Signifying the nonlocality of NOON states using Einstein-Podolsky-Rosen steering inequalities

R. Y. Teh, L. Rosales-Zárate, B. Opanchuk, and M. D. Reid

Centre for Quantum and Optical Science, Swinburne University of Technology, Melbourne, Australia

We construct Einstein-Podolsky-Rosen (EPR) steering signatures for the nonlocality of the entangled superposition state described by $\frac{1}{\sqrt{2}}\{|N\rangle|0\rangle + |0\rangle|N\rangle\}$, called the two-mode NOON state. The signatures are a violation of an EPR steering inequality based on an uncertainty relation. The violation confirms an EPR steering between the two modes and involves certification of an intermode correlation for number, as well as quadrature phase amplitude measurements. We also explain how the signatures certify an Nth order quantum coherence, so the system (for larger N) can be signified to be in a superposition of states distinct by a mesoscopic value of quantum number in each mode. Finally, we examine the limitations imposed for lossy scenarios, discussing how experimental realizations may be possible for $N = 2,3$.

I. INTRODUCTION

The generation and signification of a macroscopic quantum superposition state is an outstanding challenge. Schrödinger explained that according to quantum mechanics it is conceptually possible for a macroscopic system (like a cat) to become entangled with a microscopic one in such a way that a superposition of two macroscopically distinct states is created [1]. Schrödinger pointed out the paradoxical nature of such a macroscopic system: a superposition of two macroscopically distinguishable states cannot be interpreted as being in one or the other of the states until measured. In realistic scenarios, couplings to external environments make the generation of macroscopic superposition states difficult but simpler mesoscopic realizations are feasible [2–4].

One of the most interesting realizations is the two-mode NOON state [5–16]:

$$|\psi_{\text{NOON}}\rangle = \frac{1}{\sqrt{2}}\{|N\rangle|0\rangle + e^{i\phi}|0\rangle|N\rangle\}. \qquad (1)$$

Here, N boson particles (or photons) are in a superposition of being either in the first mode (denoted a) or the second mode (denoted b). The modes may correspond to different spatial paths. Denoting the creation and destruction operators for the two modes by $\hat{a},\hat{a}^{\dagger}$ and $\hat{b},\hat{b}^{\dagger}$, $|n\rangle|m\rangle$ is the eigenstate of numbers $\hat{n}_a = \hat{a}^{\dagger}\hat{a}$ and $\hat{n}_b = \hat{b}^{\dagger}\hat{b}$ with eigenvalues n and m, respectively. Experiments have used spontaneous parametric down conversion to generate photonic NOON states for N up to 5 [5,8–13]. Recent experiments achieve Hong-Ou-Mandel interference with atoms (for $N = 2$) [17] and proposals exist for Bose-Einstein condensates (BEC) [18]. NOON states are typically signified by way of interference fringes or fidelity [8–13,15].

As $N \to \infty$, the NOON state is a superposition of two states with macroscopically different values of quantum number $\hat{n}$ in each mode. While genuinely macroscopic systems (like a cat) would involve many degrees of freedoms (for example, many modes) [19], the NOON state (similar to other single and two-mode states studied in the literature [3,4]) nonetheless provides a simple model for the Schrödinger cat paradox, as $N \to \infty$. The NOON state superposition (1) can therefore elucidate aspects of the transition from microscopic to macroscopic.

In order to quantify the transition, we refer to the state (1) as an "N-scopic superposition."

Our motivation is to investigate the *nonlocality* of the NOON state (or of an approximate NOON state that may be generated experimentally). While nonlocality between two microscopic systems (corresponding to $N = 1$) has been experimentally certified using Bell inequalities [20], relatively little is known about nonlocality between more mesoscopic systems [21]. In particular, it is an important goal to experimentally verify the nonlocality of an entangled state like that described by Schrödinger, where the system is in a superposition of two mesoscopically distinguishable states. In this paper, we derive a set of *Einstein-Podolsky-Rosen (EPR) steering inequalities* [22–24] based on the number-phase uncertainty relation

$$\Delta\hat{n}\Delta\hat{P}^N \geqslant \tfrac{1}{2}|\langle[\hat{n},\hat{P}^N]\rangle|, \qquad (2)$$

where $\hat{n}$ is the mode number and $\hat{P}$ is the mode quadrature amplitude (defined below). We show how violation of these inequalities can be used to demonstrate the nonlocality of the NOON state for arbitrary N. By examining realistic scenarios for NOON states where losses are present, we suggest feasible tests for $N = 2,3$.

The detection of an EPR steering nonlocality between two optical systems consisting of many photons has been experimentally verified [25], but this does not in itself imply the type of entangled state considered by Schrödinger: By contrast, we are able to show that the violation of the EPR steering inequalities as predicted for the NOON state certifies the N-scopic nature of the entanglement of (1), which involves a superposition of number states distinct by N quanta.

EPR steering has been established as a distinct type of nonlocality, different to both Bell's nonlocality and entanglement [22,23,26]. "Steering" is the term used by Schrödinger [27] to describe the effect where an observer at one location can apparently change the quantum state at another—the effect Einstein called "spooky action-at-a-distance" [28]. Some EPR steering and Bell inequalities have been derived for NOON states [29–31]. For $N = 1$ this led to the experimental verification of the Bell nonlocality of a single photon [31]. More recently, steering inequalities for $N = 1$ have been used to give conclusive proof of the "collapse of the wave

function" [32,33]. The EPR steering inequalities for larger N may therefore open a way to investigate such effects for a mesoscopic superposition state.

Most steering and Bell inequalities derived to date use either number or quadrature phase amplitude measurements. The proposal of this paper combines number and quadrature phase amplitude measurements. This gives two advantages. First, the number measurements are useful in optimizing violation of the inequalities for entangled two-mode systems over a range of field intensities where there is a perfect number correlation between the modes (as with NOON states). Second, the inequalities are based on variances and provide a simple method in nonideal scenarios with small losses to demonstrate that the nonlocality observed in the experiment is indeed due to a superposition of states distinct by $\sim N$ quanta.

Summary of paper. Our proposed EPR steering inequalities are derived in Secs. II and III of this paper. For $N = 2$ and $\phi \neq 0$ we show that a suitable signature for the steering nonlocality of a NOON state is the violation of the EPR steering inequality

$$\Delta_{\inf} \hat{n}_b \Delta_{\inf} (\hat{P}_b^2) \geqslant |\langle \hat{C}_b \rangle|_{\inf}/2, \tag{3}$$

where $\hat{C}_b = 2\hat{X}_{b,\pi/4}^2 - \hat{X}_b^2 - \hat{P}_b^2$. Here we define the rotated quadrature phase amplitudes for mode b as $\hat{X}_{b,\theta} = \hat{X}_b \cos\theta + \hat{P}_b \sin\theta$ and $\hat{P}_{b,\theta} = -\hat{X}_b \sin\theta + \hat{P}_b \cos\theta$, where $\hat{X}_b = \hat{b} + \hat{b}^\dagger$, $\hat{P}_b = (\hat{b} - \hat{b}^\dagger)/i$. Also $\hat{X}_a = \hat{a} + \hat{a}^\dagger$, $\hat{P}_a = (\hat{a} - \hat{a}^\dagger)/i$. The $\Delta_{\inf} \hat{n}_b$ is the uncertainty in the prediction for $\hat{n}_b$ based on measurement of $\hat{n}_a$. Similarly $\Delta_{\inf} \hat{P}_b^2$ is the uncertainty in $\hat{P}_b^2$ based on the measurement $\hat{X}_a$; and $|\langle \hat{C}_b \rangle|_{\inf}$ is the magnitude of the mean value of $\hat{C}_b$ based on the measurement $\hat{X}_a$. An EPR steering inequality is obtained by replacing the quantities of an uncertainty relation [in this case (2)] with their predicted ("inferred") values [25,34,35]. In Sec. II, we summarize the local hidden state (LHS) model developed by Wiseman, Jones, and Doherty [22]. Using the methods of Cavalcanti *et al.* [23], we prove that (3) is a steering inequality the violation of which falsifies LHS models, so that steering of the mode b (by measurements on the mode a) can be confirmed.

In Secs. II and III, we provide similar inequalities for arbitrary N, including one for odd N and $\phi = 0$. Specifically, EPR steering of the mode b is confirmed if

$$E_N^{(p)} = \frac{\Delta_{\inf} \hat{n}_b \Delta_{\inf} \hat{P}_b^N}{\frac{1}{2}|\langle [\hat{n}_b, \hat{P}^N] \rangle|_{\inf}} < 1 \tag{4}$$

or

$$E_N^{(x)} = \frac{\Delta_{\inf} \hat{n}_b \Delta_{\inf} \hat{X}_b^N}{\frac{1}{2}|\langle [\hat{n}_b, \hat{X}_b^N] \rangle|_{\inf}} < 1. \tag{5}$$

For the ideal NOON state, $\Delta_{\inf} \hat{n}_b = 0$ and the usefulness of the inequality depends on whether the denominator is nonzero. We show that the first criterion is useful provided $\cos\phi \neq 0$ for N odd or $\sin\phi \neq 0$ for N even, and the second criterion is useful for all N provided $\sin\phi \neq 0$. We explain in Sec. VII for N up to 3 how the denominator of the inequality can be measured via homodyne detection. For $N = 1$ the inequality becomes straightforwardly

$$\Delta_{\inf} \hat{n}_b \Delta_{\inf} \hat{P}_b < |\langle \hat{X}_b \rangle|_{\inf}/2. \tag{6}$$

The cases of $N = 1$ and $N = 2$ are analyzed in detail in Secs. IV and V. The explanation of how the steering

inequalities signify an N-scopic superposition state is given in Sec. VIII.

The inequalities (3)–(5) involve measurement of number ($\hat{n}_a$, $\hat{n}_b$) and hence have the drawback of low detection efficiencies (in the photonic case). In the first instance, we propose that the correlation be established by postselection of the events where a total of N quanta (photons) are detected at the sites of both modes. A second problem is distinguishing between the detection of two and one photons at a given site. Here, beam splitters or photon number-resolving detectors could be used [10,13] in conjunction with postselection over events where a total N photons is counted. The measurement of observables $\hat{X}^N$, $\hat{P}^N$ is achievable via optical homodyne techniques that are highly efficient. Nonetheless, we explain in Secs. VI, VII, and IX that losses have a significant effect (measurement efficiencies of $\eta > 0.94$ are required for $N = 3$) and that care needs to be taken to avoid possible loopholes created by asymmetrical losses for the number and quadrature measurements.

II. EPR STEERING INEQUALITIES BASED ON UNCERTAINTY RELATIONS

In this section we give the formal derivation of the EPR steering criteria summarized in the Introduction. We show in Sec. III how we can use the inequalities to detect EPR steering for a NOON state.

A. EPR steering inequalities

EPR steering is verified as a failure of local hidden state models (LHS). The LHS model was pioneered in the papers by Wiseman, Jones, and Doherty [22] and is based on the local hidden variable models considered by Bell [36]. We define two subsystems A and B and consider spacelike separated measurements on each of them. The measurements are described quantum mechanically by observables $\hat{X}_A(\theta)$ and $\hat{X}_B(\phi)$ (respectively) and the outcomes are given by the numbers $X_A(\theta)$ and $X_B(\phi)$ (written without the "hats"). Here θ and ϕ denote the measurement choice at the locations A and B. To prove Bell's nonlocality, one falsifies a description of the statistics based on a local hidden variable model, where the averages are given as

$$\langle X_B(\phi) X_A(\theta) \rangle = \int_\lambda d\lambda \, P(\lambda) \langle X_B(\phi) \rangle_\lambda \langle X_A(\theta) \rangle_\lambda. \tag{7}$$

Here $\int_\lambda P(\lambda) d\lambda = 1$ so that the $P(\lambda)$ is a probability density (or probability if the integral is replaced by a discrete summation, as explained in Bell's papers [36]). The λ denotes a set of variables $\{\lambda\}$ that take the role of the hidden variables as postulated in Bell's model. The $\langle X_A \rangle_\lambda$ denotes the average of the results X_A for the system in the particular hidden variable state denoted by λ; and similarly for $\langle X_B \rangle_\lambda$. The $P(\lambda)$ is independent of the θ and ϕ. The factorization that occurs for the moments in the integrand is due to the assumption of "locality" [36].

To prove EPR steering of subsystem B, we need to falsify a description of the statistics based on a local hidden state (LHS)

model where the averages are given as [22,23]

$$\langle X_B(\phi) X_A(\theta) \rangle = \int_\lambda d\lambda \, P(\lambda) \langle X_B(\phi) \rangle_{\lambda,\rho} \langle X_A(\theta) \rangle_\lambda. \quad (8)$$

Here an extra condition is placed on the average $\langle X_B \rangle_\lambda$. The ρ subscript denotes that the average is to be consistent with that of a *quantum density operator* ρ_λ^B. This is the case for all choices ϕ of measurement at B. For example, if $X_B(\theta) = X_B$ and $X_B(\pi/2) = P_B$ then the statistics for the LHS model must be consistent with a local uncertainty principle namely $\langle (X_B - \langle X_B \rangle)^2 \rangle_\lambda \langle (P_B - \langle P_B \rangle)^2 \rangle_\lambda \geqslant 1$. The ρ_λ^B is an example of a local quantum state (for site B). No such constraint is made for the moments $\langle X_A(\theta) \rangle_\lambda$, written without the subscript.

In this paper we consider three quantum observables defined through the uncertainty relation:

$$\Delta \hat{\sigma}_B^X \Delta \hat{\sigma}_B^Y \geqslant |\langle \hat{\sigma}_B^Z \rangle|/2. \quad (9)$$

Following the approach given in Refs. [25,34] used to derive a criterion for the EPR paradox [26] and also for EPR steering [22,25], we consider the average conditional uncertainty $\Delta_{\inf} \sigma_B^X$ defined by

$$\left(\Delta_{\inf} \hat{\sigma}_B^X \right)^2 = \sum_{x_j^A} P(x_j^A) \left[\Delta \left(\sigma_B^X | x_j^A \right) \right]^2. \quad (10)$$

Here, we denote the possible results of a measurement $\hat{X}_A$ at A by $\{x_j^A\}$. $P(x_j^A)$ is the probability for obtaining the result x_j^A. The uncertainty (10) is a measure of the (average) uncertainty in the *inferred* value [which we take to be the mean of the conditional distribution $P(\sigma_B^X | x_j^A)$] for a measurement $\hat{\sigma}_B^X$ at B given a measurement $\hat{X}_A$ at A. Specifically, $[\Delta(\sigma_B^X | x_j^A)]^2$ is the variance of the conditional distribution $P(\sigma_B^X | x_j^A)$. We define similarly

$$\left(\Delta_{\inf} \hat{\sigma}_B^Y \right)^2 = \sum_{y_j^A} P(y_j^A) \left[\Delta \left(\sigma_B^X | y_j^A \right) \right]^2, \quad (11)$$

noting that the $\{y_i\}$ is the set of results for a measurement $\hat{Y}_A$ made at A to infer the value of the measurement of $\hat{\sigma}_B^Y$ at B. Further, we define an (average) inferred value for the modulus of the mean of measurement of $\hat{\sigma}_B^Z$ given a measurement $\hat{Z}_A$ at A as

$$|\langle \hat{\sigma}_B^Z \rangle|_{\inf} = \sum_{z_j^A} P(z_j^A) |\langle \sigma_B^Z \rangle_{z_j^A}|. \quad (12)$$

Here $\langle \sigma_B^Z \rangle_{z_j^A}$ is the mean of the conditional distribution $P(\sigma_B^Z | z_j^A)$ and the $\{z_j\}$ is the set of values for a measurement $\hat{Z}_A$ at A, that we use to infer outcomes for $\hat{\sigma}_B^Z$. Using these definitions, we can prove the following result [35].

Result (1): The EPR steering inequality. The LHS model (8) implies the inequality

$$\left(\Delta_{\inf} \hat{\sigma}_B^X \right) \left(\Delta_{\inf} \hat{\sigma}_B^Y \right) \geqslant |\langle \hat{\sigma}_B^Z \rangle|_{\inf}/2. \quad (13)$$

Hence violation of this inequality (called an EPR steering inequality) implies failure of the LHS model [Eq. (8)], and therefore steering of system B by (measurements at A). The proof is given in Appendix A.

III. STEERING INEQUALITIES FOR THE NOON STATE

To arrive at a steering signature for a NOON state, we consider the three observables for each mode: number $\hat{n}$, and the two quadrature phase amplitudes $\hat{X}$ and $\hat{P}$. Specifically, $\hat{n}_a = \hat{a}^\dagger \hat{a}$, $\hat{X}_a = \hat{a} + \hat{a}^\dagger$, and $\hat{P}_a = (\hat{a} - \hat{a}^\dagger)/i$, and $\hat{n}_b = \hat{b}^\dagger \hat{b}$, $\hat{X}_b = \hat{b} + \hat{b}^\dagger$, and $\hat{P}_b = (\hat{b} - \hat{b}^\dagger)/i$. Where the notation is clear, we omit the "hat" for these operators. Using the Result (1) given by Eq. (13), we can write down EPR steering criteria associated with the three observables: we certify EPR steering (of B by A) if either one of the following hold:

$$\Delta_{\inf} n_b \Delta_{\inf} \left(P_b^N \right) < \left| \langle [n_b, P_b^N] \rangle \right|_{\inf}/2 \quad (14)$$

and

$$\Delta_{\inf} n_b \Delta_{\inf} \left(X_b^N \right) < \left| \langle [n_b, X_b^N] \rangle \right|_{\inf}/2. \quad (15)$$

Here, $\Delta_{\inf} n_b$ refers to the average uncertainty of the result for n_b given a measurement $\hat{O}_n$ at A, as defined by (10). Similarly, $\Delta_{\inf} P_b^N$ refers to the average uncertainty of the result for P_b^N given a measurement $\hat{O}_p$ at A. The $\Delta_{\inf} X_b^N$ refers to the average uncertainty of the result for X_b^N given a measurement $\hat{O}_x$ at A. The $|\langle \hat{C} \rangle|_{\inf}$ where $\hat{C} = [n_b, P_b^N]$ (or $[n_b, X_b^N]$) is defined similarly, by (12), as the average value of the modulus of the expectation value of $\hat{C}$ conditioned on a measurement $\hat{O}_c$ at A. The steering inequalities of this paper take $\hat{O}_n = \hat{n}_a$, $\hat{O}_p = \hat{X}_a$, $\hat{O}_x = \hat{X}_a$, and $\hat{O}_c = \hat{X}_a$. The motivation for this choice is explained in Sec. IV.

To evaluate the right side of the inequalities (14) and (15), we determine the commutation relations: $[n, X] = -iP$ and $[n, P] = iX$. By ordering the P's to be always on the left of the X's and since $[X, P] = 2i$, we arrive at the commutation relation $[X, P^k] = 2ikP^{k-1}$. It can be shown that $[n, P^N] = iN\{P^{N-2}[PX + (N-1)i]\}$ and $[n, X^N] = -iN\{X^{N-2}[XP - (N-1)i]\}$. We use this result to further evaluate the right side of the steering inequalities. Most generally, the right side of the steering inequality (14) can be written

$$\left| \langle [n_b, P_b^N] \rangle \right|_{\inf} = N \left| \langle P_b^{N-1} X_b + i(N-1)P_b^{N-2} \rangle \right|_{\inf}, \quad (16)$$

so that the procedure is to measure the modulus of the expectation value of the measurement $\hat{C} = P_b^{N-1} X_b + i(N-1)P_b^{N-2}$ made on mode b, given a specific result for a measurement $\hat{O}_c$ is made on mode a, and then take the weighted average as defined by (12). We discuss methods for measuring $P_b^{N-1} X_b$ where $N = 1, 2, 3$ in Sec. VII below.

To investigate whether the steering inequalities will be useful for the NOON states (1) with phase ϕ, we evaluate the prediction for the right side of the steering inequality (14) in the general NOON case. We will take $\hat{O}_c$ to be the measurement X_a and denote the result of that measurement by x. We find

$$
\begin{aligned}
\left| \langle [n_b, P_b^N] \rangle \right|_{\inf} &= N \left| \langle P_b^{N-2}(P_b X_b + (N-1)i) \rangle \right|_{\inf} \\
&= N |(\langle b^N \rangle + (-1)^{N+1} \langle b^{\dagger N} \rangle|_{\inf} \\
&= N\sqrt{N!} \left| \frac{[e^{i\phi} + (-1)^{N+1} e^{-i\phi}]}{2} \right| \\
&\quad \times \int_{-\infty}^{\infty} |\langle x|0 \rangle \langle x|N \rangle| \, dx, \quad (17)
\end{aligned}
$$

where $|x\rangle$ are the eigenstates of X. The cases $N = 1$ and $N = 2$ are presented in Secs. IV and V below. We find similarly

$$\left|\langle[n_b, X_b^N]\rangle\right|_{\inf} = N\left|\langle X_b^{N-2}[X_b P_b - (N-1)i]\rangle\right|_{\inf}$$

$$= N\left|-\langle b^N\rangle + \langle(b^\dagger)^N\rangle\right|_{\inf}$$

$$= N\sqrt{N!}\,|\sin\phi|\int_{-\infty}^{\infty}|\langle x|0\rangle\langle x|N\rangle|\,dx. \quad (18)$$

Now we determine when each of the steering criteria (14) and (15) will be useful. For the NOON state, the mode numbers are always correlated, and we observe that $\Delta_{\inf}n_b = 0$. Hence either of the steering criteria (14) and (15) will be effective to detect steering in NOON states, *provided* that the right side of the inequality is not zero, and provided the variances $\Delta_{\inf}(X_b^N)$, $\Delta_{\inf}(P_b^N)$ are finite. Since the integral $\int_{-\infty}^{\infty}|\langle x|0\rangle\langle x|N\rangle|\,dx$ is nonzero for the NOON state, we see from the expressions (17) and (18) that the condition for the right side of the inequalities (14) and (15) to be nonzero is, for N odd, $\cos\phi \neq 0$ and $\sin\phi \neq 0$, respectively; for N even, $\sin\phi \neq 0$ in both cases.

To summarize, we rewrite the EPR steering criteria (14) and (15) as

$$E_N^{(p)} = \frac{\Delta_{\inf}n_b \Delta_{\inf}P_b^N}{\frac{1}{2}\left|\langle[n_b, P_b^N]\rangle\right|_{\inf}} < 1 \quad (19)$$

and

$$E_N^{(x)} = \frac{\Delta_{\inf}n_b \Delta_{\inf}X_b^N}{\frac{1}{2}\left|\langle[n_b, X_b^N]\rangle\right|_{\inf}} < 1. \quad (20)$$

Steering is obtained if $E_N^{(x/p)} < 1$. Either criterion is sufficient to certify an EPR paradox, or EPR steering. For the NOON state $|\psi_{\text{NOON}}\rangle = \frac{1}{\sqrt{2}}\{|N\rangle|0\rangle + e^{i\phi}|0\rangle|N\rangle\}$ the first criterion is useful provided $\cos\phi \neq 0$ for N odd or $\sin\phi \neq 0$ for N even, and the second criterion is useful for all N provided $\sin\phi \neq 0$. We comment that the right side of the steering inequalities (14) and (15) needs to be *measured* in the experiment. We examine how this can be done below in Sec. VII, finding that cases of low N are much more accessible to experiment. We also point out that except where $N = 1$ or 2, the equivalence of the first two lines in Eqs. (17) and (18) holds only for the expectation values as calculated for the ideal NOON state (1).

In Sec. VI, we will evaluate predictions for the nonideal case where loss is present. To complete the prediction for the steering inequalities with loss present, we also need to calculate $\Delta_{\inf}(P^N)$, $\Delta_{\inf}(X^N)$. In this paper, we use $\hat{O}_x = \hat{O}_p = \hat{X}_a$ as the measurement on mode a. As above, we take x to be the result of the measurement X_a. We evaluate

$$\Delta_{\inf}^2(P_b^N) = \int_{-\infty}^{\infty} P(x)\left\{\Delta(P_b^N|x)\right\}^2 dx$$

$$= \int_{-\infty}^{\infty} P(x)\left[\langle P_b^{2N}\rangle_x - \langle P_b^N\rangle_x^2\right]dx, \quad (21)$$

where $\langle\ldots\rangle_x$ denotes the expectation value conditioned on the result x, as defined for (12). The $\langle P_b^{2N}\rangle_x$ and $\langle P_b^N\rangle_x$ can be expressed in terms of the momentum representation functions
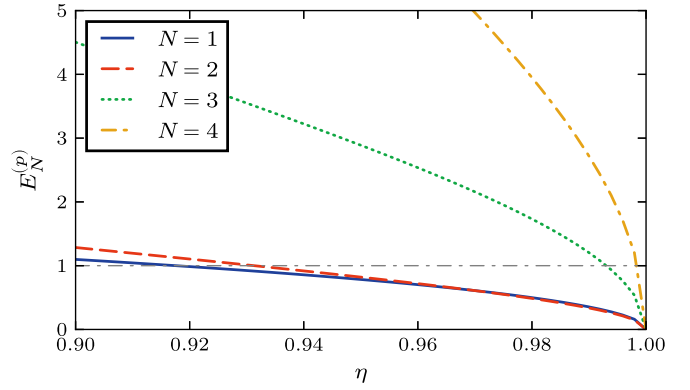

FIG. 1. Predictions for EPR steering of the NOON states. EPR steering is observed when $E_N^{(p)} < 1$. It is assumed that the two-mode NOON state is created and that each mode is then (independently) subjected to losses. Loss at each mode is modeled by a beam splitter coupling as described in Sec. VI. Here the beam splitter transmission efficiencies are $\eta = \eta_a = \eta_b$. We select the NOON state (1) with $\phi = 0$ for N odd and $\phi = \pi/2$ for N even.

$\langle p|N\rangle$ as shown in Appendix B. Similarly,

$$\Delta_{\inf}^2(X_b^N) = \int_{-\infty}^{\infty} P(x)\left\{\Delta(X_b^N|x)\right\}^2 dx$$

$$= \int_{-\infty}^{\infty} P(x)\left[\langle X_b^{2N}\rangle_x - \langle X_b^N\rangle_x^2\right]dx. \quad (22)$$

The $\langle X_b^N\rangle_x$ and $\langle X_b^{2N}\rangle_x$ can be solved in terms of the harmonic-oscillator wave functions $\langle x|N\rangle$ (27) as shown in Appendix B and explained for $N = 1, 2$ below. We have introduced the shorthand notation $\Delta^2 x \equiv (\Delta x)^2$ to avoid overuse of brackets. We have solved for the effect of loss on the NOON states using the methods outlined in Sec. VI and the results for the steering inequalities are plotted in Fig. 1.

IV. SPECIAL CASE OF $N = 1$

Steering for the case of $N = 1$ has been proposed by Jones and Wiseman [32] and experimentally achieved by Fuwa *et al.* [33]. The inequalities used in those papers verified steering in the high efficiency limit based on homodyne detection, thus giving a firm experimental proof of the nonlocality of the NOON ($N = 1$) state. Here, we outline the application of the steering inequalities (14) and (15) for this case.

For $N = 1$, the relevant Heisenberg uncertainty relations are $\Delta n \Delta P \geq |\langle X\rangle|/2$ and $\Delta n \Delta X \geq |\langle P\rangle|/2$. We see from (13) that a criterion sufficient to certify EPR steering of mode b by measurements on mode a is

$$\Delta_{\inf}n_b \Delta_{\inf}P_b < |\langle X_b\rangle|_{\inf}/2. \quad (23)$$

The inequality $\Delta_{\inf}n_b \Delta_{\inf}X_b < |\langle P_b\rangle|_{\inf}/2$ is also a steering criterion. Note we can also define the corresponding criteria for steering of the a mode by interchanging the a and b indices. The quantities have been defined above in Secs. II and III.

The choice of measurements $\hat{O}_n$, $\hat{O}_c$, $\hat{O}_p$, $\hat{O}_x = \hat{X}_a$ to be made on the mode a [as defined for Eqs. (14) and (15)] is generally so as to optimize the criterion for a given state, but is otherwise not explicitly specified in the criterion. Here, the choice of $\hat{O}_n = \hat{n}_a$ is crucial because it takes advantage

of the correlation of number between the two modes of the NOON state, to allow precisely that $\Delta_{\text{inf}} n_b = 0$. The criterion (23) is then predicted to be satisfied for any finite $\Delta_{\text{inf}} P_b$, provided $|\langle X_b \rangle|_{\text{inf}} \neq 0$. For the choice of $\hat{O}_c$, $\hat{O}_p$, we focus on quadrature phase amplitude measurements because they are readily measurable experimentally. For $\Delta_{\text{inf}} P_b$, we select $\hat{O}_p = \hat{X}_a$, but we note in the Appendices B–D that the result is not particularly sensitive to this choice. On the other hand, without a suitable measurement on mode a, $|\langle X_b \rangle|_{\text{inf}}$ will vanish. We find below that the measurement $\hat{X}_a$ on a does not completely collapse the state b, and the resulting superposition predicts a nonzero result for $|\langle X_b \rangle|_{\text{inf}}$. With this motivation, we take $\hat{O}_n = \hat{n}_a$, $\hat{O}_p = \hat{X}_a$, $\hat{O}_x = \hat{X}_a$, and $\hat{O}_c = \hat{X}_a$.

We examine the NOON state $|\psi\rangle = \frac{1}{\sqrt{2}}\{|N\rangle|0\rangle + |0\rangle|N\rangle\}$ ($\phi = 0$) and restrict therefore to the steering criterion (23). The measurement of n_a will enable a perfectly accurate prediction for the number n_b, so that $\Delta_{\text{inf}} n = 0$. Taking $\hat{O}_c = X_a$ we evaluate the mean of X_b (or P_b) at b, given a result x for measurement of X_a at A. This enables us to evaluate $|\langle X_b \rangle|_{\text{inf}}$ and $|\langle P_b \rangle|_{\text{inf}}$ for a valid steering criterion. If we measure X_a with result x, the normalized reduced wave function is (we denote the eigenstate of X for mode a by $|x\rangle$)

$$|\psi\rangle_x = \frac{\langle x|N\rangle|0\rangle + \langle x|0\rangle|N\rangle}{\sqrt{|\langle x|N\rangle|^2 + |\langle x|0\rangle|^2}}. \qquad (24)$$

Thus we write the reduced density operator as

$$\rho_{\text{red},x} = \frac{1}{2P(x)}\{|\langle x|N\rangle|^2|0\rangle\langle 0| + |\langle x|0\rangle|^2|N\rangle\langle N| \\ + \langle 0|x\rangle\langle x|N\rangle|0\rangle\langle N| + \langle N|x\rangle\langle x|0\rangle|N\rangle\langle 0|\}, \quad (25)$$

where the probability distribution for obtaining a result x for X_a is

$$P(x) = \tfrac{1}{2}\{|\langle x|0\rangle|^2 + |\langle x|N\rangle|^2\}. \qquad (26)$$

Here $\langle x|n\rangle$ are the standard oscillator wave functions

$$\langle x|n\rangle = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}}\frac{2^{\frac{1}{4}}}{\sqrt{c}}e^{-\frac{x^2}{c^2}}H_n\left(\frac{\sqrt{2}}{c}x\right) \qquad (27)$$

involving Hermite polynomials H_n and derived using that $\hat{x} = \frac{c}{2}(\hat{a} + \hat{a}^\dagger)$, $\hat{p} = \frac{c}{2i}(\hat{a} - \hat{a}^\dagger)$. In this paper we have taken $c = 2$. Now we see that the mean for X_b given the result x for X_a is

$$\langle X_b \rangle_x = \text{Tr}(\rho_{\text{red},x}X_b) \\ = \frac{1}{2P(x)}\{\langle 0|x\rangle\langle x|N\rangle\langle N|X_b|0\rangle \\ + \langle N|x\rangle\langle x|0\rangle\langle 0|X_b|N\rangle\} \qquad (28)$$

and similarly

$$\langle P_b \rangle_x = \text{Tr}(\rho_{\text{red},x}P_b) \\ = \frac{1}{2P(x)}\{\langle 0|x\rangle\langle x|N\rangle\langle N|P_b|0\rangle| \\ + \langle N|x\rangle\langle x|0\rangle\langle 0|P_b|N\rangle\}. \qquad (29)$$

In fact the mean $\langle X_b \rangle_x$ will be nonzero only for $N = 1$, in which case the steering criterion (23) is satisfied because $\Delta_{\text{inf}} n_b = 0$ (and $\Delta_{\text{inf}} P_b \neq \infty$). Hence the inequality (23) is a suitable steering criterion for $N = 1$. Specifically, following

the definition (12), we evaluate

$$|\langle X_b \rangle|_{\text{inf}} = \int_{-\infty}^{\infty} P(x)|\langle X_b \rangle_x|dx = \sqrt{\frac{2}{\pi}}, \qquad (30)$$

where $\langle X_b \rangle_x$ is the conditional quantity between two modes, as defined in (12). To complete the prediction for the steering inequality, we calculate a suitable value for $\Delta_{\text{inf}} P_b$ by selecting the measurement at A to be X_a. We denote the result of that measurement by x. Then the reduced density operator is $\rho_{\text{red},x}$ as above, which for $N = 1$ gives

$$(\Delta(P_b|x))^2 = \frac{1}{2P(x)}\{|\langle x|1\rangle|^2 + 3|\langle x|0\rangle|^2\} \qquad (31)$$

and thus

$$\Delta_{\text{inf}}^2 P_b = \int_{-\infty}^{\infty} P(x)\{\Delta(P_b|x)\}^2 dx \\ = \frac{1}{2}\int_{-\infty}^{\infty}\{|\langle x|1\rangle|^2 + 3|\langle x|0\rangle|^2\}dx = 2, \qquad (32)$$

where $\Delta^2 x \equiv (\Delta x)^2$. We obtain an EPR steering when $E_1^{(p)} \equiv \frac{\Delta_{\text{inf}} n_b \Delta_{\text{inf}} P_b}{|\langle X_b \rangle|_{\text{inf}}/2} < 1$. For the ideal NOON state with no losses, $E_1^{(p)} = 0$ and the steering is always detectable via this criterion. The situation with loss is studied in Sec. VI and presented in Fig. 1. Efficiencies $\eta > 0.92$ are required to detect the steering.

V. SPECIAL CASE OF $N = 2$

We now examine the details for the NOON state with $N = 2$ which represents an important case potentially accessible to experiment, in view of recent advances [13,17,33]. First, $[n,X^2] = -i(XP + PX) = -2iXP - 2 = 2(a^{\dagger 2} - a^2)$. Similarly, $[n,P^2] = i(XP + PX) = -2(a^{\dagger 2} - a^2)$. The steering criteria are

$$\Delta_{\text{inf}} n \Delta_{\text{inf}}(X^2) < |\langle a^{\dagger 2} - a^2 \rangle|_{\text{inf}} \qquad (33)$$

and

$$\Delta_{\text{inf}} n \Delta_{\text{inf}}(P^2) < |\langle a^{\dagger 2} - a^2 \rangle|_{\text{inf}}. \qquad (34)$$

For the NOON state (1) with $N = 2$ we obtain

$$|\langle a^{\dagger 2} - a^2 \rangle|_{\text{inf}} = \int P(x)|\langle \hat{a}^2|x\rangle - \langle(\hat{a}^\dagger)^2|x\rangle|\,dx \\ = \sqrt{2}|\sin\phi|\int |\langle x|0\rangle\langle x|2\rangle|\,dx \\ = 2\sqrt{\frac{2}{e\pi}}|\sin\phi| = 0.968|\sin\phi|.$$

Both the steering criteria (33) and (34) become useful for the NOON state with $\phi = \pi/2$. We show in Appendix B by integration of the Hermite polynomials that $\Delta_{\text{inf}}(X^2) = 3.18$ and $\Delta_{\text{inf}}(P^2) = 3.18$. For the ideal case with no detection loss, $\Delta_{\text{inf}} n = 0$ and the steering for the NOON state with $N = 2$ is detectable using either criterion. Comparing with the results for $N = 1$, we see that the prediction for the ratio of the right to left sides of the steering inequalities decreases for $N = 2$. We expect the criteria will be more difficult to satisfy at higher N in nonideal cases. Details of the calculations for arbitrary N are given in Appendices B and C. The effect of the losses is studied below in Sec. VI and the results are shown in Fig. 1.
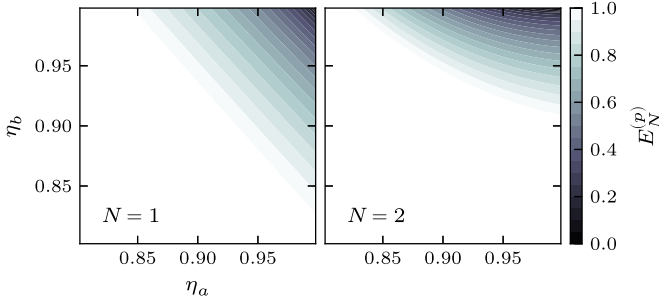
FIG. 2. Contour plot shows the effect of loss on the EPR steering: EPR steering is observed when $E_N^{(p)} < 1$. The η_a and η_b are the efficiencies for detection of mode a and b, respectively.

VI. INCLUDING LOSSES

Signatures of the NOON state superposition are known to be fragile to losses. We examine the effect of loss on the signatures proposed here, by using a simple model for loss. We couple each mode a and b to second independent fields taken as single modes and initially in independent vacuum states, following the beam splitter model introduced for the study of the decoherence of a macroscopic superposition state by Yurke and Stoler [3]. We thus evaluate the moments of detected fields with boson operators a_{det}, b_{det} given by

$$a_{\text{det}} = \sqrt{\eta_a}a + \sqrt{1 - \eta_a}a_v,$$
$$b_{\text{det}} = \sqrt{\eta_b}b + \sqrt{1 - \eta_b}b_v. \quad (35)$$

Here the a_v and b_v are destruction operators for independent external vacuum modes that couple to the modes of the NOON state. These external modes model the presence of an external environment into which quanta can be lost from the a and b modes. The amount of coupling for each mode is determined by the efficiency factors η_a and η_b, respectively. The $\eta_{A/B} = 1$ indicates zero loss; low $\eta_{A/B}$ indicates high loss. The model is effective for optical NOON states where thermal noise can be neglected. The full calculation is explained in Appendix D. We find for $N = 1$ and $\phi = 0$

$$E_1^{(p)} \equiv \frac{\Delta_{\text{inf}} n \, \Delta_{\text{inf}} P}{|\langle X \rangle|_{\text{inf}}/2}$$
$$= 2\left[\frac{\eta_b(\eta_a + \eta_b - 2)}{2(\eta_a - 2)}(1 + \eta_b)\right]^{\frac{1}{2}} \Big/ \left[\sqrt{\frac{2}{\pi}}\sqrt{\eta_a\eta_b}\right]. \quad (36)$$

The expressions for higher N are more complex but are explained in Appendix D and evaluated numerically. Figure 1 shows $E_N^{(p)}$ versus η, for the case of symmetrical efficiency $\eta = \eta_a = \eta_b$. The criterion for EPR steering is satisfied for $N = 1$ provided $\eta > 0.92$ but as expected for the NOON state, the cutoff efficiency increases sharply for higher N. For $N = 2$ there is asymmetrical dependence on η_a and η_b as evident by the contour plots of Fig. 2. The signature appears more sensitive to the efficiency η_B of mode b. Such asymmetrical sensitivity depending on the steering direction has been noted previously [37,38].

We note that the model (35) describes losses that occur *prior* to detection. It is assumed that the subsequent detection

process gives no further loss. Alternatively, if the beam splitter is to model detection losses, then the losses would need to be assumed *identical* for each of the detection processes (number or homodyne). In reality, for low N the numbers n_a, n_b are usually detected via counting techniques where the efficiency of detection is often small. On the other hand, the quadratures X and P are measured via homodyne detection where efficiencies are high (at least for optical fields). This creates a situation where the loss coefficient η is dependent on the choice of measurement made at each site, which we point out can *create loopholes* in the use of the signature for a practical experiment if not considered carefully [36,39]. We discuss this further in the Conclusion.

VII. MEASUREMENT

We next consider how to experimentally *measure* the moments on the right side of the steering inequalities (14) and (15). For $N = 1$ this is straightforward as explained in Sec. IV. For $N = 2$, on examining the expressions (17) and (18), we see we need to measure $\langle[n, P^2]\rangle = \langle XP + PX \rangle$. We define the measurable rotated quadrature phase amplitudes as $X_\theta = X \cos(\theta) + P \sin(\theta)$ and $P_\theta = -X \sin(\theta) + P \cos(\theta)$. Hence $X_{\pi/4} = \frac{1}{\sqrt{2}}\{X + P\}$ and $P_{\pi/4} = \frac{1}{\sqrt{2}}\{-X + P\}$ and we note that $\langle X_{\pi/4}^2 \rangle = \langle X^2 + P^2 + XP + PX \rangle/2$. Thus we can deduce either $\langle XP \rangle$ or $\langle PX \rangle$ by measuring the moments $\langle X^2 \rangle$, $\langle P^2 \rangle$, and $\langle X_{\pi/4}^2 \rangle$. The steering criteria (4) and (5) for $N = 2$ can be written as (here we drop the subscripts b for convenience)

$$\Delta_{\text{inf}} n \, \Delta_{\text{inf}}(P^2) < |\langle[n, P^2]\rangle|_{\text{inf}}/2$$
$$= \left|\langle X_{\pi/4}^2 - X^2/2 - P^2/2 \rangle\right|_{\text{inf}} \quad (37)$$

and

$$\Delta_{\text{inf}} n \, \Delta_{\text{inf}}(X^2) < |\langle[n, X^2]\rangle|_{\text{inf}}/2$$
$$= \left|\langle X_{\pi/4}^2 - X^2/2 - P^2/2 \rangle\right|_{\text{inf}}. \quad (38)$$

The moments of X, P, and $X_{\pi/4}$ are each measurable using homodyne detection.

For $N = 3$, we see from (17) and (18) that we need to measure $[n, P^3] = 3\langle P^2 X + 2iP \rangle$, the other measurements being straightforward. Expanding gives

$$\langle X_{\pi/4}^3 - P_{\pi/4}^3 \rangle = \frac{1}{\sqrt{2}}(\langle X^3 \rangle + 6i\langle P \rangle + 3\langle P^2 X \rangle).$$

Hence we can measure $\langle X_{\pi/4}^3 \rangle, \langle P_{\pi/4}^3 \rangle, \langle X^3 \rangle, \langle P \rangle$ and consequently infer the value of $\langle P^2 X \rangle$. Specifically, the steering inequalities become

$$\Delta_{\text{inf}} n \, \Delta_{\text{inf}}(P^3) < |\langle[n, P^3]\rangle|_{\text{inf}}/2$$
$$= \left|\langle \sqrt{2}(X_{\pi/4}^3 - P_{\pi/4}^3) - X^3 \rangle\right|_{\text{inf}}/2 \quad (39)$$

and

$$\Delta_{\text{inf}} n \, \Delta_{\text{inf}}(X^3) < |\langle[n, X^3]\rangle|_{\text{inf}}/2$$
$$= \left|\langle \sqrt{2}(X_{\pi/4}^3 + P_{\pi/4}^3) - P^3 \rangle\right|_{\text{inf}}/2. \quad (40)$$

We comment that the inequalities (37)–(40) are valid as a sufficiency test of EPR steering for *all* states, i.e., we do not assume ideal NOON states.

VIII. DISCUSSION

To conclude, we discuss an obvious question, which is how to ensure in an experiment that the observed steering is due to the quantum coherence of the NOON superposition, as opposed to an alternative microscopic effect that might arise from superpositions of number states distinct by less than N quanta? This is an important question where losses are present, because then the outcomes for number measurements can be different to zero and N.

First, the answer is clear in the ideal case of a two-mode system that generates only outcomes zero or N for the number measurements. The density operator can then be written in terms of four basis states $|0\rangle|0\rangle, |0\rangle|N\rangle, |N\rangle|0\rangle, |N\rangle|N\rangle$. The violation of an EPR steering inequality is also confirmation of an entangled state, and in this case that can only imply entanglement involving the mesoscopically distinguishable basis states. The violation of the steering inequality confirms the presence of an Nth-order off-diagonal matrix element (i.e., $\langle 0|\langle N|\rho|0\rangle|N\rangle \neq 0$). The details are straightforward and given in Appendix E.

In experiments where loss or noise is present, the distribution p_n for number $\hat{n}_b$ will include outcomes other than zero and N. It is not then clear whether an observation of EPR steering is a result of the superposition of states such as $|M'\rangle|n\rangle$ and $|M\rangle|m\rangle$ where $M - M' \sim N$, or the result of less interesting superpositions where $M \sim M'$.

The problem of determining whether the system has an Nth-order quantum coherence (defined as $\langle 0|\langle N|\rho|0\rangle|N\rangle \neq 0$) is nontrivial [40–43]. However, the following approach based on the steering inequality may be useful. The outcomes for number at mode a are either $n_a > 0$ or $n_a = 0$. The distribution for the outcome n_b of number at mode b given any result $n_a > 0$ is (for small losses) a "hill" centered near (or at) zero. The distribution for n_b given the result $n_a = 0$ is a hill centered near N. The mean and variance of each of the two hills is measurable and denoted by $\langle n_b \rangle_1$, $(\Delta \hat{n}_b)_1^2$ and $\langle n_b \rangle_2$, $(\Delta \hat{n}_b)_2^2$ respectively. For small losses, each of the two variances will be *small*.

We suppose that the experimentalist has measured a violation of the EPR steering inequality $\Delta_{\inf}\hat{n}_b \Delta_{\inf}(\hat{P}_b^N) \geqslant |\langle \hat{C}_b \rangle|_{\inf}/2$ where $(\Delta_{\inf}\hat{n}_b)^2 = \sum_{n_a} P(n_a)(\Delta(n_b|n_a))^2$ and $P(n_a)$ is the probability of outcome n_a. We note that where the conditional distribution for n_b given n_a is uniform for all $n_a > 0$, this will imply violation of the new but similar inequality

$$\left[P_1(\Delta \hat{n}_b)_1^2 + P_2(\Delta \hat{n}_b)_2^2 \right] \left(\Delta_{\inf} \hat{P}_b^N \right)^2 \geqslant \frac{1}{4} |\langle \hat{C}_b \rangle|_{\inf}^2. \quad (41)$$

Here we specify as selected in (3) that the inferred values for $\hat{P}_b^N$ and $\hat{C}_b$ are calculated using the same observable at mode a. Here P_1 is the probability of $n_a > 0$ and we have assumed $(\Delta \hat{n}_b)_1^2 = \Delta(n_b|n_a)$ for $n_a > 0$. Similarly, P_2 is the probability that $n_a = 0$ and $(\Delta \hat{n}_b)_2^2 = \Delta(n_b|n_a)$ for $n_a = 0$. We note that the loss model of Sec. VI predicts the distributions to be uniform, but if this is not the case then the inequality can be measured directly.

It is shown in Appendix F that violation of the inequality (41) is a negation of the mixture

$$\rho = P_1 \rho_1^{ab} + P_2 \rho_2^{ab}, \quad (42)$$

where ρ_1^{ab} and ρ_2^{ab} are two-mode density operators with a mean and variance for $\hat{n}_b$ given by $\langle n_b \rangle_1$, $(\Delta n_b)_1^2$ and $\langle n_b \rangle_2$, $(\Delta n_b)_2^2$ respectively. The negation is for *all* mixtures of the form (42), which includes where ρ_i^{ab} can be a superposition of number states. However the spread of number states involved in the superposition is constrained by the small variances associated with each ρ_i^{ab}. The ρ_1^{ab} and ρ_2^{ab} each have a variance for $\hat{n}_b$ that is narrower than the variance of the distribution given by the NOON superposition state. In other words, the violation of the inequality (41) can only be consistent with a density operator ρ involving superpositions $|\psi_{\sup}\rangle$ of states distributed over *both* hills.

IX. CONCLUSION

The particular steering inequalities we present in this paper involve measurements of number as well as quadrature phase amplitude correlation. Number measurements often entail poor efficiencies. It would seem feasible to perform in the first instance an experiment based on postselection of the events where a total of N quanta (e.g., photons) are detected across both sites. The problem of distinguishing multiple from single photon counts at a given location require photon-number-resolving detectors, or could be handled with N-photon counts being evaluated using multiple beam splitters [10,13].

The experiment for $N = 2$ would be a demonstration of a higher order (more mesoscopic) nonlocality than for $N = 1$ and would seem not unrealistic given the high efficiencies available with homodyne detection. Our calculations show that $\eta > 0.94$ is required. Care is needed to model the homodyne inefficiency as a *loss before detection*, and this small amount of loss must therefore *also* enter into the evaluation of the number correlation, to avoid the well-documented possible loopholes associated with losses that depend on measurement choices. The experiment for $N = 1$ is feasible. Such an experiment would complement that performed recently by Fuwa *et al.* [33] based on a different EPR steering inequality.

Finally, we point out that the steering inequalities (4) and (5) might be useful for detecting steering in other two-mode systems, especially where there is an intermode photon number correlation so that $\Delta_{\inf}n_b = 0$. For instance, we can apply the first-order inequality $\Delta_{\inf}\hat{n}_b \Delta_{\inf}\hat{P}_b < |\langle \hat{X}_b \rangle|_{\inf}/2$ [Eq. (6)] to the two-mode squeezed state. Denoting the two-mode squeeze parameter by r, the solutions for this state give $\Delta_{\inf}n_b = 0$ for all r. Further, it is well known that there is an EPR correlation between the quadrature phase amplitudes of the two modes for all r [25,34], so that $|\langle \hat{X}_b \rangle|_{\inf} \neq 0$ and $\Delta_{\inf}\hat{P}_b \to 0$ as $r \to \infty$. While steering has been experimentally achieved for this state via the alternative EPR steering inequality $\Delta_{\inf}\hat{X}_b \Delta_{\inf}\hat{P}_b < 1$ [34,44,45], it is quite possible that the use of the steering inequality with the number correlation $\Delta_{\inf}n_b = 0$ (which is valid for all r) may provide advantages in some regimes.

APPENDIX A: PROOF OF RESULT (1)

We will assume that the LHS model holds, for which moments are given by

$$
\langle X_A(\theta) X_B(\phi) \rangle = \int_\lambda d\lambda \, P(\lambda) \langle X_A(\theta) \rangle_{\lambda,\rho} \langle X_B(\phi) \rangle_\lambda
$$
$$
\equiv \sum_R P_R \langle X_A(\theta) \rangle_{R,\rho} \langle X_B(\phi) \rangle_R. \quad (A1)
$$

Here we give two alternative (but equivalent) notations for the hidden variable-type parameters, denoting the continuous variable option by the symbol λ as in Bell's work and the discrete option by R. The proof is unchanged whether we use integrals (λ) or discrete summations (R).

We consider the inference variance $(\Delta_{\inf} \sigma_A^X)^2$. Based on the definitions given in Sec. III, we see that $\sum_{x_j^B} P(x_j^B) \{\Delta(\sigma_A^X | x_j^B)\}^2 = \sum_{x_j^B} P(x_j^B) \sum_{\sigma_A^X} P(\sigma_A^X | x_j^B) \{\sigma_A^X - \langle \sigma_A^X \rangle_{x_j^B}\}^2$, which we can reexpress as $\sum_{x_j^B, \sigma_A^X} P(x_j^B, \sigma_A^X) \{\sigma_A^X - \langle \sigma_A^X \rangle_{x_j^B}\}^2$ and hence as $\sum_R P_R \sum_{x_j^B, \sigma_A^X} P_R(x_j^B, \sigma_A^X) \{\sigma_A^X - \langle \sigma_A^X \rangle_{x_j^B}\}^2$. This follows using that for a probabilistic (hidden variable) mixture $P(x_j^B, \sigma_X^A) = \sum_R P_R P_R(x_j^B, \sigma_X^A)$. Now we note that $\langle (x - \delta)^2 \rangle \geqslant \langle (x - \langle x \rangle)^2 \rangle$ where δ is any number. Hence the expression becomes bounded from below, and we can simplify further to show that

$$
\sum_R P_R \sum_{x_j^B, \sigma_A^X} P_R(x_j^B, \sigma_A^X) \{\sigma_A^X - \langle \sigma_A^X \rangle_{x_j^B}\}^2
$$
$$
\geqslant \sum_R P_R \sum_{x_j^B} P_R(x_j^B) \{\Delta_R(\sigma_A^X | x_j^B)\}^2
$$
$$
= \sum_R P_R \{\Delta_{\inf,R} \sigma_A^X\}^2.
$$

Here, the subscripts R imply that the probabilities, averages, and variances are with respect to the state R and we have used that $\{\Delta_R(\sigma_A^X | x_j^B)\}^2 = \sum_{\sigma_A^X} P_R(\sigma_A^X | x_j^B) \{\sigma_A^X - \langle \sigma_A^X \rangle_{x_j^B, R}\}^2$. We note that the symbol λ is used alternatively to R in the main text, to describe that the variables may also be continuous. The proof follows similarly in either case. Now, if we assume the separability between the bipartition $A - B$ for each state R, in accordance with the LHS model (8), then

$$
P_R(x_j^B, \sigma_A^X) = P_R(x_j^B) P_R(\sigma_A^X). \quad (A2)
$$

This implies $\langle \sigma_A^X \rangle_{x_j^B, R} = \langle \sigma_A^X \rangle_R$ and $\{\Delta_R(\sigma_A^X | x_j^B)\}^2 = (\Delta_R \sigma_A^X)^2$. Then we find, on using $\sum_{x_j^B} P_R(x_j^B) = 1$, that we can write $\{\Delta_{\inf,R} \sigma_A^X\}^2 = \{\Delta_R \sigma_A^X\}^2$. Thus, on applying the

Cauchy-Schwarz inequality, we see that

$$
\Delta_{\inf}^2 \sigma_A^X \Delta_{\inf}^2 \sigma_A^Y \geqslant \left(\sum_R P_R \{\Delta_R \sigma_A^X\}^2 \right) \left(\sum_R P_R \{\Delta_R \sigma_A^Y\}^2 \right)
$$
$$
\geqslant \left(\sum_R P_R \{\Delta_R \sigma_A^X\} \{\Delta_R \sigma_A^Y\} \right)^2,
$$

where we define $\Delta_{\inf}^2 \sigma_A^X \equiv (\Delta_{\inf} \sigma_A^X)^2$ and

$$
\Delta_{\inf}^2 \sigma_A^Y \equiv (\Delta_{\inf} \sigma_A^Y)^2 = \sum_{y_j^B} P(y_j^B) \{\Delta(\sigma_A^X | y_j^B)\}^2
$$

noting that the $\{y_j\}$ is the set of results for a measurement y made at B to infer the value of the measurement of σ_A^Y at A. We consider an LHS model (8) where we assume the states at A are local *quantum* states, so that we can use quantum uncertainty relations to derive a final steering inequality: e.g., $\{\Delta_R(\sigma_A^X)\}\{\Delta_R(\sigma_A^Y)\} \geqslant |\langle \sigma_A^Z \rangle_R|/2$ for any quantum state denoted by R. Using the above results, the LHS model implies

$$
(\Delta_{\inf} \sigma_A^X)(\Delta_{\inf} \sigma_A^Y) \geqslant \sum_R P_R \{\Delta_R \sigma_A^X\} \{\Delta_R \sigma_A^Y\}
$$
$$
\geqslant \sum_R P_R (|\langle \sigma_A^Z \rangle_R|/2).
$$

However, for a separable model, we know that $\langle \sigma_A^Z \rangle_{z_j^B, R} = \langle \sigma_A^Z \rangle_R$ and hence

$$
\sum_{z_j^B} P(z_j^B) \sum_R P_R |\langle \sigma_A^Z \rangle_{z_j^B, R}| = \sum_R P_R \sum_{z_j^B} P(z_j^B) |\langle \sigma_A^Z \rangle_R|
$$
$$
= \sum_R P_R |\langle \sigma_A^Z \rangle_R|,
$$

where here the $\{z_j\}$ is the set of results for a measurement z at B, that we use to infer results for σ_A^Z. Hence

$$
(\Delta_{\inf} \sigma_A^X)(\Delta_{\inf} \sigma_A^Y) \geqslant \sum_{z_j^B} P(z_j^B) \sum_R P_R |\langle \sigma_A^Z \rangle_{z_j^B, R}|/2
$$
$$
= \sum_{z_j^B} P(z_j^B) |\langle \sigma_A^Z \rangle_{z_j^B}|/2.
$$

We have used (for states constrained by the LHS model),

$$
\langle \sigma_A^Z \rangle_{z_j^B} = \sum_{\sigma_A^Z} \sigma_A^Z P(\sigma_A^Z | z_j^B)
$$
$$
= \sum_{\sigma_A^Z} \sigma_A^Z \sum_R P_R P_R(\sigma_A^Z | z_j^B)
$$
$$
= \sum_R P_R \langle \sigma_A^Z \rangle_{z_j^B, R}.
$$

Defining $|\langle \sigma_A^Z \rangle|_{\inf} = \sum_{z_j^B} P(z_j^B) |\langle \sigma_A^Z \rangle_{z_j^B}|$, we see finally that the LHS model implies $(\Delta_{\inf} \sigma_A^X)(\Delta_{\inf} \sigma_A^Y) \geqslant |\langle \sigma_A^Z \rangle|_{\inf}/2$. Violation of this inequality implies failure of the LHS model, and therefore implies steering of A by B. The result is steering of B by A if the A and B indices are exchanged (as in the main text). This completes the proof. ∎

APPENDIX B: EVALUATION OF INFERRED VARIANCES

Here we will evaluate the inferred uncertainties $\Delta_{\text{inf}}(X^N)$ and $\Delta_{\text{inf}}(P^N)$ for the NOON state given in Eq. (1). We first consider $X \equiv X_b$ and evaluate $\Delta_{\text{inf}}^2(X_b^N) \equiv [\Delta_{\text{inf}}(X_b^N)]^2$, which is given by (22). The terms of the form $\langle X_b^n \rangle_{\text{inf},x} \equiv \langle X_b^n | x \rangle$, with $n = N$ or $n = 2N$, are evaluated using the reduced density operator $\rho_{\text{red},x}$:

$$\rho_{\text{red},x} = \tfrac{1}{2}\{|\langle x|N\rangle|^2 |0\rangle\langle 0| + e^{-i\phi}\langle x|N\rangle\langle 0|x\rangle |0\rangle\langle N|$$
$$+ |\langle x|0\rangle|^2 |N\rangle\langle N| + e^{i\phi}\langle x|0\rangle\langle N|x\rangle |N\rangle\langle 0|\} \quad (B1)$$

and the fact that operators $\hat{X}$ and $\hat{P}$ can be described in terms of a complete set of projectors as $\hat{X}_B^n = \int_{-\infty}^{\infty} x_B^n |x_B\rangle\langle x_B| \, dx_B$ and $\hat{P}_B^n = \int_{-\infty}^{\infty} p_B^n |p_B\rangle\langle p_B| \, dp_B$. Therefore, we get

$$\langle X^n \rangle_{\text{inf},x} = \text{Tr}(\rho_{\text{red},x} X^n)$$
$$= \frac{1}{2P(x)}\left[|\langle x|N\rangle|^2 \int x_B^n |\langle x_B|0\rangle|^2 dx_B \right.$$
$$+ e^{-i\phi}\langle x|N\rangle\langle 0|x\rangle \int x_B^n \langle N|x_B\rangle\langle x_B|0\rangle \, dx_B$$
$$+ e^{i\phi}\langle x|0\rangle\langle N|x\rangle \int x_B^n \langle 0|x_B\rangle\langle x_B|N\rangle \, dx_B$$
$$\left. + |\langle x|0\rangle|^2 \int x_B^n |\langle x_B|N\rangle|^2 \, dx_B \right], \quad (B2)$$

where $P(x) = \tfrac{1}{2}[|\langle x|0\rangle|^2 + |\langle x|N\rangle|^2]$ is the probability of measuring X_A and getting outcome x and $\langle x|N\rangle$ are the harmonic-oscillator functions given in Eq. (27). The value for $\Delta_{\text{inf}}(X_b^N)$ is obtained on evaluating the expressions of $\langle X^n \rangle_{\text{inf},x}$, with $n = N$ or $2N$, and substituting on the expression given in Eq. (22). Similarly, we evaluate the inferred variance of $P \equiv P_b$, which is given by (21). Using the reduced density operator $\rho_{\text{red},x}$ given above we find

$$\langle P^n \rangle_{\text{inf},x} = \frac{1}{2P(x)}\left[|\langle x|N\rangle|^2 \int p_B^n |\langle p_B|0\rangle|^2 dp_B \right.$$
$$+ e^{-i\phi}\langle x|N\rangle\langle 0|x\rangle \int p_B^n \langle 0|p_B\rangle\langle p_B|N\rangle \, dp_B$$
$$+ e^{i\phi}\langle x|0\rangle\langle N|x\rangle \int p_B^n \langle N|p_B\rangle\langle p_B|0\rangle \, dp_B$$
$$\left. + |\langle x|0\rangle|^2 \int p_B^n |\langle p_B|N\rangle|^2 \, dp_B \right]. \quad (B3)$$

To evaluate, we first consider $N = 2$. We let $\phi = \pi/2$:

$$\langle X^2 \rangle_{\text{inf},x} = \frac{1}{2P(x)}\left[|\langle x|2\rangle|^2 \int x_B^2 |\langle x_B|0\rangle|^2 dx_B \right.$$
$$\left. + |\langle x|0\rangle|^2 \int x_B^2 |\langle x_B|2\rangle|^2 \, dx_B \right]$$
$$= 1 + \frac{8}{3 - 2x^2 + x^4}$$

and

$$\langle X^4 \rangle_{\text{inf},x} = 3 + \frac{72}{x^4 - 2x^2 + 3},$$

where we have used that $P(x) = \frac{e^{-\frac{x^2}{2}}}{2\sqrt{2\pi}}(\frac{(2x^2-2)^2}{8} + 1)$. On performing the integration using the above results we get for the $N = 2$ state that $\Delta_{\text{inf}}^2(X^2) = 10.1351$ and $\Delta_{\text{inf}}(X^2) = 3.18356$. Similarly we evaluate $\Delta_{\text{inf}}^2(P_b^N)$:

$$\langle P^2 \rangle_{\text{inf},x} = 1 + \frac{8}{3 - 2x^2 + x^4},$$
$$\langle P^4 \rangle_{\text{inf},x} = 3 + \frac{72}{x^4 - 2x^2 + 3}.$$

These results are the same as for X, since for this value of angle $e^{i\phi} = i = -e^{-i\phi}$, and also

$$\int p_B^n \langle 0|p_B\rangle\langle p_B|2\rangle \, dp_B = \int p_B^n \langle 2|p_B\rangle\langle p_B|0\rangle \, dp_B$$
$$= \int x_B^n \langle 0|x_B\rangle\langle x_B|2\rangle \, dx_B,$$

so that the second and third terms of Eq. (B3) cancel. We obtain for $N = 2$ that $\Delta_{\text{inf}}^2(P_b^2) = 10.1351$.

Continuing for higher N, we obtain for $N = 3$, $\Delta_{\text{inf}}^2(P_b^3) = 477.081$ and $\Delta_{\text{inf}}(P_b^3) = 21.8422$; for $N = 4$, $\Delta_{\text{inf}}^2(P_b^4) = 10982.8$ and $\Delta_{\text{inf}}(P_b^4) = 104.799$; and for $N = 5$, $\Delta_{\text{inf}}^2(P_b^5) = 795639$ and $\Delta_{\text{inf}}(P_b^5) = 891.986$. Identical results are obtained for the inferred variances in X^N.

APPENDIX C: EVALUATION OF $|\langle [n_b, X_b^N] \rangle|_{\text{inf}}$ AND $|\langle [n_b, P_b^N] \rangle|_{\text{inf}}$

The expressions for the terms $|\langle [n_b, X_b^N] \rangle|_{\text{inf}}$ and $|\langle [n_b, P_b^N] \rangle|_{\text{inf}}$ are calculated from Eq. (16) and the first line of Eq. (18). Using that $X_b = b + b^\dagger$ and $P_b = (b - b^\dagger)/i$, we note that on evaluating the expectation value for the NOON states given in Eq. (1), the only nonzero contributions involve terms of the form $\langle b^N \rangle$ and $\langle b^{\dagger N} \rangle$:

$$|\langle [n_b, X_b^N] \rangle|_{\text{inf}} = N| - \langle b^N \rangle + \langle b^{\dagger N} \rangle|_{\text{inf}},$$
$$|\langle [n_b, P_b^N] \rangle|_{\text{inf}} = N|\langle b^N \rangle + (-1)^{N+1}\langle b^{\dagger N} \rangle|_{\text{inf}}. \quad (C1)$$

We evaluate $\langle b^N \rangle = \text{Tr}[\rho_{\text{red},x} b^N]$ and $\langle b^{\dagger N} \rangle = \text{Tr}[\rho_{\text{red},x} (b^\dagger)^N]$ using the reduced density matrix given in Eq. (B1):

$$\langle \hat{b}^N \rangle_{\text{inf},x} = \frac{e^{i\phi}}{2P(x)}\sqrt{N!}\langle x|0\rangle\langle x|N\rangle,$$
$$\langle (\hat{b}^\dagger)^N \rangle_{\text{inf},x} = \frac{e^{-i\phi}}{2P(x)}\sqrt{N!}\langle x|0\rangle\langle x|N\rangle. \quad (C2)$$

On integrating over all possible values we get

$$|\langle [n_b, X_b^N] \rangle|_{\text{inf}} = N\sqrt{N!}|\sin\phi| \int_{-\infty}^{\infty} |\langle x|N\rangle\langle 0|x\rangle| \, dx.$$

For $|\langle [n_b, P_b^N] \rangle|_{\text{inf}}$ the expression for N odd is given by

$$|\langle [n_b, P_b^N] \rangle|_{\text{inf}} = N\sqrt{N!}|\cos\phi| \int_{-\infty}^{\infty} |\langle x|N\rangle\langle 0|x\rangle| \, dx,$$

while for N even we find

$$\left|\langle [n_b, P_b^N]\rangle\right|_{\mathrm{inf}} = N\sqrt{N!}\,|\sin\phi|\int_{-\infty}^{\infty}|\langle x|N\rangle\langle 0|x\rangle|\,dx. \tag{C3}$$

We obtain for $N = 2$ that $\left|\langle [n_b, P_b^N]\rangle\right|_{\mathrm{inf}} = 1.93577$ with $\phi = \pi/2$; for $N = 3$, $\left|\langle [n_b, P_b^N]\rangle\right|_{\mathrm{inf}} = 4.53$ with $\phi = 0$; for $N = 4$, $\left|\langle [n_b, P_b^N]\rangle\right|_{\mathrm{inf}} = 11.2024$ with $\phi = \pi/2$; and for $N = 5$, $\left|\langle [n_b, P_b^N]\rangle\right|_{\mathrm{inf}} = 29.5504$ with $\phi = 0$.

APPENDIX D: INCLUDING LOSSES

The detected fields $\hat{a}_{\mathrm{det}}$, $\hat{b}_{\mathrm{det}}$ are given by

$$a_{\mathrm{det}} = \sqrt{\eta_a}\,a + \sqrt{1-\eta_a}\,a_v, \quad a_{\mathrm{loss}} = -\sqrt{1-\eta_a}\,a + \sqrt{\eta_a}\,a_v,$$

with similar definitions for the mode operators b_{det} and b_{loss}. Using these transformations it is possible to write the operators a, b and hence the NOON state $|\psi\rangle$ of Eq. (1) in terms of $a_{\mathrm{det}}^\dagger$, $a_{\mathrm{loss}}^\dagger$, $b_{\mathrm{det}}^\dagger$, and $b_{\mathrm{loss}}^\dagger$. We will denote the vacuum state for all four modes by $|0\rangle$. The density operator $\rho = |\psi\rangle\langle\psi|$ can then also be expressed in terms of these operators. Since we are not interested in the modes a_{loss} and b_{loss} (which we label A,loss and B,loss) we take the trace over the states of the loss mode to evaluate $\rho' \equiv \mathrm{Tr}_{A,\mathrm{loss};B,\mathrm{loss}}\rho$. After using the binomial expansion for terms such as $(\sqrt{\eta_a}\,a_{\mathrm{det}}^\dagger - \sqrt{(1-\eta_a)}\,a_{\mathrm{loss}}^\dagger)$ and performing the trace, the reduced density operator for the detected modes is

$$\rho' = \frac{1}{2}\Bigg[\sum_s \binom{N}{N-s}(\eta_a)^{N-s}(1-\eta_a)^s\,|N-s\rangle_{A,\mathrm{det}}\langle N-s|\otimes|0\rangle_{B,\mathrm{det}}\langle 0| + (\sqrt{\eta_a\eta_b})^N e^{-i\phi}\,|N\rangle_{A,\mathrm{det}}\langle 0|\otimes|0\rangle_{B,\mathrm{det}}\langle N|$$

$$+ (\sqrt{\eta_a\eta_b})^N e^{i\phi}\,|0\rangle_{A,\mathrm{det}}\langle N|\otimes|N\rangle_{B,\mathrm{det}}\langle 0| + \sum_s \binom{N}{N-s}(\eta_b)^{N-s}(1-\eta_b)^s\,|0\rangle_{A,\mathrm{det}}\langle 0|\otimes|N-s\rangle_{B,\mathrm{det}}\langle N-s|\Bigg]. \tag{D1}$$

1. Calculating $\Delta_{\mathrm{inf}}^2(P_b^N)$ and $\Delta_{\mathrm{inf}}^2(X_b^N)$

The $\Delta_{\mathrm{inf}}^2(P_b^N)$ and $\Delta_{\mathrm{inf}}^2(X_b^N)$ are the inferred variances of quantities P_b^N and X_b^N due to a measurement in X_a. These are given by (21) and (22). We evaluate these inferred variances using the density operator for modes a_{det} and b_{det} given in Eq. (D1). For the inferred variances we evaluate the density operator ρ'', where we consider that the mode A,det is in the state $|x\rangle$. This density operator is given by

$$\rho'' = \frac{|x\rangle_{A,\mathrm{det}}\langle x|\rho'|x\rangle_{A,\mathrm{det}}\langle x|}{P(x)}$$

$$= \frac{1}{2P(x)}\Bigg[\sum_s \binom{N}{N-s}(\eta_a)^{N-s}(1-\eta_a)^s\,\langle x|N-s\rangle_{Ad}\langle N-s|x\rangle|x\rangle_{Ad}\langle x|\otimes|0\rangle_{Bd}\langle 0|$$

$$+ (\sqrt{\eta_a\eta_b})^N e^{-i\phi}\,\langle x|N\rangle_{Ad}\langle 0|x\rangle|x\rangle_{Ad}\langle x|\otimes|0\rangle_{Bd}\langle N| + (\sqrt{\eta_a\eta_b})^N e^{i\phi}\,\langle x|0\rangle_{Ad}\langle N|x\rangle|x\rangle_{Ad}\langle x|\otimes|N\rangle_{Bd}\langle 0|$$

$$+ \sum_s \binom{N}{N-s}(\eta_b)^{N-s}(1-\eta_b)^s\,\langle x|0\rangle_{Ad}\langle 0|x\rangle|x\rangle_{Ad}\langle x|\otimes|N-s\rangle_{Bd}\langle N-s|\Bigg],$$

where

$$P(x) = \mathrm{Tr}[|x\rangle_{A,\mathrm{det}}\langle x|\rho'|x\rangle_{A,\mathrm{det}}\langle x|]$$

$$= \frac{1}{2}\Bigg[\sum_s \binom{N}{N-s}(\eta_a)^{N-s}(1-\eta_a)^s\,|\langle x|N-s\rangle|^2 + \sum_s \binom{N}{N-s}(\eta_b)^{N-s}(1-\eta_b)^s\,|\langle x|0\rangle|^2\Bigg]. \tag{D2}$$

Here we are using the following notation for the modes: $Ad \equiv A$,det and $Bd \equiv B$,det. In order to compute $\Delta^2(P_b^N|x)$ and $\Delta^2(X_b^N|x)$, we trace out the A,det mode to get the reduced density operator for B,det mode:

$$\rho_{\mathrm{red,det},x} = Tr_{A,\mathrm{det}}(\rho'')$$

$$= \frac{1}{2P(x)}\Bigg[\sum_s \binom{N}{N-s}(\eta_a)^{N-s}(1-\eta_a)^s\,\langle x|N-s\rangle_{Ad}\langle N-s|x\rangle|0\rangle_{Bd}\langle 0|$$

$$+ (\sqrt{\eta_a\eta_b})^N e^{-i\phi}\,\langle x|N\rangle_{Ad}\langle 0|x\rangle|0\rangle_{Bd}\langle N| + (\sqrt{\eta_a\eta_b})^N e^{i\phi}\,\langle x|0\rangle_{Ad}\langle N|x\rangle|N\rangle_{Bd}\langle 0|$$

$$+ \sum_s \binom{N}{N-s}(\eta_b)^{N-s}(1-\eta_b)^s\,\langle x|0\rangle_{Ad}\langle 0|x\rangle|N-s\rangle_{Bd}\langle N-s|\Bigg]. \tag{D3}$$

The inferred variances are defined as

$$\Delta^2\left(X_b^N|x\right) = \left\langle\left(X_b^N\right)^2|x\right\rangle - \left\langle X_b^N|x\right\rangle^2, \quad \Delta^2\left(P_b^N|x\right) = \left\langle\left(P_b^N\right)^2|x\right\rangle - \left\langle P_b^N|x\right\rangle^2. \tag{D4}$$

Next we evaluate $\langle X_b^n|x\rangle = \mathrm{Tr}[\rho_{\mathrm{red,det},x} X_b^n]$ and $\langle P_b^n|x\rangle = \mathrm{Tr}[\rho_{\mathrm{red,det},x} P_b^n]$ using the density operator given in Eq. (D3) obtaining

$$\begin{aligned}
\langle X_b^n|x\rangle = \frac{1}{2P(x)}\Bigg[&\sum_s \binom{N}{N-s}(\eta_a)^{N-s}(1-\eta_a)^s \langle x|N-s\rangle_{Ad}\langle N-s|x\rangle \int x_B^n \langle 0|x_B\rangle\langle x_B|0\rangle\,dx_B \\
&+(\sqrt{\eta_a\eta_b})^N e^{-i\phi}\langle x|N\rangle_{Ad}\langle 0|x\rangle \int x_B^n \langle N|x_B\rangle\langle x_B|0\rangle dx_B + (\sqrt{\eta_a\eta_b})^N e^{i\phi}\langle x|0\rangle_{Ad}\langle N|x\rangle \int x_B^n \langle 0|x_B\rangle\langle x_B|N\rangle dx_B \\
&+\sum_s \binom{N}{N-s}(\eta_b)^{N-s}(1-\eta_b)^s \langle x|0\rangle_{Ad}\langle 0|x\rangle \int x_B^n \langle N-s|x_B\rangle\langle x_B|N-s\rangle\,dx_B \Bigg],
\end{aligned} \tag{D5}$$

$$\begin{aligned}
\langle P^n\rangle_{\mathrm{inf},x} = \frac{1}{2P(x)}\Bigg[&\sum_s \binom{N}{N-s}(\eta_a)^{N-s}(1-\eta_a)^s \langle x|N-s\rangle_{A,\mathrm{det}}\langle N-s|x\rangle \int p_B^n \langle 0|p_B\rangle\langle p_B|0\rangle\,dp_B \\
&+(\sqrt{\eta_a\eta_b})^N e^{-i\phi}\langle x|N\rangle_{Ad}\langle 0|x\rangle \int p_B^n \langle N|p_B\rangle\langle p_B|0\rangle dp_B + (\sqrt{\eta_a\eta_b})^N e^{i\phi}\langle x|0\rangle_{Ad}\langle N|x\rangle \int p_B^n \langle 0|p_B\rangle\langle p_B|N\rangle\,dp_B \\
&+\sum_s \binom{N}{N-s}(\eta_b)^{N-s}(1-\eta_b)^s \langle x|0\rangle_{A,\mathrm{det}}\langle 0|x\rangle \int p_B^n \langle N-s|p_B\rangle\langle p_B|N-s\rangle\,dp_B \Bigg].
\end{aligned} \tag{D6}$$

The value of the corresponding variances for $\Delta^2(X_b^N|x)$ and $\Delta^2(P_b^N|x)$ of Eqs. (D4) is evaluated using the expressions given in Eqs. (D5) and (D6) considering $n = N$ or $n = 2N$.

2. Inferred variances $\Delta_{\mathrm{inf}}^2(n_b)$ including losses

$\Delta_{\mathrm{inf}}^2(n_b)$ is the inferred variance of n_b due to a measurement in n_a. In order to evaluate this variance we will consider that the outcome in n_a is m. We define $P(m)$ as the probability for obtaining the result m for n_a. Next, we evaluate the reduced density operator ρ_m for the modes A,det and B,det given that the outcome is m:

$$\begin{aligned}
\rho_m &= \frac{1}{P(m)}[|m\rangle_{Ad}\langle m|\rho'|m\rangle_{Ad}\langle m|] \\
&= \left[\binom{N}{m}\eta_a^m(1-\eta_a)^{N-m}|m\rangle_{Ad}\langle m| \otimes |0\rangle_{Bd}\langle 0| + \sum_s \binom{N}{N-s}\eta_b^{N-s}(1-\eta_b)^s|0\rangle_{Ad}\langle 0| \otimes |N-s\rangle_{Bd}\langle N-s| \right]\Big/[2P(m)],
\end{aligned}$$

where

$$P(m) = \mathrm{Tr}[|m\rangle_{A,\mathrm{det}}\langle m|\rho'|m\rangle_{A,\mathrm{det}}\langle m|] = \frac{1}{2}\binom{N}{m}\eta_a^m(1-\eta_a)^{N-m} + \frac{1}{2}. \tag{D7}$$

In order to write the last line we have used that $\sum_s^N \binom{N}{N-s}\eta_b^{N-s}(1-\eta_b)^s = 1$.

Next we evaluate $\langle n_B\rangle_{\mathrm{inf},m} = \mathrm{Tr}[\rho_m n_B]$ and $\langle n_B^2\rangle_{\mathrm{inf},m} = \mathrm{Tr}[\rho_m n_B^2]$ obtaining

$$\langle n_B\rangle_{\mathrm{inf},m} = \frac{1}{2}\frac{\sum_s \binom{N}{N-s}\eta_b(1-\eta_b)^s\delta_{m,0}(N-s)}{P(n_A=m)}, \quad \langle n_B^2\rangle_{\mathrm{inf},m} = \frac{1}{2}\frac{\sum_s \binom{N}{N-s}\eta_b(1-\eta_b)^s\delta_{m,0}(N-s)^2}{P(n_A=m)}.$$

Since $n_A = m = 0$ is the only nonzero contribution for the statistical moments we obtain

$$\langle n_B\rangle_{\mathrm{inf},0} = \frac{1}{2}\frac{N\eta_b}{P(n_A=0)}, \quad \langle n_B^2\rangle_{\mathrm{inf},0} = \frac{1}{2}\frac{\eta_b(N-N\eta_b+N^2\eta_b)}{P(n_A=0)}, \quad P(n_A=0) = \frac{1}{2}[(1-\eta_a)^N+1].$$

Using the above results we evaluate the inferred variance for $m = 0$, which we denote by $\Delta_{\mathrm{inf}}^2 n_{b,0}$:

$$\Delta_{\mathrm{inf}}^2 n_{b,0} = \frac{\eta_b(N-N\eta_b)+N\eta_b(1-\eta_a)^N(1-\eta_b+N\eta_b)}{[(1-\eta_a)^N+1]^2}.$$

In order to evaluate the variance of the inferred value n_B, we sum over all possible values of m obtaining

$$\Delta^2 n_{\mathrm{inf}} = \sum_m^N P(n_A=0)\Delta^2 n_{\mathrm{inf},m=0} = \frac{\eta_b(N-N\eta_b)+N(1-\eta_a)^N(\eta_b-\eta_b^2+N\eta_b^2)}{2[(1-\eta_a)^N+1]}. \tag{D8}$$

3. Evaluation of $|\langle[n_b, X_b^N]\rangle|_{\inf}$ and $|\langle[n_b, P_b^N]\rangle|_{\inf}$

Full evaluation of the terms $|\langle[n_b, X_b^N]\rangle|_{\inf}$ and $|\langle[n_b, P_b^N]\rangle|_{\inf}$ [given by Eqs. (16) and (18)] reveals that for the lossy system and for $N \leqslant 5$:

$$\left|\left[n_b, X_b^N\right]\right\rangle\big|_{\inf} = N| - \langle b^N\rangle + \langle b^{\dagger N}\rangle|_{\inf}, \quad \left|\left[n_b, P_b^N\right]\right\rangle\big|_{\inf} = N|\langle b^N\rangle + (-1)^{N+1}\langle b^{\dagger N}\rangle|_{\inf}.$$

We evaluate $\langle b^N\rangle = \text{Tr}[\rho_{\text{red,det},x} b^N]$ and $\langle b^{\dagger N}\rangle = \text{Tr}[\rho_{\text{red,det},x}(b^\dagger)^N]$ using the reduced density matrix given in Eq. (D3) and performing the corresponding trace, we obtain

$$\langle b^N\rangle_{\inf,x} = \frac{1}{2P(x)}(\sqrt{\eta_a \eta_b})^N e^{i\phi}\langle x|N\rangle\langle 0|x\rangle\sqrt{N!}$$

and $\langle(b^\dagger)^N\rangle_{\inf,x} = (\langle b^N\rangle_{\inf,x})^*$. Thus the expressions obtained are identical to (C2) and (C3) but replacing $\sqrt{N!}$ with $C_\eta = \sqrt{N!}(\sqrt{\eta_a \eta_b})^N$.

APPENDIX E: PROOF OF Nth-ORDER QUANTUM COHERENCE AND ENTANGLEMENT

The question is how to prove experimentally that the system is indeed in a superposition of the two number states $|N\rangle|0\rangle$ and $|0\rangle|N\rangle$ that are distinct by N quanta in each mode. For such a state, the density matrix ρ has a nonzero off-diagonal element:

$$\langle 0|\langle N|\rho|0\rangle|N\rangle \neq 0. \tag{E1}$$

We refer to the nonzero term (E1) as an Nth-*order quantum coherence*. The presence of this term distinguishes the superposition of the two states $|N\rangle|0\rangle$, $|0\rangle|N\rangle$ from a classical mixture of the two states.

In the ideal scenario, the experiment generates only outcomes zero or N for the number measurements n_a or n_b. It is then straightforward to show that any violation of an EPR steering inequality is also a signature of an Nth-order quantum coherence (E1). The objective is to construct the density operator ρ for the system and to prove that necessarily (E1) holds. Since there are only two outcomes for each mode, any viable two-mode density operator could be written in terms of four basis states $|0\rangle|0\rangle, |0\rangle|N\rangle, |N\rangle|0\rangle, |N\rangle|N\rangle$. Supposing an EPR steering inequality to be violated, this will negate the LHS model given by (8) and therefore also any fully separable quantum model

$$\rho = \sum_R P_R \rho_a \rho_b, \tag{E2}$$

where ρ_a and ρ_b are density matrices for the single modes a and b [22]. Thus the system cannot be in any mixture of the basis states $|0\rangle|0\rangle, |0\rangle|N\rangle, |N\rangle|0\rangle, |N\rangle|N\rangle$, which are separable states. There are only certain remaining possibilities for ρ and these require nonzero off-diagonal elements. For example, for the NOON state the results for number measurements $\hat{n}$ would be either zero or N in one mode, correlated with N or zero in the other mode. Assuming that there is a nonzero probability for the outcome for $|0\rangle|N\rangle$ and $|N\rangle|0\rangle$, this ensures that the off-diagonal term $\langle 0|\langle N|\rho|0\rangle|N\rangle$ is nonzero. Also, the failure of the separable model (E2) ensures the system cannot be in the product state $(|N\rangle + |0\rangle)(|N\rangle + |0\rangle)/2$. This implies the system is the entangled superposition of states $|0\rangle|N\rangle$ and $|N\rangle|0\rangle$.

APPENDIX F: PROOF OF INEQUALITY (41)

First we prove the the uncertainty relation

$$(\Delta \hat{n}_b)^2 \left(\Delta_{\inf} \hat{P}_b^N\right)^2 \geqslant \tfrac{1}{4}|\langle \hat{C}_b\rangle|_{\inf}^2, \tag{F1}$$

which holds for any two-mode state. We follow the methods used in Refs. [41]. The variance is defined as $(\Delta \hat{n}_b)^2 = \sum_{n_b} P(n_b)(n_b - \langle n_b\rangle)^2$ (denoting the outcomes of $\hat{n}_b$ by n_b). We can consider marginals and joint distributions for the measurements on both modes a and b. Thus we write $(\Delta \hat{n}_b)^2 = \sum_{n_b, p_a} P(n_b, p_a)(n_b - \langle n_b\rangle)^2$ and then $(\Delta \hat{n}_b)^2 = \sum_{n_b, p_a} P(n_b|p_a)P(p_a)(n_b - \langle n_b\rangle)^2$. Thus

$$(\Delta \hat{n}_b)^2 = \sum_{p_a} P(p_a) \sum_{n_b} P(n_b|p_a)(n_b - \langle n_b\rangle)^2 \geqslant \sum_{p_a} P(p_a) \sum_{n_a} P(n_b|p_a)(n_b - \langle n_b\rangle_{p_a})^2 = \sum_{p_a} P(p_a)(\Delta(n_b|p_a))^2,$$

where $(\Delta(n_b|p_a))^2 = \sum_{n_a} P(n_b|p_a)(n_b - \langle n_b\rangle_{p_a})^2$ and $\langle n_b\rangle_{p_a}$ is the mean of the conditional distribution $P(n_b|p_a)$. For each p_a we have defined the distribution $P(n_b|p_a)$ as $P_{p_a}(n_b)$ and we see that the quantity $\sum_{n_b} P_{p_a}(n_b)(n_b - X)^2$, where X is any

constant, is minimized by the choice $X = \langle n_b \rangle_{p_a} = \sum_{n_b} P_{p_a}(n_b) n_b$. Next, we write

$$(\Delta \hat{n}_b)^2 \left(\Delta_{\inf} \hat{P}_b^N\right)^2 \geqslant \left\{ \sum_{p_a} P(p_a)(\Delta(n_b|p_a))^2 \right\} \left\{ \sum_{p_a} P(p_a)\left(\Delta\left(P_b^N|p_a\right)\right)^2 \right\}$$

$$\geqslant \left| \sum_{p_a} P(p_a) \Delta(n_b|p_a) \Delta\left(P_b^N|p_a\right) \right|^2 \geqslant \frac{1}{4} \left| \sum_{p_a} P(p_a)|\langle \hat{C}_b \rangle_{p_a}| \right|^2 = \frac{1}{4}|\langle \hat{C}_b \rangle|_{\inf}^2,$$

where we apply the Cauchy-Schwarz inequality and use the uncertainty relation (2) that holds for the state of b conditioned on the measurement result p_a of mode a. This proves (F1). Then we can say that for the mixture ρ^{ab} of (42) [using that for a mixture it is true that [46] $(\Delta O)_\rho^2 \geqslant \sum_i P_i (\Delta O)_i^2$ where O is any quantum observable, and also true that $(\Delta_{\inf} P_b^N)^2 \geqslant \sum_i P_i (\Delta_{\inf} P_b^N)_i^2$ [47]]:

$$\left\{ \sum_i P_i (\Delta \hat{n}_b)_i^2 \right\} \left(\Delta_{\inf} \hat{P}_b^N\right)^2 \geqslant \left\{ \sum_i P_i (\Delta n_b)_i^2 \right\} \left\{ \sum_i P_i \left(\Delta_{\inf} P_b^N\right)_i^2 \right\} \geqslant \left| \sum_i P_i (\Delta n_b)_i \left(\Delta_{\inf} P_b^N\right)_i \right|^2 \geqslant \frac{1}{4} \left| \sum_i P_i |\langle \hat{C}_b \rangle|_{\inf,i} \right|^2,$$

$$(\text{F2})$$

where we use the Cauchy-Schwarz inequality and that the uncertainty relation (2) holds for each ρ_i^{ab}. Now we see that $\sum_{p_a} P(p_a)|\langle \hat{C}_b \rangle_{p_a}| = \sum_{p_a} P(p_a)| \sum_{c_b} C_b P(C_b|p_a)|$. If the system is described by the mixture ρ^{ab} then

$$\langle C_b \rangle_{p_a} = \sum_{c_b} C_b P(C_b|p_a) = \sum_{C_b} C_b \frac{P(C_b, p_a)}{P(p_a)} = \sum_{C_b} C_b \sum_i P_i \frac{P_i(C_b, p_a)}{P(p_a)} = \sum_i P_i \sum_{C_b} C_b \frac{P_i(p_a)}{P(p_a)} P_i(C_b|p_a), \quad (\text{F3})$$

where the subscript i denotes the probabilities for the component ρ_i^{ab}. We can write

$$\left| \sum_i P_i \sum_{C_b} C_b \frac{P_i(p_a)}{P(p_a)} P_i(C_b|p_a) \right| \leqslant \sum_i P_i \frac{P_i(p_a)}{P(p_a)} \left| \sum_{C_b} C_b P_i(C_b|p_a) \right|.$$

Thus from (F3)

$$\sum_{p_a} P(p_a)|\langle C_b \rangle_{p_a}| \leqslant \sum_i P_i \sum_{p_a} P_i(p_a) \left| \sum_{C_b} C_b P_i(C_b|p_a) \right| = \sum_i P_i |\langle \hat{C}_b \rangle|_{\inf,i},$$

where $|\langle \hat{C}_b \rangle|_{\inf,i} = \sum_{p_a} P_i(p_a)| \sum_{C_b} C_b P_i(C_b|p_a)|$. Thus we have proved that $\sum_i P_i |\langle \hat{C}_b \rangle|_{\inf,i} \geqslant \sum_{p_a} P(p_a)|\langle C_b \rangle_{p_a}|$. Hence we can write from (F2):

$$\left\{ \sum_i P_i (\Delta n_b)_i^2 \right\} \left(\Delta_{\inf} P_b^N\right)^2 \geqslant \frac{1}{4} \left| \sum_i P_i \left| \langle \hat{C}_b \rangle|_{\inf,i} \right|^2 \geqslant \frac{1}{4} \left| \sum_{p_a} P(p_a) \left| \langle C_b \rangle_{p_a} \right| \right|^2 = \frac{1}{4}|\langle \hat{C}_b \rangle|_{\inf}^2.$$

This proves the inequality (41).   ∎

[1] E. Schroedinger, Naturwiss. **23**, 807 (1935).

[2] W. H. Zurek, Phys. Today **44**, 36 (1991); A. O. Caldeira and A. J. Leggett, Phys. Rev. A **31**, 1059 (1985); D. F. Walls and G. J. Milburn, *ibid.* **31**, 2403 (1985).

[3] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).

[4] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **77**, 4887 (1996); C. Monroe *et al.*, Science **272**, 1131 (1996); J. Friedman *et al.*, Nature (London) **406**, 43 (2000); D. Leibfried *et al.*, *ibid.* **438**, 639 (2005); A. Ourjoumtsev *et al.*, *ibid.* **448**, 784 (2007).

[5] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987); Z. Y. Ou, L. J. Wang, X. Y. Zou, and L. Mandel, Phys. Rev. A **41**, 566 (1990); **42**, 2957 (1990).

[6] J. P. Dowling, Contemp. Phys. **49**, 125 (2008).

[7] H. Lee, P. Kok, and J. P. Dowling, J. Mod. Opt. **49**, 2325 (2002);

[8] P. Kok, H. Lee, and J. P. Dowling, Phys. Rev. A **65**, 052104 (2002); J. Fiurášek, *ibid.* **65**, 053818 (2002).

[9] P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature (London) **429**, 158 (2004).

[10] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Nature (London) **429**, 161 (2004).

[11] I. Afek, O. Ambar, and Y. Silberberg, Science **328**, 879 (2010).

[12] Y. Israel, I. Afek, S. Rosen, O. Ambar, and Y. Silberberg, Phys. Rev. A **85**, 022115 (2012).

[13] H. Kim, H. S. Park, and S.-K. Choi, Opt. Express **17**, 19720 (2009).

[14] R. Kruse, L. Sansoni, S. Brauner, R. Ricken, C. S. Hamilton, I. Jex, and C. Silberhorn, Phys. Rev. A **92**, 053841 (2015); J. Lugani, S. Ghosh, and K. Thyagarajan, *ibid.* **83**, 062333 (2011).

[14] A. N. Boto, P. Kok, D. S. Abrams, S. L. Braunstein, C. P. Williams, and J. P. Dowling, Phys. Rev. Lett. **85**, 2733 (2000); Y.-H. Kim and Y. H. Shih, Found. Phys. **29**, 1849 (1999).

[15] M. D'Angelo, M. V. Chekhova, and Y. Shih, Phys. Rev. Lett. **87**, 013602 (2001).

[16] R. W. Boyd and J. P. Dowling, Quant. Inf. Process. **11**, 891 (2012).

[17] R. J. Lewis-Swan and K. V. Kheruntsyan, Nat. Commun. **5**, 3752 (2014); S. S. Hodgman, R. G. Dall, A. G. Manning, K. G. H. Baldwin, and A. G. Truscott, Science **331**, 1046 (2011).

[18] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A **55**, 4318 (1997); J. I. Cirac, M. Lewenstein, K. Molmer, and P. Zoller, *ibid.* **57**, 1208 (1998); D. Gordon and C. M. Savage, *ibid.* **59**, 4623 (1999); J. Dunningham and K. Burnett, J. Mod. Opt. **48**, 1837 (2001); C. Lee, Phys. Rev. Lett. **97**, 150402 (2006); T. J. Haigh, A. J. Ferris, and M. K. Olsen, Opt. Commun. **283**, 3540 (2010); L. D. Carr, D. R. Dounas-Frazer, and M. A. Garcia-March, Europhys. Lett. **90**, 10005 (2010); H. Cable, F. Laloë, and W. J. Mullin, Phys. Rev. A **83**, 053626 (2011); Q. Y. He, P. D. Drummond, M. K. Olsen, and M. D. Reid, *ibid.* **86**, 023626 (2012).

[19] See *Macroscopic Quantumness: Theory and Applications in Optical Sciences*, special issue of Opt. Commun. **337**, 1 (2015).

[20] B. Hensen *et al.*, Nature (London) **526**, 682 (2015); M. Giustina *et al.*, Phys. Rev. Lett. **115**, 250401 (2015); L. K. Shalm *et al.*, *ibid.* **115**, 250402 (2015).

[21] N. D. Mermin, Phys. Rev. D **22**, 356 (1980); P. D. Drummond, Phys. Rev. Lett. **50**, 1407 (1983); M. Zukowski, Phys. Lett. A **177**, 290 (1993); A. Peres, Found. Phys. **29**, 589 (1999); H. Weinfurter and M. Zukowski, Phys. Rev. A **64**, 010102 (2001); M. D. Reid, Phys. Rev. Lett. **84**, 2765 (2000); M. Zukowski and C. Brukner, *ibid.* **88**, 210401 (2002); D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, *ibid.* **88**, 040404 (2002); M. D. Reid, W. J. Munro, and F. De Martini, Phys. Rev. A **66**, 033801 (2002); M. Eibl, S. Gaertner, M. Bourennane, C. Kurtsiefer, M. Zukowski, and H. Weinfurter, Phys. Rev. Lett. **90**, 200403 (2003); A. Lamas-Linares, J. Howell, and D. Bouwmeester, Nature (London) **412**, 887 (2001).

[22] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007); S. J. Jones, H. M. Wiseman, and A. C. Doherty, Phys. Rev. A **76**, 052116 (2007).

[23] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A **80**, 032112 (2009).

[24] D. Smith *et al.*, Nat. Commun. **3**, 625 (2012); B. Wittmann *et al.*, New J. Phys. **14**, 053030 (2012); A. J. Bennet, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, Phys. Rev. X **2**, 031003 (2012).

[25] M. D. Reid *et al.*, Rev. Mod. Phys. **81**, 1727 (2009).

[26] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[27] E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935); **32**, 446 (1936).

[28] M. Born, *Born-Einstein Letters, 1916-1955: Friendship, Politics and Physics in Uncertain Times* (Palgrave MacMillan, London, 2005).

[29] P. Chowdhury, T. Pramanik, A. S. Majumdar, and G. S. Agarwal, Phys. Rev. A **89**, 012104 (2014); P. Chowdhury, T. Pramanik, and A. S. Majumdar, *ibid.* **92**, 042317 (2015).

[30] C. F. Wildfeuer, A. P. Lund, and J. P. Dowling, Phys. Rev. A **76**, 052101 (2007); F. Toppel, M. V. Chekhova, and G. Leuchs, arXiv:1607.01296 [quant-ph] (2016); I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acin, and G. Adesso, Phys. Rev. Lett. **115**, 210401 (2015).

[31] S. M. Tan, D. F. Walls, and M. J. Collett, Phys. Rev. Lett. **66**, 252 (1991); L. Hardy, *ibid.* **73**, 2279 (1994); J. Dunningham and V. Vedral, *ibid.* **99**, 180404 (2007); S. A. Babichev, J. Appel, and A. I. Lvovsky, *ibid.* **92**, 193601 (2004).

[32] S. J. Jones and H. M. Wiseman, Phys. Rev. A **84**, 012110 (2011).

[33] M. Fuwa, S. Takeda, M. Zwierz, H. M. Wiseman, and A. Furusawa, Nat. Commun. **6**, 6665 (2015).

[34] M. D. Reid, Phys. Rev. A **40**, 913 (1989).

[35] M. D. Reid, Phys. Rev. A **88**, 062338 (2013).

[36] J. S. Bell, Physics **1**, 195 (1964); J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978); N. Brunner *et al.*, Rev. Mod. Phys. **86**, 419 (2014).

[37] Q. Y. He and M. D. Reid, Phys. Rev. A **88**, 052121 (2013); S. Kiesewetter, Q. Y. He, P. D. Drummond, and M. D. Reid, *ibid.* **90**, 043805 (2014); L. Rosales-Zárate *et al.*, J. Opt. Soc. Am. B **32**, A82 (2015).

[38] S. L. W. Midgley, A. J. Ferris, and M. K. Olsen, Phys. Rev. A **81**, 022101 (2010); V. Händchen *et al.*, Nat. Photon. **6**, 596 (2012).

[39] P. Pearle, Phys. Rev. D **2**, 1418 (1970).

[40] E. G. Cavalcanti and M. D. Reid, Phys. Rev. Lett. **97**, 170405 (2006); C. Marquardt, U. L. Andersen, G. Leuchs, Y. Takeno, M. Yukawa, H. Yonezawa, and A. Furusawa, Phys. Rev. A **76**, 030101 (2007).

[41] E. G. Cavalcanti and M. D. Reid, Phys. Rev. A **77**, 062108 (2008); B. Opanchuk, L. Rosales-Zárate, R. Y. Teh, and M. D. Reid, arXiv:1609.06028 (2016).

[42] B. Yadin and V. Vedral, Phys. Rev. A **93**, 022122 (2016).

[43] F. Fröwis, P. Sekatski, and W. Dür, Phys. Rev. Lett. **116**, 090801 (2016).

[44] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Phys. Rev. Lett. **68**, 3663 (1992).

[45] M. D. Reid and P. D. Drummond, Phys. Rev. Lett. **60**, 2731 (1988).

[46] H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68**, 032103 (2003).

[47] M. J. Mallon, M. D. Reid, and M. K. Olsen, J. Phys. B: At. Mol. Opt. Phys. **41**, 015501 (2008).