Quantum phase estimation and quantum counting with qudits

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(Received 19 August 2016; published 7 October 2016)

Implementations of the quantum phase estimation and quantum counting algorithms with qudits are proposed. The construction of the basic building blocks of the algorithms is described in detail and the fidelity of the quantum counting algorithm with qubits and qudits is simulated numerically. Along with the exponential increase of the size of the target register, the simulations with qudits demonstrate significantly higher probability of finding the number of solutions than qubits and more consistent performance for a different number of solutions.

DOI: 10.1103/PhysRevA.94.042307

I. INTRODUCTION

The algorithm for the quantum search in an unstructured database invented by Grover [1] is one of the major milestones in the development of quantum computation. Search algorithms in databases are widely used in statistical data processing, e.g., in searching for the maximum or minimum element or for an item corresponding to other distinct criteria. The Grover search has been demonstrated in nuclear-magnetic-resonance (NMR) experiments with two [2–6] and three [7] qubits and in several other physical platforms with two qubits: linear optics [8], trapped ions [9], cavity quantum electrodynamics (QED) [10], etc. Because the physical mechanism of Grover's search is wave amplitude amplification, the algorithm has been demonstrated also in individual Rydberg atoms with eight different levels [11] and in classical Fourier optics with 32 elements [12].

Without knowing in advance the number of solutions, Grover's algorithm cannot be used. To this end, the exact number of elements, satisfying the search criteria, can be found by the quantum counting algorithm [13,14]. The latter has been experimentally demonstrated by Jones and Mosca [15] and Lee *et al.* [16] in NMR.

A key subroutine of quantum counting is the quantum phase estimation algorithm [17,18]. The goal of the algorithm is to find an eigenvalue of a unitary operator while the system remains in the corresponding eigenstate. Abrams and Lloyd [19] have shown how the eigenvalues of a unitary operator can be found by using the quantum Fourier transform. Travaglione and Milburn [20] have proposed a scheme for finding eigenstates with trapped-ion qubits. Quantum phase estimation has been demonstrated in an NMR experiment by Lee *et al.* [16].

The quantum phase estimation algorithm is an essential part of Shor's quantum factoring algorithm [21]. Shor's factoring has been tested in numerous experiments. Some examples are the factorization of the number 15 by Vandersypen *et al.* [22] in NMR, and by Politi *et al.* [23] in linear optics. The number 21 has been factored by Peng *et al.* [24] in NMR and by Martin-Lopez *et al.* [25] using a photonic circuit with qubit recycling. The largest number factored experimentally by a quantum system is 143 by Xu *et al.* [26] in NMR using adiabatic quantum computation (rather than Shor's algorithm).

Quantum algorithms are designed for qubits—two-state quantum systems. In this paper, we propose implementations of the quantum phase estimation and quantum counting algorithms with qudits—quantum systems with d states [27]. We scrutinize the construction of the basic elements of the algorithms using ideas from an earlier proposal for quantum search with qudits [28]. In particular, we demonstrate numerically that the qudit quantum counting significantly outperforms the original quantum counting with qubits.

The motivation of the present work derives from the growing interest in qutrits—three-state quantum systems—and qudits in general. Beyond the obvious exponential increase of the computational Hilbert space qudits offer a number of advantages over qubits. For example, qudits allow one to construct new types of quantum protocols [29,30] and entanglement [31], noise-resistant Bell inequalities [32], larger violations of nonlocality [33], more secure quantum communication [34,35], and optimization of the Hilbert space dimensionality vs control complexity [36]. Moreover, efficient recipes for construction of the most general unitary transformations of qutrits [37,38] and qudits [39] have been proposed.

This paper is organized as follows. In Sec. II, the quantum phase estimation algorithm is discussed. We review the implementation of this algorithm with qubits in Sec. II A, and we propose the qudit implementation in Sec. II B. In Sec. III A we review the quantum amplitude amplification routine in qubit quantum counting, and in Sec. III C we introduce the qudit quantum counting. In Sec. III D, we present numerical simulations of quantum counting with qubits and qudits. Finally, the conclusions are summarized in Sec. IV.

II. QUANTUM PHASE ESTIMATION

In this section, the quantum phase estimation algorithm with qubits is reviewed, and its implementation with qudits is presented.

A. Quantum phase estimation with qubits

Consider an arbitrary unitary operator $\hat{U}$ with an eigenvector $|u\rangle$ corresponding to an eigenvalue $e^{2\pi i\phi}$, where ϕ is an unknown phase. The phase estimation algorithm finds this ϕ [17]. In order to apply the algorithm, we need a black box, which can calculate U^{2^j}, where j is an integer. For this purpose the algorithm uses two registers: a control register A and a

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target register B. The phase ϕ can be expressed, with accuracy r, by the r-bit binary fraction [40],

$$\phi = 0.\phi_1\phi_2\ldots\phi_r = \frac{\phi_1}{2} + \frac{\phi_2}{4} + \cdots + \frac{\phi_r}{2^r}. \quad (1)$$

The control register A contains k qubits. Depending on the required accuracy of estimation of ϕ and the required probability of success $1-\epsilon$ (with ϵ being the admissible error), k can be calculated from [17]

$$k \geqq r + \left\lceil \log_2\left(2 + \frac{1}{2\epsilon}\right)\right\rceil, \quad (2)$$

where $\lceil x \rceil$ is the ceiling (round-up) of x and r is the required number of bits (r-bit precision) to estimate ϕ. The initial condition of register A is $|0\rangle$.

The target register B has n qubits with n being the number of qubits required to store $|u\rangle$. The initial state of register B is $|u\rangle$.

The quantum phase estimation algorithm proceeds as follows. The first step of the algorithm is a Hadamard gate on each of the qubits in register A in order to obtain an equal superposition of all possible basis states [17,40]. The second step is to apply the sequence of controlled unitary operators U^{2^j} on register B, with $j = 0,1,2,\ldots,k-1$. The ensuing state of register A is [17,40]

$$2^{-k/2}\big(|0\rangle + e^{2\pi i 2^{k-1}\phi}|1\rangle\big)_1\big(|0\rangle + e^{2\pi i 2^{k-2}\phi}|1\rangle\big)_2\cdots$$
$$\cdots\big(|0\rangle + e^{2\pi i 2^{0}\phi}|1\rangle\big)_k = 2^{-k/2}\sum_{l=0}^{2^k-1} e^{2\pi i l\phi}|l\rangle. \quad (3)$$

The expression (3) can be written by using binary fractions of ϕ as follows:

$$2^{-k/2}(|0\rangle + e^{2\pi i 0.\phi_k}|1\rangle)_1(|0\rangle + e^{2\pi i 0.\phi_{k-1}\phi_k}|1\rangle)_2\cdots$$
$$\cdots(|0\rangle + e^{2\pi i 0.\phi_1\phi_2\ldots\phi_{k-1}\phi_k}|1\rangle)_k. \quad (4)$$

The third step of the algorithm is the application of the inverse Fourier transform to register A [17]. The outcome of it is $|\phi_1\phi_2\ldots\phi_k\rangle$. The fourth step is to measure the state of register A. Register B remains in state $|u\rangle$ throughout the process because it is an eigenstate of U.

B. Quantum phase estimation with qudits

Here we propose a recipe to build a quantum phase estimation algorithm by using qubits in the control register A and qudits in the target register B. Assume that the control-$\hat{U}$ gate has just one control qubit and one target qudit. The control-$\hat{U}$ gate is a unitary transformation which is applied on the target register, depending on the state of the control register. It acts as follows:

$$|0\rangle|m\rangle \to |0\rangle|m\rangle, \quad (5a)$$
$$|1\rangle|m\rangle \to |1\rangle\hat{U}|m\rangle. \quad (5b)$$

Hence,

$$\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|m\rangle \to \tfrac{1}{\sqrt{2}}|0\rangle|m\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\hat{U}|m\rangle. \quad (6)$$

If a qudit with dimension $d > 2$ is used instead of the control qubit, the state of the target qudit will change only if the control qudit is in state $|d-1\rangle$,

$$|l\rangle|m\rangle \to |l\rangle|m\rangle \quad (l \neq d-1), \quad (7a)$$
$$|d-1\rangle|m\rangle \to |d-1\rangle\hat{U}|m\rangle. \quad (7b)$$

Hence, we find (with $l \neq d-1$)

$$\tfrac{1}{\sqrt{2}}(|l\rangle + |d-1\rangle)|m\rangle \to \tfrac{1}{\sqrt{2}}|l\rangle|m\rangle + \tfrac{1}{\sqrt{2}}|d-1\rangle\hat{U}|m\rangle. \quad (8)$$

Therefore, only two states of the qudit (for example, state $|d-1\rangle$ paired with any other state) could be used in the phase estimation algorithm. Hence using qudits in the control register A does not provide any benefit over qubits. However, using qudits in the target register B, which is used to store the information, gives an exponential growth of the database by a factor $(d/2)^n$ compared to qubits.

The steps in the quantum phase estimation with qudits are the same as those in the algorithm with qubits, the only difference being that the control (A) and the target (B) registers have different dimensions.

III. QUANTUM COUNTING ALGORITHM

A. Grover search and quantum counting with qubits

The Grover search is a quantum algorithm for searching in an unstructured database [1]. The initial state of the system is an equal superposition of all states,

$$|a\rangle = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}|l\rangle. \quad (9)$$

This state can be constructed by preparing all qubits in state $|0\rangle$ and applying Hadamard gates to all of them.

A crucial part of the algorithm is the Grover operator $\hat{G} = \hat{R}\hat{O}$. It consists of a quantum oracle $\hat{O} = 2|s\rangle\langle s| - \hat{I}$ (which has the ability to recognize the solution $|s\rangle$) and a Householder reflection $\hat{R} = \hat{I} - 2|a\rangle\langle a|$, which performs a reflection about the average vector $|a\rangle$ (also known as "inversion about the mean"). Grover's search finds a solution after N_g repeated applications of the Grover operator $\hat{G}$ on the initial state $|a\rangle$, i.e., $\hat{G}^{N_g}|a\rangle \approx |s\rangle$, where the number of iterations is

$$N_g = \left\lfloor \frac{\pi}{2\theta}\right\rfloor \underset{N\gg M}{\approx} \left\lfloor \frac{\pi}{4}\sqrt{\frac{N}{M}}\right\rfloor, \quad (10)$$

where $\lfloor x \rfloor$ is the floor function (the integer part of x), and

$$\theta = 2\arcsin\sqrt{\frac{M}{N}}. \quad (11)$$

Here M is the number of possible solutions to the search problem. The inequality $M < N/2$ must be fulfilled for the Grover's iteration to succeed. For large ratios $N/M \gg 1$, Grover's fidelity is very close to 1. Alternatively, one can use the deterministic version of Grover's search, which uses complex $\hat{R}$ and $\hat{O}$, for which the fidelity is exactly equal to unity [41–43], i.e., $\hat{G}^{N_g}|a\rangle = |s\rangle$.

The probability for finding the sought state $|s\rangle$ is a sinusoidal function of the number of iterations, hence it is essential to know the Grover's number N_g. Therefore, it is required to know the exact number of solutions M satisfying the search conditions.

The number of solutions M can be found by the quantum counting algorithm, which uses a single application of the quantum phase estimation algorithm. Quantum counting uses the eigenvectors and eigenvalues of the Grover iteration operator $\hat{G}$. To this end, let us denote the superposition of all $N - M$ states (i.e., database elements) that do not meet the search criteria by $|\alpha\rangle$, and the superposition of all M states that are solutions by $|\beta\rangle$,

$$|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum\nolimits_l |l\rangle, \quad |\beta\rangle = \frac{1}{\sqrt{M}} \sum\nolimits_l |l\rangle, \tag{12}$$

where the first sum runs over all nonsolutions and the second sum over all solutions to the search problem. Obviously,

$$|a\rangle = \cos\frac{\theta}{2}\,|\alpha\rangle + \sin\frac{\theta}{2}\,|\beta\rangle. \tag{13}$$

State $|a\rangle$ can be expressed also as [44]

$$|a\rangle = \tfrac{1}{\sqrt{2}}(e^{-i\theta/2}\,|+\rangle + e^{i\theta/2}\,|-\rangle), \tag{14}$$

where

$$|\pm\rangle = \tfrac{1}{\sqrt{2}}(\,|\alpha\rangle \pm i\,|\beta\rangle). \tag{15}$$

It is a matter of simple algebra to show that $\hat{G}\,|+\rangle = e^{-i\theta}\,|+\rangle$ and $\hat{G}\,|-\rangle = e^{i\theta}\,|-\rangle$. This means that $|+\rangle$ and $|-\rangle$ are eigenvectors of $\hat{G}$ with eigenvalues $e^{-i\theta}$ and $e^{i\theta}$, respectively. Therefore the problem of finding the number of solutions M to the search problem translates to the problem of finding the eigenvalues of $\hat{G}$, i.e., the value of θ, because [cf. Eq. (11)]

$$M = N\sin^2(\theta/2). \tag{16}$$

The angle θ can be found by using the quantum phase estimation algorithm with $\hat{G}$ for $\hat{U}$ and $|a\rangle = \hat{H}^{\otimes n}\,|00\cdots0\rangle$ for $|u\rangle$, where $\hat{H}^{\otimes n}$ is the Hadamard gate applied to all qubits. The algorithm gives as solutions the angles θ and $-\theta$ (modulo 2π). If the register A contains k qubits and the register B contains n qubits, θ is found with r-bits precision and the probability of success is at least $1-\epsilon$, if k satisfies Eq. (2).

The size of the registers is finite, hence we obtain an estimate of θ, rather than θ itself. Likewise, instead of the number of solutions to the search problem M, an estimate for M is found.

B. Quantum search with qudits

The qudit-based Grover search [28] begins with the system initialized in the equal-weight coherent superposition $|a\rangle$ of all basis states [cf. Eq. (9)], but with arbitrary relative phases. The superposition $|a\rangle$ is obtained by preparing all qudits in their state $|0_j\rangle$ ($j = 1,2,\ldots,n$) and applying a single-qudit transformation $\hat{F}$ on all of them. This transformation $\hat{F}$ generalizes the Hadamard gate $\hat{H}$, and only demands to drive the single-qudit state $|0_j\rangle$ into an equal-weight superposition state in all qubits ($j = 1,2,\ldots,n$),

$$\hat{F}\,|0_j\rangle = \sum_{q=0}^{d-1} \xi_q\,|q_j\rangle, \tag{17}$$

with $|\xi_q| = d^{-1/2}$. Hence the first column of the matrix representation of $\hat{F}$ must contain elements of equal moduli.

An example of $\hat{F}$ is the discrete Fourier transform $\mathcal{F}$ (DFT). (The Hadamard gate $\hat{H}$ is the two-dimensional DFT.) However, it is not necessary to assume that $\hat{F}$ is the DFT because the construction of DFT for $d > 2$ may be very

TABLE I. Results of numerical simulations of the quantum counter for the two states of register B—corresponding to θ and $2\pi - \theta$—which have the highest probability to be obtained (third column). Register A contains from 5 to 8 qubits. Register B contains from 3 to 5 qubits or qutrits. The number of sought elements is $M = 3$ in all cases. The estimate for M listed in the last column differs from this number due to the finite size of the registers.

		Qubits		
Number of qubits in A	Number of qubits in B	States in B	Total probability	Estimate of M
5	4	5 and 27	0.549	3.555
6	3	13 and 51	0.533	2.839
6	4	9 and 55	0.953	2.925
6	5	6 and 58	0.676	2.696
7	3	27 and 101	0.931	3.028
7	4	18 and 110	0.819	2.925
8	4	36 and 220	0.413	2.925
		Qutrits		
Number of qubits in A	Number of qutrits in B	States in B	Total probability	Estimate of M
5	4	2 and 30	0.998	3.083
6	3	7 and 57	0.981	3.064
6	4	4 and 60	0.990	3.083
6	5	2 and 62	0.793	2.335
7	3	14 and 114	0.925	3.064
7	4	8 and 120	0.961	3.083
8	4	16 and 240	0.852	3.083

demanding. Moreover, the DFT demands specific phase relations between its elements, while no such relations in $\hat{F}$ are required; it is only necessary to use the same $\hat{F}$ in all steps. In general, there are numerous suitable choices for $\hat{F}$ [28]. There are many techniques for construction of these gates; for example, efficient methods exist for trapped ions [45–47].

The qudit search proceeds via repeated applications of the Grover operator $\hat{G}$, which has the same form $\hat{G} = \hat{R}\hat{O}$ as for qubits. The only difference is that in the reflection-about-the-average operator $\hat{R} = \hat{H}^{\otimes n}\hat{O}_0\hat{H}^{\otimes n}$ (where $\hat{O}_0$ is the conditional sign-flip operator of the initial state $|0_1 0_2 \cdots 0_n\rangle$), the Hadamard gate $\hat{H}$ is replaced by $\hat{F}$: $\hat{R} = \hat{F}^{\otimes n}\hat{O}_0(\hat{F}^{\dagger})^{\otimes n}$ [28].

C. Quantum counting with qudits

Building upon the qudit quantum search described above here we use a similar idea to design the qudit quantum counting algorithm. The latter uses two registers, A and B. Register A consists of k qubits and register B of n qudits. The initial state of the registers is $|0\rangle_A |0\rangle_B = |00\cdots 0\rangle$.

The algorithms begin with the creation of an equal-weight superposition of all basis states. Starting from the initial state $|0\rangle_A |0\rangle_B$, we use the Hadamard gate $\hat{H}$ on register A, and the operator $\hat{F}$ on register B that transforms each qudit to an equal superposition of its states with dimension d,

$$\hat{H}^{\otimes k} |0\rangle_A \hat{F}^{\otimes n} |0\rangle_B = \frac{1}{\sqrt{2^k d^n}} \sum_{j=0}^{2^k-1} |j\rangle_A \sum_{l=0}^{d^n-1} |l\rangle_B. \tag{18}$$

The qudit quantum counting algorithm proceeds similarly to the qubit quantum counting described above, with the exception of the Grover operator $\hat{G}$. As in the qudit Grover search, the reflection-about-the-mean operator $\hat{R}$ in the Grover operator $\hat{G} = \hat{R}\hat{O}$ is constructed by using the operator $\hat{F}$ instead of the Hadamard gate $\hat{H}$,

$$\hat{R} = \hat{F}^{\otimes n}\hat{O}_0(\hat{F}^{\dagger})^{\otimes n}, \tag{19}$$

where, as before, $\hat{O}_0$ is the conditional sign-flip operator of the initial state $|00\cdots 0\rangle$.

Next, qudit quantum phase estimation of the eigenvalues of $\hat{G}$ is performed. The measurement of the state of register A returns the values of θ and $2\pi - \theta$. Then the number of solutions M can be calculated from Eq. (16).

D. Examples

The results of numerical simulations of quantum counting with qubits and qutrits are shown in Table I. The control register A contains 5 to 8 qubits, the target register B contains 3 to 5 qubits or qutrits, and the number of sought elements is 3. It can be seen from the table that when qutrits are used instead of qubits in register B, the probability to find the number of solutions increases dramatically.

In order to find the solution to the quantum counting problem, we proceed as follows. After numerical iteration of the quantum counting algorithm with either qubits or qutrits we arrive at a probability distribution among the 2^k states of the control register A, as the one shown in Fig. 1 for the case when register A contains 5 qubits, register B contains 4 qubits (top frame) or 4 qutrits (bottom frame), and the number of sought elements is 3. Then we identify the positions of the two symmetric maxima, which give the values θ and $2\pi - \theta$. (Note that the 2^k states span the range $[0,2\pi]$, with state $|00\cdots 0\rangle$ corresponding to $\theta = 0$ and state $|11\cdots 1\rangle$ corresponding to $\theta = 2\pi - 2\pi/2^k$.) For example, the positions of the two maxima in the bottom frame of Fig. 1 are 2 and 30, from which we find the value $\theta = 2\pi \times 2/2^5 = \pi/8$, and hence the number of solutions is $M = 3^4 \sin^2(\theta/2) \approx 3.083$, as seen in Table I. The probability of finding the angles θ (i.e., obtaining state $|2\rangle$) and $2\pi - \theta$ (i.e., obtaining state $|30\rangle$), obtained after numerical iteration of the quantum counting algorithm, is about 27.4% with 4 qubits and 49.9% with 4 qutrits in register B. The total probability (the sum of the two probabilities for θ and $2\pi - \theta$) for finding the number of solutions is 54.9% with 4 qubits and far greater, 99.8% with 4 qutrits—the numbers listed in Table I. As Fig. 1 shows, for qubits the population is distributed among several states in the control register A. For qudits, the population is concentrated essentially in the two states that correspond to θ and $2\pi - \theta$, hence the success probability of the quantum counting algorithm is far greater.

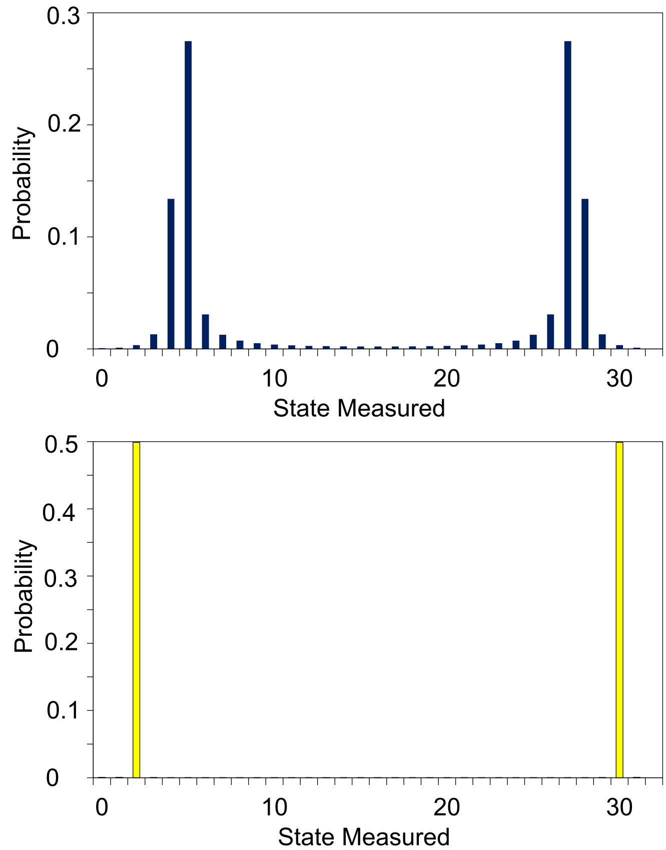

FIG. 1. Simulations of quantum counting with qubits (top) and qutrits (bottom) in register B. Plotted are the populations of the states in the control register A. Register A contains 5 qubits (i.e., $2^5 = 32$ states), while register B contains 4 qubits (top) or 4 qutrits (bottom). The number of sought elements is 3. The total probability to find the angle θ or $2\pi - \theta$ is approximately 0.549 with qubits and 0.998 with qutrits.

Figure 2 presents a similar simulation for 6 control qubits and 3 target qubits (top) or qutrits (bottom). Similar enhancement of the success probability for qutrits over qubits is observed.

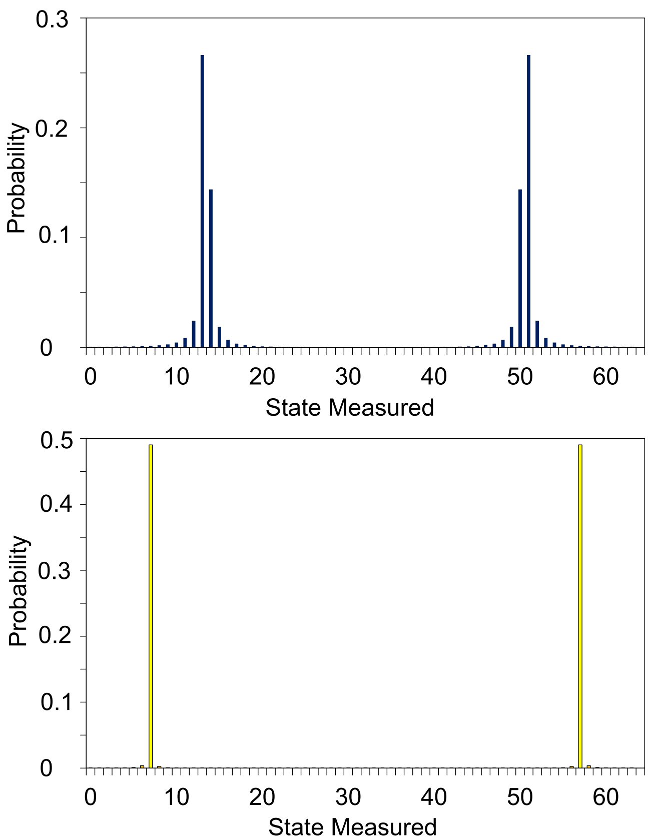

FIG. 2. Simulations of quantum counters with qubits (top) and with qutrits (bottom). Plotted are the populations of the states in the control register A. Register A contains 6 qubits (i.e., $2^6 = 64$ states), while register B contains 3 qubits (top) or 3 qutrits (bottom). The number of sought elements is 3. The total probability to find the angle θ or $2\pi - \theta$ is approximately 0.533 with qubits and 0.981 with qutrits.

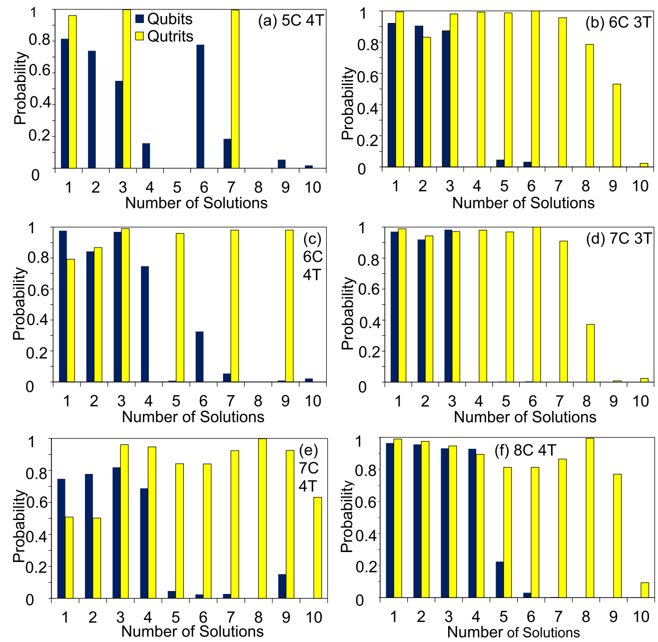

FIG. 3. Numerically simulated efficiency of quantum counters with different number of qubits in the control register A and qudits—qubits (dark blue bars) or qutrits (light yellow bars)—in the target register B, vs the number of solutions M. (a) 5 qubits in A, 4 qudits in B; (b) 6 qubits in A, 3 qudits in B; (c) 6 qubits in A, 4 qudits in B; (d) 7 qubits in A, 3 qudits in B; (e) 7 qubits in A, 4 qudits in B; (f) 8 qubits in A, 4 qudits in B.

Figure 3 presents numerical simulations of the quantum counting success probability for various numbers of qubits in register A and qubits or qutrits in register B versus the number of solutions M. For the small number of qubits in the control register A [frame (a)], the quantum counting algorithm behaves erratically, due to the large step in θ (and hence in the number of solutions M). Indeed, for $k = 5$ qubits in register A and $n = 4$ qubits or qutrits in register B, the step in θ is $2\pi/2^k = \pi/16$, and the step in M of Eq. (16) is too large to identify the value of M with certainty. As the number of control qubits increases, the step in θ decreases and the quantum counting algorithm becomes more consistent in finding the number of solutions. The qutrit implementation delivers more consistent results, with higher probability than the qubit implementation. The latter works relatively well only for a small number of solutions M, while the qutrit implementation remains reliable for much larger M.

In general, the quantum counting algorithm performs well if the number of states in register A, i.e., 2^k, is significantly larger than the number of states in register B, i.e., d^n, so that the step in the value of M is sufficiently small ($\Delta M \ll 1$). At the same time, the sizes of the registers A and B should be sufficiently large in order to find the phase θ with good accuracy. Therefore, the conditions for the qudit quantum counting algorithm are

$$2^k \gg d^n, \quad k \gg 1, \quad n \gg 1. \tag{20}$$

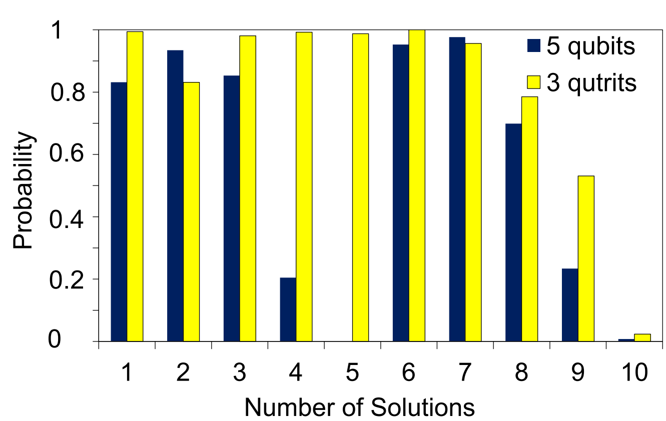

FIG. 4. Simulations of quantum counting with 5 qubits (dark blue bars) or 3 qutrits (light yellow bars) in register B vs the number of solutions M to the quantum counting problem. Plotted are the populations of the states in the control register A with the highest probability for each value of M (i.e., the solutions to the quantum counting problem). Register A contains 6 qubits. Despite the similar number of elements in register B ($2^5 = 32$ for qubits vs $3^3 = 27$ for qutrits) the qutrit implementation outperforms the one with qubits.

These conditions are fulfilled sufficiently well only in frames (b), (d), (e), and (f) in Fig. 3, which is clearly reflected in the observed quantum counting efficiencies. Moreover, because the size of the control register A is present in the sine function in Eq. (16), which has slow variation for small values of its argument and fast variation for larger values, the quantum counting algorithm performs best for small values of θ, i.e., for a small number of solutions M.

We note that the qudit quantum counter performs significantly better than a qubit quantum counter with a similar register size. For example, the quantum counter with 6 qubits in A and 3 qutrits in B (meaning $3^3 = 27$ elements in B) considerably outperforms the quantum counter with 6 qubits in A and 5 qubits in B (meaning $2^5 = 32$ elements in B), as seen in Fig. 4.

IV. CONCLUSION

In this paper, we proposed implementations of the quantum phase estimation and quantum counting algorithms with qudits. Qudits are used in the target register, while the control register still uses qubits because only two states are needed for the conditional operations. We described the construction of the basic building blocks of the algorithms, following ideas from an earlier proposal for the Grover search with qudits [28]. Numerical simulations with qubits replaced by qutrits in the target register demonstrate an increase of the quantum counting efficiency and a higher probability of finding the number of solutions. Besides the exponential increase of the size of the target register qudits offer further advantages over qubits in terms of more consistent performance.

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