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Topological origin of universal few-body clusters in Efimov physics

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Efimov physics is renowned for the self-similar spectrum featuring the universal ratio of one eigenenergy to its neighbor. Even more esoteric is the numerically unveiled fact that every Efimov trimer is accompanied by a pair of tetramers. Here we demonstrate that this hierarchy of universal few-body clusters has a topological origin by identifying the numbers of universal three- and four-body bound states with the winding numbers of the renormalization-group limit cycle in theory space. The finding suggests a topological phase transition in mass-imbalanced few-body systems which should be tested experimentally.

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Critical phenomena are the prime example of universality in which different physical systems are grouped into universality classes sharing the same critical exponents. Here, the universality classes of critical phenomena are categorized by the fixed points of the renormalization-group (RG) flow which represent the continuous scale invariance of the second-order phase transition [1]. Yet another characteristic of the RG flow, which represents the universal *discrete* scale invariance, can be found in the Efimov effect [2] and its four-body extension [3–5], in which resonantly interacting three and four bosons form an infinite series of three- and four-body bound states that feature the self-similar spectrum: the three-body bound states (Efimov trimers) are related to one another by a scaling factor of $(22.7)^2$, and each Efimov trimer is accompanied by two four-body bound states. This discrete scale invariance makes the Efimov physics a quintessential example of the RG limit cycle [6,7], where an RG flow forms a periodic circle rather than converges to a fixed point.

Historically, Thomas [8] found that the three-body energy is unbounded from below for a resonant two-body interaction, the origin of which was shown [9] to be the short-range information that can be encapsulated in a single three-body parameter. Efimov [2] generalized the Thomas theorem to the infinitely many infrared energy levels, the existence of which was proved mathematically by Amado and Noble [10,11]. The universal nature of the Efimov trimers together with its connection to the RG limit cycle was elucidated by Bedaque *et al.* [12,13] based on effective field theory. The universality of the Efimov physics has been studied in a variety of systems including mass-imbalanced fermions [14,15], particles in mixed dimensions [16], nucleons [17], magnons [18], and macromolecules [19]. Experimentally, since the first observation in ultracold Cs atoms [20], Efimov trimers have been realized in various bosonic atoms [21–26], three-component fermionic atoms [27,28], and mass-imbalanced Bose-Fermi mixtures [29–31]. The four-body extension of Efimov trimers and its universality have also been studied based on the Fadeev-Yakubovski equations [3,4,32], the correlated Gaussian hyperspherical method [5], and the Alt-Grassberger-Sandhas equations [33–36]. The tetramers were recently observed experimentally in Cs atoms [37].

In this Rapid Communication we address the unresolved question of how the universal four-body physics is related to the RG limit cycle, and in particular, whether the embedding of the limit cycle in theory space unveils the information of universal Efimov physics. In the following, we answer these questions by demonstrating that the one-to-two ratio between the number of Efimov trimers and that of the associated tetramers can be understood as the topological winding numbers of the RG limit cycle in theory space, as conjectured previously [38]. In revealing this connection, we have used the functional renormalization group (FRG) to deal with the nonperturbative situation in Efimov physics.

Previous FRG analyses [39–41] on the resonantly interacting four bosons suffer spurious four-body bound states due to the pointlike approximation where the momentum dependence of Green's functions are almost disregarded. To avoid such an artifact, we take into account the full momentum dependence of Green's functions with the separable pole approximation [42–45]. We obtain the RG flows of the three- and four-body coupling constants $g_3(k)$ and $g_4(k)$, which are defined as the three- and four-body one-particle irreducible (1PI) vertices of the flowing action [46] with loop corrections down to the RG-cutoff scale of k. Here, $g_3(k)$ and $g_4(k)$ can be interpreted as the effective three- and four-body interaction strengths at the energy scale of k, and therefore, the k dependencies of $g_3(k)$ and $g_4(k)$ represent how three- and four-body physics vary with the scale transformation. We will see in the following that the one-dimensional trajectory of $(g_3(k), g_4(k))$ parametrized by k forms a loop in theory space, and that the winding of the loop in theory space determines the number of universal bound states.

In Fig. 1(a), $g_3(k)$ and $g_4(k)$ at the unitarity limit are plotted against the cutoff k, which demonstrates that $g_3(k)$ and $g_4(k)$ both have the log-periodic k dependencies to form an RG limit cycle. Furthermore, $g_4(k)$ flows twice from $-\infty$ to ∞ every time $g_3(k)$ flows from $-\infty$ to ∞, indicating that there appear two four-body bound states associated with one Efimov trimer, since the emergence of bound states is connected with the divergent coupling constants $g_3(k)$ and $g_4(k)$ [39]. We also notice that the flow of g_4 accompanies that of g_3, indicating the nonnecessity of any four-body parameter in accordance with Ref. [3]. To see how the limit cycle is embedded in theory

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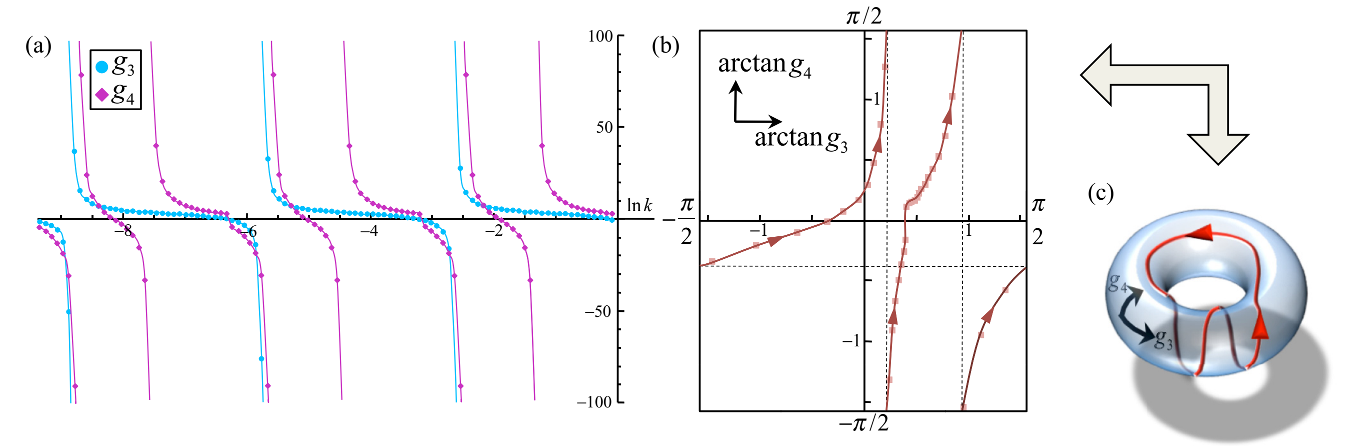

FIG. 1. (a) Cutoff k dependencies of the three-body and four-body coupling constants $g_3(k)$ and $g_4(k)$ at the unitarity limit $1/a = 0$. The abscissa shows the logarithm of the cutoff k. The blue dots and the purple diamonds show $g_3(k)/2$ and $g_4(k)/250$, where the multiplication factors $1/2$ and $1/250$ are introduced to display the two flows simultaneously. (b) RG flow in theory space of the three-body and four-body coupling constants. The abscissa shows $\arctan(g_3/16)$ and the ordinate shows $\arctan(g_4/6250)$, where we use arctan to display diverging coupling constants, and g_3 and g_4 are multiplied by 1/16 and 1/6250 for the sake of simultaneous display. The brown curve shows the RG limit cycle obtained by eliminating the cutoff k dependence of $g_3(k)$ and $g_4(k)$ from (a). If we glue the edges of the figure to form a torus, the limit cycle winds once in the g_3 direction and twice in the g_4 direction, revealing a topological nature. (c) Schematic illustration of the topological feature of the limit cycle described in (b). (c) is adapted from Fig. 5 of Ref. [38] with modifications.

space, we plot the one-parameter trajectory of $(g_3(k), g_4(k))$ on the g_3-g_4 plane in Fig. 1(b). A topological nature of the limit cycle emerges if we regard the g_3-g_4 plane as a torus by gluing the opposite sides of the edges of the plane: The RG limit cycle forms a closed loop, which winds the torus twice in the g_4 direction and once in the g_3 direction as schematically illustrated in Fig. 1(c). These results suggest that the universal numbers of the three- and four-body bound states are, in fact, the topological winding numbers of the RG limit cycle. Mathematically, the topological number is defined by the first homotopy group π_1, which classifies the way of embedding a closed loop in a larger space according to the number of windings of the closed loop in the larger space. In our setup, a closed loop is embedded in the theory space spanned by the three- and four-body coupling constants g_3 and g_4, which should be regarded as a torus T^2 so that the limit cycle forms a closed loop. The topological number is, then, defined by $\pi_1(T^2)$ which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ is the additive group of integers, i.e., the topological winding number is a pair of integers (n,m). Thus the limit cycle belongs to a homotopy class $(n,m) = (1,2) \in \mathbb{Z} \times \mathbb{Z}$, reflecting the universal numbers of the three- and four-body bound states. Our finding shows that the geometrical property of how the RG flow is embedded in theory space indeed contains the information about the universal property of the Efimov physics.

We now present our theoretical framework for obtaining the results. To make semianalytic calculations of the four-body sector possible, we use an effective field theory that exactly reproduces two- and three-body observables of identical bosons with the contact interaction. For this purpose, we consider the following action:

$$S[\psi,\phi] := \int_P \psi^*(P)(ip^0 + \mathbf{p}^2 - \mu_1)\psi(P) + \int_P \phi^*(P)\left[-\frac{1}{16\pi}\sqrt{\frac{ip^0}{2} + \frac{\mathbf{p}^2}{4} - \mu_1} - \mu_2\right]\phi(P)$$
$$- \int_{P P_2 P_1} G_\psi\left(\frac{P}{3} + P_2 + P_1\right)\phi^*\left(\frac{2P}{3} + P_2\right)\psi^*\left(\frac{P}{3} - P_2\right)\psi\left(\frac{P}{3} - P_1\right)\phi\left(\frac{2P}{3} + P_1\right), \quad (1)$$

where P denotes the four-momentum consisting of Matsubara frequency p^0 and momentum $\mathbf{p}$, $\int_P := \int \frac{d^4p}{(2\pi)^4}$, and $G_\psi(P) := (ip^0 + \mathbf{p}^2 - \mu_1)^{-1}$. In Eq. (1), ψ describes a particle, and ϕ describes a dimer. Throughout this Rapid Communication, we employ the units $\hbar = 2m = 1$, where m $(2m)$ is the mass of a particle (a dimer). In our model, we have reduced the Yukawa coupling between a particle and a dimer in the ordinary two-channel model of identical bosons [12,47] to the particle exchange interaction, as in the third term on the right-hand side of Eq. (1). As we show below, our model exactly reproduces the dimer propagator and the Skorniakov-Ter-Martirosian equation [48] for the three-body scattering.

Based on this model, we have performed the FRG analysis that is governed by the Wetterich equation [46]:

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\tilde{\partial}_k \ln\left(\frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi(P)\delta\Phi(P)} + R_{\Phi,k}(P)\right), \quad (2)$$

where $\Phi(P)=(\psi(P),\psi^*(P),\phi(P),\phi^*(P))$, and $\Gamma_k[\Phi]$ is the flowing action which is defined as the 1PI effective action of the cutoff k dependent action $S_k := S + \int_P \psi^*(P)R_{\psi,k}(P)\psi(P) + \int_P \phi^*(P)R_{\phi,k}(P)\phi(P)$ and reduces in the high-energy limit $k \to \Lambda$ to the action S and in the low-energy limit $k \to 0$ to the quantum effective action Γ that is defined as the Legendre transform of the Schwinger functional. The symbol Tr implies the sum over momenta, Matsubara frequencies, and field species. The derivative $\tilde{\partial}_k$ acts only on the regulators $R_{\Phi,k} = R_{\psi,k}, R_{\phi,k}$, which are chosen as

$$R_{\psi,k}(Q)=\frac{k^2}{c^2},\qquad R_{\phi,k}(Q)=\frac{\sqrt{k^2-\mathbf{q}^2}}{16\pi}\Theta(k^2-\mathbf{q}^2), \tag{3}$$

where Θ is the unit-step function and c is a constant to be specified later. The continuous regulators facilitate numerical calculations.

To focus on the three- and four-body physics, we perform the vertex expansion [49] of Eq. (2) with respect to the field variables to derive the RG equations for 1PI vertices:

$$\begin{aligned}\Gamma_k[\psi,\psi^*,\phi,\phi^*] :=& \int_P \psi^*(P)G^{-1}_{\psi,k}(P)\psi(P) + \int_P \phi^*(P)\Gamma^{(2)}_k(P)\phi(P)\\ &+\int_{\substack{P_1,P_2\\P_1',P_2'}}(2\pi)^4\delta^{(4)}(P_1+P_2-P_2'-P_1')\Gamma^{(3)}_k(P_1P_2;P_2'P_1')\phi^*(P_1)\psi^*(P_2)\psi(P_2')\phi(P_1')\\ &+\frac{1}{(2!)^2}\int_{\substack{P_1,P_2,P_3\\P_1',P_2',P_3'}}(2\pi)^4\delta^{(4)}(P_1+P_2+P_3-P_3'-P_2'-P_1')\Gamma^{(4)}_k(P_1P_2P_3;P_3'P_2'P_1')\\ &\times\phi^*(P_1)\psi^*(P_2)\psi^*(P_3)\psi(P_3')\psi(P_2')\phi(P_1')+\cdots,\end{aligned} \tag{4}$$

where $\Gamma^{(n)}_k$ is the 1PI vertex that represents the correlation of n particles at the RG cutoff scale of k. By substituting Eq. (4) into Eq. (2), we obtain the exact FRG equation of $\Gamma^{(n)}_k$. Since we are interested in the three- and four-body physics, we have only to consider the 1PI vertices up to $n=4$. Indeed, the exact FRG equation of $\Gamma^{(n)}_k$ is closed up to n [50], showing that the n-body physics in the vacuum is not affected by the $(n+1)$-body physics.

We first consider the one-, two-, and three-body sectors separately to obtain the limit cycle of the three-body coupling constant g_3. We find that the β function of the one-body sector vanishes [38,39,47], because the self-energy correction is absent in the particle vacuum. Therefore, the cutoff k dependent one-body inverse propagator is given by

$$G^{-1}_{\psi,k}(P)=G^{-1}_{\psi}(P)=ip^0+\mathbf{p}^2-\mu_1. \tag{5}$$

Concerning the two-body sector, the exact FRG equation corresponding to Fig. 2(a) can also be solved analytically. Together with the renormalization condition $\mu_2+\frac{\Lambda}{8\sqrt{2}\pi c}=\frac{1}{16\pi a}=0$ at the unitarity limit $a=\pm\infty$, the exact two-body sector is given by

$$\Gamma^{(2)}_k(P)=\frac{1}{16\pi}\sqrt{\frac{ip^0}{2}+\frac{\mathbf{p}^2}{4}-\mu_1}, \tag{6}$$

where we follow the trick of Ref. [51] to set the constant c in Eq. (3) to be $c=\infty$, thereby integrating out the ψ field first and then the ϕ field in the RG flow. We note that Eq. (6) is consistent with the expression of the dimer self-energy [47,52]. The inverse dimer propagator thus becomes

$$G^{-1}_{\phi,k}(P)=R_{\phi,k}(P)+\frac{1}{16\pi}\sqrt{\frac{ip^0}{2}+\frac{\mathbf{p}^2}{4}-\mu_1}. \tag{7}$$

The three-body sector can also be solved exactly. The exact FRG equation for the three-body sector corresponding to Fig. 2(b) can be analytically integrated with respect to the cutoff k, resulting in an integral equation corresponding to Fig. 2(c). We note that the integral

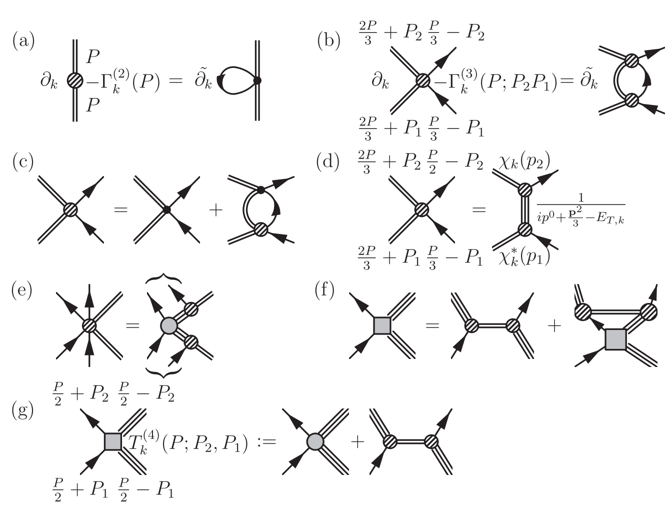

FIG. 2. Diagrammatic expressions for the exact FRG equations for (a) the two-body sector and (b) the three-body sector. The black dot in (a) shows the bare particle exchange interaction introduced in Eq. (1), the solid curve shows the propagator G_ψ of a particle, and the double line shows the propagator $G_{\phi,k}$ of a dimer introduced in Eq. (7). (c) Integral form of the exact FRG equation for the three-body sector. Decomposition of (d) the three-body 1PI vertex and (e) the four-body 1PI vertex. The curly brackets indicate symmetrization with respect to external lines. (f) Integral form of the exact FRG equation for the four-body sector. (g) Definition of the particle-trimer scattering amplitude $T^{(4)}_k$ represented as a square vertex.

equation reduces in the infrared limit $k \to 0$ to the Skorniakov-Ter-Martirosian equation [48]. Since the s-wave component of the scattering amplitude makes the dominant contribution to the low-energy Efimov physics, we perform a projection onto $\Gamma_k^{(3)}(p_2,p_1) = 2\pi \int d\,\cos\theta_{p_2p_1}\Gamma_k^{(3)}(ip^0 = k^2 + 3\mu_1;\mathbf{p}_2,\mathbf{p}_1)$, and define the dimensionless three-body coupling constant by $g_3 := -k^2\Gamma_k^{(3)}(p_2 = 0, p_1 = 0)$, the divergence of which is connected with the appearance of Efimov trimers. By solving the FRG equation corresponding to Fig. 2(c), we obtain the exact k dependence of $g_3(k)$ in Fig. 1(a).

Concerning the four-body sector, we cannot perform an exact FRG calculation since the four-body sector requires the knowledge of the full momentum dependence of the particle-dimer scattering amplitude, which is too complicated to deal with directly. Our strategy is to take into account the dominant intermediate state in the total particle-dimer scattering process by making use of the Källén-Lehmann spectral representation. By focusing on the s-wave component of the particle-dimer scattering amplitude, we decompose $-\Gamma_k^{(3)}(ip^0,\mathbf{p};p_2,p_1)$ as

$$-\Gamma_k^{(3)}(ip^0,\mathbf{p};p_2p_1) = \frac{\chi_k(p_2)\chi_k^*(p_1)}{ip^0 + \frac{\mathbf{p}^2}{3} - E_{T,k}}, \tag{8}$$

which is diagrammatically represented in Fig. 2(d). In Eq. (8), $E_{T,k}$ is the binding energy of an intermediate Efimov state and $\chi_k(p)$ is the Bethe-Salpeter wave function of the Efimov trimer. Equation (8) means that we replace the particle-dimer scattering process by a propagation process of a relevant Efimov state. This approximation is based on the separable pole approximation [42–45], which respects the position and the residue of the bound-state pole of the subsystem amplitude. The rationale for the approximation is the pole dominance of the subsystem amplitude as first pointed out by Lovelace [53]. The separable approximation and its extensions are extensively studied [54–61] in four ^{4}He atoms and an α particle, and are found to reproduce four-body binding energies for a given interparticle interaction obtained by other methods (see, e.g., Refs. [62,63]). We should note, however, that our one-term separable pole approximation does not have a hard evidence for its validity in accurate four-body calculations, yet the approximation enables us to deal with a nonperturbative four-body FRG equation with the full momentum dependence of correlation functions.

The function χ_k and $E_{T,k}$ can be determined from the Bethe-Salpeter equation which is obtained by substituting Eq. (8) into the three-body equation as depicted in Fig. 2(c). If we choose the intermediate Efimov state such that $E_{T,k} \gg k^2$, the Bethe-Salpeter equation reduces to an analytically solvable one given in Ref. [64], and the resulting χ_k and $E_{T,k}$ are given by

$$\chi_k(p) = A\frac{\sin\left[s_0\mathrm{arcsinh}\left(\frac{\sqrt{3}p}{\sqrt{2E_{T,k}}}\right)\right]}{p/\sqrt{E_{T,k}}}, \tag{9}$$

$$E_{T,k} = 6\Lambda^2 e^{-2n(k)\pi/s_0}, \tag{10}$$

where $s_0 \simeq 1.00624$ is Efimov's scaling parameter, $n(k)$ is a nonnegative integer, Λ is an ultraviolet cutoff, and $A \simeq 5.00858$ is the normalization factor determined through the procedure developed in Ref. [67]. To take into account the relevant intermediate Efimov state that satisfies $E_{T,k} \gg k^2$, we choose the integer $n(k)$ in Eq. (10) as

$$n(k) = \left\lfloor \frac{s_0}{2\pi}\ln\frac{6\Lambda^2}{k^2} \right\rfloor, \tag{11}$$

where $\lfloor x \rfloor$ gives the largest integer less than or equal to x.

Based on Eqs. (8)–(11), the four-body sector is greatly simplified. Following Ref. [66], we decompose the four-body 1PI vertex as depicted in Fig. 2(e). The resulting four-body FRG equation can be integrated with respect to k, resulting in a simple form as depicted in Fig. 2(f). We note that the integrated four-body FRG equation possesses the same structure as the Alt-Grassberger-Sandhas equation [42,43], except that our equation does not involve the dimer-dimer scattering process. The neglect of the dimer-dimer process may be justified again by the pole dominance of the subsytem amplitude, which is the rationale for the separable pole approximations such as the generalized unitary pole approximation [45] and the energy-dependent pole approximation [44]. Because dimer states are absent at the unitarity limit, the contribution of the dimer-dimer process to the entire four-body process is expected to be small.

As in the three-body sector, we perform an s-wave projection onto $T_k^{(4)}(p_2,p_1) = 2\pi \int d\,\cos\theta_{p_2p_1}T_k^{(4)}(ip^0 = e^{2\pi/s_0}k^2 + 4\mu_1;\,p_2,p_1)$ and define the dimensionless four-body coupling constant by $g_4 := \sqrt{E_{T,k}}\,T_k^{(4)}(p_2 = 0, p_1 = 0)$, the divergence of which is identified with the appearance of tetramers. By solving the FRG equation given in Fig. 2(f), we obtain the nonperturbative RG flow of g_4.

By evaluating the energies of an Efimov trimer and the associated two tetramers from the values of the cutoff $k = k_{3b}$ and $k = k_{4b}^{(1)}, k_{4b}^{(2)}$ at which g_3 and g_4 diverge, respectively, we find the following relations:

$$\frac{k_{4b}^{(1)}}{k_{3b}} = 1.11, \qquad \frac{k_{4b}^{(2)}}{k_{3b}} = 3.66, \tag{12}$$

which agree reasonably with the exact ratio between the energy of an Efimov trimer and that of associated two tetramers $\sqrt{\frac{E_{4b}^{(1)}}{E_{3b}}} = 1.00113$, $\sqrt{\frac{E_{4b}^{(2)}}{E_{3b}}} = 2.14714$ [35].

In conclusion, we have demonstrated that the numbers of universal few-body clusters are, in fact, the topological winding numbers of the RG flow trajectory in theory space. The one-to-two correspondence in the numbers of Efimov trimers and tetramers is a consequence of the torus topology in the space of the three-body and four-body coupling constants, where the trajectory winds twice in the g_4 direction every time it winds in the g_3 direction. Because of the topological nature of the limit cycle, we expect that the numbers of universal few-body clusters are protected against small variations of particle masses and dimensions that respect the discrete scale invariance. Our result demonstrates that geometrical properties of how a "renormalized trajectory" [1] is embedded in the entire theory space contains the information of the universal physics characterized by the renormalized trajectory.

A closely related question is a topological phase transition in the four-body physics. If we introduce a mass imbalance in a four-particle system, there is a situation in which the number of four-body companions accompanying an Efimov trimer changes according to the mass ratio [65]. Our present analysis suggests that such a situation may be regarded as a new type of topological phase transition in theory space at which the winding number of the RG limit cycle changes. Now that Efimov trimers are observed in mass-imbalanced mixtures of certain atomic species [29–31], the topological phase transition in the four-body physics should be within experimental reach.

Another important question is the predicted instability of tetramers accompanying an Efimov trimer. As pointed out in Ref. [34], a tetramer can, in principle, decay into a composite of a particle and a deep Efimov trimer, making the tetramer unstable. One may think that such an instability rounds off the diverging behavior of the four-body coupling constant g_4 by introducing an imaginary part to g_4, thereby removing the topological feature of the RG limit cycle. However, we find that even if we take the instability into account by introducing a deep Efimov trimer so that a tetramer can decay into the trimer state, the imaginary part of g_4 remains zero and the divergence of g_4 survives.

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[1] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
[2] V. Efimov, Phys. Lett. B **33**, 563 (1970); Sov. J. Nucl. Phys. **12**, 589 (1971).
[3] L. Platter, H.-W. Hammer, and Ulf-G. Meissner, Phys. Rev. A **70**, 052101 (2004).
[4] H.-W. Hammer and L. Platter, Eur. Phys. J. A **32**, 113 (2007).
[5] J. von Stecher, J. P. D'Incao, and C. H. Greene, Nat. Phys. **5**, 417 (2009).
[6] K. G. Wilson, Phys. Rev. D **3**, 1818 (1971).
[7] S. D. Glazek and K. G. Wilson, Phys. Rev. Lett. **89**, 230401 (2002).
[8] L. H. Thomas, Phys. Rev. **47**, 903 (1935).
[9] G. S. Danilov, Sov. Phys. JETP **13**, 349 (1961); J. Exp. Theor. Phys. **40**, 498 (1961).
[10] R. D. Amado and J. V. Noble, Phys. Lett. **35**, 25 (1971).
[11] R. D. Amado and J. V. Noble, Phys. Rev. D **5**, 1992 (1972).
[12] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999).
[13] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A **646**, 444 (1999).
[14] V. Efimov, Nucl. Phys. A **210**, 157 (1973).
[15] D. S. Petrov, Phys. Rev. A **67**, 010703 (2003).
[16] Y. Nishida and S. Tan, Phys. Rev. A **79**, 060701 (2009).
[17] H.-W. Hammer and L. Platter, Ann. Rev. Nucl. Part. Sci. **60**, 207 (2010).
[18] Y. Nishida, Y. Kato, and C. D. Batista, Nat. Phys. **9**, 93 (2013).
[19] T. Pal, P. Sadhukhan, and S. M. Bhattacharjee, Phys. Rev. Lett. **110**, 028105 (2013).
[20] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nögerl, and R. Grimm, Nature (London) **440**, 315 (2006).
[21] S. Knoop, F. Ferlaino, M. Mark, M. Berninger, H. Schöbel, H.-C. Nägerl, and R. Grimm, Nat. Phys. **5**, 227 (2009).
[22] M. Zaccanti, B. Deissler, C. D'Errico, M. Fattori, M. Jona-Lasinio, S. Müller, G. Roati, M. Inguscio, and G. Modugno, Nat. Phys. **5**, 586 (2009).
[23] N. Gross, Z. Shotan, S. Kokkelmans, and L. Khaykovich, Phys. Rev. Lett. **103**, 163202 (2009).
[24] S. E. Pollack, D. Dries, and R. G. Hulet, Science **326**, 1683 (2009).
[25] B. Huang, L. A. Sidorenkov, R. Grimm, and J. M. Hutson, Phys. Rev. Lett. **112**, 190401 (2014).
[26] M. Kunitski, S. Zeller, J. Voigtsberger, A. Kalinin, L. Ph. H. Schmidt, M. Schöffler, A. Czasch, W. Schöllkopf, R. E. Grisenti, T. Jahnke, D. Blume, and R. Dörner, Science **348**, 551 (2015).
[27] J. H. Huckans, J. R. Williams, E. L. Hazlett, R. W. Stites, and K. M. O'Hara, Phys. Rev. Lett. **102**, 165302 (2009).
[28] S. Nakajima, M. Horikoshi, T. Mukaiyama, P. Naidon, and M. Ueda, Phys. Rev. Lett. **105**, 023201 (2010).
[29] G. Barontini, C. Weber, F. Rabatti, J. Catani, G. Thalhammer, M. Inguscio, and F. Minardi, Phys. Rev. Lett. **103**, 043201 (2009).
[30] R. Pires, J. Ulmanis, S. Häfner, M. Repp, A. Arias, E. D. Kuhnle, and M. Weidemüller, Phys. Rev. Lett. **112**, 250404 (2014).
[31] S.-K. Tung, K. Jiménez-García, J. Johansen, C. V. Parker, and C. Chin, Phys. Rev. Lett. **113**, 240402 (2014).
[32] M. R. Hadizadeh, M. T. Yamashita, L. Tomio, A. Delfino, and T. Frederico, Phys. Rev. Lett. **107**, 135304 (2011).
[33] A. Deltuva, Phys. Rev. A **82**, 040701(R) (2010).
[34] A. Deltuva, Eur. Phys. Lett. **95**, 43002 (2011).
[35] A. Deltuva, Few-Body Syst. **54**, 569 (2013).
[36] A. Deltuva, Few-Body Syst. **54**, 1517 (2013).
[37] F. Ferlaino, S. Knoop, M. Berninger, W. Harm, J. P. D'Incao, H.-C. Nägerl, and R. Grimm, Phys. Rev. Lett. **102**, 140401 (2009).
[38] Y. Horinouchi and M. Ueda, Phys. Rev. Lett. **114**, 025301 (2015).
[39] R. Schmidt and S. Moroz, Phys. Rev. A **81**, 052709 (2010).
[40] B. Jaramillo Ávila and M. C. Birse, Phys. Rev. A **88**, 043613 (2013).
[41] B. Jaramillo Ávila and M. C. Birse, Phys. Rev. A **92**, 023601 (2015).
[42] E. O. Alt, P. Grassberger, and W. Sandhas, Phys. Rev. C **1**, 85 (1970).
[43] P. Grassberger and W. Sandhas, Nucl. Phys. B **2**, 181 (1967).
[44] S. Sofianos, N. J. McGurk, and H. Fiedeldey, Nucl. Phys. A **318**, 295 (1979).

[45] A. Casel, H. Haberzettl, and W. Sandhas, Phys. Rev. C **25**, 1738 (1982).
[46] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
[47] E. Braaten and H.-W. Hammer, Phys. Rep. **428**, 259 (2006).
[48] G. V. Skornyakov and K. A. Ter-Martirosyan, Zh. Eksp. Teor. Fiz. **31**, 775 (1956) [Sov. Phys. JETP **4**, 648 (1956)].
[49] T. R. Morris, Phys. Lett. B **334**, 355 (1994).
[50] S. Floerchinger, Nucl. Phys. A **927**, 119 (2014).
[51] S. Diehl, H. C. Krahl, and M. Scherer, Phys. Rev. C **78**, 034001 (2008).
[52] S. Moroz, S. Floerchinger, R. Schmidt, and C. Wetterich, Phys. Rev. A **79**, 042705 (2009).
[53] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[54] S. Sofianos, H. Fiedeldey, and H. Haberzettl, Phys. Rev. C **22**, 1772 (1980).
[55] A. C. Fonseca, H. Haberzettl, and E. Cravo, Phys. Rev. C **27**, 939 (1983).
[56] S. A. Sofianos, H. Fiedeldey, H. Haberzettl, and W. Sandhas, Phys. Rev. C **26**, 228 (1982).
[57] J. A. Tjon, Phys. Lett. B **63**, 391 (1976).
[58] V. F. Kharchenko and V. E. Kuzmichev, Nucl. Phys. A **183**, 606 (1972); Phys. Lett. B **42**, 328 (1972).
[59] S. Nakaichi-Maeda and T. K. Lim, Phys. Rev. A **28**, 692 (1983).
[60] A. C. Fonseca, Phys. Rev. C **30**, 35 (1984).
[61] I. M. Narodetsky, E. S. Galpern, and V. N. Lyakhovitsky, Phys. Lett. **46**, 51 (1973); I. M. Narodetsky, Nucl. Phys. A **221**, 191 (1974).
[62] H. Kamada and W. Glöckle, Nucl. Phys. A **548**, 205 (1992).
[63] M. R. Hadizadeh, Lauro Tomio, and S. Bayegan, Phys. Rev. C **83**, 054004 (2011); M. R. Hadizadeh and S. Bayegan, Few-Body Syst. **40**, 171 (2007).
[64] A. O. Gogolin, C. Mora, and R. Egger, Phys. Rev. Lett. **100**, 140404 (2008).
[65] D. Blume and Y. Yan, Phys. Rev. Lett. **113**, 213201 (2014).
[66] Y. Tanizaki, Prog. Theor. Exp. Phys. **2013**, 113A01 (2013).
[67] D. Lurié, A. J. Macfarlane, and Y. Takahashi, Phys. Rev. **140**, B1091 (1965).