2n}$  is generated by any LE of the form  $[\mathbf{V}, \mathbf{\Gamma}', -\alpha(\mathbf{\Gamma}' + \mathbf{\Gamma}'^{\top})]$ , for all matrices  $\mathbf{\Gamma}'$  AS such that  $(\mathbf{\Gamma}' + \mathbf{\Gamma}'^{\top}) \leq 0$ . Using the relation (29), we are able to obtain a covariant LE  $[\mathbf{V}', \mathbf{\Gamma}_p, \mathbf{D}_p]$  with

$$\mathbf{V}' = \alpha\mathbf{S}\mathbf{S}^{\top}, \quad \mathbf{\Gamma}_p = \mathbf{S}\mathbf{\Gamma}'\mathbf{S}^{-1}, \quad \mathbf{D}_p = -\alpha\mathbf{S}(\mathbf{\Gamma}' + \mathbf{\Gamma}'^{\top})\mathbf{S}^{\top}. \quad (76)$$

It is interesting to note that, recalling Corollary 3, the value of  $\alpha$  only depends on the reservoir structure (it does not depend on the Hamiltonian of the system) used to prepare the initial state  $\mathbf{V} = \alpha\mathbf{l}_{2n}$ ; conveniently the Hamiltonian at this stage can be taken as zero. Another remarkable fact is that any one-mode

Gaussian state ( $n = 1$ ) is included in the scheme provided by Eq. (76): these states have a CM written as  $\mathbf{V}'$  in Eq. (76) with  $\alpha = \mu_1 \geq 1$  and  $\mathbf{S} \in \text{Sp}(2, \mathbb{R})$ , as a consequence of the Williamson theorem.

In Ref. [12], the authors established specific conditions for a system to be driven to a pure steady state when evolving under the LME subjected to the restrictions in Eq. (4). Equation (76) with  $\alpha = 1$  constitutes a simple connection with some of their results.

## Examples

### 1. Two-mode thermal squeezed states

Consider the following CM:

$$\mathbf{V}' = (2\bar{n} + 1)\mathbf{S}_r\mathbf{S}_r^{\top}, \quad (77)$$

with

$$\mathbf{S}_r := \left[ \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \oplus \begin{pmatrix} \cosh r & -\sinh r \\ -\sinh r & \cosh r \end{pmatrix} \right], \quad (78)$$

where  $r \geq 0$  is the squeezing parameter and  $\bar{n} \geq 0$  is the mean number of thermal photons of both modes. Note that  $\mathbf{S}_r \in \text{Sp}(4, \mathbb{R})$ . Following the criteria (21) and (24), the state in Eq. (77) will be respectively entangled if and only if  $r > \ln(2\bar{n} + 1)$  and steerable if and only if  $r > \cosh^{-1}(2\bar{n} + 1)$ .

If one wants to engineer a reservoir with the steady state given in Eq. (77), it is possible to apply the scheme in Eq. (76). For this purpose, one needs a reservoir structure able to produce a steady state of the form  $\mathbf{V} = \alpha\mathbf{l}_{2n}$  with  $\alpha = (2\bar{n} + 1)$ . An example of a system with this steady state is the one in Sec. VB with  $\bar{N}_1 = \bar{N}_2$ . The system in Sec. VA can also be used to this end, but now, besides the condition  $\bar{N}_1 = \bar{N}_2$ , one needs to take  $\kappa = 0$ —this condition is necessary to guarantee that  $\mathbf{\Gamma} + \mathbf{\Gamma}^{\top} \leq 0$ , as required in Corollary 3. Recalling that the Hamiltonian dynamics does not affect the value of  $\alpha$  in Eq. (76), one can use either  $\omega_1 = \omega_2 = \kappa = 0$  and  $\zeta_1 = \zeta_2 := \zeta$  in Eq. (42), or  $\varpi_1 = \varpi_2 = \Omega = 0$  and  $\zeta_1 = \zeta_2 := \zeta$  in Eq. (53) to obtain a reservoir structure with  $\mathbf{\Gamma}' = -\frac{\zeta}{2}\mathbf{l}_4$  and  $\mathbf{D}' = (2\bar{n} + 1)\zeta\mathbf{l}_4$ . This structure produces the steady state  $\mathbf{V} = (2\bar{n} + 1)\mathbf{l}_{2n}$ .

Now, one needs to apply the covariance relation (76) and determine the engineered reservoir with matrices  $\mathbf{\Gamma}_p$  and  $\mathbf{D}_p$  through the symplectic matrix (78), i.e.,

$$\mathbf{\Gamma}_p = -\frac{\zeta}{2}\mathbf{l}_4, \quad \mathbf{D}_p = (2\bar{n} + 1)\zeta\mathbf{S}_r\mathbf{S}_r^{\top}. \quad (79)$$

From Eq. (29), one can see that  $\lambda' = \mathbf{S}_r\lambda$ , and the Lindblad operators in Eq. (4) for  $\lambda'$  are characteristics of a squeezed thermal bath.

### 2. Optical parametric oscillator steady states

The steady state in Eq. (66) is a pure steady state if and only if  $\epsilon_1 = -\epsilon_2$  [12]. Under these conditions and if we define  $\epsilon := \sqrt{(\kappa + \epsilon_2)/(\kappa - \epsilon_2)}$ , the CM of this steady state is written as

$$\mathbf{V}' = \mathbf{S}_p\mathbf{S}_p^{\top}, \quad \mathbf{S}_p := \begin{pmatrix} \frac{1+\epsilon}{2} & \frac{1-\epsilon}{2} \\ \frac{1-\epsilon}{2} & \frac{1+\epsilon}{2} \end{pmatrix} \oplus \begin{pmatrix} \frac{1+\epsilon}{2\epsilon} & \frac{\epsilon-1}{2\epsilon} \\ \frac{\epsilon-1}{2\epsilon} & \frac{1+\epsilon}{2\epsilon} \end{pmatrix}, \quad (80)$$

which is entangled for any value of  $\epsilon$ . Obviously, if one wants to produce this pure state as a steady state of an OPO, the only

step is to produce an OPO satisfying the mentioned conditions. On the other side, it is not possible to produce it by using the covariance rules in Eq. (76) for an OPO, since Corollary 3 requires  $\mathbf{\Gamma} + \mathbf{\Gamma}^\top \geq 0$ , which is not the case for  $\mathbf{\Gamma}$  in Eq. (62) with  $\epsilon_1 = -\epsilon_2$ .

By the Williamson theorem, the symplectic spectrum of the CM of the pure state in Eq. (80) is the identity matrix [30]. If one can choose suitable values of the parameters in Eqs. (61) and (62) such that  $\mathbf{\Gamma}'$  and  $\mathbf{D}'$  satisfy the LE  $[I_4, \mathbf{\Gamma}', \mathbf{D}']$ , then Eq. (75) can be applied. In fact, this happens when  $\epsilon_1 = \epsilon_2 = 0$ . Thus, preparing an OPO system such that this last condition holds, Eq. (75) becomes

$$[\mathbf{S}_p \mathbf{S}_p^\top, \mathbf{S}_p \mathbf{\Gamma}' \mathbf{S}_p^{-1}, \mathbf{S}_p \mathbf{D}' \mathbf{S}_p^\top]. \quad (81)$$

with  $\mathbf{D}'$  as in Eq. (61) and

$$\mathbf{\Gamma}' = \begin{bmatrix} -\frac{\kappa}{2} & 0 \\ -\kappa & -\frac{\kappa}{2} \end{bmatrix} \oplus \begin{bmatrix} -\frac{\kappa}{2} & 0 \\ -\kappa & -\frac{\kappa}{2} \end{bmatrix}. \quad (82)$$

Note that the matrices  $\mathbf{\Gamma}'$  and  $\mathbf{D}'$  in this example do not have the invariant structure in Eq. (35). Note also that the same pure state can be engineered by using thermal baths. The recipe for this case is just Eq. (79) but replacing  $\mathbf{S}_r$  by  $\mathbf{S}_p$  and using  $\bar{n} = 0$ .

An entangled and steerable (with respect to both partitions) mixed state can also be prepared by following the same recipe. Suppose that one wants to create the below state as a steady state of an OPO-covariant-LE:

$$\mathbf{V}' = \mathbf{S}_p^{-1} \mathbf{\Lambda} \mathbf{S}_p^{-\top}, \quad (83)$$

with  $\mathbf{S}_p$  defined in Eq. (80) and

$$\mathbf{\Lambda} = \text{Diag}[1, (1 - \epsilon)^{-1}, 1, (1 + \epsilon)^{-1}]. \quad (84)$$

This matrix is the solution for the LE  $[\mathbf{\Lambda}, \mathbf{\Gamma}', \mathbf{D}']$  with  $\mathbf{D}'$  in Eq. (61) and  $\mathbf{\Gamma}'$  in Eq. (62) both with  $\kappa = 1$ ,  $\epsilon_1 = 0$ , and  $\epsilon_2 = \epsilon$ . By the same recipe as before, the LE in Eq. (75) has the above  $\mathbf{V}'$  as solution if we replace  $\mathbf{S}$  by  $\mathbf{S}_p$  and the mentioned matrices  $\mathbf{\Gamma}'$  and  $\mathbf{D}'$ .

## VII. FINAL REMARKS

Symmetries and properties of the Lyapunov equation were used to classify the features of the steady state of a LME with a quadratic Hamiltonian and linear Lindblad operators. The connection with the Lyapunov equation eases the characterization of the state, a task that is typically difficult when performed using the master equation directly.

For Gaussian steady states, we focused on known bona fide relations for the covariance matrix of a state. Specifically, we considered conditions for the classicality, separability, and steerability of Gaussian states. We remark, however, that the extension for any other bona fide relation is straightforward and can be performed following the lines presented here. For instance, we can refer to the characterization of tripartite entanglement given in Ref. [31]. We also analyze the consequences for the covariance matrix of a steady state when a transformation symmetry of the Lyapunov equation is performed.

We focused our examples on systems with one or two degrees of freedom, which has enabled us to compare the results of our corollaries with the results extracted directly from the covariance matrix of the system after solving the Lyapunov equation. However, our results are applicable to systems with a generic number of degrees of freedom. For large systems, in particular in the absence of symmetries, numerical solutions might be needed to find the covariance matrix of the steady state; for instance, the systems considered in Ref. [32]. In this situation, instabilities associated with the algorithms for solving Lyapunov equations may arise [33]. The robustness of the analytical results shows the advantage with respect to either perturbations of the systems parameters or preparation imprecisions. In other words, our results are advantageous since one does not need to solve a Lyapunov equation to know some of the system properties or symmetries. In addition, our results are suitable for the engineering of (a reservoir leading to a specific) steady state of a LME having suitable properties and symmetries.

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## APPENDIX A: NOTATIONS AND DEFINITIONS

Throughout the text we use some mathematical objects whose notations are defined here:

(1)  $\text{Mat}(m, \mathbb{K})$ : set of all  $m \times m$  square matrices over the field  $\mathbb{K}$ .

(2)  $\text{GL}(m, \mathbb{K}) := \{\mathbf{M} \in \text{Mat}(m, \mathbb{K}) \mid \det \mathbf{M} \neq 0\}$ : general linear group over field  $\mathbb{K}$ .

(3)  $\text{Sp}(2m, \mathbb{R}) := \{\mathbf{M} \in \text{Mat}(2m, \mathbb{R}) \mid \mathbf{M} \mathbf{J} \mathbf{M}^\top = \mathbf{J}\}$ : real symplectic group.

(4)  $\text{O}(m) := \{\mathbf{M} \in \text{Mat}(m, \mathbb{R}) \mid \mathbf{M} \mathbf{M}^\top = \mathbf{I}_m\}$ : real orthogonal group.

(5)  $\mathbf{I}_m$ : Identity matrix in  $\text{Mat}(m, \mathbb{K})$ .

(6)  $\mathbf{0}_m$ : zero matrix in  $\text{Mat}(m, \mathbb{K})$ .

(7)  $\text{Spec}_{\mathbb{K}}(\mathbf{M}) := \{\nu_1, \dots, \nu_l\}$  is the spectrum of  $\mathbf{M} \in \text{Mat}(m, \mathbb{K})$ . It is the set of its eigenvalues  $\nu_k \in \mathbb{K}$ ,  $\forall k$  and  $l \leq m$ .

(8)  $\mathbf{M}^\top$ : transposition of  $\mathbf{M}$ .  $\mathbf{M}^{-\top}$ : inverse of  $\mathbf{M}^\top$ .

(9)  $\mathbf{M}^*$ : complex conjugation of the elements of  $\mathbf{M}$ .

(10)  $\text{In}(\mathbf{M}) := (n_+, n_0, n_-)(\mathbf{M})$ : Inertia index, i.e., the triple containing the number of eigenvalues of  $\mathbf{M}$  with positive ( $n_+$ ), null ( $n_0$ ), and negative ( $n_-$ ) real part. Note that, if  $\mathbf{M} \in \text{Mat}(m, \mathbb{K})$ , then  $m = n_+ + n_0 + n_-$ .

In what follows,  $\mathbf{M} \in \text{Mat}(m, \mathbb{K})$  and  $\mathbf{M} = \mathbf{M}^\dagger$ :

(1)  $\mathbf{M} > 0$  ( $\mathbf{M} < 0$ ): positive (negative) definiteness of  $\mathbf{M}$ , i.e., all its eigenvalues are positive (negative).

(2)  $\mathbf{M} \geq 0$  ( $\mathbf{M} \leq 0$ ): Positive (negative) semidefiniteness of  $\mathbf{M}$ , i.e., all its eigenvalues are non-negative (nonpositive). In this text, the statement  $\mathbf{M} \geq 0$  ( $\mathbf{M} \leq 0$ ) means that  $\mathbf{M}$  can, but not necessarily, have null eigenvalues. This is the same as say that the set of matrices such that  $\mathbf{M} > 0$  ( $\mathbf{M} < 0$ ) is a subset of those satisfying  $\mathbf{M} \geq 0$  ( $\mathbf{M} \leq 0$ ).

It is noteworthy that, following our definitions, the sum of two positive (negative) semidefinite matrices is positive (negative) semidefinite, i.e., the sum will have non-negative (nonpositive) eigenvalues. In addition, the sum of two positive (negative) definite matrices is positive (negative) definite.

### APPENDIX B: ON THE $P$ -REPRESENTABILITY OF STATES

Due to the absence of a proof in the literature, this appendix is devoted to prove that the necessary and sufficient condition for  $P$ -representability of a  $n$ -mode Gaussian state is Eq. (17).

A quantum state  $\hat{\rho}$  is  $P$ -representable, by definition, if it can be written as a convex and regular sum of coherent states

through the Glauber–Sudarshan  $P$ -function [34]:

$$\hat{\rho} = \int P(\zeta) |\zeta\rangle \langle \zeta| d^{2n} \zeta, \quad (\text{B1})$$

where  $\zeta \in \mathbb{R}^{2n}$  and  $|\zeta\rangle$  is a coherent state.

The sufficient condition is proved in Ref. [13] for two mode Gaussian states, i.e.,  $n = 2$  in Eq. (B1). The extension for any  $n$ -mode state (not only the Gaussians) follows the same recipe: using the definition of the CM [Eq. (8)] with the  $\hat{\rho}$  in Eq. (B1), Eq. (17) follows immediately.

To prove the necessary condition (only for Gaussian states), we choose two Gaussian states  $\hat{\rho}$  and  $\hat{\rho}_0$ , with the respective CMs  $\mathbf{V}$  and  $\mathbf{V}_0$  such that  $\mathbf{V} \geq \mathbf{V}_0$ . These states can be related through a Gaussian noise channel [35]:

$$\hat{\rho} = \frac{1}{(\pi \hbar)^n} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{\hbar} \zeta \cdot \Delta^{-1} \zeta}}{\sqrt{\text{Det} \Delta}} \hat{T}_\zeta \hat{\rho}_0 \hat{T}_\zeta^\dagger d^{2n} \zeta. \quad (\text{B2})$$

The operators  $\hat{T}_\zeta$  are the Weyl displacement operators [21], and  $\Delta := \mathbf{V} - \mathbf{V}_0 \geq 0$ . Mathematically speaking, Eq. (B2) express the very known fact that the convolution of two Gaussian functions is a Gaussian function. If we choose  $\mathbf{V}_0 = \mathbf{l}_{2n}$  and  $\hat{\rho}_0$  as a vacuum state, the positive-semidefiniteness of  $\Delta$  implies relation (17), as we should prove.

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