

Device-independent dimension tests in the prepare-and-measure scenarioJamie Sikora,^{1,2} Antonios Varvitsiotis,^{1,2,3} and Zhaohui Wei^{1,2,3,*}¹*Centre for Quantum Technologies, National University of Singapore, Singapore*²*MajuLab, CNRS-UNS-NUS-NTU International Joint Research Unit, UMI 3654, Singapore*³*School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore*

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Analyzing the dimension of an unknown quantum system in a device-independent manner, i.e., using only the measurement statistics, is a fundamental task in quantum physics and quantum information theory. In this paper, we consider this problem in the prepare-and-measure scenario. Specifically, we provide a lower bound on the dimension of the prepared quantum systems, which is a function that only depends on the measurement statistics. Furthermore, we show that our bound performs well in several examples. In particular, we show that our bound provides insights into the notion of dimension witness, and we also use it to show that the sets of restricted-dimensional prepare-and-measure correlations are not always convex.

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Introduction. In the device-independent paradigm, one tries to understand the properties of an unknown (classical or quantum) system based only on the correlations resulting from measurements performed on the system [1–14]. In this work, we consider the problem of lower bounding the dimension of a uncharacterized quantum system in a device-independent manner. This problem is quite interesting from the viewpoints of both physics and quantum computation, and has attracted much attention [1–10]. Indeed, the dimension of a quantum system is a fundamental physical property and is also widely regarded as a valuable computational resource, as one always tries to implement an algorithm or protocol with the smallest dimension possible.

This question was first considered in the Bell scenario [1–4] where a quantum system is shared by two parties, each performing local measurements on their own subsystems, illustrated in Fig. 1(a). The corresponding set of measurement statistics is called a *Bell correlation*. Then the task is, for a given Bell correlation, to lower bound the dimension of the underlying quantum system. In [1], the dimension witness approach was introduced to address this problem. A (*linear*) *d*-dimensional witness is defined as a hyperplane that contains all Bell correlations that can be generated using *d*-dimensional quantum systems in one of its half spaces. However, though providing very strong and intuitive physical insights, dimension witnesses can suffer from two apparent drawbacks. First, dimension witnesses do not always give a lower bound on the dimension of the underlying quantum system as a direct function of the correlation data. Second, identifying dimension witnesses amounts to characterizing the complicated structure of quantum correlations with restricted dimensions, which is often a very challenging task.

Another approach was recently introduced in an attempt to overcome these difficulties. Specifically, a new lower bound on the dimension of a quantum system needed to generate a Bell correlation was given in [3]. This bound is easy to calculate as it is a simple function of the Bell correlation and is tight in many cases.

The dimension witness approach was later generalized to the prepare-and-measure (PM) scenario, which is simpler and more general [5]. Unlike the Bell scenario, the PM scenario does not involve entanglement and this makes it easier to implement experimentally [7,8,11]. In the PM scenario, one party, the *preparer*, prepares one of finitely many quantum states, then the other party, the *measurer*, performs one of finitely many measurements on the state; see Fig. 1(b). The corresponding set of measurement statistics is called a *PM correlation*. Similar to the Bell scenario, a very important and natural problem is to lower bound the dimension of the quantum system required to generate a given PM correlation. For this, the approach of linear dimension witness was generalized to the PM scenario in [5], where the preparer and the measurer share classical public randomness. The case where the devices are independent was considered in [6], where one needs to use nonlinear dimension witnesses.

Despite these encouraging results, dimension witnesses in the PM scenario suffer from similar drawbacks as those for the Bell scenario, which restricts their applicability. Indeed, some are very specific, e.g., the dimension witnesses discussed in [5,6] apply only for the case of binary measurements. Consequently, it is highly desirable to identify a lower bound for the PM scenario, analogous to that in [3], that is applicable to PM correlations with arbitrary parameters. In this work, we provide such a lower bound. To achieve this, we first transfer the target PM scenario to a corresponding Bell scenario, and then apply the bound given in [3], leading to a lower bound for PM correlations. We show that this lower bound performs very well on some interesting applications, e.g., quantum random access coding, and it also gives insights into the concept of dimension witness. Specifically, we show that the dimension witness provided in [9] can be obtained as a direct consequence of our lower bound. Furthermore, we also use our lower bound to prove that the sets of restricted-dimensional PM correlations are not always convex.

Scenarios. A two-party Bell scenario consists of two parties, Alice and Bob, that are in separate locations, and share a quantum state ρ acting on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. Alice and Bob each have a (local) measurement apparatus acting on their respective subsystems; see Fig. 1(a). A Bell correlation r is the collection of the joint conditional probabilities $r(a,b|x,y)$, i.e., the

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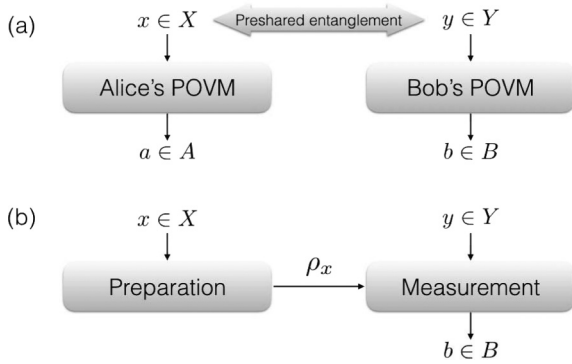


FIG. 1. (a) A Bell scenario. (b) A prepare-and-measure scenario.

probability Alice and Bob get output $(a, b) \in A \times B$ when they use measurement settings $(x, y) \in X \times Y$. In [3], it was shown that for a given Bell correlation $r = r(a, b|x, y)$, both d_1 and d_2 are lower bounded by the following two quantities:

$$\max_{y, y'} \left[\sum_{b, b'} \min_x \left(\sum_a \sqrt{r(a, b|x, y)} \sqrt{r(a, b'|x, y')} \right)^2 \right]^{-1}, \quad (1)$$

$$\max_{x, x'} \left[\sum_{a, a'} \min_y \left(\sum_b \sqrt{r(a, b|x, y)} \sqrt{r(a', b|x', y)} \right)^2 \right]^{-1}. \quad (2)$$

As mentioned above, a PM scenario has one party preparing a quantum state and the other measuring it, thus the outcome probabilities are only seen on one side. The preparer can generate one out of N possible states, denoted by ρ_x , where $x \in X = \{1, 2, \dots, N\}$. The measurer can choose one of M different measurements to perform, indexed by $y \in Y = \{1, \dots, M\}$. Each measurement y consists of the operators $\{\Pi_b^y\}$, where $b \in B = \{1, \dots, K\}$ denotes the measurement outcome. The probability of getting outcome b when measurement y is performed on quantum state ρ_x can thus be expressed as $p(b|x, y) = \text{Tr}(\rho_x \Pi_b^y)$.

In this paper, we focus on the following problem: For a given PM correlation p , what is the smallest dimension of a quantum system that is necessary to generate it? Throughout, we denote this quantity by $\mathcal{D}(p)$. Note that one can always generate $p(b|x, y)$ if one chooses ρ_x to be the computational basis state $|x\rangle\langle x|$ and Π_b^y to be the diagonal measurement operator $\sum_{z \in X} p(b|z, y) |z\rangle\langle z|$. This proves that $\mathcal{D}(p) \leq N$ for all PM correlations p . However, lower bounding $\mathcal{D}(p)$ for a PM correlation p is a much more interesting and challenging task.

Deriving our lower bound. In this section, we will prove the following theorem as the main result of the current paper.

Theorem. For any PM correlation p , we have that $\mathcal{D}(p)$ is lower bounded by

$$\left[\sum_{x, x'} q_x q_{x'} \min_y \left(\sum_b \sqrt{p(b|x, y)} \sqrt{p(b|x', y)} \right)^2 \right]^{-1}, \quad (3)$$

for any probability distribution $q = (q_x)$ over X .

Proof. Let $d = \mathcal{D}(p)$. Suppose p can be realized by the quantum states $\{\rho_x\}_x$ and the measurements $\{\Pi_b^y\}_b$ acting on

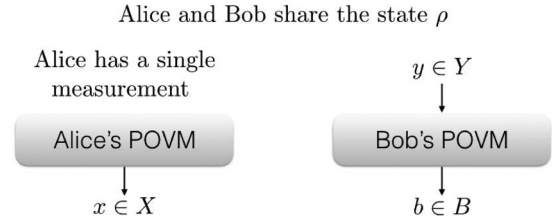


FIG. 2. Transforming a PM scenario to a Bell scenario.

\mathbb{C}^d . Consider the following Bell scenario where Alice and Bob share the state

$$\rho \equiv \sum_x q_x |x\rangle\langle x| \otimes \rho_x$$

acting on $\mathbb{C}^N \otimes \mathbb{C}^d$. Here, $\{|x\rangle\}_x$ is the computational basis of \mathbb{C}^N , and q is some fixed probability distribution over X . Furthermore, the two parties perform the following measurements: Alice measures the subsystem on \mathbb{C}^N in the computational basis and Bob measures in the same way as he does in the PM scenario (so the measurement outcome sets of Alice and Bob are X and B , respectively). See Fig. 2.

The Bell correlation corresponding to the strategy described above is given by $r(x, b|y) = q_x p(b|x, y)$. Applying the lower bounds (1) and (2) to the Bell correlation r , we get two lower bounds on $\mathcal{D}(p)$. It can be shown that one of the bounds always dominates the other, so in the theorem we only give the larger of the two, which completes the proof. ■

Some remarks on the lower bound (3) are in order. Obviously, since $\mathcal{D}(p)$ is integral, we can round the quantity (3) up if it is not an integer. Also one would naturally want to identify a probability distribution q that maximizes (3). In general, finding the probability distribution maximizing (3) corresponds to minimizing a nonconvex quadratic function over the set of probability vectors (called the simplex). It is known that solving such a problem is NP-hard [15]. However, any probability vector q does yield a lower bound on $\mathcal{D}(p)$. In this work, we mostly consider the uniform distribution, i.e., $q_x = 1/N$ for all $x \in X$. As it turns out, this simple choice is often sufficient to give tight bounds. Later we give an application where one can analytically prove this is optimal.

Lastly, whenever we have at most four possible preparations, i.e., $N \leq 4$, there exists a tractable algorithm to find the distribution q that maximizes (3). Specifically, in this case, the problem amounts to solving a semidefinite program, and this can be done in polynomial time [16]. For this, we use the fact that quadratic optimization over the simplex can be expressed as a linear conic programming problem over the cone of completely positive matrices [17]. An n -by- n matrix is *completely positive* if it can be written as $\sum_{i=1}^k x_i x_i^\top$ (for some $k \geq 1$) where $x_i \in \mathbb{R}_+^n$. However, for $n \leq 4$, an n -by- n matrix is completely positive if and only if it is positive semidefinite and has non-negative entries.

Before discussing applications of our lower bound, we first show it can be tight, even when q is uniform. Consider the toy example where $X = \{0, 1\}^M$, $Y = \{1, \dots, M\}$, $B = \{0, 1\}$, and the PM correlation given by

$$p(b|x, y) \equiv \delta_{b, x_y}, \quad (4)$$

where δ is the Kronecker delta function and x_y denotes the y th bit of the bit string $x \in \{0,1\}^M$. Intuitively, this means that given an encoding of the bit string x , measurement can pick out any of the bits perfectly, and thus can discriminate the quantum states received perfectly. In this case, our lower bound (3) yields $\mathcal{D}(p) \geq 2^M = N$, when q is uniform. This is tight since $\mathcal{D}(p) \leq N$ for any PM correlation.

In the toy example above, we see that there is no way to compress the states, that is, to reduce the dimension below the trivial bound of N . It turns out that there is a sufficient condition which follows easily from our main theorem. Since it is always true that

$$\sum_x \min_y \left(\sum_b \sqrt{p(b|x,y)} \sqrt{p(b|x',y)} \right)^2 = N, \quad (5)$$

by setting q to be uniform and applying our lower bound (3), we get the following sufficient condition for the impossibility of quantum compressibility, i.e., $\mathcal{D}(p) = N$:

$$\forall x \neq x', \quad \exists y, \forall b \quad \text{it holds that } p(b|x,y) p(b|x',y) = 0. \quad (6)$$

If (6) holds, p cannot be (quantumly) compressed.

Insights for dimension witness. We now introduce two examples to show that our lower bound provides insights into the concept of dimension witness.

We have mentioned that when the preparer and the measurer in a PM scenario are independent, nonlinear dimension witnesses have been proposed [6,11]. Suppose we have a PM correlation p in the setting of $X = \{0,1,2,3\}$, $Y = \{0,1\}$, and $B = \{0,1\}$. Then the determinant of the 2×2 matrix

$$W_2 = \begin{bmatrix} p(0|0,0) - p(0|1,0) & p(0|2,0) - p(0|3,0) \\ p(0|0,1) - p(0|1,1) & p(0|2,1) - p(0|3,1) \end{bmatrix}$$

is a nonlinear dimension witness [6]. In [6], it was pointed out that if $\det(W_2) = 2$ (the largest value possible), then $\mathcal{D}(p) \geq 4$. It can easily be shown that this also follows from our sufficient condition for noncompressibility. This is because when the determinant is 2, all the entries of W_2 must be 1 or -1 . Then one can check that the conditions (6) are met, and we get $\mathcal{D}(p) = 4$.

For our second example, we recover the dimension witness in [9]. Specifically, we show something slightly more general, that for $K = 2$ and any M, N , we have that

$$\sum_{x,x'} \max_y |p(1|x,y) - p(1|x',y)|^2 \leq \left(1 - \frac{1}{d}\right) N^2 \quad (7)$$

is a quadratic dimension witness for $d < N$. This can be easily shown by applying the Fuchs–van de Graaf inequalities [18] to our lower bound (3) (with q uniform).

Note that in [9], the dimension witness (7) was derived in the special case $M = N(N - 1)/2$. In this case, the measurements y in (7) are fixed and labeled by $y = (x, x'), x > x'$. It turns out that in this case, (7) can be tight [9].

Relation to the positive semidefinite rank (PSD rank). We first consider the case $M = 1$, i.e., the measurer has only one choice of measurement. For this, we introduce the N -by- K row stochastic matrix Q with entries given by $p(b|x)$. In this case, it is known [19] that $\mathcal{D}(p)$ is

equal to the PSD rank of Q , denoted $\text{rank}_{\text{psd}}(Q)$, defined as the least integer $c \geq 1$ such that there exist $c \times c$ positive semidefinite matrices $E_1, \dots, E_N, F_1, \dots, F_K$ satisfying $Q_{x,b} = \text{Tr}(E_x F_b)$, for all x, b . The PSD rank is an important quantity in computer science and mathematical optimization [20–22].

We can expand this idea to more measurements in a few ways. First, in [19], it was shown that $\text{rank}_{\text{psd}}(Q) \geq \sum_b \max_x Q_{x,b}$ (this holds for any entrywise non-negative matrix Q). Applying this to each y individually, we get

$$\mathcal{D}(p) \geq \max_y \sum_b \max_x p(b|x,y), \quad (8)$$

which was also observed in [23]. Notice that the lower bound (8) is always upper bounded by K . Thus, our lower bound (3) can vastly outperform this bound, as can be seen by the PM correlation (4). Second, we can consider the matrix Q to the KM -by- N matrix Q' with entries given by $Q'_{(y,b),x} = p(b|x,y)$. By definition, it follows that $\mathcal{D}(p) \geq \text{rank}_{\text{psd}}(Q')$. Then using the technique to lower bound the PSD rank introduced in [19], we can recover (3), giving an alternative proof of our main result.

Quantum random access codes. We now consider the PM correlations arising from an information task known as random access coding. The goal here is to encode M bits into a quantum state of hopefully small dimension such that the measurer can choose any of the M bits to learn with high probability. Moreover, consider a PM scenario with $X = \{0,1\}^M, Y = \{1, \dots, M\}$, and $B = \{0,1\}$, where the PM correlation is given by

$$p(b|x,y) \equiv \begin{cases} \beta & \text{if } b = x_y \\ 1 - \beta & \text{if } b \neq x_y. \end{cases} \quad (9)$$

Here, $\beta \geq 1/2$ is the success probability of learning the y th bit correctly. There are well-known examples where a single qubit can encode two bits with $\beta \approx 0.8536$ (see [24,25]) or three bits with $\beta \approx 0.7887$ (see [26,27]). We see that for both of these examples, the bound (3) is tight (when q is uniform). In fact, it can be shown that q being uniform is the optimal choice when applying our lower bound (3) for this case of random access codes [28].

We now ask the question of how the dimension is affected when β changes. We compare our bound to Nayak's bound [29], which is essentially optimal and states that a random access code requires at least $[1 - H(\beta)]M$ qubits, where H is the binary entropy function $H(\beta) \equiv -\beta \log_2(\beta) - (1 - \beta) \log_2(1 - \beta)$. In other words, Nayak's bound can be expressed as $\mathcal{D}(p) \geq \lceil 2^{[1 - H(\beta)]M} \rceil$.

For small values of M , our bound behaves very well by being quite close to Nayak's bound. For example, for $M = 2$, Nayak's bound beats our bound for $\beta \in (0.8900, 0.9083) \cup (0.9674, 0.9714)$. For all other values of β , we get the same bound. Thus, our bound performs very well and is close to optimal in this setting.

However, Nayak's bound is concerned with the worst case probability of correctly decoding a bit. Therefore, one can easily construct other PM correlations where our lower bound is greater. For example, if we alter only a few of the states in a random access code such that for some x , some of the bits are

decoded with a very small success probability, Nayak’s bound will approach 1 (as the binary entropy will approach 1). On the other hand, our bound can still be large as it deals with all of the outcome probabilities independently.

Witnessing the nonconvexity of restricted-dimensional PM correlations. We now study the sets of restricted-dimensional PM correlations, that is, the sets $D_d \equiv \{p : \mathcal{D}(p) \leq d\}$, for some fixed integer $d \geq 1$. It was first pointed out in [5] that there exist choices for d for which D_d is not convex. We now show that this can be proved easily using the lower bound (3).

For this, consider the PM scenario with $X = \{0, 1\}^2$, $Y = \{1, 2\}$, and $B = \{0, 1, 2\}$. For $i = 1, 2$, define

$$p_i(b|(x_1, x_2), y) \equiv \begin{cases} \delta_{b, x_i} & \text{if } y = i \\ \delta_{b, 2} & \text{if } y \neq i. \end{cases} \quad (10)$$

Intuitively, this is similar to our toy example (4) where one wants to perfectly decode one of two bits, except that the measurer can output “I am not certain” (as indicated by the outcome “2”). Then, p_i is the PM correlation where the preparer chooses two bits (x_0, x_1) , sends x_i , and the measurer learns and outputs x_i if and only if $y = i$ (and outputs “2” otherwise). For this reason, it is clear that each p_i is in D_2 .

The convex combination is more than just sending a random choice of x_1 or x_2 , as the measurer’s positive operator-valued

measures (POVMs) need to be taken into account. It turns out that any qubit encoding is not possible as our lower bound (3) applied to $\frac{1}{2}p_1 + \frac{1}{2}p_2$ (with q uniform) is equal to $16/7 > 2$. Therefore, $\frac{1}{2}p_1 + \frac{1}{2}p_2 \notin D_2$, witnessing that D_2 is not convex.

Conclusions. In this work, we derived a lower bound for the dimension of any quantum system that can produce a given PM correlation, which is applicable to any choice of parameters, and has nontrivial applications. Specifically, we showed that our lower bound provides insights for the notion of dimension witness. We also used the bound to prove that the set of restricted-dimension PM correlations is not always convex. Due to the generality of the PM scenario, we believe that our lower bound will lead to more nontrivial applications and will provide insights into the study of device-independent quantum processing tasks.

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