

Quantum-speed-limit bounds in an open quantum evolution

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Quantum mechanics dictates bounds for the minimal evolution time between predetermined initial and final states. Several of these quantum-speed-limit (QSL) bounds were derived for nonunitary dynamics using different approaches. Here, we perform a systematic analysis of the most common QSL bounds in the damped Jaynes-Cummings model, covering the Markovian and non-Markovian regimes. We show that only one of the analyzed bounds cleaves to the essence of the QSL theory outlined in the pioneer works of Mandelstam and Tamm and of Margolus and Levitin in the context of unitary evolutions. We also show that all QSL bounds analyzed reflect the fact that in our model non-Markovian effects speed up quantum evolution. However, it is not possible to infer Markovian or non-Markovian behavior of the dynamics by analyzing only the QSL bounds.

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Knowing the fundamental limits that quantum mechanics imposes on the maximum speed of evolution between two distinguishable states is of utmost importance for quantum communication [1], computation [2], metrology [3], and many other areas of quantum physics. In particular, the presence of decoherence [4,5] makes the estimation of the minimal duration time of a process of key value in the design of quantum control protocols and in the implementation of quantum information tasks.

The quantum-speed-limit (QSL) time, τ , is defined as the minimal time a quantum system needs to evolve between an initial and a final state separated by a given predetermined distance [6,7]. The pioneering work on this subject was conducted by Mandelstam and Tamm (MT) [8], who derived a bound for the evolution time of a system between two pure orthogonal states through a unitary dynamics generated by a time-independent Hamiltonian \hat{H} . The resulting lower bound for the evolution time was given as $t \geq \tau^{\text{MT}} \equiv \hbar\pi/2\langle\Delta\hat{H}\rangle$, where $\langle\Delta\hat{H}\rangle = \sqrt{\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2}$ denotes the mean standard deviation of the energy of the system. Several years later, Margolus and Levitin (ML) [9,10] studied the same problem and arrived at a different bound, i.e., $t \geq \tau^{\text{ML}} \equiv \pi\hbar/2\langle\hat{H}\rangle$, where $\langle\hat{H}\rangle$ is the mean energy. Therefore, for unitary dynamics connecting two orthogonal pure states, the bound for the quantum speed limit is not unique and the result was usually given by combining these two independent bounds and looking for the tightest: $t \geq \max[\tau^{\text{MT}}; \tau^{\text{ML}}]$ [11].

For nonunitary dynamics the extension of the MT approach was given in [6] using the Bures fidelity [12–14] between the initial and the final states. From their approach two bounds can be extracted, which we call τ_i^{min} and τ_i^{av} .

The first minimal evolution time, τ_i^{min} , corresponds to the time required by the process to traverse a distance equal to the geodesic length between the two states $\hat{\rho}_0$ and $\hat{\rho}_t$. This time can be estimated with little information on the dynamics and could depend on the actual time t only implicitly through state $\hat{\rho}_t$.

The second QSL bound, τ_i^{av} , involves a definition of the average speed of evolution, $\mathcal{V}_t^{\text{av}}$ (in frequency units), calculated in terms of the quantum Fisher information along the evolution path. Both QSL bounds, τ_i^{min} and τ_i^{av} , are tight for an evolution along the geodesic path between the initial and the final states. This continuous-in-time tightness feature is important to engineering evolutions that achieve the minimal time of evolution set by quantum mechanics. However, here we show that the explicit dependence of the average velocity, $\mathcal{V}_t^{\text{av}}$, on the actual evolution time t , makes τ_i^{av} an inconsistent estimate of the minimal evolution time. This is shown in the well-known damped Jaynes-Cummings (DJC) model. On the contrary, τ_i^{min} , gives a finite estimate of the minimal evolution time for all times at which the asymptotic state is essentially reached.

Other QSL bounds have been given in the literature for nonunitary evolutions [7,15–17]. Some of them [7,16] are also based on the definition of velocities, \mathcal{V}_t (in frequency units), which depend explicitly on the actual evolution time t . We show that all these bounds also give inconsistent estimates of the minimal evolution time. In the case of the QSL bound in [7], we have also demonstrated another drawback: it does not show continuous-in-time saturation, i.e., an evolution path where the bound is tight for all times. Thus, we argue that for nonunitary evolutions, τ_i^{min} is, within the analyzed QSL bounds, the only one that sticks close to the essence of the QSL theory. This essence is to estimate not the actual evolution time, but the minimal time needed to connect two states separated by a given distance.

Another interesting aspect of the QSL bounds for open systems that was recently discussed in the literature is their connection with the non-Markovian character of the

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nonunitary evolutions [7,18,19]. In fact, it was suggested in Ref. [7] that one of their proposed QSL bounds could include enough information on the dynamics to be correlated with the Markovianity or non-Markovianity of the system evolution. In particular, it was remarked that the non-Markovian effects associated with the information backflow from the environment could lead to faster quantum evolution and hence to shorter QSL times. Similar statements were made in [18] and [19]. We can say that the statement that the non-Markovianity speeds up the quantum evolution and that this feature can be inferred from the behavior of the QSL bounds is widespread [15]. Here, we consider the DJC model in the rotated-wave approximation (RWA) but with a detuning between the peak frequency of the spectral density and the transition frequency of the qubit whose dynamics can be tuned from essentially Markovian to non-Markovian. We found that in the DJC model the non-Markovian effects indeed speed up the quantum evolution. Comparing all the QSL bounds analyzed, over a wide range of parameters that control the system, with a measure of the non-Markovianity of the evolution, we show that all of them are systematically smaller in the region of parameters corresponding to non-Markovian effects with respect to their values in the region of parameters corresponding to Markovian behavior of the dynamics. In this sense we can say that the QSL bounds analyzed reflect the speedup of quantum evolution due to non-Markovian effects in the DJC model. However, we have shown that the converse is not true, so there are regions of parameters that cannot be associated with non-Markovian behavior of the dynamics where the QSL bounds are as small as in the region of parameters where the dynamics is essentially non-Markovian. Therefore, it is not possible to infer the speedup of quantum evolution due to non-Markovian effects from the QSL bounds analyzed.

The paper is organized as follows. In Sec. II we summarize the three approaches to deriving the QSL bounds treated in this work and analyze the conditions for their saturation. Next, in Sec. III we review the model used to test our statements: the DJC model for a zero-temperature reservoir within the RWA, whose dynamics can be tuned from Markovian to non-Markovian regimes. Our results are reported in Sec. IV, and in Sec. V we conclude with some final remarks.

II. QUANTUM-SPEED-LIMIT BOUNDS FOR OPEN SYSTEMS

A desirable feature for any QSL time bound is to be tight. This means that there is always an evolution that allows its saturation. Here, we summarize the derivation of the QSL bounds given in [6], [7], and [16] and we briefly analyze the conditions for their saturation. In particular, we focus on whether or not a continuous-in-time saturation exists, i.e., an evolutionary path that, for every time, saturates the bound.

A. QSL bounds in terms of the quantum Fisher information

The approach in [6] is based on the Bures fidelity [13] between the initial and the final states, i.e.,

$$F_B(\hat{\rho}_0, \hat{\rho}_t) = \text{Tr}(\sqrt{\sqrt{\hat{\rho}_0} \hat{\rho}_t \sqrt{\hat{\rho}_0}}). \quad (1)$$

The authors prove that, among all the metrics based on the Bures fidelity, the tightest lower bound for the Bures length [20], $\int_0^t \sqrt{\mathcal{F}_Q(t')}/4 dt'$, is given by the Bures angle, $\arccos(F_B[\hat{\rho}_0, \hat{\rho}_t])$ [12,14], i.e.,

$$\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t) \equiv \arccos(F_B(\hat{\rho}_0, \hat{\rho}_t)) \leq \int_0^t \sqrt{\mathcal{F}_Q(t')}/4 dt'. \quad (2)$$

Here, $\mathcal{F}_Q(t)$, is the quantum Fisher information along the path determined by the system evolution and its square root is proportional to the instantaneous speed of separation between two neighboring states. Eq. (2) implies that the length of the geodesic that connects $\hat{\rho}(0)$ with $\hat{\rho}(t)$ is always shorter than the length of the actual path.

The geometric interpretation of Eq. (2) allows us to set up two types of minimal evolution time for two states separated by a given predetermined distance. The first, which we call τ_t^{\min} , corresponds to the time it takes the system to travel (along the actual evolution path) the same length as the geodesic's length between the two states, i.e.,

$$\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t) = \int_0^{\tau_t^{\min}} \sqrt{\mathcal{F}_Q(t')}/4 dt'. \quad (3)$$

It is important to realize that knowing $\mathcal{F}_Q(t)$ along the path, in principle, requires less information than knowing exactly the actual dynamics of the system. In this way, this QSL time follows the essence of the QSL theory because, knowing the initial and final state but not knowing the actual evolution time t , we can estimate a lower bound for the evolution time. This is well illustrated, for example, for any unitary evolution generated by a time-independent Hamiltonian, where $\mathcal{F}_Q(t) = 4\langle(\Delta \hat{H})^2\rangle_{\hat{\rho}_0}/\hbar^2$ for all times. So, in this case we only need the variance of the energy of the system to estimate the bound,

$$\tau_t^{\min} = \hbar \mathcal{L}(\hat{\rho}_0, \hat{\rho}_t) / \sqrt{\langle(\Delta \hat{H})^2\rangle_{\hat{\rho}_0}}, \quad (4)$$

that for orthogonal pure states, i.e., $\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t) = \pi/2$, is equal to τ^{MT} . The QSL bound τ_t^{\min} allows us to define the speed-limit “velocity” (in frequency units),

$$\mathcal{V}_t^{\min} \equiv \frac{\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t)}{\tau_t^{\min}}, \quad (5)$$

which depends on t only implicitly through the final state $\hat{\rho}_t$.

The second QSL bound comes directly from rearranging Eq. (2),

$$t \geq \frac{\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t)}{\mathcal{V}_t^{\text{av}}} \equiv \tau_t^{\text{av}}, \quad (6)$$

where we define the “average speed of evolution” as

$$\mathcal{V}_t^{\text{av}} \equiv (1/t) \int_0^t \sqrt{\mathcal{F}_Q(t')}/4 dt'. \quad (7)$$

In the case of unitary evolution generated by a time-independent Hamiltonian we have that $\mathcal{V}_t^{\text{av}} = \sqrt{\langle(\Delta \hat{H})^2\rangle_{\hat{\rho}_0}}/\hbar$ does not depend on the actual time of evolution t , and $\tau_t^{\text{av}} = \tau_t^{\min}$. For nonunitary evolutions the times τ_t^{av} and τ_t^{\min} do not need to be equal, and in general, $\mathcal{V}_t^{\text{av}}$ depends explicitly on t , contrary to the velocity \mathcal{V}_t^{\min} in Eq. (5). Later we show, in a specific system, that $\tau_t^{\min} < \tau_t^{\text{av}}$ and the explicit dependence of $\mathcal{V}_t^{\text{av}}$ on t makes τ_t^{av} an inconsistent estimate of the minimal evolution time between $\hat{\rho}_0$ and $\hat{\rho}_t$.

It is clear, from the geometric character of the inequality in Eq. (2) that the saturation $\tau = \tau_t^{\min}$ or $\tau = \tau_t^{\text{av}}$ is only possible whenever the system evolution is through a geodesic path, so in this case we have $\tau = \tau_t^{\min} = \tau_t^{\text{av}}$ for all values of t . Thus, both bounds, $\tau = \tau_t^{\min}$ and $\tau = \tau_t^{\text{av}}$, are continuously tight, i.e., their saturation is continuously in the variable t along the evolutions over geodesics.

B. QSL bounds in terms of different operator norms

Deffner and Lutz [7] derived three different QSL bounds for a pure initial state $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$ employing the von Neumann trace inequality for operators. As in Ref. [6] their approach also uses the Bures angle, $\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t) = \arccos(\sqrt{\langle\psi_0|\hat{\rho}_t|\psi_0\rangle})$, in order to measure the predetermined distance between the initial and the final states. The derivation can be summarized as follows. First, from the time derivative of the Bures angle and using that $x \leq |x|$, we can arrive at

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq |\langle\psi_0|\dot{\hat{\rho}}_t|\psi_0\rangle| = |\text{Tr}(\hat{\rho}_0 \dot{\hat{\rho}}_t)|. \quad (8)$$

Next, we use the von Neumann trace inequality for Hilbert-Schmidt class operators,¹

$$|\text{Tr}(\hat{\rho}_0 \dot{\hat{\rho}}_t)| \leq \sigma_1(t) = \|\dot{\hat{\rho}}_t\|_{\text{op}}, \quad (9)$$

where $\sigma_1(t)$ is the largest singular value of $\dot{\hat{\rho}}_t$, and because this operator is Hermitian, $\sigma_1(t)$ is equal to its operator norm, denoted $\|\dots\|_{\text{op}}$. Together with the inequality, Eq. (9), we use the set of inequalities for trace class operators,

$$\|\hat{A}\|_{\text{op}} \leq \|\hat{A}\|_{\text{hs}} \leq \|\hat{A}\|_{\text{tr}}, \quad (10)$$

where $\|\hat{A}\|_{\text{tr}} \equiv \text{Tr}(\sqrt{\hat{A}^\dagger \hat{A}}) = \sum_i \sigma_i$ is the trace norm and $\|\hat{A}\|_{\text{hs}} \equiv \sqrt{\text{Tr}(\hat{A}^\dagger \hat{A})} = \sqrt{\sum_i \sigma_i^2}$ is the Hilbert-Schmidt norm. Gathering all the inequalities the authors arrive at

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \|\dot{\hat{\rho}}_t\|_{\text{op}} \leq \|\dot{\hat{\rho}}_t\|_{\text{hs}} \leq \|\dot{\hat{\rho}}_t\|_{\text{tr}}, \quad (11)$$

and integrating over time, we finally obtain

$$\begin{aligned} \sin^2(\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t)) &\leq \int_0^t \|\dot{\hat{\rho}}_{t'}\|_{\text{op}} dt' \\ &\leq \int_0^t \|\dot{\hat{\rho}}_{t'}\|_{\text{hs}} dt' \leq \int_0^t dt \|\dot{\hat{\rho}}_t\|_{\text{tr}}. \end{aligned} \quad (12)$$

These inequalities are valid for any density operator evolution, and in the same way as Eq. (2), Eq. (12) serves as the starting point to derive QSL bounds if we define

$$\mathcal{V}_t^{\text{op, tr, hs}} \equiv (1/t) \int_0^t \|L_{t'}(\hat{\rho}_{t'})\|_{\text{op, tr, hs}} dt'. \quad (13)$$

Then the three QSL bounds derived in [7] are

$$t \geq \tau_t^{\text{op, tr, hs}} = \frac{\sin^2[\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t)]}{\mathcal{V}_t^{\text{op, tr, hs}}}. \quad (14)$$

¹For two Schmidt class operators \hat{A} and \hat{B} the von Neumann trace inequality is $\text{Tr}(\hat{A}\hat{B}) \leq \sum_i \sigma_i \lambda_i$, where the sum is over the singular values, σ_i and λ_i , of the operators, \hat{A} and \hat{B} , respectively, in descending order, $\sigma_1 \geq \sigma_2 \geq \dots$ and $\lambda_1 \geq \lambda_2 \geq \dots$ [21].

Because $\mathcal{V}_t^{\text{op}} \leq \mathcal{V}_t^{\text{hs}} \leq \mathcal{V}_t^{\text{tr}}$, the greater QSL bound is τ_t^{op} . Later we show, in a specific system, that $\tau_t^{\text{op}} > \tau_t^{\min}$, and the explicit dependence of $\mathcal{V}_t^{\text{op}}$ on t makes τ_t^{op} also an inconsistent estimate of the minimal evolution time between $\hat{\rho}_0$ and $\hat{\rho}_t$.

We note that the inequalities in Eq. (12) do not have a clear geometric interpretation, so the conditions for their saturation [which leads to the saturation of the QSL bounds in Eq. (14)] are not as evident. In the case of τ_t^{op} , the saturation corresponds to

$$\sin^2(\mathcal{L}(\hat{\rho}_0, \hat{\rho}_t)) = \int_0^t dt \|\dot{\hat{\rho}}_t\|_{\text{op}}. \quad (15)$$

In order to have saturation over a given evolution path, we need to satisfy the equalities in Eqs. (8) and (9) for all times t . So, the mean $\langle\psi_0|\dot{\hat{\rho}}_t|\psi_0\rangle = \text{Tr}(\hat{\rho}_0 \dot{\hat{\rho}}_t)$ should be positive along the path. Let us suppose that this is the case, so now we want to see if it is possible to saturate Eq. (9) at all times t , i.e., $\text{Tr}(\hat{\rho}_0 \dot{\hat{\rho}}_t) = \sigma_1(t) \equiv \|\dot{\hat{\rho}}_t\|_{\text{op}} > 0$ along some evolutionary path. In order to see that this is not possible, we first observe that the von Neumann trace inequality $\text{Tr}(\hat{\rho}_0 \dot{\hat{\rho}}_t) \leq \sigma_1(t)$ is saturated along an evolution path iff $\hat{\rho}_0$ and $\dot{\hat{\rho}}_t$ are simultaneously unitarily diagonalizable for all evolution times. This means that $\sigma_1(t)$ must be the eigenvalue of $\dot{\hat{\rho}}_t$ associated with the time-independent common eigenvector, $|\psi_0\rangle$, of $\hat{\rho}_t$ and $\hat{\rho}_0$. Therefore, the structure of the evolved state should be

$$\hat{\rho}_t = \left(1 + \int_0^t \sigma_1(t') dt'\right) \hat{\rho}_0 + \hat{A}_t, \quad (16)$$

where \hat{A}_t has a support in the subspace orthogonal to the subspace spanned by $\hat{\rho}_0 \equiv |\psi_0\rangle\langle\psi_0|$. But because we assume that Eq. (8) is saturated at all times, we have $\sigma_1(t) > 0$ at all times. So, $\hat{\rho}_t$ in Eq. (16) is not a physical state for all $t > 0$, because otherwise we would have, for the probability of finding the evolved state in the initial state,

$$\text{Tr}(\hat{\rho}_0 \hat{\rho}_t) = 1 + \int_0^t \sigma_1(t') dt' > 1, \quad (17)$$

where we use that $\hat{\rho}_0 \hat{A}_t = 0$ at all times. Therefore it is not possible to find an evolutionary path where Eq. (9) is saturated at all times if Eq. (8) is also saturated at all times. The saturation $t = \tau_t^{\text{op}}$ can be possible only for certain times t along a given path of the system evolution. This contrasts clearly with $t = \tau_t^{\text{av}} = \tau_t^{\min}$, which is a continuous-in-time saturation along a geodesic evolutionary path.

C. QSL bound using the notion of quantumness

The derivation of a QSL bound in [16] follows a very different approach based on the notion of ‘‘quantumness’’. Quantification of the nonclassical character of a quantum system has recently attracted much attention [22,23]. In particular, the notion of quantumness associated with the noncommutativity of the algebra of observables [22,23] was defined as

$$\begin{aligned} Q(\hat{\rho}_a, \hat{\rho}_b) &= 2\|[\hat{\rho}_a, \hat{\rho}_b]\|_{\text{hs}}^2 \\ &= -4\text{Tr}[(\hat{\rho}_a \hat{\rho}_b)^2 - \hat{\rho}_a^2 \hat{\rho}_b^2], \end{aligned} \quad (18)$$

such that $0 \leq Q(\hat{\rho}_a, \hat{\rho}_b) \leq 1$. Note that $Q(\hat{\rho}_a, \hat{\rho}_b) = 0$ iff $[\hat{\rho}_a, \hat{\rho}_b] = 0$ [22,23], which means that $\hat{\rho}_a$ and $\hat{\rho}_b$ are diagonal

in the same basis. In that sense $Q(\hat{\rho}_a, \hat{\rho}_b)$ is a witness of the coherences that state $\hat{\rho}_b$ has in the basis of eigenstates of $\hat{\rho}_a$, and vice versa. Therefore, in system evolution, the quantumness, $Q(\hat{\rho}_0, \hat{\rho}_t)$, as a function of time, monitors the generation of coherences in the evolved state $\hat{\rho}_t$, in the eigenstate basis of the initial state $\hat{\rho}_0$.

Contrary to the approaches described in the previous sections, in order to get a QSL bound, the approach in [16] does not use explicitly any distance between the initial and the final state. Instead, from the definition of the quantumness $Q(\hat{\rho}_0, \hat{\rho}_t)$, the authors use the Cauchy-Schwarz inequality, i.e., $|\text{Tr}(\hat{A}^\dagger \hat{B})| \leq \|\hat{A}\|_{\text{hs}} \|\hat{B}\|_{\text{hs}}$, to obtain

$$\frac{|\dot{Q}(\hat{\rho}_0, \hat{\rho}_t)|}{\sqrt{Q(\hat{\rho}_0, \hat{\rho}_t)}} \leq 2\sqrt{2} \|\dot{[\hat{\rho}_0, \hat{\rho}_t]}\|_{\text{hs}}, \quad (19)$$

where $\dot{Q}(\hat{\rho}_0, \hat{\rho}_t) = 4 \text{Tr}(\hat{A}_t^\dagger \hat{B}_t)$, with $\hat{A}_t \equiv [\hat{\rho}_0, \hat{\rho}_t]$ and $\hat{B}_t \equiv [\hat{\rho}_0, \dot{\hat{\rho}}_t]$. Now, for the integration in time of the left-hand side of Eq. (19), they use that

$$\int_0^t \frac{|\dot{Q}(\hat{\rho}_0, \hat{\rho}_{t'})|}{\sqrt{Q(\hat{\rho}_0, \hat{\rho}_{t'})}} dt' \geq \left| \int_0^t \frac{dQ}{\sqrt{Q'}} \right| = 2\sqrt{Q(\hat{\rho}_0, \hat{\rho}_t)}. \quad (20)$$

Therefore, they finally obtain

$$\sqrt{Q(\hat{\rho}_0, \hat{\rho}_t)/2} \leq \int_0^t \|\dot{[\hat{\rho}_0, \hat{\rho}_{t'}]}\|_{\text{hs}} dt'. \quad (21)$$

A QSL bound, τ_t^{quant} , can be set up from the inequality in Eq. (21) in the same way that τ_t^{av} , was set up from the inequality in Eqs. (2) or the bounds, $\tau_t^{\text{op, tr, hs}}$, from the inequalities in Eq. (12), i.e.,

$$t \geq \tau_t^{\text{quant}} = \frac{\sqrt{Q(\hat{\rho}_0, \hat{\rho}_t)/2}}{\mathcal{V}_t^{\text{quant}}}, \quad (22)$$

where we define the time-average velocity in frequency units,

$$\mathcal{V}_t^{\text{quant}} \equiv \frac{1}{t} \int_0^t \|\dot{[\hat{\rho}_0, \hat{\rho}_{t'}]}\|_{\text{hs}} dt'. \quad (23)$$

In order to have saturation in Eq. (21), therefore $t = \tau_t^{\text{quant}}$, for all times over a given evolutionary path, we need to satisfy the equalities in Eqs. (19) and (20) for all times t . Let us suppose that the rate of change of the quantumness, $\dot{Q}(\hat{\rho}_0, \hat{\rho}_t)$, is positive along the evolutionary path, so the equality, Eq. (20), is saturated along the path. This means that the rate of generation of coherences in $\hat{\rho}_t$, in the basis of eigenstates of $\hat{\rho}_0$, is positive at all times: something that could be possible. In order to saturate Eq. (19) at all times along some evolutionary path, we need that

$$\hat{B}_t = \xi_t \hat{A}_t, \quad (24)$$

with ξ_t a real function of time. Because we assume that $\dot{Q}(\hat{\rho}_0, \hat{\rho}_t) = 4\xi_t \text{Tr}(\hat{A}_t^\dagger \hat{A}_t) \geq 0$ we have that $\xi_t \geq 0$ at all times. This means that (i) $\hat{\rho}_t = \xi_t \hat{\rho}_0$ or (ii) $\hat{\rho}_0$ and $\hat{\rho}_t - \xi_t \hat{\rho}_0$ are diagonal in the same basis, at all times along some evolutionary path. Option (i) is not possible because, imposing the normalization condition on the evolved state, we arrive at $\int_0^t \xi_{t'} dt' = 0$ at all times, a condition that cannot be satisfied unless $\xi_t = 0$ at all times. But $\xi_t = 0$ at all times, corresponds to the trivial evolution where the evolved state remains equal to $\hat{\rho}_0$ at all times. However, condition (ii) can be satisfied, for example, in

the cases of quasiclassical models consisting of evolved states diagonal in the eigenbasis of the initial state $\hat{\rho}_0$ at all times, with only their eigenvalues changing along the evolutionary path [24]. Therefore, the QSL bound, τ_t^{av} , in principle, can be saturated continuously in time along some evolutionary paths.

III. THE JAYNES-CUMMINGS MODEL FOR A ZERO-TEMPERATURE RESERVOIR

In this section, we present a simple physical model that will serve as a platform to study all the QSL bounds presented in the previous section. We consider the exactly solvable damped Jaynes-Cummings model for a two-level system interacting with a bosonic quantum reservoir at zero temperature, in both the resonant and the detuning regimes [5,25–30]. The Hamiltonian of the system is given by $\hat{H} = \hat{H}_0 + \hat{H}_I$. The free Hamiltonian of the qubit and the modes of the reservoir is $\hat{H}_0 = \omega_0 \hat{\sigma}_z + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k$, while $\hat{H}_I = \sum_k g_k \hat{b}_k \hat{\sigma}_+ + g_k^* \hat{b}_k^\dagger \hat{\sigma}_-$ is the interaction Hamiltonian between them (g_k is the coupling strength between the qubit and mode k). Here, ω_0 , is the energy difference between the two levels of the system, $\hat{\sigma}_\pm$ are the increasing and decreasing operators for the qubit, and $\hat{\sigma}_z$ is a Pauli operator. The operators, \hat{b}_k^\dagger and \hat{b}_k , are the creation and annihilation operators for the bosonic modes whose frequencies are ω_k . In the limit of an infinite number of reservoir modes and a smooth spectral density, this model leads to the reduced qubit's evolution given by the exact master equation,

$$\begin{aligned} \dot{\hat{\rho}}_t = & -\frac{is_t}{2} [\hat{\sigma}_z, \hat{\rho}_t] \\ & + \gamma_t \left(\hat{\sigma}_- \hat{\rho}_t \hat{\sigma}_+ - \frac{1}{2} \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}_t - \frac{1}{2} \hat{\rho}_t \hat{\sigma}_+ \hat{\sigma}_- \right), \end{aligned} \quad (25)$$

with $s_t = -2\text{Im}\{\dot{G}(t)/G(t)\}$ and $\gamma_t = -2\text{Re}\{\dot{G}(t)/G(t)\}$ the time-dependent Lamb shift and the decay rate, respectively [5]. The solution of this master equation is given by the channel [5,28]

$$\hat{\rho}_t = \Lambda_t[\hat{\rho}_0] = \begin{bmatrix} |G(t)|^2 \rho_{ee} & G(t) \rho_{eg} \\ G(t)^* \rho_{eg}^* & 1 - |G(t)|^2 \rho_{ee} \end{bmatrix}, \quad (26)$$

where the initial state of the qubit is

$$\hat{\rho}_0 = \begin{bmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{eg}^* & 1 - \rho_{ee} \end{bmatrix} \quad (27)$$

in the basis, $|z; \pm\rangle$, of eigenstates of the free Hamiltonian of the qubit. The function, $G(t)$, is the solution to the equation $\dot{G}(t) = -\int_0^t d\tau f(t-\tau)G(\tau)$, with $G(0) = 1$ and where $f(t-\tau)$ is the two-point correlation function of the reservoir, i.e., the Fourier transform of the spectral density $J(\omega)$. For a Lorentzian spectral density, $J(\omega) = \gamma_0 \lambda^2 / 2\pi [(\omega - \omega_c)^2 + \lambda^2]$ (λ is its width, ω_c is its peak frequency, and γ_0 is an effective coupling constant), we obtain the result [30]

$$f(t) = \frac{1}{2} \gamma_0 \lambda e^{-\lambda|t|(1-i\delta/\lambda)}, \quad (28)$$

with $\delta = \omega_0 - \omega_c$ the detuning between the peak frequency of the spectral density and the transition frequency of the qubit.

Therefore,

$$G(t) = e^{-\frac{\lambda t}{2}(1-i\frac{\delta}{\lambda})} \left[\frac{\lambda}{\Omega} \left(1 - i\frac{\delta}{\lambda} \right) \sinh\left(\frac{\Omega t}{2}\right) + \cosh\left(\frac{\Omega t}{2}\right) \right], \quad (29)$$

where $\Omega = \lambda\sqrt{(1-i\delta/\lambda)^2 - 2\gamma_0/\lambda}$ and the time-dependent decay rate is

$$\gamma_t = \gamma_0 \Re e \left(\frac{2 \sinh\left(\frac{\Omega t}{2}\right)}{\frac{\Omega}{\lambda} \cosh\left(\frac{\Omega t}{2}\right) + (1-i\frac{\delta}{\lambda}) \sinh\left(\frac{\Omega t}{2}\right)} \right). \quad (30)$$

Note that if we measure the time in units of $1/\lambda$, the function $G(t)$, and therefore the decay rate γ_t , depends on only two parameters, i.e., γ_0/λ and δ/λ .

An important feature of the DJC model is that different regimes of the parameters γ_0/λ and δ/λ can be associated with Markovian and non-Markovian effects on the evolution. In the limit $\gamma_0/\lambda \ll 1$ and $\delta/\lambda \ll 1$, we get, for the decay rate, $\gamma_t = \gamma_0/(1 + \coth(\lambda t/2))$, which is a strictly increasing positive function of time that, when $\lambda t \gg 1$, corresponds to $\gamma_t \sim \gamma_0$. Because $\gamma_t = \gamma_0/(1 + \coth(\lambda t/2)) \geq 0$ at all times, Eq. (25) is a Markovian master equation [27] in the regime $\gamma_0/\lambda \ll 1$ and $\delta/\lambda \ll 1$. However, away from this regime, in order to check the Markovianity or non-Markovianity of the dynamics, it is necessary to monitor the distinguishability of any two states along the evolution. This is because the accepted notion of Markovianity that we use here is based on the idea that for Markovian processes any two quantum states become less and less distinguishable under the dynamics, leading to a continuous loss of information into the environment [27].

The trace norm of the difference, $\hat{\rho}_1 - \hat{\rho}_2$, is used to define the trace distance,

$$D(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{2} \|\hat{\rho}_1 - \hat{\rho}_2\|_{\text{tr}} = \frac{1}{2} \text{Tr} |\hat{\rho}_1 - \hat{\rho}_2|, \quad (31)$$

which is a measure of the distance between the two quantum states [13]. This measure has the nice property that it can be interpreted as a measure of the distinguishability of $\hat{\rho}_1$ and $\hat{\rho}_2$ [31]. Therefore, based on the trace distance, the characterization of the non-Markovian character of a quantum process, given by the map $\hat{\rho}_t = \Lambda_t[\hat{\rho}_0]$, can be stated as follows: a quantum map $\hat{\rho}_t = \Lambda_t[\hat{\rho}_0]$ is non-Markovian if and only if there is a pair of initial states, $\hat{\rho}_{0,1}$ and $\hat{\rho}_{0,2}$, such that the trace distance between the corresponding evolved states increases at a certain time t , i.e.,

$$\sigma(t, \hat{\rho}_{0,1}, \hat{\rho}_{0,2}) \equiv \frac{d}{dt} D(\Lambda_t[\hat{\rho}_{0,1}], \Lambda_t[\hat{\rho}_{0,2}]) > 0, \quad (32)$$

where $\sigma(t, \hat{\rho}_{0,1}, \hat{\rho}_{0,2})$ denotes the rate of change of the trace distance at time t corresponding to the initial pair of states [31]. For a non-Markovian process, information must flow from the environment to the system for some interval of time, and thus we must have $\sigma > 0$ for this time interval. A good measure of non-Markovianity of the channel should witness the total increase in the distinguishability over the whole time evolution, i.e., the total amount of information flowing from the environment back into the system. This suggests defining a measure $\mathcal{N}(\Lambda_t)$ for the non-Markovianity of a quantum

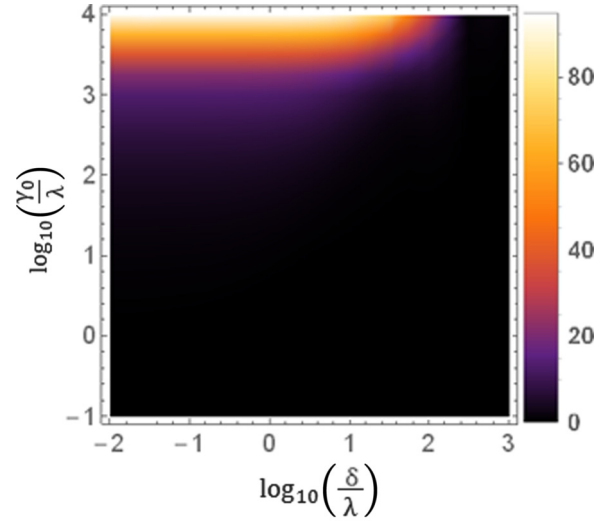


FIG. 1. Density plot of the non-Markovianity of the channel corresponding to the DJC model, measured by the expression in Eq. (33) for the initial states, $\hat{\rho}_e = |x; +\rangle\langle x; +|$ and $\hat{\rho}_g = |x; -\rangle\langle x; -|$, and a total time of evolution such that $\lambda t = 1000$. See the text for details.

process through [27]

$$\mathcal{N}(\Lambda_t) = \max_{\{\hat{\rho}_{0,1}, \hat{\rho}_{0,2}\}} \mathcal{N}(\Lambda_t, \hat{\rho}_{0,1}, \hat{\rho}_{0,2}), \quad (33)$$

with

$$\mathcal{N}(\Lambda_t, \hat{\rho}_{0,1}, \hat{\rho}_{0,2}) = \int_{\sigma>0} dt \sigma(t, \hat{\rho}_{0,1}, \hat{\rho}_{0,2}). \quad (34)$$

For a general process the maximization over the initial states $\hat{\rho}_{0,1}$ and $\hat{\rho}_{0,2}$, in $\mathcal{N}(\Lambda_t)$, is a difficult task. However for the DJC model considered here, when $\delta \neq 0$, it was shown in [31] that $\mathcal{N}(\Lambda_t) = \mathcal{N}(\Lambda_t, \hat{\rho}_e, \hat{\rho}_g)$, where $\hat{\rho}_e = |x; +\rangle\langle x; +|$ and $\hat{\rho}_g = |x; -\rangle\langle x; -|$, with $|x; \pm\rangle$ the eigenstates of the Pauli operator $\hat{\sigma}_x$. In Fig. 1 we show the behavior of the measure $\mathcal{N}(\Lambda_t)$ as a function of the parameters γ_0/λ and δ/λ , which control the DJC model.

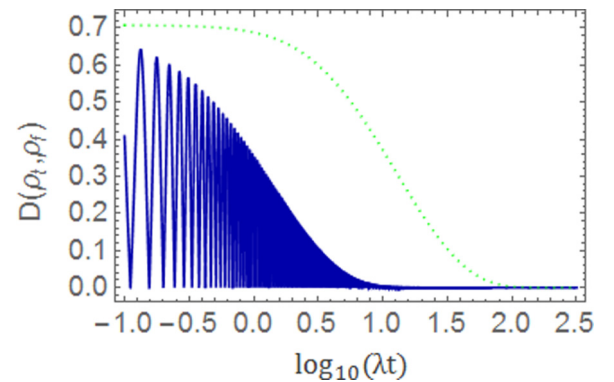


FIG. 2. Trace distance $D(\hat{\rho}_t, \hat{\rho}_f)$ between the final stationary state $\hat{\rho}_f = |z; -\rangle\langle z; -|$ and the evolved state $\hat{\rho}_t$ of the qubit in the DJC model, as a function of the scaled time λt . The dotted green line corresponds to the Markovian regime, with $\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 0.1$, and the solid blue line to the non-Markovian regime with $\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 10^4$. In both cases the initial state of the evolution is $\hat{\rho}_i = |x; +\rangle\langle x; +|$.

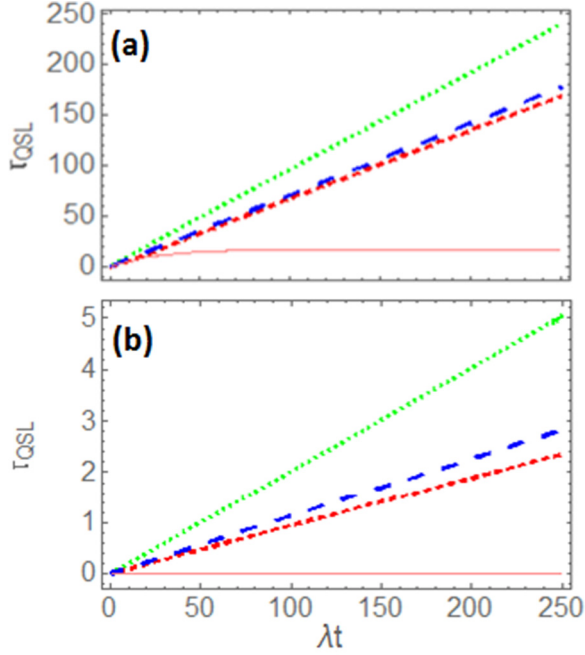


FIG. 3. QSL bounds as a function of the final time of the evolution corresponding to the DJC model, starting with the initial state of the qubit $\hat{\rho}_i = |x; +\rangle\langle x; +|$. (a) An example in the Markovian regime with $\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 0.1$; (b) an example in the non-Markovian regime with $\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 10^4$. The dotted green line represents τ_t^{quant} ; the short-dashed red line, τ_t^{op} ; the long-dashed blue line, τ_t^{av} and the solid red line, τ_t^{min} .

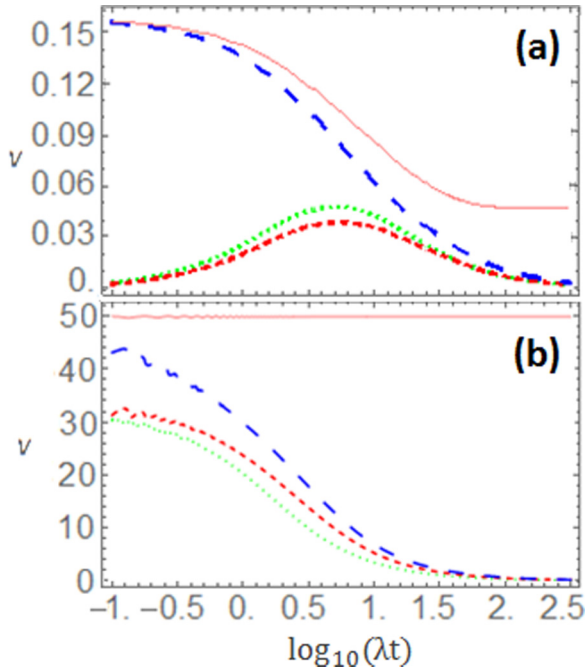


FIG. 4. The average speed as a function of the final time of the evolution in the DJC model, with the initial state of the qubit $\hat{\rho}_i = |x; +\rangle\langle x; +|$. (a) An example in the Markovian regime with $\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 0.1$; (b) an example in the non-Markovian regime with $\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 10^4$. The dotted green line represents $\mathcal{V}_t^{\text{quant}}$ [Eq. (23)]; the short-dashed red line, $\mathcal{V}_t^{\text{op}}$ [Eq. (13)]; the long-dashed blue line, $\mathcal{V}_t^{\text{av}}$ [Eq. (6)]; and the solid red line, $\mathcal{V}_t^{\text{min}}$ [Eq. (5)].

IV. RESULTS: QSL BOUNDS IN THE DJC MODEL

The DJC model is a very suitable framework in which to analyze all the QSL bounds discussed in the previous section. Our goal is to examine which of the bounds stay close to the essence of the QSL theory, giving consistent estimates for the minimal evolution time to reach the final state from the initial one within the framework of open quantum evolutions.

The reduced evolution of the qubit in the DJC model in Eq. (26) has a stationary state for all values of the parameters δ/λ and γ_0/λ . Indeed, no matter which is the initial state and due to the fact that $\lim_{t \rightarrow \infty} G(t) = 0$, the asymptotic final state is $\hat{\rho}_f = |z; -\rangle\langle z; -|$. The speed at which an evolved state approaches the stationary state is different in the Markovian versus non-Markovian regimes. This is clearly shown in Fig. 2, where we plot the trace distance, $D(\hat{\rho}_t, \hat{\rho}_f)$, between the evolved state of the qubit $\hat{\rho}_t$ and its stationary state $\hat{\rho}_f$ as a function of time for two different parameters that control the environment and its interaction with the qubit. The initial state is $\hat{\rho}_i = |x; +\rangle\langle x; +|$, however, similar results were obtained for any other $\hat{\rho}_i$ (not shown). We see that in the Markovian regime ($\delta/\lambda = 0.1$ and $\gamma_0/\lambda = 0.1$) the stationary state is reached at times $\lambda t > 100$, while in the non-Markovian regime ($\delta/\lambda =$

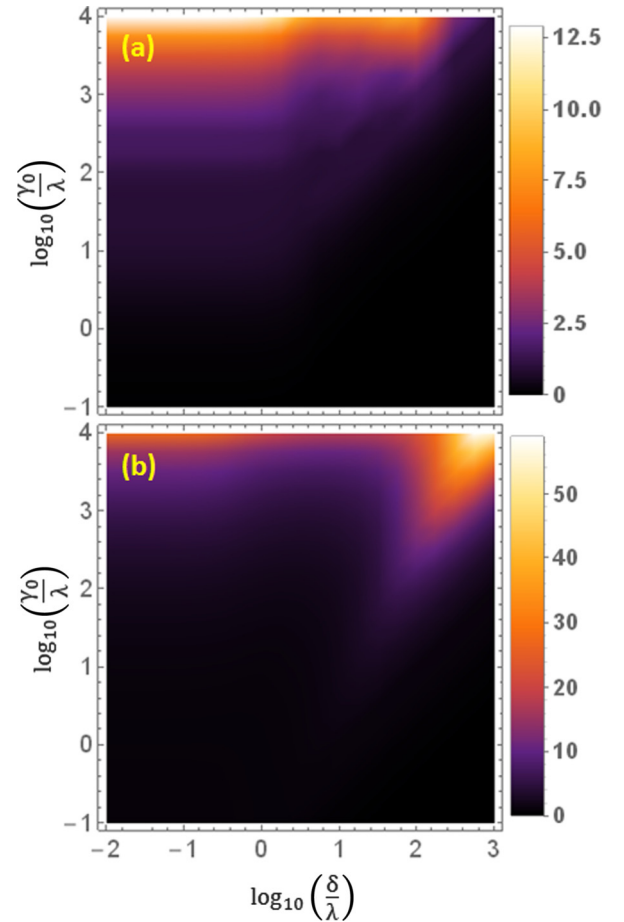


FIG. 5. Density plot of the non-Markovianity over a state evolution path in the DJC model, measured by the expression in Eq. (35), calculated from the initial state, $\hat{\rho}_e = |x; +\rangle\langle x; +|$. The time of evolution is (a) $\lambda t = 1$ and (b) $\lambda t = 100$.

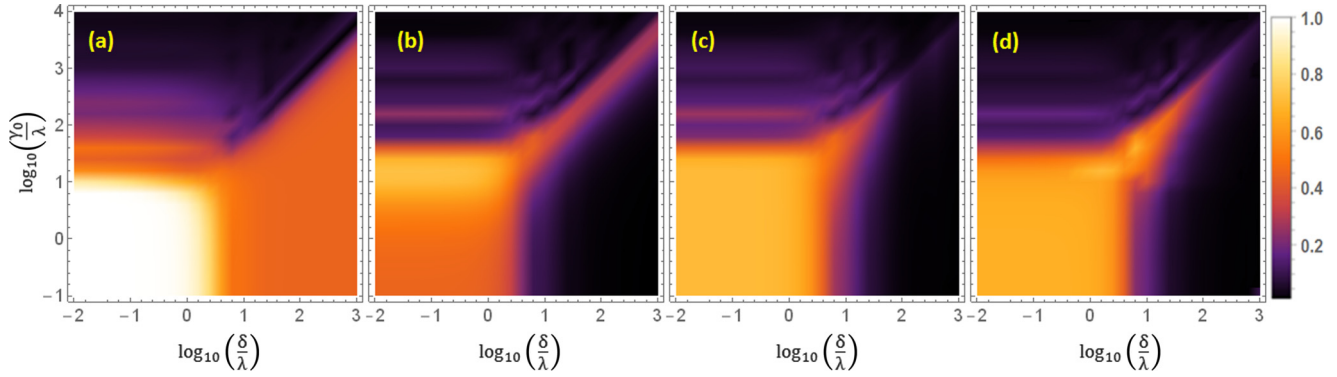


FIG. 6. Density plot of the QSL bounds as a function of the parameters γ_0/λ and δ/λ . The initial state is $\hat{\rho}_i = |x; +\rangle\langle x; +|$ and the final time of evolution is $\lambda t = 1$. (a) τ_t^{quant} , (b) τ_t^{op} , (c) τ_t^{av} , and (d) τ_t^{min} . See text for details

0.1 and $\gamma_0/\lambda = 10\,000$) the final state $\hat{\rho}_f$ is approached at earlier times ($\lambda t \approx 16$). This shows the speedup of evolution in the non-Markovian regime.

Let us now consider the behavior of the different QSL bounds as a function of the final time of evolution λt shown in Fig. 3. We remark that equivalent results were obtained for any other initial pure state (not shown). We can appreciate in Fig. 3 that for times $\lambda t > 100$ when, in either the Markovian or the non-Markovian regime, the qubit has reached the stationary state $\hat{\rho}_f$ (see Fig. 2), only the bound τ_t^{min} remains constant. The other bounds grow approximately linear. This behavior is due to the fact that, in the denominator of the definitions of τ_t^{av} , τ_t^{op} , and τ_t^{quant} [Eqs. (6), (14), and (22), respectively], the “average velocities” (in frequency units), $\mathcal{V}_t^{\text{av}}$, $\mathcal{V}_t^{\text{op}}$, and $\mathcal{V}_t^{\text{quant}}$ appear, which depend on the actual evolution time t . These average velocities go to 0 when the stationary state is achieved, while the quantities in the numerator of the definitions of the bounds remain constant. This is shown in Fig. 4, where we plot $\mathcal{V}_t^{\text{av}}$, $\mathcal{V}_t^{\text{op}}$, and $\mathcal{V}_t^{\text{quant}}$ as a function of the evolution time λt , and we also plot $\mathcal{V}_t^{\text{min}}$, which is defined in Eq. (5).

The results shown in Figs. 3 and 4 clearly show that none of the bounds, τ_t^{av} , τ_t^{op} , and τ_t^{quant} , give a consistent estimate of

the minimal time to achieve the final state $\hat{\rho}_f = |z; -\rangle\langle z; -|$ starting from the initial state $\hat{\rho}_i = |x; +\rangle\langle x; +|$. Moreover, the average velocities, $\mathcal{V}_t^{\text{av}}$, $\mathcal{V}_t^{\text{op}}$, and $\mathcal{V}_t^{\text{quant}}$, show the same asymptotic behavior as the instant speed of evolution, given by $\sqrt{\mathcal{F}_Q(\tau)}/4$, which for $\lambda\tau > 100$ also goes to 0. This fact goes against the essence of the QSL theory, which pursues the estimation of the speed-limit velocity of the evolution between two states. On the contrary, τ_t^{min} gives a consistent estimate of the minimal time needed to reach $\hat{\rho}_f$ from $\hat{\rho}_i$ and, also, provides a quantum speed limit of evolution.

Although we have shown that only one of the QSL bounds presented in Sec. II gives a reliable estimate of the minimum evolution time, we now study the connection of these bounds with the non-Markovianity character of the evolution [7,18,19].

The measure $\mathcal{N}(\Lambda_t)$ in Eq. (33) is suitable for characterizing the degree of non-Markovianity of a quantum channel Λ_t . However, in order to establish a possible link between the QSL bounds and the non-Markovian effects of the dynamics it is more appropriate to define a measure of non-Markovianity over the actual trajectory of the system, i.e., from the initial state $\hat{\rho}_0$ to the final one $\hat{\rho}_t$, which enters into the definition of

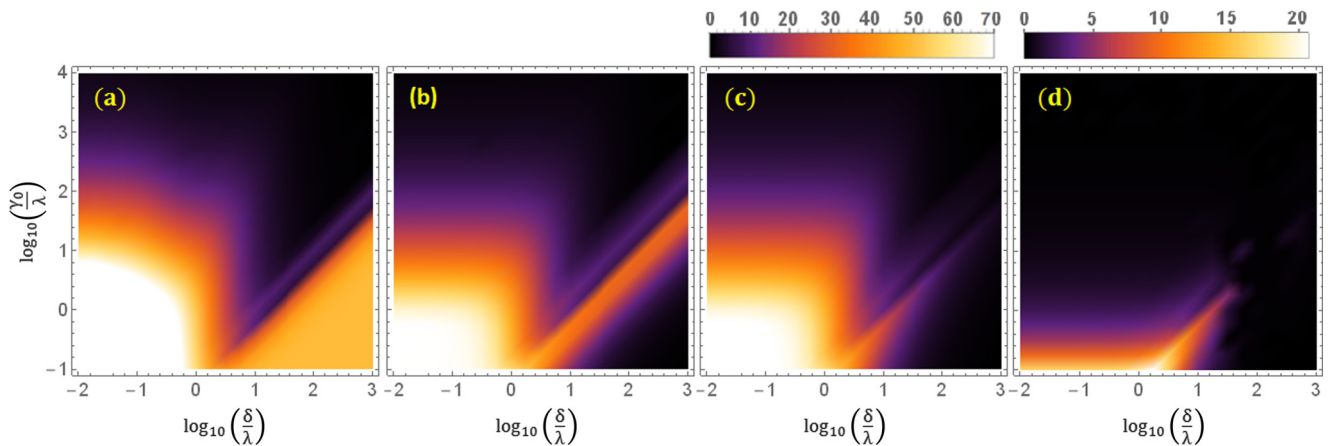


FIG. 7. Density plot of the QSL bounds as a function of the parameters γ_0/λ and δ/λ . The initial state is $\hat{\rho}_i = |x; +\rangle\langle x; +|$ and the final time of evolution is $\lambda t = 100$. (a) τ_t^{quant} , (b) τ_t^{op} , (c) τ_t^{av} , and (d) τ_t^{min} . The density color scale in the range 0–70 is used in (a)–(c), while the one in the range 0–20 is used in (d). This shows that τ_t^{min} is a better estimate of the minimal evolution time between state $\hat{\rho}_i$ and state $\hat{\rho}_{\lambda t=100}$.

the QSL bounds. In this way, we define

$$\begin{aligned}\tilde{\mathcal{N}}(t; \Lambda_t, \hat{\rho}_0) &= \int_{0, \bar{\sigma} > 0}^t \bar{\sigma}(t', \hat{\rho}_0, \hat{\rho}_{t'}) dt' \\ &= \int_0^t \frac{|\bar{\sigma}(t', \hat{\rho}_0, \hat{\rho}_{t'})| + \bar{\sigma}(t', \hat{\rho}_0, \hat{\rho}_{t'})}{2} dt',\end{aligned}\quad (35)$$

which depends on the final time t , and where

$$\bar{\sigma}(t, \hat{\rho}_0, \hat{\rho}_t) \equiv \frac{d}{dt} D(\hat{\rho}_0, \Lambda_t[\hat{\rho}_0]).\quad (36)$$

In Fig. 5 we show the density plot of $\tilde{\mathcal{N}}(t; \Lambda_t, \hat{\rho}_0)$ as a function of the parameters γ_0/λ and δ/λ for an initial state $\hat{\rho}_i = |x; +\rangle\langle x; +|$ and two final evolution times: $\lambda t = 1$ [Fig. 5(a)] and $\lambda t = 100$ [Fig. 5(b)]. Comparing Fig. 1 and Fig. 5, we can see similar qualitative behavior of the two measures of non-Markovianity as a function of the two parameters, γ_0/λ and δ/λ , that control the dynamics of the channel.

In order to compare the non-Markovianity measure $\tilde{\mathcal{N}}$ with the QSL bounds we compute them for the same region of parameters γ_0/λ and δ/λ and also considering the initial state $\hat{\rho}_i = |x; +\rangle\langle x; +|$. In Fig. 6 we show the QSL bounds calculated for a final state at $\lambda t = 1$, and in Fig. 7 for a final state at $\lambda t = 100$. The region of large $\tilde{\mathcal{N}}$ in Fig. 5(a) corresponds to small values of all of the QSL bounds in Fig. 6. The same result can be observed comparing Fig. 5(b) and Fig. 7. In this sense, large non-Markovianity implies small QSL bounds. This is a manifestation of the speedup of the quantum evolution in the non-Markovian regime that we show in Fig. 2. But looking at the value of the QSL bounds for different values of the parameters γ_0/λ and δ/λ , it is not possible to infer which are the parameter regions of non-Markovian behavior of the channel. For example, the region of small values of the QSL bounds in the lower-right corner of Figs. 6(b)–6(d) and the region of intermediate values in Fig. 6(a) do not correspond to the region of parameters with high values of the measure $\tilde{\mathcal{N}}$ in Fig. 5(a). Exactly the same analysis can be done for the case $\lambda t = 100$, which is plotted in Figs. 5(b) and 7.

V. CONCLUSIONS

Two quantum states are not perfectly distinguishable unless their supports do not overlap. This means that states that are close in Hilbert space are less distinguishable, so the distance between states fixes the degree of distinguishability between them. Therefore, in order to connect with a physical evolution two states with some fixed degree of distinguishability, it is necessary to go at least the same distance that separates the two states. This is the origin of the minimal time of evolution settled by quantum mechanics. The quantum-speed-limit (QSL) theory is devoted to establishing lower bounds of

this minimal time of evolution, and its origin dates back to the pioneering works of Mandelstam and Tamm and of Margolus and Levitin on unitary evolutions connecting pure states. It is important to note that the loss of distinguishability between near-neighbor states in quantum mechanics is intrinsic and has nothing to do with the precision of the measurement apparatus used to distinguish them. This contrasts with the classical case, where the states of the system are given by points in the phase space, whose distinguishability is not related to the distance between them.

A reasonable requirement that any expression corresponding to a QSL bound for the minimal time of evolution between two states must satisfy is that if we apply the formula in the context of a given dynamics, the result must be close to the minimal time of evolution, and not to the actual time of evolution between the states (unless the bound has been saturated). In this work we have analyzed the QSL bounds for the minimal time of evolution in open quantum systems [6,7,16] and have shown that only one, given in [6], effectively verifies this basic requirement. This was done using the damped Jaynes-Cummings model, which, for any initial state, has the same stationary state. So, we have revealed that the QSL bounds in [7] and [16] grow indefinitely with the actual evolution time, while the final state is essentially reached at finite times. On the contrary, the QSL bound in [6] remains constant for any time greater than the time at which the stationary state is essentially reached. We have also demonstrated that, contrary to the QSL bounds in [6] and [16], the QSL bound in [7] cannot be saturated continuously in time along a quantum evolution path.

In relation to the possible link between non-Markovian effects and the behavior of QSL bounds we found that all of the analyzed bounds have lower values in a parameter region that matches the parameter region where the speedup of quantum evolution due to non-Markovian effects takes place in the damped Jaynes-Cummings model. However, we have also shown that there is a parameter region of lower values of all the analyzed bounds that does not correspond to the region of non-Markovian effects on the evolution. In this sense, we have demonstrated, with a counterexample, that the statement that non-Markovian effects on quantum evolution can be studied through the QSL bounds is false.

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