Optimal quantum cloning based on the maximin principle by using *a priori* information

Peng Kang,[1] Hong-Yi Dai,[2] Jia-Hua Wei,[1] and Ming Zhang[1,*]
[1]*College of Mechatronic Engineering and Automation, National University of Defense Technology, Changsha 410073, China*
[2]*College of Science, National University of Defense Technology, Changsha 410073, China*
(Received 30 March 2016; revised manuscript received 7 July 2016; published 5 October 2016)

We propose an optimal $1 \rightarrow 2$ quantum cloning method based on the maximin principle by making full use of *a priori* information of amplitude and phase about the general cloned qubit input set, which is a simply connected region enclosed by a "longitude-latitude grid" on the Bloch sphere. Theoretically, the fidelity of the optimal quantum cloning machine derived from this method is the largest in terms of the maximin principle compared with that of any other machine. The problem solving is an optimization process that involves six unknown complex variables, six vectors in an uncertain-dimensional complex vector space, and four equality constraints. Moreover, by restricting the structure of the quantum cloning machine, the optimization problem is simplified as a three-real-parameter suboptimization problem with only one equality constraint. We obtain the explicit formula for a suboptimal quantum cloning machine. Additionally, the fidelity of our suboptimal quantum cloning machine is higher than or at least equal to that of universal quantum cloning machines and phase-covariant quantum cloning machines. It is also underlined that the suboptimal cloning machine outperforms the "belt quantum cloning machine" for some cases.

DOI: 10.1103/PhysRevA.94.042304

I. INTRODUCTION

In contrast with classical states, quantum states drawn from a set that contains at least two nonorthogonal states cannot be cloned perfectly based on the no-cloning theorem [1,2]. However, we can approximately clone an arbitrary state with proper fidelity, or perfectly clone linearly independent quantum states with certain probabilities [3]. The development of quantum cloning has gone on for over 30 years, and some theoretical [4–28] and experimental progress [29–33] has been made in this domain.

For the sake that quantum cloning could be applied to quantum computation and quantum information [3,34], such as the security analysis of quantum key distribution (QKD) protocols [35] and quantum cloning attacks to QKD [36,37], a growing number of works have appeared in this realm recently. The universal quantum cloning machine (UQCM) acting on the whole Bloch sphere with one input and two identical outputs is presented by Bužek and Hillery [4], and the optimality of the UQCM is demonstrated by Bruß *et al.* [5]. Furthermore, the $1 \rightarrow 2$ UQCM has also been extended to $N \rightarrow M$ cases [6–9] and some other cases [10–12]. Bruß*et al.* [13] presented the phase-covariant quantum cloning machine (PCQCM), of which the input state is restricted in the "equator" of the Bloch sphere, and the fidelity of the PCQCM is higher than that of the UQCM. The more general situations of PCQCM also have been studied [14–19]. Hu *et al.* [20] analyzed the problem of $1 \rightarrow 2$ approximate quantum clonings for the quantum state between two latitudes on the Bloch sphere, namely, the "belt quantum cloning machine" (BQCM). Bartkiewicz and Miranowicz [21] found an optimal quantum cloning machine, which clones qubits of arbitrary symmetrical distribution around the Bloch vector with the highest fidelity. The probabilistic quantum cloning machine was first proposed by Duan and Guo [22]. Some interesting research about quantum cloning was also presented in Refs. [25–28]. However, there are still many important and open problems to be taken into account for quantum cloning.

The $1 \rightarrow 2$ quantum cloning machines acting on the Bloch sphere are the simplest, but they are very important ones. When the input states span the whole Bloch sphere, the UQCM [5] is the optimal one of the $1 \rightarrow 2$ quantum cloning machines, and the cloning fidelity always equals $5/6$. For the input set in a belt of the Bloch sphere, namely, in which the quantum states are distributed between two latitudes on the Bloch sphere, one can use the BQCM [20] to perform quantum cloning and obtain a better cloning quality than by using the UQCM by taking advantage of the amplitude information of the input set. Moreover, if the input set lies on the "equator" of the Bloch sphere, the PCQCM [13] is advisable to get higher fidelity than the UQCM. From the BQCM and PCQCM, we find that *a priori* amplitude information can be utilized to improve the quality of quantum cloning, and these aforementioned quantum cloning machines could not make use of *a priori* phase information.

It is quite important to study the state-dependent cloning problem when *a priori* information of the cloned state is known but not exact. We would like to ask the following question: How do we improve the cloning fidelity in terms of *a priori* amplitude and phase information of the general input set? We suggest an optimal scheme based on the maximin principle to perform quantum cloning by making full use of *a priori* information of the general input set. Our scheme may be of great use for quantum-cloning-based feedback controls, which are very different from both coherent feedback controls and measurement-based feedback controls.

It is well known that the quantum coherent feedback and quantum-measurement-based feedback approaches have been widely studied and several works [38–41] have recently compared these two kinds of quantum feedback controls. However, there exists an alternative approach of quantum feedback control [42], which can be regarded as quantum-cloning-based feedback control. In Ref. [42] a cloning machine

*Corresponding author: zhangming@nudt.edu.cn

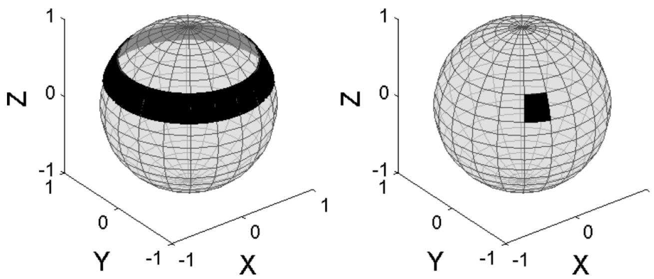

FIG. 1. The belt area on the Bloch sphere shows the set of qubits that are studied in the BQCM, while the block regime is the qubit set we want to clone in an optimal way.

is served to obtain the feedback signal instead of feeding back precisely the process output.

Without loss of generality, an input set on the Bloch sphere could be expressed as (the block regime shown in Fig. 1)

$$S(\alpha_1,\alpha_2;\phi_1,\phi_2) = \{|\psi\rangle = \alpha|0\rangle + \beta e^{i\phi}|1\rangle, \quad |\alpha_1 \leqslant \alpha \leqslant \alpha_2, \phi_1 \leqslant \phi \leqslant \phi_2\}, \tag{1}$$

where the (relative) phase factor $\phi \in [0,2\pi]$, and the real amplitude factors α and β satisfy $\alpha^2 + \beta^2 = 1$. In addition, they are restricted by $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$ and $0 \leqslant \phi_1 \leqslant \phi_2 \leqslant 2\pi$. In our optimal $1 \to 2$ quantum cloning method, *a priori* amplitude and phase information $(\alpha_1,\alpha_2;\phi_1,\phi_2)$ of the cloned input set could be used to design the optimal quantum cloning machine based on the maximin principle; it is reasonable to call such an optimal cloner an "optimal maximin quantum cloning machine" (OMQCM). We would like to point out that the fidelity of the OMQCM is optimal for input set $S(\alpha_1,\alpha_2;\phi_1,\phi_2)$ among all quantum cloning machines in theory. Furthermore, under some proper conditions on the construction of the OMQCM, the explicit form of the suboptimal maximin quantum cloning machine (SMQCM) is presented in detail. By comparing, we find that the SMQCM is better than or the same as the UQCM and the PCQCM in terms of fidelity, and it outperforms the BQCM for some cases. In particular, we expect that the smaller the input set, i.e., the more information of the input set that is given, the better one can clone each of its states [13].

The rest of this paper is organized as follows. In Sec. II, we present the $1 \to 2$ quantum cloning method based on the maximin principle by using *a priori* amplitude and phase information of the input states. By restricting the structure of the OMQCM, we obtain the concrete form of the SMQCM in Sec. III. From the numerical examples in Sec. IV, one can find that our SMQCM can outperform some former proposals, such as the UQCM, the PCQCM, and the BQCM. At last we conclude with Sec. V. Full details of the optimization procedure are included in Appendices A and B.

II. OPTIMAL QUANTUM CLONING METHOD

Let us begin with a brief statement of the $1 \to 2$ quantum cloning process acting on the Bloch sphere [5]. A quantum machine can be described as a unitary operator U:

$$|\psi\rangle|0\rangle|X\rangle \to U|\psi\rangle|0\rangle|X\rangle,$$

where $|\psi\rangle$ is an arbitrary input qubit of system 1, the state of system 2 is given by a blank qubit $|0\rangle$, and the auxiliary system x is in the state $|X\rangle$, of which the dimension is not restricted. We denote

$$|\Psi\rangle = U|\psi\rangle|0\rangle|X\rangle.$$

Then the density matrix of the whole system can be written as

$$\rho_{12x}^{\text{out}} = |\Psi\rangle\langle\Psi|.$$

By taking a partial trace, we can obtain the reduced density matrices for systems 1 and 2, respectively:

$$\rho_1 = \text{Tr}_{2x}\left(\rho_{12x}^{\text{out}}\right), \qquad \rho_2 = \text{Tr}_{1x}\left(\rho_{12x}^{\text{out}}\right).$$

Usually, with the symmetry requirement that two outputs are identical [5], namely, $\rho_1 = \rho_2$, the unitary operator U could be defined by

$$U|0\rangle|0\rangle|X\rangle = a_0|00\rangle|A_0\rangle + b_0(|01\rangle + |10\rangle)|B_0\rangle + c_0|11\rangle|C_0\rangle,$$
$$U|1\rangle|0\rangle|X\rangle = a_1|11\rangle|A_1\rangle + b_1(|10\rangle + |01\rangle)|B_1\rangle + c_1|00\rangle|C_1\rangle, \tag{2}$$

where the coefficients a_i, b_i, and c_i are complex, and the capital letters A_i, B_i, and C_i refer to output ancilla states, with $i = 0, 1$. We do not specify the dimension of the ancilla; for $|A_i\rangle$, $|B_i\rangle$, and $|C_i\rangle$, the only condition is that they are normalized: $\langle A_i|A_i\rangle = \langle B_i|B_i\rangle = \langle C_i|C_i\rangle = 1$.

Due to the unitarity of operator U, Eq. (2) must satisfy the following conditions:

$$|a_0|^2 + 2|b_0|^2 + |c_0|^2 = 1,$$
$$|a_1|^2 + 2|b_1|^2 + |c_1|^2 = 1,$$
$$c_1^* a_0\langle C_1|A_0\rangle + b_1^* b_0\langle B_1|B_0\rangle + b_0^* b_1\langle B_0|B_1\rangle + a_1^* c_0\langle A_1|C_0\rangle = 0, \tag{3}$$

where * denotes the conjugation. The cloned fidelity is denoted as

$$F = \langle\psi|\rho_1|\psi\rangle. \tag{4}$$

In order to obtain the OMQCM, we need to derive the concrete parameters a_i, b_i, and c_i of operator U, shown as Eq. (2). It is noted that the parameters for the OMQCM in our method are obtained based on the maximin principle by using *a priori* information about the amplitude and phase of the input set $S(\alpha_1,\alpha_2;\phi_1,\phi_2)$, given by Eq. (1). The concrete implementation procedures of this method are presented as follows:

Step 1. From an arbitrary input set $S(\alpha_1,\alpha_2;\phi_1,\phi_2)$ in Eq. (1), we can always obtain the minimal fidelity F_0 by searching amplitude factor α and phase factor ϕ spaces:

$$F_0 = \min_{\alpha;\phi}(F) \quad \text{such that } 0 \leqslant \alpha_1 \leqslant \alpha \leqslant \alpha_2 \leqslant 1, \quad 0 \leqslant \phi_1 \leqslant \phi \leqslant \phi_2 \leqslant 2\pi. \tag{5}$$

Actually, F_0 is an expression on the parameters a_i, b_i, and c_i and ancilla states $|A_i\rangle$, $|B_i\rangle$, and $|C_i\rangle$ of the cloning machine, with $i = 0,1$.

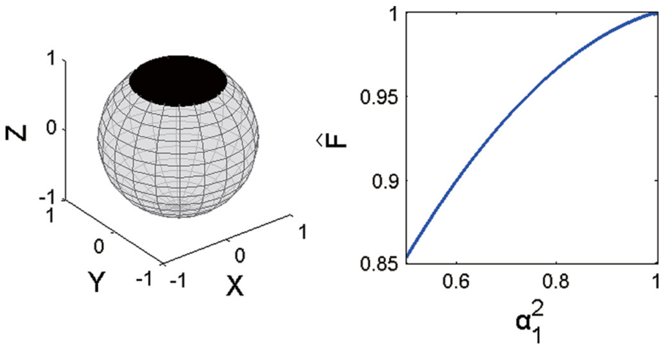

FIG. 2. Left: The input set $S\{\alpha_1,1;-\pi,\pi\}$ with $\frac{1}{\sqrt{2}} < \alpha_1 < 1$ (the Bloch sphere hat). Right: The optimal fidelity $\hat{F}$ function of α_1^2.

Step 2. By tuning the parameters and ancilla states, we can maximize F_0:

$$\begin{aligned}\hat{F} = &\max_{a_i,\ldots;|A_i\rangle,\ldots} (F_0)\\ \text{such that } &|a_0|^2 + 2|b_0|^2 + |c_0|^2 = 1,\\ &|a_1|^2 + 2|b_1|^2 + |c_1|^2 = 1,\\ &c_1^* a_0 \langle C_1|A_0\rangle + b_1^* b_0 \langle B_1|B_0\rangle\\ &+ b_0^* b_1 \langle B_0|B_1\rangle + a_1^* c_0 \langle A_1|C_0\rangle = 0,\\ &\langle A_i|A_i\rangle = \langle B_i|B_i\rangle = \langle C_i|C_i\rangle = 1. \end{aligned} \tag{6}$$

Parameters a_i, b_i, and c_i and ancilla states $|A_i\rangle$, $|B_i\rangle$, and $|C_i\rangle$ of the OMQCM could be designed in the optimization process. As we can see, this optimal method makes full use of *a priori* information about the amplitude and phase of input set $S(\alpha_1,\alpha_2;\phi_1,\phi_2)$. Theoretically, the fidelity of our OMQCM is highest in terms of the maximin principle.

In general, we may encounter the optimal quantum cloning problem when the cloned states are in the neighborhood of a given quantum state. We obtain a perfect solution to it by our optimal scheme. First of all, for input set $S(\alpha,\alpha;-\pi,\pi)$ with $\frac{1}{\sqrt{2}} < \alpha < 1$, the optimal quantum cloning machine derived by our method is same as the result of Ref. [31]:

$$\begin{aligned} U|00\rangle &= |00\rangle,\\ U|10\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \end{aligned} \tag{7}$$

Moreover, for input states in a spherical cap, namely, $S(\alpha_1,1;-\pi,\pi)$ with $\frac{1}{\sqrt{2}} < \alpha_1 < 1$, the optimal quantum cloning machine is still the one shown in Eq. (7). The details of the proof are given in Appendix A. The distribution of the input set and the fidelity function of parameter α_1^2 are depicted in Fig. 2. It is shown that the more exact information we have on the input set (the bigger parameter α_1), the better quantum cloning quality we obtain. (It should emphasized that the optimal fidelity of our machine is bigger than $\frac{1}{2} + \sqrt{\frac{1}{8}}$, that of the PCQCM.)

III. SUBOPTIMAL MAXIMIN QUANTUM CLONING

In order to get an intuitive understanding, it is necessary to obtain the analytic form of our machine. While the analytic solution for the OMQCM is difficult to acquire for general cases, a reasonable compromise strategy is used to derive the analytic expression of the SMQCM (a quantum cloning machine whose construction is restricted compared with the OMQCM). Inspired by the UQCM and PCQCM [5,13], we restrict the construction of the SMQCM to the simplified form below:

$$\begin{aligned} U|0\rangle|0\rangle|X\rangle &= a|00\rangle|A\rangle + b(|01\rangle + |10\rangle)|A_\perp\rangle + c|11\rangle|A\rangle,\\ U|1\rangle|0\rangle|X\rangle &= a|11\rangle|A_\perp\rangle + b(|10\rangle + |01\rangle)|A\rangle + c|00\rangle|A_\perp\rangle, \end{aligned} \tag{8}$$

where $\langle A|A\rangle = \langle A_\perp|A_\perp\rangle = 1$, $\langle A|A_\perp\rangle = 0$, and the real factors a, b, and c satisfy the normalization condition

$$a^2 + 2b^2 + c^2 = 1. \tag{9}$$

Compared with the OMQCM, the construction-restricted conditions of the SMQCM can be presented as

$$\begin{aligned} a_0 &= a_0^* = a_1 = a_1^* = a,\\ b_0 &= b_0^* = b_1 = b_1^* = b,\\ c_0 &= c_0^* = c_1 = c_1^* = c,\\ |A_0\rangle &= |B_1\rangle = |C_0\rangle = |A\rangle,\\ |A_1\rangle &= |B_0\rangle = |C_1\rangle = |A_\perp\rangle. \end{aligned} \tag{10}$$

Then, the fidelity shown in Eq. (4) is rewritten as

$$F = a^2 + b^2 + 2\alpha^2(1-\alpha^2)[-a^2 + c^2 + 2ab + 2bc\cos(2\phi)].$$

By setting $\eta = \cos(2\phi)$, $p = 2\alpha^2(1-\alpha^2)$, the expression of fidelity could be further simplified as

$$F = a^2 + b^2 + p(-a^2 + c^2 + 2ab + 2bc\eta). \tag{11}$$

Naturally, Eqs. (5) and (6) would be reduced to

$$\begin{aligned} F_0 = &\min_{p;\eta} a^2 + b^2 + p(-a^2 + c^2 + 2ab + 2bc\eta)\\ \text{such that } &0 \leqslant p_1 \leqslant p \leqslant p_2 \leqslant \frac{1}{2}\\ &-1 \leqslant \eta_1 \leqslant \eta \leqslant \eta_2 \leqslant 1 \end{aligned} \tag{12}$$

and

$$\begin{aligned} \hat{F} = &\max_{a,b_i,c}(F_0)\\ &\text{such that } a^2 + 2b^2 + c^2 = 1, \end{aligned} \tag{13}$$

where

$$\begin{aligned} p_1 &= \min_{\alpha_1 \leqslant \alpha \leqslant \alpha_2} 2\alpha^2(1-\alpha^2), \quad \eta_1 = \min_{\phi_1 \leqslant \phi \leqslant \phi_2} \cos(2\phi),\\ p_2 &= \max_{\alpha_1 \leqslant \alpha \leqslant \alpha_2} 2\alpha^2(1-\alpha^2), \quad \eta_2 = \max_{\phi_1 \leqslant \phi \leqslant \phi_2} \cos(2\phi). \end{aligned} \tag{14}$$

It should be pointed out that $\hat{F}$ is the minimal fidelity of the SMQCM for the input set $S(\alpha_1,\alpha_2;\phi_1,\phi_2)$.

We could obtain the analytic expression for the SMQCM by solving Eqs. (12) and (13). Moreover, the solving process in our proposal can be interpreted as three steps:

(1) According to the signs of the coefficients of η and p in Eq. (12), the parameter space $a^2+2b^2+c^2=1$ is decomposed into nine subspaces (i,j) with $i,j=1,2,3$. Then, we derive the analytic expression F_0^{ij} in each subspace (i,j), respectively.

(2) In subspace (i,j), from the corresponding expression F_0^{ij}, we can solve Eq. (13) by using the Kuhn-Tucker method [43] to obtain $\hat{F}^{ij}$ and the parameters (a,b,c).

(3) We select the biggest one from the set $\{\hat{F}_{ij}|i,j=1,2,3\}$ as $\hat{F}$. It is worth pointing out that parameters (a,b,c) corresponding to the biggest $\hat{F}_{ij}$ can be used to obtain the SMQCM.

A. Expressions of $F_0^{i,j}$ in nine parameter subspaces

From Eq. (12), one can find that the expression of F_0 is related to the signs of the coefficients of p and η. First, from the signals of the coefficient of η, the whole parameter space $a^2+2b^2+c^2=1$ would be decomposed into three subspaces 1, 2, and 3, namely, $2bc>0$, $2bc=0$, and $2bc<0$. In addition, we have

if $\quad 2bc>0$

$$F_0=\min_{p_1\leqslant p\leqslant p_2} a^2+b^2+p(-a^2+c^2+2ab+2bc\eta_1),$$

if $\quad 2bc=0$

$$F_0=\min_{p_1\leqslant p\leqslant p_2} a^2+b^2+p(-a^2+c^2+2ab),$$

if $\quad 2bc<0$

$$F_0=\min_{p_1\leqslant p\leqslant p_2} a^2+b^2+p(-a^2+c^2+2ab+2bc\eta_2).$$

Furthermore, by discussing the sign of the coefficient of p, each of the three parameter subspaces could be decomposed into three smaller subspaces. Thus the total parameter space has been decomposed into nine subspaces (i,j) with $i,j=1,2,3$. The corresponding expressions of F_0^{ij} in the nine subspaces are listed in Table I, where

$$g_1=a^2+2b^2+c^2-1,\quad g_2=2bc,$$
$$g_3=-a^2+c^2+2ab+2bc\eta_1,\quad g_4=-a^2+c^2+2ab,$$
$$g_5=-a^2+c^2+2ab+2bc\eta_2. \tag{15}$$

B. Optimization for F_0^{ij} by using the Kuhn-Tucker method

In this section, we solve Eq. (13) in each of the nine parameter subspaces, respectively. For subspace (i,j) with $i,j=1,2,3$, the objective function is expression of F_0^{ij}, and the constraint conditions are corresponding subspace conditions. Moreover, we could solve them by using the Kuhn-Tucker method. The explicit optimization procedure in subspace (1,1) is elaborated in Appendix B. Similarly to subspace (1,1), the results for the other eight subspaces can be obtained in the same way. For convenience, expressions of $\hat{F}_{ij}$ $(i,j=1,2,3)$ are displayed in Table II.

TABLE I. Nine subspaces (i,j) and corresponding expressions of $F_0^{i,j}$.

Subspaces	Subspace conditions	Expressions of $F_0^{i,j}$
(1,1)	$g_2>0,\ g_3>0,\ g_1=0$	$F_0^{11}=a^2+b^2+p_1g_3$
(1,2)	$g_2>0,\ g_3=0,\ g_1=0$	$F_0^{12}=a^2+b^2$
(1,3)	$g_2>0,\ g_3<0,\ g_1=0$	$F_0^{13}=a^2+b^2+p_2g_3$
(2,1)	$g_2=0,\ g_4>0,\ g_1=0$	$F_0^{21}=a^2+b^2+p_1g_4$
(2,2)	$g_2=0,\ g_4=0,\ g_1=0$	$F_0^{22}=a^2+b^2$
(2,3)	$g_2=0,\ g_4<0,\ g_1=0$	$F_0^{23}=a^2+b^2+p_2g_4$
(3,1)	$g_2<0,\ g_5>0,\ g_1=0$	$F_0^{31}=a^2+b^2+p_1g_5$
(3,2)	$g_2<0,\ g_5=0,\ g_1=0$	$F_0^{32}=a^2+b^2$
(3,3)	$g_2<0,\ g_5<0,\ g_1=0$	$F_0^{33}=a^2+b^2+p_2g_5$

Parameters μ' and μ in Table II are denoted in the Appendix, and $\varepsilon(x)$ is a unit step function. For more detailed discussions, we refer to Appendix B, in which the corresponding parameters (a,b,c) and $\hat{F}_{ij}$ of the cloning machine in subspace (i,j) are presented in detail.

C. The ultimate solution

Following the procedure in Secs. III A and III B, the maximin problem has been solved in nine subspaces, and we get the expression $\hat{F}_{ij}$ in subspace (i,j). Once the amplitude and phase information of the input set is given by $(\alpha_1,\alpha_2;\phi_1,\phi_2)$, the exact values of $\hat{F}_{ij}$ could be derived, and we can always choose the biggest one, $\hat{F}=\max F_{ij}$, from the set $\{\hat{F}_{ij}|i,j=1,2,3\}$. We should emphasize that the biggest fidelity $\hat{F}=\max\hat{F}_{ij}$ corresponds to the optimal parameters (a,b,c); then we obtain the concrete form of the SMQCM.

IV. COMPARISONS AND DISCUSSIONS

To illustrate explicitly the SMQCM in this paper, we present some numerical examples to demonstrate that the SMQCM is better than or the same as the UQCM (and the PCQCM) in terms of fidelity, and it outperforms the BQCM for some cases.

A. Comparisons between the SMQCM and the UQCM

Suppose that we know nothing about the input states on the Bloch sphere, namely, $\alpha_1=0$, $\alpha_2=1$, $\phi_1=-\pi$, and $\phi_2=\pi$; one can make use of the UQCM to perform a

TABLE II. Expressions of $\hat{F}_{ij}$ in nine subspaces (i,j).

Subspaces	Expressions of $\hat{F}_{i,j}$
(1,1)	$\hat{F}_{11}=-\mu'\varepsilon(g_2)\varepsilon(g_3)$
(1,2)	$\hat{F}_{12}=(a^2+b^2)\varepsilon(g_2)$
(1,3)	$\hat{F}_{13}=-\mu'\varepsilon(g_2)\varepsilon(-g_3)$
(2,1)	$\hat{F}_{21}=-\mu\varepsilon(g_4)$
(2,2)	$\hat{F}_{22}=\frac{5}{6}$
(2,3)	$\hat{F}_{23}=-\mu\varepsilon(-g_4)$
(3,1)	$\hat{F}_{31}=-\mu'\varepsilon(-g_2)\varepsilon(g_5)$
(3,2)	$\hat{F}_{32}=(a^2+b^2)\varepsilon(-g_2)$
(3,3)	$\hat{F}_{33}=-\mu'\varepsilon(-g_2)\varepsilon(-g_5)$

TABLE III. Examples for comparisons between the SMQCM and the UQCM.

$S(\alpha_1,\alpha_2;\phi_1,\phi_2)$	$(p_1,p_2;\eta_1,\eta_2)$	$\hat{F}$
$S(0,1;-\pi,\pi)$	$(0,\frac{1}{2};-1,1)$	$\hat{F}_{22}=\frac{5}{6}$
$S((\frac{2+\sqrt{3}}{4})^{\frac{1}{2}},1;-\frac{\pi}{4},\frac{\pi}{4})$	$(0,\frac{1}{8};0,1)$	$\hat{F}_{23}=\frac{11+\sqrt{11}}{16}$

cloning task with a fidelity of $\frac{5}{6}$. Moreover, we can arrive at $\hat{F}=\hat{F}_{22}=\frac{5}{6}$ by our SMQCM, for which the parameters (a,b,c) are given in Eq. (B11). Hence, the SMQCM is the same as the UQCM when we do not know *a priori* information about the amplitude and phase of the input set completely. On the other hand, if the input set is smaller than the Bloch sphere, for example, $\alpha_1=(\frac{2+\sqrt{3}}{4})^{\frac{1}{2}}$, $\alpha_2=1$, $\phi_1=-\frac{\pi}{2}$, and $\phi_2=\frac{\pi}{2}$, we would find that the fidelity $\hat{F}$ of our machine is equal to $\hat{F}_{23}=\frac{11+\sqrt{11}}{16}\approx 0.895>\frac{5}{6}=0.8333$. Meanwhile, the corresponding parameters can be presented as $b=\sqrt{\frac{1}{22+6\sqrt{11}}}$, $a=(3+\sqrt{11})\sqrt{\frac{1}{22+6\sqrt{11}}}$, and $c=0$. From this case, we can see that our SMQCM is better than the UQCM [5] (see Table III).

B. Comparisons between the SMQCM and the PCQCM

In Table IV, $t_1^1\leqslant t_2^1<(\frac{2-\sqrt{2}}{4})^{\frac{1}{2}}$, $t_1^2\leqslant(\frac{2-\sqrt{2}}{4})^{\frac{1}{2}}\leqslant t_2^2$, $(\frac{2-\sqrt{2}}{4})^{\frac{1}{2}}<t_1^3\leqslant t_2^3$, and $r_1^i=\min_{t_1^i\leqslant t\leqslant t_2^i}2t^2(1-t^2)$, $r_2^i=\max_{t_1^i\leqslant t\leqslant t_2^i}2t^2(1-t^2)$, with $i=1,2,3$.

For convenience we choose the "equator" in the x-z plane instead of the x-y equator [13]. If the input set is the prime meridian of the Bloch sphere, namely, $\alpha_1=0$, $\alpha_2=1$, and $\phi_1=\phi_2=0$, the fidelity of the PCQCM equals $\frac{1}{2}+\sqrt{\frac{1}{8}}$. By using our method, we can get the concrete parameters $a=\frac{1}{2}+\sqrt{\frac{1}{8}}$, $b=\sqrt{\frac{1}{8}}$, and $c=\frac{1}{2}-\sqrt{\frac{1}{8}}$ for the SMQCM, and $\hat{F}=\hat{F}_{12}=\frac{1}{2}+\sqrt{\frac{1}{8}}$. Thus, the SMQCM is the same as the PCQCM [13] in this case.

When the input set lies in the prime meridian of the Bloch sphere, and its amplitude is restricted by $0<\alpha_1\leqslant\alpha_2<1$, it could be discussed in three cases:

(1) If $0<\alpha_1\leqslant\alpha_2<(\frac{2-\sqrt{2}}{4})^{\frac{1}{2}}$, one can find that $p_2\in(0,\frac{1}{4})$, and $\hat{F}$ is equal to $\hat{F}_{13}=\frac{1+\sqrt{1-4p_2+8(p_2)^2}}{2}$, which is bigger than the fidelity of the PCQCM. Meanwhile, the parameters of the SMQCM can be presented as

$$b^2=\frac{(p_2+\mu')^2(p_2-1-\mu')^2}{p_2^2(p_2+\mu')^2+[2(p_2+\mu')^2+(p_2\eta_1)^2](p_2-1-\mu')^2},$$
$$a=\frac{p_2}{p_2-1-\mu'}b,$$
$$c=-\frac{p_2\eta_1}{p_2+\mu'}b, \tag{16}$$

where $\mu'=-\frac{1+\sqrt{1-4p_2+8(p_2)^2}}{2}$.

(2) When $0\leqslant\alpha_1\leqslant(\frac{2+\sqrt{2}}{4})^{\frac{1}{2}}\leqslant\alpha_2$, $\hat{F}=\hat{F}_{12}$ and is equal to $\frac{1}{2}+\sqrt{\frac{1}{8}}$, and the SMQCM reduces to the PCQCM [13].

(3) From Table IV, we can obtain that $\hat{F}=\hat{F}_{11}=\frac{1+\sqrt{1-4p_1+8(p_1)^2}}{2}$ and $p_1>\frac{1}{4}$. Thus, $\hat{F}$ is always bigger than $\frac{1}{2}+\sqrt{\frac{1}{8}}$. By replacing p_2 with p_1 in Eq. (16), we obtain the (a,b,c) of the SMQCM.

According to the aforementioned discussions, we would like to point out that the SMQCM is better than or the same as the PCQCM.

TABLE IV. Examples for comparisons between the SMQCM and the PCQCM.

$S(\alpha_1,\alpha_2;\phi_1,\phi_2)$	$(p_1,p_2;\eta_1,\eta_2)$	$\hat{F}$
$S(0,1;0,0)$	$(0,\frac{1}{2};1,1)$	$\hat{F}_{12}=\frac{1}{2}+\sqrt{\frac{1}{8}}$
$S(t_1^1,t_2^1;0,0)$	$(r_1^1,r_2^1;1,1)$	$\hat{F}_{13}=\frac{1+\sqrt{1-4r_2^1+8(r_2^1)^2}}{2}$
$S(t_1^2,t_2^2;0,0)$	$(r_1^2,r_2^2;1,1)$	$\hat{F}_{12}=\frac{1}{2}+\sqrt{\frac{1}{8}}$
$S(t_1^3,t_2^3;0,0)$	$(r_1^3,r_2^3;1,1)$	$\hat{F}_{11}=\frac{1+\sqrt{1-4r_1^3+8(r_1^3)^2}}{2}$

TABLE V. Example for comparisons between the SMQCM and the BQCM.

$S(\alpha_1,\alpha_2;\phi_1,\phi_2)$	$(p_1,p_2;\eta_1,\eta_2)$	$\hat{F}$
$S((\frac{2}{5})^{\frac{1}{2}},(\frac{3}{5})^{\frac{1}{2}};-\frac{\pi}{3},\frac{\pi}{3})$	$(\frac{12}{25},\frac{1}{2};\frac{1}{2},1)$	$\hat{F}_{11}\approx 0.8858$

C. Comparisons between the SMQCM and the BQCM

The BQCM takes full advantage of the amplitude information of the input set and can get a higher fidelity than the UQCM, but it could not utilize the phase information. In our scheme, both the amplitude and the phase information can be used to design the cloning machine; thus we can get a better cloning machine than the BQCM. For instance, if the input states lie in $S((\frac{2}{5})^{\frac{1}{2}},(\frac{3}{5})^{\frac{1}{2}};-\frac{\pi}{3},\frac{\pi}{3})$, one can find that $\hat{F}=\hat{F}_{11}=0.8858$, and $a\approx 0.6503$, $b\approx 0.4956$, and $c\approx 0.2931$. Meanwhile, we should point out that the mean fidelity of the BQCM for this block area is smaller than $\frac{1}{2}+\sqrt{\frac{1}{8}}\approx 0.8535$ [20]. Hence, the SMQCM can outperform the BQCM for some cases (see Table V).

V. CONCLUSIONS

In summary, we present an optimal $1\to 2$ quantum cloning method for states on the Bloch sphere based on the maximin principle. This method can take full advantage of *a priori* information about the amplitude and phase of the input set. To design the parameters of the optimal maximin quantum cloning machine, we obtain the optimal quantum cloning machine for input states in the Bloch spherical cap, and the fidelity $\hat{F}$ is bigger than $\frac{1}{2}+\sqrt{\frac{1}{8}}$. Moreover, by restricting the structure of the OMQCM, we obtain the concrete form of the sub-optimal maximin quantum cloning machine. The theoretical analysis and numerical examples explicitly demonstrate that the cloning machine derived from our proposal can be used to clone quantum qubits with higher fidelity. In contrast to the preceding schemes [5,13,20], our proposal has the following advantages. First, the cloned qubit input set in our method is $S(\alpha_1,\alpha_2;\phi_1,\phi_2)$, given in terms of four parameters, and the input sets exploited in the UQCM, PCQCM, and BQCM can be just considered special cases of the four-parameter input set. Thus, our scheme has wider application than others. Second, the proposal makes full use of *a priori* information about the amplitude and phase of the input set with the aid of the maximin principle, while the PCQCM and the BQCM only use the amplitude information of input. In theory, the cloning quality of the OMQCM is better than or the same as that of the PCQCM and the BQCM for an arbitrary input set S shown in Eq. (1). Third, it is exemplified that the SMQCM could outperform the UQCM, PCQCM, and BQCM in terms of fidelity, even through the SMQCM is a suboptimal cloning machine of our optimal method. In fact, the UQCM and the PCQCM can be regarded as special cases of the SMQCM; when the input set is smaller, the cloning fidelity of our machine is higher than $\frac{5}{6}$ or $\frac{1}{2}+\sqrt{\frac{1}{8}}$. The SMQCM also outperforms the BQCM in some cases. It will be significant to extend our proposal to $N \to M$ quantum cloning in higher-level systems or to mixed-state quantum cloning.

ACKNOWLEDGMENTS

This work is supported by the Program for National Science Foundation of P. R. China (Grants No. 61273202, No. 61134008, and No. 61673389).

APPENDIX A: OPTIMAL QUANTUM CLONING MACHINE FOR STATES IN A SPHERICAL CAP

To make it easy, we first consider the input set $S(\alpha_1,1;-\pi,\pi)$ with $\frac{1}{\sqrt{2}}<\alpha_1<1$, and denote $\hat{F}_0(T_U)=\min_{|\psi\rangle\in S} F$ with $T_U(|\psi\rangle\langle\psi|)=\mathrm{Tr}_{2,x}(U|\psi 0X\rangle\langle\psi 0X|U^\dagger)$. The optimal quantum cloning translation is $\hat{U}$ which satisfies $\hat{F}_0(T_{\hat{U}})=\max_U \hat{F}_0(T_U)=\hat{F}$. The average translation of $T_{\hat{U}}$ is $\bar{T}_{\hat{U}}$;

$$\bar{T}_{\hat{U}}(|\psi\rangle\langle\psi|)=\frac{1}{2\pi}\int_{-\pi}^{\pi}U_\phi^\dagger T_{\hat{U}}(U_\phi|\psi\rangle\langle\psi|U_\phi^\dagger)U_\phi\,d\phi$$

for any pure state $|\psi\rangle$ and all unitary phase-shift operators $U_\phi=\exp[-i/2(\sigma_z-1)\phi]$, where $\phi\in[-\pi,\pi]$ and σ_z is the Pauli operator diag$\{1,-1\}$. Then we prove that $\hat{F}_0(\bar{T}_{\hat{U}})=\hat{F}_0(T_{\hat{U}})$; namely, the fidelity of the optimal quantum cloning machine for the input set $S(\alpha_1,1;-\pi,\pi)$ with $\frac{1}{\sqrt{2}}<\alpha_1<1$ is independent of the phase of states $|\psi\rangle\in S$.

For any pure state $|\psi\rangle$, we have

$$\begin{aligned}\mathrm{Tr}[|\psi\rangle\langle\psi|\bar{T}_{\hat{U}}(|\psi\rangle\langle\psi|)]&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}[\sigma_{U_\phi}T_{\hat{U}}(\sigma_{U_\phi})]\,d\phi\\&\geqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}F_0(T_{\hat{U}})=F_0(T_{\hat{U}}),\end{aligned}$$

where $\sigma_{U_\phi}=U_\phi|\psi\rangle\langle\psi|U_\phi^\dagger$, hence $F_0(\bar{T}_{\hat{U}})\geqslant F_0(T_{\hat{U}})$. By definition of $F_0(T_{\hat{U}})$ we also have $F_0(T_{\hat{U}})\geqslant F_0(\bar{T}_{\hat{U}})$, i.e., $F_0(T_{\hat{U}})=F_0(\bar{T}_{\hat{U}})$.

This means the fidelity of the optimal quantum cloning machine is the same for any state in the set $S(\alpha_1,1;-\pi,\pi)$. By using Kraus decompositions, we find the upper merit $T_{\hat{U}}$ should have the following form:

$$\begin{pmatrix}(1-\tau_1)|\beta|^2+\tau_1|\alpha|^2 & \tau_2\alpha\beta^*\\ \tau_2^*\alpha^*\beta & (1-\tau_1)|\alpha|^2+\tau_1|\beta|^2\end{pmatrix},\tag{A1}$$

where τ_i is complex for $i=1,2$. The proof is similar to one in the appendix of Ref. [13], and we do not explore it in this paper. The fidelity of the OMQCM will be

$$\begin{aligned}F=&|\alpha|^4(|a_0|^2+|b_0|^2)+|\alpha|^2|\beta|^2(1-|a_1|^2-|b_1|^2)\\&+|\beta|^4(|a_1|^2+|b_1|^2)+|\alpha|^2|\beta|^2(1-|a_0|^2-|b_0|^2)\\&+|\alpha|^2|\beta|^2(a_0b_1^*\langle B_1|A_0\rangle+b_0a_1^*\langle A_1|B_0\rangle)\\&+|\alpha|^2|\beta|^2(a_0^*b_1\langle A_0|B_1\rangle+b_0^*a_1\langle B_0|A_1\rangle).\end{aligned}\tag{A2}$$

Comparing ρ_1 and the form of $T_{\hat{U}}$ in Eq. (A1), we arrive at

$$\begin{aligned}&a_0c_1^*\langle C_1|A_0\rangle+b_0b_1^*\langle B_1|B_0\rangle=0,\\&c_0a_1^*\langle A_1|C_0\rangle+b_0b_1^*\langle B_1|B_0\rangle=0,\\&a_0b_0^*\langle B_0|A_0\rangle+b_0c_0^*\langle C_0|B_0\rangle=0,\\&a_1b_1^*\langle B_1|A_1\rangle+b_1c_1^*\langle C_1|B_1\rangle=0,\\&b_1c_0^*\langle C_0|B_1\rangle+c_1b_0^*\langle B_0|C_1\rangle=0.\end{aligned}\tag{A3}$$

If we want to maximize F, we must set $\tilde{a}\bar{b}^*\langle\bar{B}|\tilde{A}\rangle+\tilde{b}\bar{a}^*\langle\bar{A}|\tilde{B}\rangle=\bar{a}\tilde{b}^*\langle\tilde{B}|\bar{A}\rangle+\bar{b}\tilde{a}^*\langle\tilde{A}|\bar{B}\rangle=|a||\tilde{b}|+|b||\tilde{a}|$, and we have

$$\begin{aligned}F=&|\alpha|^4(|a_0|^2+|b_0|^2)+|\alpha|^2|\beta|^2(1-|a_1|^2-|b_1|^2)\\&+|\beta|^4(|a_1|^2+|b_1|^2)+|\alpha|^2|\beta|^2(1-|a_0|^2-|b_0|^2)\\&+2|\alpha|^2|\beta|^2(|a_0||b_1|+|b_0||a_1|).\end{aligned}\tag{A4}$$

Combining Eqs. (5) and (A3), constraints are displayed as follows:

$$\begin{aligned}h_1&=|a_0|^2+2|b_0|^2+|c_0|^2-1=0,\\h_2&=|a_1|^2+2|b_1|^2+|c_1|^2-1=0,\\h_3&=a_0c_1^*\langle C_1|A_0\rangle+b_0b_1^*\langle B_1|B_0\rangle=0,\\h_4&=c_0a_1^*\langle A_1|C_0\rangle+b_0b_1^*\langle B_1|B_0\rangle=0,\\h_5&=a_0b_0^*\langle B_0|A_0\rangle+b_0c_0^*\langle C_0|B_0\rangle=0,\\h_6&=a_1b_1^*\langle B_1|A_1\rangle+b_1c_1^*\langle C_1|B_1\rangle=0,\\h_7&=b_1c_0^*\langle C_0|B_1\rangle+c_1b_0^*\langle B_0|C_1\rangle=0.\end{aligned}\tag{A5}$$

The independent variables are the absolute values of the coefficients $a_i,b_i,\ldots$, their phases, the absolute values of the scalar products of the ancilla states, and their phases, which we denote

$$\begin{aligned}a_0&=|a_0|e^{i\delta_{a_0}},\\\langle A_0|B_1\rangle&=|\langle A_0|B_1\rangle|e^{i\delta_{A_0B_1}}.\end{aligned}$$

Using the method of Lagrange multipliers, we derive the optimal cloning machine; namely, we should solve the system of equations

$$\frac{\partial F}{\partial |a_0|} + \sum_{i=1}^{7} \lambda_i \frac{\partial h_1}{\partial |a_0|} = 0,$$
$$\frac{\partial F}{\partial |b_0|} + \sum_{i=1}^{7} \lambda_i \frac{\partial h_1}{\partial |b_0|} = 0, \tag{A6}$$
$$\vdots$$
$$h_i = 0, \qquad i = 1,\dots,8,$$

where h_i denotes constraints, and the Lagrange multipliers are λ_i.

Taking the partial derivative with respect to $|c_0|$, one can obtain

$$2\lambda_1|c_0| + \lambda_4 a_1^* e^{i\delta_{c_0}}\langle A_1|C_0\rangle + \lambda_5 b_0 e^{-i\delta_{c_0}}\langle C_0|B_0\rangle + \lambda_7 b_1 e^{-i\delta_{c_0}}\langle C_0|B_1\rangle = 0. \tag{A7}$$

From the derivatives with respect to $|\langle A_1|C_0\rangle|$, $|\langle C_0|B_0\rangle|$, and $|\langle C_0|B_1\rangle|$, we arrive at

$$\lambda_4|a_1||c_0| = 0,$$
$$\lambda_5|b_0||c_0| = 0,$$
$$\lambda_7|b_1||c_0| = 0. \tag{A8}$$

After multiplying Eq. (A7) by $|c_0|$, we find

$$\lambda_1|c_0|^2 = 0. \tag{A9}$$

In the same way, we obtain

$$\lambda_2|c_1|^2 = 0. \tag{A10}$$

First of all, we assume $\lambda_1\lambda_2 \neq 0$, so $|c_0| = |c_1| = 0$. Considering h_3 in Eq. (A5), we get $b_0 b_1^*\langle B_1|B_0\rangle = 0$, namely, $|b_0| = 0$, $|b_1| = 0$ or $\langle B_1|B_0\rangle = 0$.

If $|b_0| = 0$, we have $|a_0| = 1$ and

$$F = |\alpha|^4 + |\beta|^4 + |\beta|^2[(|\alpha|^2 - |\beta|^2)|b_1|^2 + 2|\alpha|^2|b_1|].$$

Since $|\alpha|^2 \geqslant |\beta|^2$ and $|b_1| \leqslant \frac{1}{\sqrt{2}}$, we find $|b_1| = \frac{1}{\sqrt{2}}$ corresponds a maximum of F:

$$F = 1 - (\sqrt{2}-1)|\beta|^4 - \left(\frac{3}{2} - \sqrt{2}\right)|\beta|^2. \tag{A11}$$

If $|b_1| = 0$, we have $|a_1| = 1$ and

$$F = |\alpha|^4 + |\beta|^4 + |\alpha|^2[(|\beta|^2 - |\alpha|^2)|b_0|^2 + 2|\alpha|^2|b_0|].$$

We find that if $\frac{|\beta|^2}{|\alpha|^2-|\beta|^2} \geqslant \frac{1}{\sqrt{2}}$, then $|b_0| = \frac{1}{\sqrt{2}}$ corresponds a maximum $F = 1 - (\sqrt{2}-1)|\alpha|^4 - (\frac{3}{2} - \sqrt{2})|\alpha|^2 \leqslant \frac{1}{2} + \frac{\sqrt{2}}{4}$, whereas if $\frac{|\beta|^2}{|\alpha|^2-|\beta|^2} < \frac{1}{\sqrt{2}}$, then $|b_0| = \frac{|\beta|^2}{|\alpha|^2-|\beta|^2}$ corresponds to a maximum $F = 1 + |\alpha|^2|\beta|^2(\frac{|\beta|^2}{|\alpha|^2-|\beta|^2} - 2) < 1 - (\sqrt{2} - 1)|\beta|^4 - (\frac{3}{2} - \sqrt{2})|\beta|^2$.

If $|b_0| = |b_1| = 0$ and $\langle B_1|B_0\rangle = 0$, we obtain $F < 1 - (\sqrt{2}-1)|\beta|^4 - (\frac{3}{2} - \sqrt{2})|\beta|^2$.

Similarly, when $\lambda_1 = \lambda_2 = 0$, we derive a contradiction. When $\lambda_1 = |c_1| = 0$ ($\lambda_2 = |c_0| = 0$) and $\lambda_2|c_0| \neq 0$ ($\lambda_1|c_1| \neq 0$), we find $F < 1 - (\sqrt{2}-1)|\beta|^4 - (\frac{3}{2} - \sqrt{2})|\beta|^2$.

Now we can conclude that for the input set $S(\alpha,\alpha;-\pi,\pi)$ with $\alpha > \frac{1}{\sqrt{2}}$, the optimal quantum cloning machine is

$$U|00\rangle = |00\rangle,$$
$$U|10\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{A12}$$

The fidelity $\hat{F}$ is

$$\hat{F} = 1 - (\sqrt{2}-1)|\beta|^4 - \left(\frac{3}{2} - \sqrt{2}\right)|\beta|^2.$$

It is easy to find that when the input set is $S(\alpha_1,1;-\pi,\pi)$ with $\frac{1}{\sqrt{2}} < \alpha_1 < 1$, the optimal quantum cloning machine is shown as Eq. (A12), and when the input set is $S(0,\alpha_2;-\pi,\pi)$ with $0 < \alpha_2 < \frac{1}{\sqrt{2}}$, the optimal quantum cloning machine is

$$U|00\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$U|10\rangle = |11\rangle. \tag{A13}$$

APPENDIX B: OPTIMIZATION FOR THE SMQCM IN NINE SUBSPACES

Equation (13) in subspace (1,1) can be expressed as

$$\hat{F}_{11} = \max_{a,b,c} a^2 + b^2 + p_1(-a^2 + c^2 + 2ab + 2bc\eta_1)$$
$$\text{such that} \quad g_1 = a^2 + 2b^2 + c^2 - 1 = 0,$$
$$g_2 = 2bc > 0,$$
$$g_3 = -a^2 + c^2 + 2ab + 2bc\eta_1 > 0. \tag{B1}$$

By the Kuhn-Tucker method we can derive the parameters (a,b,c) of the cloning machine in subspace (1,1). First of all, in order to simplify the solving process, we ignore inequality constraints and solve the system of equations

$$\frac{\partial \hat{F}_{11}}{\partial a} + \mu_1 \frac{\partial g_1}{\partial a} = 0,$$
$$\frac{\partial \hat{F}_{11}}{\partial b} + \mu_1 \frac{\partial g_1}{\partial b} = 0,$$
$$\frac{\partial \hat{F}_{11}}{\partial c} + \mu_1 \frac{\partial g_1}{\partial c} = 0,$$

where μ_1 is the Lagrange multiplier. Moreover, the above equations can be presented as

$$2a - 2p_1 a + 2p_1 b + \mu_1 2a = 0,$$
$$2b + 2p_1 a + 2p_1 c\eta_1 + \mu_1 4b = 0,$$
$$2p_1 c + 2p_1 b\eta_1 + \mu_1 2c = 0. \tag{B2}$$

For Eq. (B1), if $p_1 = 1 + \mu_1$ or $p_1 = -\mu_1$, we would find that $\hat{F}_{11} \leqslant \frac{5}{6}$, and we can choose the UQCM to perform the quantum cloning task in this case. Thus, we just consider the situations $p_1 \neq 1 + \mu_1$ and $p_1 \neq -\mu_1$; meanwhile, one can obtain that

$$a = \frac{p_1}{p_1 - 1 - \mu_1} b,$$
$$c = -\frac{p_1 \eta_1}{p_1 + \mu_1} b. \tag{B3}$$

By substituting Eq. (B3) into g_1 in Eq. (B1), we can get

$$b^2 = \frac{(p_1+\mu_1)^2(p_1-1-\mu_1)^2}{(p_1)^2(p_1+\mu_1)^2 + [2(p_1+\mu_1)^2 + (p_1\eta_1)^2](p_1-1-\mu_1)^2}. \tag{B4}$$

With $p_1 \neq 1 + \mu_1$ and $p_1 \neq -\mu_1$, we derive $b \neq 0$. Moreover, by inserting Eq. (B3) into the second formula in Eq. (B2), we obtain

$$(1+2\mu_1)[(p_1-1-\mu_1)(p_1+\mu_1) + (p_1)^2] + \left(1-\eta_1^2\right)(p_1)^2(p_1-1-\mu_1) = 0. \tag{B5}$$

If we set $k_1 = \mu_1 + \frac{1}{2}$, Eq. (B5) would become

$$(k_1)^3 + \frac{-1+4p_1-8(p_1)^2-2\left(\eta_1^2-1\right)(p_1)^2}{4}k_1 + \frac{2\left(\eta_1^2-1\right)(p_1)^3 - \left(\eta_1^2-1\right)(p_1)^2}{4} = 0. \tag{B6}$$

This is a standard cubic equation, which could be solved by using a formula of finding roots on cubic equations. Subsequently, we could derive expressions of μ_1, b, a, and c.

Furthermore, if the inequality constraints $g_2 > 0$ and $g_3 > 0$ are satisfied, the upper discussion is valid. By inserting expressions of b, a, and c into Eq. (B1), we can find

$$\hat{F}_{11} = \max(-\mu_1) = \max\left(-k_1 + \tfrac{1}{2}\right).$$

If the inequality constraints $g_2 > 0$ and $g_3 > 0$ are not satisfied, the upper discussion is invalid, and we denote

$$\hat{F}_{11} = 0,$$

namely,

$$\hat{F}_{11} = -{\mu_1}'\varepsilon(g_2)\varepsilon(g_3), \tag{B7}$$

where $-{\mu_1}' = \max(-\mu_1) = \frac{1}{2} + \min_{\{i|k_1^i=(k_1^i)^*\}} k_1^i$ $(i = 1,2,3)$, k_1^i is the root of the cubic equation in Eq. (B6), and $\varepsilon(x)$ is unit step function. According to the aforementioned analysis, this process is displayed as follows:

$$\begin{aligned}
&(k_1)^3 + \frac{-1+4p_1-8(p_1)^2-2\left(\eta_1^2-1\right)p_1^2}{4}k_1 + \frac{2\left(\eta_1^2-1\right)(p_1)^3 - \left(\eta_1^2-1\right)(p_1)^2}{4} = 0\\
&k_1^{'} = \min_{\{i|k_1^i=(k_1^i)^*\}} k_1^i \quad (i = 1,2,3)\\
&\mu_1^{'} = k_1^{'} - \tfrac{1}{2}\\
&b^2 = \frac{(p_1+\mu_1^{'})^2(p_1-1-\mu_1^{'})^2}{(p_1)^2(p_1+\mu_1^{'})^2 + [2(p_1+\mu_1^{'})^2 + (p_1\eta_1)^2](p_1-1-\mu_1^{'})^2}\\
&a = \frac{p_1}{p_1-1-\mu_1^{'}}b\\
&c = -\frac{p_1\eta_1}{p_1+\mu_1^{'}}b\\
&\hat{F}_{11} = -{\mu_1}'\varepsilon(g_2)\varepsilon(g_3),
\end{aligned} \tag{B8}$$

where k_1^i $(i = 1,2,3)$ are roots of the cubic equation in Eq. (B6). Solving these equations in Eq. (B8) successively, we obtain the solution for the optimization problem in subspace $(1,1)$. It should be noted that the parameters (a,b,c) correspond to the quantum cloning machine in subspace $(1,1)$.

The optimization process in the other eight subspaces is similar to the case in subspace $(1,1)$. According to the quantity and form of equality constraints in subspaces (i,j) shown in Table I, the discussions in nine subspaces could be summed up in four situations $\{(1,1),(1,3),(3,1),(3,3)\}$, $\{(2,1),(2,3)\}$,

$\{(2,2)\}$, and $\{(1,2),(3,2)\}$, and we can display the summarized results as follows.

(1) Results in subspaces (i,j) $(i=1,3;\ j=1,3)$:

$$\begin{aligned} &k^3+\frac{-1+4p-8p^2-2(\eta^2-1)p^2}{4}k \\ &\quad+\frac{2(\eta^2-1)p^3-(\eta^2-1)p^2}{4}=0 \\ &k'=\min_{\{i|k^i=(k^i)^*\}}k^i \\ &\mu'=k'-\tfrac{1}{2} \\ &b^2=\frac{(p+\mu')^2(p-1-\mu')^2}{p^2(p+\mu')^2+[2(p+\mu')^2+p^2\eta^2](p-1-\mu')^2} \\ &a=\frac{p}{p-1-\mu'}b \\ &c=-\frac{p\eta}{p+\mu'}b, \end{aligned} \tag{B9}$$

where k^i $(i=1,2,3)$ are roots of the cubic equation in the first formula of Eq. (B9), and k' is the minimal real root.

(2) Results in subspaces (i,j) $(i=2;\ j=1,3)$:

$$\begin{aligned} &\mu=\frac{2p-3-\sqrt{12p^2-4p+1}}{4} \\ &b^2=\frac{p^2}{(1+2\mu)^2+2p^2} \\ &a=\frac{-(1+2\mu)}{p}b \\ &c=0. \end{aligned} \tag{B10}$$

We need to underline the fact that the fidelity of the quantum cloning machine in this form is independent of the value of the phase ϕ.

(3) Results in subspaces (i,j) $(i=2;\ j=2)$:

$$\begin{aligned} &a=\sqrt{\frac{2}{3}} \\ &b=\sqrt{\frac{1}{6}} \\ &c=0. \end{aligned} \tag{B11}$$

In this case, corresponding quantum cloning machine is same as the UQCM.

TABLE VI. Results in nine subspaces (i,j), where "\" means the result has nothing to do with the corresponding values of p or η, and g_i $(i=1,2,3,4,5)$ are given in Eq. (15).

Subspaces	(p,η)	Eq. (Bn)	Expressions of $\hat{F}(ij)$
(1,1)	(p_1,η_1)	(B9)	$\hat{F}_{11}=-\mu'\varepsilon(g_2)\varepsilon(g_3)$
(1,2)	$(\backslash,\eta_1)$	(B12)	$\hat{F}_{12}=(a^2+b^2)\varepsilon(g_2)$
(1,3)	(p_2,η_1)	(B9)	$\hat{F}_{13}=-\mu'\varepsilon(g_2)\varepsilon(-g_3)$
(2,1)	$(p_1,\backslash)$	(B10)	$\hat{F}_{21}=-\mu\varepsilon(g_4)$
(2,2)	$(\backslash,\backslash)$	(B9)	$\hat{F}_{22}=5/6$
(2,3)	$(p_2,\backslash)$	(B11)	$\hat{F}_{23}=-\mu\varepsilon(-g_4)$
(3,1)	(p_1,η_2)	(B9)	$\hat{F}_{31}=-\mu'\varepsilon(-g_2)\varepsilon(g_5)$
(3,2)	$(\backslash,\eta_2)$	(B12)	$\hat{F}_{32}=(a^2+b^2)\varepsilon(-g_2)$
(3,3)	(p_2,η_2)	(B9)	$\hat{F}_{33}=-\mu'\varepsilon(-g_2)\varepsilon(-g_5)$

(4) Results in subspaces (i,j) $(i=1,3;\ j=2)$. For cases in subspaces (1,2) and (3,2), we set $a=\sin(\varphi_1)\sin(\theta_1)$, $b=\frac{\cos(\varphi_1)}{\sqrt{2}}$, and $c=\sin(\varphi_1)\cos(\theta_1)$, with $-\pi\leqslant\varphi_1,\theta_1\leqslant\pi$, and get

$$\begin{aligned} &\frac{\cos^2(2\theta_1)}{\cos^2(2\theta_1)-2[\sin(\theta_1)+\cos(\theta_1)\eta]^2} \\ &\quad=-\frac{\sin(2\theta_1)[\sin(\theta_1)+\cos(\theta_1)\eta]}{\cos(2\theta_1)[\cos(\theta_1)-\sin(\theta_1)\eta]} \\ &\tan(\varphi_1)=\frac{-\sqrt{2}[\sin(\theta_1)+\cos(\theta_1)\eta]}{\cos(2\theta_1)} \\ &a=\sin(\varphi_1)\sin(\theta_1) \\ &b=\frac{\cos(\varphi_1)}{\sqrt{2}} \\ &c=\sin(\varphi_1)\cos(\theta_1). \end{aligned} \tag{B12}$$

Once $(\alpha_1,\alpha_2;\phi_1,\phi_2)$ of the input set is given, values of $(p_1,p_2;\eta_1,\eta_2)$ could be figured out from Eq. (14). For subspaces (i,j), we substitute values of p or η into corresponding the Eq. (Bn) presented in Table VI, with $n=9,10,11,12$. Parameters (a,b,c) of the cloning machine in subspaces (i,j) are derived by solving the corresponding Eq. (Bn) successively, and $\hat{F}_{ij}$ are also shown in Table VI. So far, we finish the optimization process and obtain optimal quantum cloning machines in nine subspaces (i,j).

[1] W. K. Wooters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[2] D. Dieks, Phys. Lett. A **92**, 271 (1982).
[3] H. Fan, Y.-N. Wang, L. Jing, J.-D. Yue, H.-D. Shi, Y.-L. Zhang, and L.-Z. Mu, Phys. Rep. **544**, 241 (2014).
[4] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[5] D. Bruß, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, Phys. Rev. A **57**, 2368 (1998).
[6] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[7] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
[8] M. Keyl and R. F. Werner, J. Math. Phys. **40**, 3283 (1999).
[9] H. Fan, K. Matsumoto, and M. Wadati, Phys. Rev. A **64**, 064301 (2001).
[10] Y. N. Wang, H. D. Shi, Z. X. Xiong, L. Jing, X. J. Ren, L. Z. Mu, and H. Fan, Phys. Rev. A **84**, 034302 (2011).
[11] A. Kay, Quantum Inf. Comput. **16**, 991 (2016).
[12] M. Studziński, P. Ćwikliński, M. Horodecki, and M. Mozrzymas, Phys. Rev. A **89**, 052322 (2014).
[13] D. Bruß, M. Cinchetti, G. M. D'Ariano, and C. Macchiavello, Phys. Rev. A **62**, 012302 (2000).

[14] C. S. Niu and R. B. Griffiths, Phys. Rev. A **60**, 2764 (1999).
[15] G. M. D'Ariano and C. Macchiavello, Phys. Rev. A **67**, 042306 (2003).
[16] F. Buscemi, G. M. D'Ariano, and C. Macchiavello, Phys. Rev. A **71**, 042327 (2005).
[17] T. Durt and B. Nagler, Phys. Rev. A **68**, 042323 (2003).
[18] G. M. D'Ariano and P. Lo Presti, Phys. Rev. A **64**, 042308 (2001).
[19] H. Fan, K. Matsumoto, X. B. Wang, and M. Wadati, Phys. Rev. A **65**, 012304 (2001).
[20] J. Z. Hu, Z. W. Yu, and X. B. Wang, Eur. Phys. J. D **51**, 381 (2009).
[21] K. Bartkiewicz and A. Miranowicz, Phys. Rev. A **82**, 042330 (2010).
[22] L. M. Duan and G. C. Guo, Phys. Lett. A **243**, 261 (1998).
[23] A. K. Pati, Phys. Rev. Lett. **83**, 2849 (1999).
[24] K. Bartkiewicz, A. Miranowicz, and S. K. Özdemir, Phys. Rev. A **80**, 032306 (2009).
[25] W. H. Zhang, L. B. Yu, Z. L. Cao, and L. Ye, Phys. Rev. A **86**, 022322 (2012).
[26] L. Jing, Y. N. Wang, H. D. Shi, L. Z. Mu, and H. Fan, Phys. Rev. A **86**, 062315 (2012).
[27] H. T. Song, S. L. Luo, N. Li, and L. N. Chang, Phys. Rev. A **88**, 042121 (2013).
[28] P. Sekatski, M. Skotiniotis, and W. Dür, Phys. Rev. A **92**, 022355 (2015).
[29] S. Fasel, N. Gisin, G. Ribordy, V. Scarani, and H. Zbinden, Phys. Rev. Lett. **89**, 107901 (2002).
[30] H. Fan, G. Weihs, K. Matsumoto, and H. Imai, Phys. Rev. A **66**, 024307 (2002).
[31] J. Fiurášek, Phys. Rev. A **67**, 052314 (2003).
[32] F. Sciarrino and F. De Martini, Phys. Rev. A **72**, 062313 (2005).
[33] K. Lemr, K. Bartkiewicz, A. Cernoch, J. Soubusta, and A. Miranowicz, Phys. Rev. A **85**, 050307 (2012).
[34] E. F. Galvao and L. Hardy, Phys. Rev. A **62**, 022301 (2000).
[35] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Phys. Rev. Lett. **88**, 127902 (2002).
[36] A. Ferenczi and N. Lütkenhaus, Phys. Rev. A **85**, 052310 (2012).
[37] K. Bartkiewicz, K. Lemr, A. Černoch, J. Soubusta, and A. Miranowicz, Phys. Rev. Lett. **110**, 173601 (2013).
[38] N. Yamamoto, Phys. Rev. X **4**, 041029 (2014).
[39] K. Jacobs, X. Wang, and H. M. Wiseman, New J. Phys. **16**, 073036 (2014).
[40] R. Hamerly and H. Mabuchi, Phys. Rev. Lett. **109**, 173602 (2012).
[41] Y. Liu, S. Shankar, N. Ofek, M. Hatridge, A. Narla, K. M. Sliwa, L. Frunzio, R. J. Schoelkopf, and M. H. Devoret, Phys. Rev. X **6**, 011022 (2016).
[42] J. L. Ting, Superlattices Microstruct. **32**, 331 (2003).
[43] H. W. Kuhn and A. W. Tucker, *Proceedings of 2nd Berkeley Symposium* (University of California Press, Berkeley, 1951), pp. 481–492.