Evolution of few-cycle pulses in nonlinear dispersive media: Velocity of the center of mass and root-mean-square duration

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Simple arithmetic dependencies of the velocity of the mass center motion and the root-mean-square duration of initially single-cycle, two-cycle, and Gaussian pulses with a random number of oscillations under the pulse envelope are derived depending on their center frequency, initial duration, and peak field amplitude, as well as on dispersive and nonlinear characteristics of homogeneous isotropic dielectric media. In media with normal group dispersion, it is shown that due to nonresonant dispersion the square of the few-cycle pulse duration increases with distance inversely proportional to the fourth power of the number of input pulse cycles. In media with normal group dispersion, the square of the pulse duration is inversely proportional to the number of input pulse cycles due to cubic nonlinearity. In media with anomalous group dispersion, it is shown that due to cubic nonlinearity, few-cycle pulse self-compression decreases with the reduction of the number of cycles in the initial pulse. This pulse self-compression effect has a threshold nature and terminates at a fixed number of cycles of the input pulse. Such a number of cycles is determined by the input intensity and the central frequency of the pulse, as well as by the dispersive and nonlinear characteristics of the medium.

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I. INTRODUCTION

The development of ultrashort-pulse lasers in the past 2 decades has led to the creation of effective systems to generate and detect few-cycle waves, including waves containing only a single cycle [1–9]. These pulses find important applications in the monitoring of ultrafast physical and chemical processes, systems of ultrahigh-speed information transmission, recording, and processing, in terahertz (THz) biophotonics [10–13]. These extremely short (few-cycle) pulses are characterized by a very wide spectrum. Traditional theoretical methods, based on the consideration of quasimonochromatic pulse envelope dynamics [14], become invalid because the concept of an "envelope" for such pulses loses its physical meaning.

Usually the analysis of few-cycle wave evolution in linear [15–17] and nonlinear [18–22] media is based on equations describing the dynamics of electric field radiation or on equations describing the dynamics of spatiotemporal spectra of radiation (a detailed overview of such works is given in Ref. [23]).

For the special case of intense few-cycle waves, theoretical analyses of their nonlinear evolution in optical media is usually time-consuming and generally requires a numerical modeling. However, in many practical situations an exhaustive analysis of changes in electric fields and spectra of optical pulses within the short distance is not required—it is sufficient to consider only dynamics of their average parameters. Integral relations for the moving velocity of center of mass and for the velocity of pulse dispersion spreading as a function of pulse temporal profile at the input of nonlinear medium were obtained in Refs. [24,25] and based on equations of optical field dynamics of broadband radiation covering a significant part of the transparent range of the optical spectrum. Kapoiko *et al.* [26] showed that for the special case of one-and-a-half-cycle pulses, such integral relations may be reduced to the form of elementary functions depending on their initial center frequency and pulse intensity, as well as on the dispersive and nonlinear characteristics of the medium. Hong *et al.* [27] considered in detail the evolution characteristics of an optical pulse "mass center" during propagation in highly noninstantaneous Kerr media. They found that the transversal motion of the pulse mass center in a highly noninstantaneous response can be approximated by a uniformly accelerated motion up to one dispersion length of the pulse.

In this article, simple dependencies in elementary functions for the velocity of the mass center motion and the dynamics of few-cycle pulse duration in media with nonresonant dispersion and cubic nonlinearity are obtained for initially single-cycle, two-cycle, and Gaussian pulses with random pulse envelope cycles. We have shown that with a decrease in pulse cycles the difference between the velocity of pulse spreading and the velocity of quasimonochromatic pulse spreading with the same center frequency can be 25% or more in a medium with normal group dispersion and instantaneous cubic nonlinearity. We have shown that with a reduction of the number of cycles in a pulse with the central frequency in the anomalous group dispersion area the effect of few-cycle pulse self-compression terminates at a fixed threshold cycle number due to cubic nonlinearity of the medium. This threshold number is determined by the central frequency of the pulse, its intensity, and the dispersive and nonlinear characteristics of the medium.

II. MASS CENTER MOTION AND EVOLUTION OF FEW-CYCLE PULSE DURATION IN NONLINEAR DIELECTRIC MEDIA

The evolution of an electric field E of a linearly polarized few-cycle pulse, which propagates in a transparent homogeneous isotropic dielectric medium, can usually be described

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by the following equation [23,28]:

$$\frac{\partial E}{\partial z} + \frac{N_0}{c}\frac{\partial E}{\partial t} - a\frac{\partial^3 E}{\partial t^3} + b\int_{-\infty}^{t} E dt' + gE^2\frac{\partial E}{\partial t} = 0, \quad (1)$$

where z is the direction of wave propagation, t is the time, c is the speed of light in a vacuum, N_0, a, and b are parameters characterizing the nonresonant dependence of the refractive index n of a medium with respect to radiation with frequency ω:

$$n(\omega) = N_0 + ca\omega^2 - \frac{cb}{\omega^2}. \quad (2)$$

Here g characterizes the instantaneous cubic field nonlinearity of the polarization response of the medium and is related to the coefficient of the nonlinear refractive index n_2 by $g = 2n_2/c$.

Equation (1) can be used to describe the propagation of few-cycle pulses because the dependence (2) correctly describes the material dispersion of isotropic dielectric media for broadband radiation, e.g., for the visible and near-infrared spectral range [29]. When describing THz radiation, Eq. (2) can be restricted to only the first two terms [19,22]. In addition, the dispersion of n_2 of optical media in the field of ultrashort pulses can, in many cases including the one mentioned above, be neglected [23], and the cubic nonlinear response, as was done for Eq. (1), can be considered instantaneous. Of course, field equation (1) can be refined for the case of more complex linear and nonlinear dispersion of the refractive index [23]. In this article, however, we restrict ourselves by the theoretical model of radiation propagation (1) to describe the dynamics of average parameters of few-cycle pulses in nonlinear medium.

Additionally, we note, that by the substitution

$$E(z,t) = \tfrac{1}{2}A(z,t)e^{i(k_0 z - \omega_0 t)} + \text{c.c.}, \quad (3)$$

where ω_0 is an arbitrary fixed frequency, k_0 is equal to $\omega_0 n(\omega_0)/c$, $n(\omega)$ is described by formula (2), and field equation (1) neglects the generation of frequency tripled radiation for the new variable $A(z,t)$ and is rewritten in the form [14,23]

$$\frac{\partial A}{\partial z} + \frac{1}{\overline{V}}\frac{\partial A}{\partial t} - \sum_{n=2}^{\infty}\beta_n\frac{i^{n+1}}{n!}\frac{\partial^n A}{\partial t^n} - i\gamma_1|A|^2A + \gamma_2\frac{\partial}{\partial t}(|A|^2A) = 0, \quad (4)$$

where $\overline{V} = (\partial k/\partial\omega)^{-1}_{\omega_0}$ is the group velocity, $\beta_n = [\partial^n k(\omega)/\partial\omega^n]_{\omega_0}$, $k = (N_0/c)\omega_0 + a\omega_0^3 - b/\omega_0$, $\gamma_1 = g\omega_0/4$, and $\gamma_2 = g/4$. In other words Eq. (1) contains the well-known Eq. (4) as a special case, where the complex amplitude of the electric field A is associated with the pulse envelope. In contrast to Eq. (4), field equation (1) also describes the generation of radiation at tripled and higher frequencies [19,22], except for the self-action of a pulse in nonlinear medium.

The velocity V of the mass center motion of an optical pulse, i.e., the first-order moment of the field distribution of radiation $\langle t\rangle = \frac{1}{W}\int_{-\infty}^{\infty} tE^2 dt$, where $W = \int_{-\infty}^{\infty} E^2 dt$ is the pulse energy, within the theoretical model is described by Eq. (1) and can be calculated by the following formula [24]:

$$V^{-1} = \frac{d\langle t\rangle}{dz} = \frac{N_0}{c} + \frac{1}{W}\left[3a\int_{-\infty}^{\infty}\left(\frac{\partial E}{\partial t}\right)^2 dt + b\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} E dt'\right)^2 dt + \frac{g}{2}\int_{-\infty}^{\infty} E^4 dt\right]. \quad (5)$$

Expression (5) can be obtained by the differentiation of the first-order moment $\langle t\rangle$ of the field distribution with respect to z, followed by the replacement of dE/dz from the wave equation (1) and the integration of the obtained relationship by parts.

The evolution of the pulse duration defined as the square root of the central second-order moment of the field distribution [30] $\tau = \sqrt{\langle\Delta t^2\rangle} = \sqrt{\frac{1}{W}\int_{-\infty}^{\infty}(t-\langle t\rangle)^2E^2dt}$ is described by the following equation [24]:

$$\tau^2 = \tau_0^2 + \left(\frac{d\langle t^2\rangle}{dz}\right)_0 z + Dz^2, \quad (6)$$

where $\tau_0 = \langle t^2\rangle_0^{1/2}$ is the input (at $z=0$) pulse duration, $(d\langle t^2\rangle/dz)_0$ characterizes at $z=0$ the spectral finiteness of the pulse, and the quantity D, characterizing the velocity of the dispersive spreading of the pulse, is calculated by the following formula [24]:

$$\begin{aligned} D &= \frac{1}{2}\frac{d^2\langle t^2\rangle}{dz^2} - \left(\frac{d\langle t\rangle}{dz}\right)^2 = \frac{9a^2}{W^2}\left\{W\int_{-\infty}^{\infty}\left(\frac{\partial^2 E}{\partial t^2}\right)^2 dt - \left[\int_{-\infty}^{\infty}\left(\frac{\partial E}{\partial t}\right)^2 dt\right]^2\right\} \\ &+ \frac{6ab}{W^2}\left[W^2 - \int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} E dt'\right)^2 dt\int_{-\infty}^{\infty}\left(\frac{\partial E}{\partial t}\right)^2 dt\right] \\ &+ \frac{b^2}{W^2}\left\{W\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} dt'\int_{-\infty}^{t'} E dt''\right)^2 dt - \left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} E dt'\right)^2 dt\right]^2\right\} \\ &+ \frac{3ag}{W^2}\left[4W\int_{-\infty}^{\infty} E^2\left(\frac{\partial E}{\partial t}\right)^2 dt - \int_{-\infty}^{\infty} E^4 dt\int_{-\infty}^{\infty}\left(\frac{\partial E}{\partial t}\right)^2 dt\right] \end{aligned}$$

$$
-\frac{bg}{W^2}\left[\frac{8W}{3}\int_{-\infty}^{\infty} tE^3\left(\int_{-\infty}^{t} Edt'\right)dt+\int_{-\infty}^{\infty} E^4dt\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} Edt'\right)^2dt\right]
+\frac{g^2}{W^2}\left[\frac{W}{3}\int_{-\infty}^{\infty} E^6dt-\frac{1}{4}\left(\int_{-\infty}^{\infty} E^4dt\right)^2\right], \tag{7}
$$

where $\langle t^2\rangle=(1/W)\int_{-\infty}^{\infty} t^2E^2dt$ is the second-order moment of the field distribution. Expression (7) can be obtained in a similar way as expression (5) by the differentiation of the second-order moment $\langle t^2\rangle$ of the field distribution with respect to z, followed by the replacement of dE/dz from the wave equation (1) and the integration of the obtained relationship by parts: once for the first derivative and once for the second derivative.

In linear media with normal group dispersion ($b=0$, $g=0$) expressions (5) and (7) are integrals of the motion of the field equation (1) for any pulse shape at the input of the medium [31–33]. In other words, from the input pulse field profile $E_0(t)$ and the medium parameters N_0 and a from Eqs. (5) and (6) it is not difficult to calculate the changes of average pulse parameters such as V and τ in the medium [31].

In linear media where the group dispersion can be anomalous ($a\neq 0,\ b\neq 0,\ g=0$), expressions (5) and (7) are integrals of the motion of Eq. (1) with additional restriction on the temporal profile of the input pulse [31]:

$$
\int_{-\infty}^{-\infty} dt'\int_{-\infty}^{t'} dt''\int_{-\infty}^{t''} Edt'''=0. \tag{8}
$$

Restriction (8) is due to the fact that dependence (2), describing the dispersion of the linear refractive index of dielectrics in the range of their transparency, is no longer correct at low frequencies, even generating negative values of n. These frequencies should not be included in the spectrum of the theoretical model, as it is required for the satisfaction of Eq. (8).

In a nonlinear medium ($a\neq 0$, $b\neq 0$, $g\neq 0$) expressions (5) and (7) with condition (8) can also be considered as integrals of motion, but at the distance [25]

$$
z\leqslant\frac{cT_0/N_0}{\max\left(\frac{ac}{N_0T_0^2},\frac{bcT_0^2}{N_0},\frac{gcE_0^2}{N_0}\right)}, \tag{9}
$$

where T_0 characterizes temporal distance between field zeros ($T_0\sim 2\pi/\langle\omega\rangle$, where $\langle\omega\rangle$ is the central frequency of radiation).

III. CALCULATION FORMULAS OF MASS CENTER DYNAMICS AND DURATION DYNAMICS FOR TYPICAL MODELS OF FEW-CYCLE PULSES

We first consider the dynamics of the average parameters of few-cycle optical pulses in a simpler case when their spectrum falls entirely into the normal group dispersion region of the medium. In this case, the dispersion model (2), as it was noted above, can be restricted only with the first two terms, and Eq. (1) takes the form of a well-known modified Korteweg-de Vries equation [23,32,33].

Assuming that the field of radiation at the entrance of the medium has the form of a Gaussian pulse,

$$
E(t)=E_0\exp\left[-\left(\frac{t}{t_0}\right)^2\right]\sin(\omega_0t), \tag{10}
$$

where E_0 is the amplitude of field cycles, ω_0 is the frequency of these cycles, and t_0 sets the pulse duration, we obtain the following for the inverse value of the velocity of the mass center motion and the square of the pulse duration in a medium from Eqs. (5), (6), (7):

$$
V^{-1}=\frac{N_0}{c}+\frac{3a\omega_0^2}{\pi^2N^2}\frac{e^{\pi^2N^2/2}(\pi^2N^2+1)-1}{e^{\pi^2N^2/2}-1}+\frac{3gE_0^2}{8\sqrt{2}}\frac{e^{\pi^2N^2/2}-4/3e^{\pi^2N^2/4}+1/3e^{-\pi^2N^2/2}}{e^{\pi^2N^2/2}-1}, \tag{11}
$$

$$
\tau^2=\tau_0^2+Dz^2, \tag{12}
$$

where the initial pulse duration

$$
\tau_0=\frac{t_0}{2}\sqrt{\frac{e^{\pi^2N^2/2}+\pi^2N^2-1}{e^{\pi^2N^2/2}-1}}, \tag{13}
$$

$$
\begin{aligned}
D={}&\frac{36a^2\omega_0^4}{\pi^4N^4}\frac{e^{\pi^2N^2}(\pi^2N^2+1/2)-e^{\pi^2N^2/2}(\pi^2N^2+2)^2/4+1/2}{\left(e^{\pi^2N^2/2}-1\right)^2}\\
&+\frac{3ag\omega_0^2E_0^2}{4\sqrt{2}\pi^2N^2}\frac{e^{\pi^2N^2/2}(\pi^2N^2+3)+2e^{\pi^2N^2/4}(\pi^2N^2+1)-(\pi^2N^2+2)-2e^{-\pi^2N^2/4}-e^{-\pi^2N^2/2}}{\left(e^{\pi^2N^2/4}+1\right)^2}\\
&+\frac{g^2E_0^4}{1152}\big[(80\sqrt{3}-81)e^{\pi^2N^2}-120\sqrt{3}e^{5\pi^2N^2/6}+216e^{3\pi^2N^2/4}-16(9+5\sqrt{3})e^{\pi^2N^2/2}+168\sqrt{3}e^{\pi^2N^2/3}\\
&-48\sqrt{3}e^{-\pi^2N^2/6}+72e^{-\pi^2N^2/4}-8\sqrt{3}e^{-\pi^2N^2/2}+(8\sqrt{3}-9)e^{-\pi^2N^2}-54\big]/\left(e^{\pi^2N^2/2}-1\right)^2,
\end{aligned} \tag{14}
$$

and $N = t_0\omega_0/\pi$ is the initial number of cycles in the pulse at the level e^{-1}.

We note that Eqs. (11) and (14) are quite complex. It should be mentioned that they can be simplified in most practical cases: for this purpose we assume that $\exp(\pi^2N^2/2) \gg 1$, and therefore $\exp(\pi^2N^2/2) - 1 \cong \exp(\pi^2N^2/2)$. Even for the case $N = 1$ one can obtain $\exp(\pi^2N^2/2) \approx 140 \gg 1$. So at $N \geqslant 1$, one can instead use simpler expressions for Eqs. (11), (13), and (14):

$$V^{-1} = \frac{N_0}{c} + 3a\omega_0^2\left(1 + \frac{1}{\pi^2N^2}\right) + \frac{3gE_0^2}{8\sqrt{2}}, \tag{15}$$

$$\tau_0 = \frac{t_0}{2}, \tag{16}$$

$$D = \frac{36a^2\omega_0^4}{\pi^2N^2}\left(1 + \frac{1}{2\pi^2N^2}\right) + \frac{3ag\omega_0^2E_0^2}{4\sqrt{2}}\left(1 + \frac{3}{\pi^2N^2}\right) + \frac{g^2E_0^4}{20}. \tag{17}$$

It is important to note that for the obtained relationships, for cases $N \to \infty$ and linear medium [nonlinear parameter $g = 0$ in Eq. (1)], expressions (15) and (17) transform into well-known expressions for the quasimonochromatic pulse [14]:

$$V_{\mathrm{lin,qm}}^{-1} = \left.\frac{d[n(\omega)\omega/c]}{d\omega}\right|_{\omega_0} = \frac{N_0}{c} + 3a\omega_0^2, \tag{18}$$

$$D_{\mathrm{lin,qm}} = \frac{\beta_2^2}{t_0^2} = \frac{36a^2\omega_0^4}{\pi^2N^2}. \tag{19}$$

The theoretical model for the limiting case of a pulse initially containing only one full field oscillation is often used [19,34]:

$$E(t) = E_0\frac{t}{t_0}\exp\left(-\frac{t^2}{t_0^2}\right), \tag{20}$$

where E_0 characterizes the input pulse amplitude and t_0 characterizes the pulse initial duration. Extremely low oscillation pulses are generated, for example, in the THz spectral range [35–37] (see example in Fig. 1). In radio physics, the model of the pulse (20) sometimes is named the "Gaussian monocycle" [38].

For the pulse with field distribution (20) at the entrance of the medium, expressions (5)–(7) take the form

$$V^{-1} = \frac{N_0}{c} + \frac{9a\omega_m^2}{2} + \frac{3gE_0^2}{32\sqrt{2}}, \tag{21}$$

where $\omega_m = \sqrt{2}/t_0$ is the frequency of the maximum spectral density of a single-cycle wave, and relation (12), where

$$\tau_0 = \frac{\sqrt{3}t_0}{2}, \tag{22}$$

$$D = \frac{27a^2\omega_m^4}{2} + \frac{27ag\omega_m^2E_0^2}{16\sqrt{2}} + \left(\frac{5}{432\sqrt{3}} - \frac{9}{2048}\right)g^2E_0^4. \tag{23}$$

It is useful to compare models of Gaussian pulses (10) by reducing the number of field cycles under the pulse envelope (20). The electric field and the modulus of the spectrum of

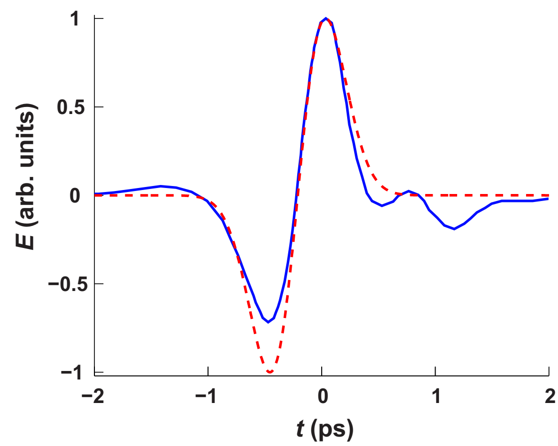

FIG. 1. The electric field E of a THz pulse, generated by the nonlinear effect of optical rectification in a crystal [37] (solid line), and the interpolation of the pulse field profile by the model of a single-cycle wave (20) (dashed line).

a Gaussian pulse for different small numbers of cycles N, including single-cycle pulses, are illustrated in Fig. 2. It is shown from Fig. 2 that, at $N = t_0\omega_0/\pi = 0.5$ and $\omega_m = \omega_0$, models of Gaussian and single-cycle waves become close,

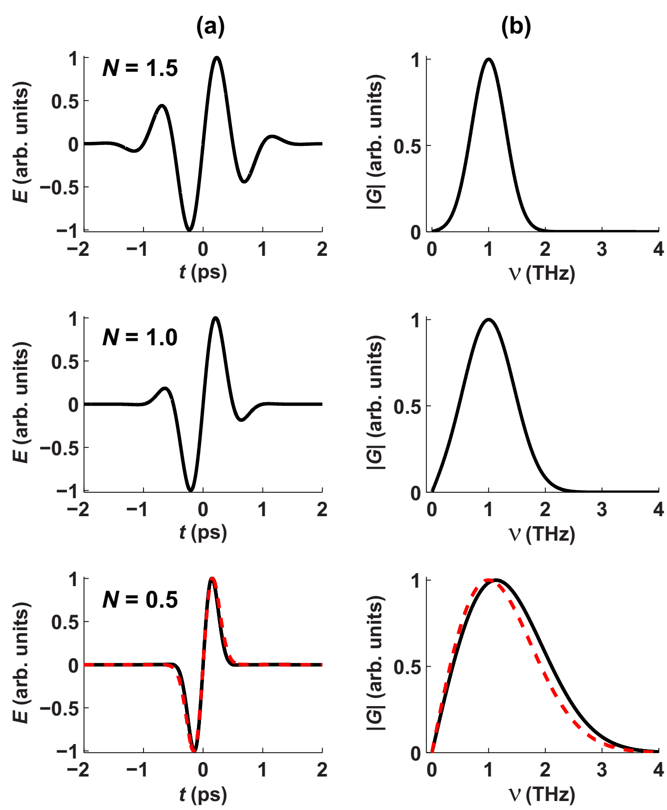

FIG. 2. The electric field E (a) and the modulus of spectrum $|G|$ (b) of a Gaussian pulse (10) with frequency $\nu_0 = \omega_0/2\pi = 1$ THz at different numbers of cycles: $N = 1.5$, $N = 1.0$, and $N = 0.5$. The electric field (a) and the modulus of spectrum $|G|$ (b) of a single-cycle pulse (20) with the same frequency $\omega_m = \omega_0$ are depicted by dashed lines.

which is why formulas (15)–(17) and (21)–(23) are close with such an N value. For clarity, pulses are illustrated for the case $\nu_0 = \omega_0/2\pi = 1.0$ THz in the figure. We note that at $N < 1$ the frequency of the maximum Gaussian pulse spectral density becomes visibly different from ν_0 and it shifts to higher frequencies [Fig. 2(b)].

It follows from expressions (12) and (17) that, in changing pulse duration with distance in a medium with normal group dispersion and instantaneous cubic nonlinearity, we can separate contributions of purely dispersive and purely nonlinear character [first and third terms in Eq. (17)], as well as their combined action [second term in Eq. (17)]. The addition with a decreasing number of cycles appears in contribution of dispersion of linear refractive index into the change of square of pulse duration, which is inversely proportional to the square of this number at a large number of cycles in the pulse. This addition is inversely proportional to the fourth power of the number of cycles. The addition with a decreasing number of cycles appears in the combined contribution of refractive index nonlinearity and dispersion. The combined contribution also does not depend on this number at the large number of cycles in a pulse. In the second case such an addition is inversely proportional to the square of the number of cycles.

It is not difficult to estimate that, when going from quasi-monochromatic radiation (10) with a large N to a single-cycle wave with $N = 0.5$, the contribution of linear refractive index dispersion into value D, which characterizes the velocity of the dispersion pulse spreading in the medium, is increased by 20%. The combined contribution of nonlinearity and dispersion into D is increased more than two times. The total change of D when going from a large number of cycles N under the envelope to $N = 0.5$, for example, when $\Delta n_{\mathrm{d}} = \Delta n_{\mathrm{nl}}$, where $\Delta n_{\mathrm{d}} = ca\omega_0^2$ characterizes dispersion and $\Delta n_{\mathrm{nl}} = (1/2)n_2 E_0^2$ characterizes the nonlinearity of the medium refractive index, is 24% [23]. This is confirmed by estimation of a more precise formula (14), which gives the change of 25%.

Let us consider now the dynamics of average parameters of pulses where the spectrum partially or entirely falls in the region of zero or anomalous group dispersion of the medium. In this case, one should use a much more complex model of dispersion of the refractive index of the medium (2).

Dispersion (2), which correctly describes the anomalous group dispersion of the medium [23,28], is due to the strong absorption of radiation in the low-frequency range [29]. Therefore, this model is not reasonable for describing the dynamics of few- or single-cycle waves in a medium because a significant part of the energy of such waves is exactly in the area of low frequencies. The requirements for radiation, where the dispersion (2) originates from the conservation of values, Eqs. (5)–(7), are formalized by condition (8). This condition is conducted naturally for the field, where a functional dependence on time is obtained in the form of the second derivative d^2/dt^2 from, for example, the field of a Gaussian pulse (10):

$$E(t) = E_0 \exp\left(-\frac{t^2}{t_0^2}\right)\frac{1}{t_0^4}\big[\big(4t^2 - 2t_0^2 - \omega_0^2 t_0^4\big) \times \sin(\omega_0 t) - 4tt_0^2\omega_0\cos(\omega_0 t)\big], \tag{24}$$

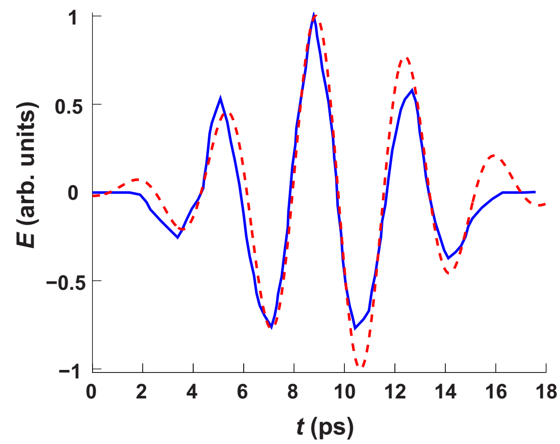

FIG. 3. The electric field E of a THz pulse generated from a time-modulated beam of relativistic electrons [39] (continuous line), and the interpolation of the pulse field profile by a theoretical model (24) (dashed line).

where E_0 characterizes the initial pulse amplitude, t_0 characterizes the pulse initial duration, and ω_0 characterizes the central frequency.

An example of the interpolation of the field profile of an electromagnetic pulse generated by a beam of relativistic electrons with dependence (24) is depicted in Fig. 3 [39].

The velocity of the mass center motion and the velocity of the dispersive spreading of the pulse (24), as it follows from expressions (5)–(7), are defined by the expression

$$V^{-1} = \frac{N_0}{c} + 3a\omega_0^2\left(1 + \frac{9}{\pi^2N^2+6}\right) + \frac{b}{\omega_0^2}\left(1 - \frac{5}{\pi^2N^2+6}\right) + \frac{3gE_0^2}{8\sqrt{2}}\left(1 + \frac{4}{\pi^2N^2+6}\right) \tag{25}$$

and relation (12), where

$$\tau_0 = \frac{t_0}{2}\sqrt{1 + \frac{4\pi^2N^2}{\pi^4N^4 + 6\pi^2N^2 + 3}}, \tag{26}$$

$$D = \frac{36a^2\omega_0^4}{\pi^2N^2}\left(1 + \frac{9}{2\pi^2N^2+24}\right) - \frac{24ab}{\pi^2N^2}\left(1 - \frac{15}{2\pi^2N^2+24}\right) + \frac{4b^2}{\pi^2\omega_0^4N^2}\left(1 - \frac{23}{2\pi^2N^2+24}\right) + \frac{3ag\omega_0^2E_0^2}{4\sqrt{2}}\left(1 + \frac{39}{\pi^2N^2+12}\right) - \frac{bgE_0^2}{4\sqrt{2}\omega_0^2}\left(1 - \frac{16}{\pi^2N^2+12}\right) + \frac{g^2E_0^4}{20}\left(1 + \frac{8}{\pi^2N^2+12}\right). \tag{27}$$

Expressions (25)–(27) are described in assumption of $N \geqslant 1$.

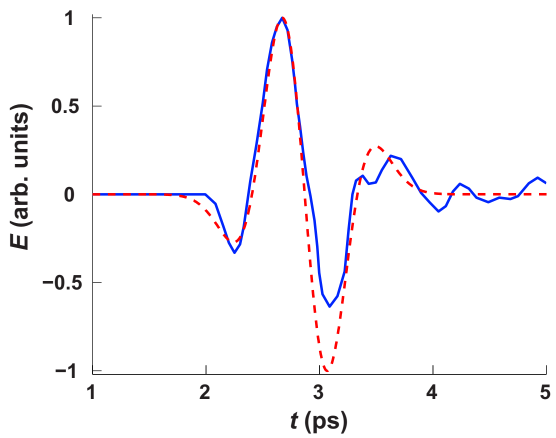

FIG. 4. The field E of a THz pulse generated during filamentation in air of a near-infrared pulse (central wavelength $\lambda_0 = 1.75\ \mu$m) [41] (continuous line), and the interpolation of a field profile of a pulse by the model of the two-cycle wave (28) (dashed line).

With a decreasing value of $N = t_0\omega_0/\pi$, the number of full cycles in a pulse (24) decreases naturally, and at $N = 2$ the pulse contains only two cycles (on level e^{-1}). With further decreasing of N, the number of cycles in a pulse remains equal to two, but the central frequency of the radiation begins to grow. For the case of a two-cycle pulse it is possible to use a simpler model [40]:

$$E(t) = E_0 \frac{2t^3 - 3t_0^2 t}{t_0^3} \exp\left(-\frac{t^2}{t_0^2}\right), \tag{28}$$

the form of which is set by the second derivative of the function (24) and therefore also satisfies the conditions set in Eq. (8).

The example of the interpolation of the field profile of an electromagnetic THz pulse generated during the filamentation of a near-infrared pulse in air by dependence (28) is depicted in Fig. 4 [41].

It can be shown that for the pulse (28), expressions (5) and (7) are reduced to the form

$$V^{-1} = \frac{N_0}{c} + \frac{21a\omega_m^2}{6} + \frac{6b}{5\omega_m^2} + \frac{4761gE_0^2}{10240\sqrt{2}}, \tag{29}$$

$$D = \frac{7a^2\omega_m^4}{2} - \frac{12ab}{5} + \frac{24b^2}{5\omega_m^2} + \frac{14283ag\omega_m^2E_0^2}{10240\sqrt{2}} - \frac{1143bgE_0^2}{12800\sqrt{2}\omega_m^2} + \frac{5}{78}g^2E_0^4. \tag{30}$$

The initial duration is $\tau_0 = \sqrt{11/20}t_0$, and the frequency of the maximum spectral density of a two-cycle pulse is $\omega_m = \sqrt{6}/t_0$.

IV. ILLUSTRATIONS OF FEW-CYCLE-PULSE EVOLUTION IN NONLINEAR DIELECTRIC MEDIA

To demonstrate the possibilities of obtained analytical relations, we will first consider the spreading of an initially single-cycle pulse in a optical medium with normal group dispersion and instantaneous cubic nonlinearity of its refractive index.

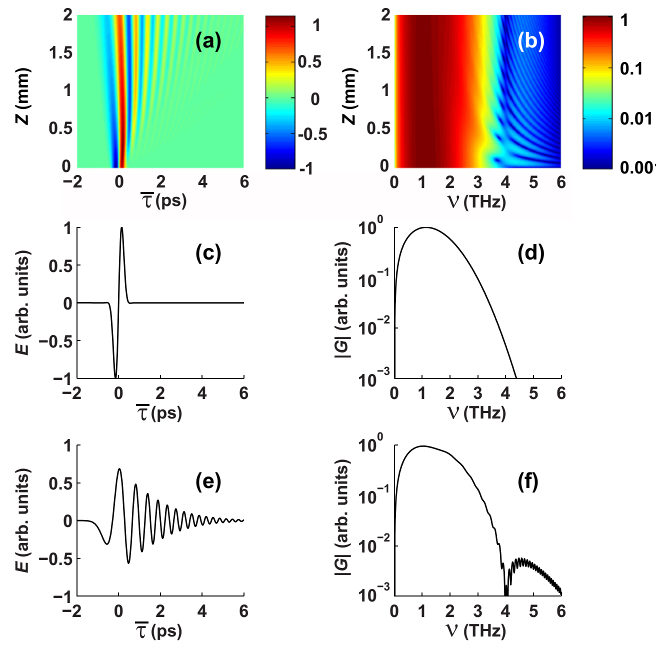

FIG. 5. Evolution of the electric field E (a) and the modulus of its spectrum $|G|$ (b) of an initially (at $z = 0$ mm) single-cycle THz pulse with field profile (c) and spectrum (d) in a medium with dispersive characteristics $N_0 = 4.73$ and $a = 2.224$ ps^3/m, corresponding to stoichiometric crystal MgO:LiNbO$_3$ [19] and induced nonlinear refractive index in the medium $\Delta n_{\rm nl} = 3.3 \times 10^{-2}$. The electric field of the pulse (e) and the modulus of its spectrum (f) at the output of nonlinear medium ($z = 2$ mm). $\overline{\tau} = t - (c/N_0)z$ is the retarded time.

Figure 5 shows calculation results of changing the field structure and the modulus of the spectrum of a single-cycle wave (20) in a medium with dispersion and nonlinearity according to Eq. (1) in the special case of $b = 0$. Here Eq. (1) is a modified Korteweg-de Vries equation. The Fourier split-step method was used during numerical calculation [14] using our department software package LBULLET 1D.

Here, radiation is considered to be "terahertz radiation" at $\nu_0 = 1/\sqrt{2}\pi t_0 = 1.0$ THz and with a corresponding central wavelength $\lambda_0 = c/\nu_0 = 0.3$ mm. The intensity of radiation I, as it was assumed, provided the induced change of the refractive index of the medium $\Delta n_{\rm nl} = (1/2)n_2E_0^2 = n_2' I = 3.3 \times 10^{-2}$. Medium parameters in calculations were considered corresponding to stoichiometric crystal MgO:LiNbO$_3$: $N_0 = 4.73$, $a = 2.224$ ps^3/m [19,31], and $n_2' = 5.4 \times 10^{-12}$ cm^2/W [42] (the large values of the coefficient of the nonlinear refractive index in the THz spectral range are also confirmed in Refs. [43,44]).

The pulse rapidly accrues new cycles and phase modulation due to dispersion and the nonlinearity of the refractive index of the medium (Fig. 5). The nonlinearity of the medium also leads to changes in the spectrum of the radiation. The radiation is generated at the spectral range from 4 to 6 THz in the medium [Fig. 5(f)] and is clearly separated from the main part of the spectrum of a wave. The maximum frequency of the spectrum of new radiation exceeds the maximum frequency of the initial spectrum over 4.5 times.

Calculated results of changes in the root-mean-square duration of an initially single-cycle pulse (20) by formulas

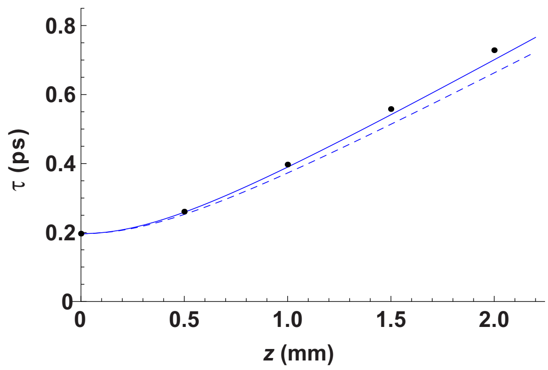

FIG. 6. Changing of the root-mean-square duration of an initially single-cycle THz pulse with distance z in the medium with dispersion characteristics $N_0 = 4.73$ and $a = 2.224$ ps^3/m, corresponding to stoichiometric crystal MgO:LiNbO$_3$ [19] with induced nonlinear refractive index $\Delta n_{\mathrm{nl}} = 3.3 \times 10^{-2}$ (solid line) and in linear medium (dashed line). Values of pulse duration are obtained by numerical calculation (see Fig. 5) according to Eq. (1) and are shown with a dotted line.

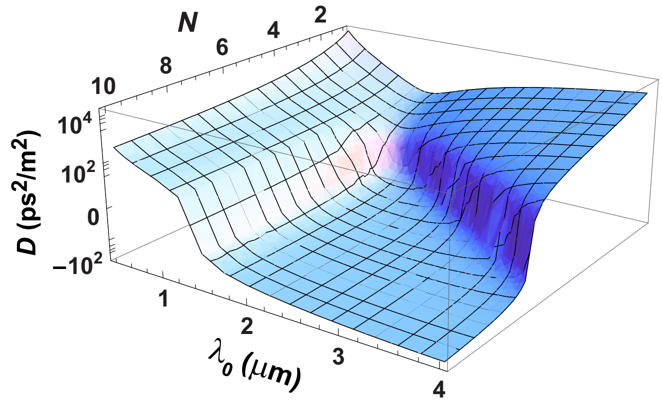

FIG. 7. Dependence of the rate of change of the square of pulse duration D (ps^2/m^2) (in logarithmic scale) with initial intensity $I_0 = 1 \times 10^{13}$ W/cm^2 during propagation of radiation in fused silica depending on the central wavelength of pulse λ_0 and initial cycle N.

(12) and (23) depending on the distance are shown in Fig. 6. In this figure, the solid line illustrates an increase in pulse duration in a nonlinear medium with normal group dispersion; the dashed line depicts this behavior in a linear case. The dotted line shows the calculation data of the pulse duration according to results of numerical modeling (see Fig. 5) by field equation (1). The parameters of radiation and the characteristics of the corresponding medium are mentioned above.

One can see from Fig. 6 that the pulse duration is increased by more than three times at the distance of 2 mm (it is about seven central wavelengths of initial radiation). The dominant effect of increasing the pulse duration for an initially few-cycle pulse is the dispersion of the medium. Calculation results of increasing the pulse duration according to the simple but approximate formulas (12) and (23) are in agreement with results corresponding to more precise but time-consuming calculations of the field equation (1), as can be seen from Fig. 6.

We now use analytical relations obtained in this work and consider a much more multivariate case of few-cycle pulses, where the spectrum may lie in the region of zero and anomalous group dispersion of nonlinear medium. In full agreement with known data [14], derived formulas (12) and (27) show that the pulse can be both broadened and compressed depending on its initial parameters and characteristics of the medium. Simple arithmetic relation makes it easy to determine the rate of change (decreasing or increasing) of the square of the pulse duration D in selected media as a function of the initial intensity, the central frequency, and the initial number of field cycles.

Calculation results for the rate of the square of change of pulse duration D (in logarithmic scale) are shown in Fig. 7 during propagation of radiation in fused silica ($N_0 = 1.45$, $a = 2.74 \times 10^{-42}$ s^3/m, $b = 3.94 \times 10^{19}$ s^{-1}m^{-1}, $n_2' = 2.9 \times 10^{-16}$ cm^2/W [45,46]), with initial intensity $I_0 = 1 \times 10^{13}$ W/cm^2 (this value of intensity corresponds to nonlinear addition to the index of refraction $\Delta n_{\mathrm{nl}} = 2.9 \times 10^{-3}$) depending on the central wavelength of pulse $\lambda_0 = 2\pi c/\omega_0$ and the initial number of cycles N.

Figure 7 demonstrates the variety of self-action scenarios for a pulse with a given initial intensity in a selected nonlinear medium with dispersion. If the central wavelength of the pulse is in the normal group dispersion of medium (it is less than the wavelength of zero group dispersion, which for fused silica has a value of $\lambda_D \approx 1.27$ μm [45]), the rate of change of the square of the pulse duration is positive: the pulse is spreading. In the region of anomalous group dispersion (the central wavelength of radiation is more than 1.27 μm at the entrance of the medium), when the number of cycles in a pulse is not very small ($N \geqslant 8$), the rate of change of the square of the pulse is negative: it corresponds to a well-known effect in nonlinear optics of self-compression of quasimonochromatic pulses in this spectral range [14]. However, as one can see from Fig. 7, with a reduction of the number of oscillations in the pulse, its self-compression is no longer observed. This threshold number of cycles in the pulse depends on the initial central wavelength of radiation for a given intensity in selected media.

Results of the numerical calculations utilizing field equation (1) which explain the above pulse's self-action scenarios are given in Fig. 8. As one can see from the figure, a pulse whose spectrum falls in the region of normal group dispersion of the medium [Fig. 8(a)] under dispersion and nonlinearity broadens significantly. The pulse duration is increased more than ten times after a propagation distance of 1 mm, wherein the pulse obtains near-linear self-phase modulation [47,48].

A pulse with the central frequency near zero group dispersion of the medium [Fig. 8(b)], broadens almost five times at this distance. This high increase in pulse duration is due to the fact that the spectrum width for the initially few-cycle pulse is comparable with its central frequency. Parts of this spectrum fall in regions with significant normal and anomalous group dispersion. Frequencies appear close to doubled frequencies in the spectrum of radiation in relation to frequencies of the initial spectrum, and the main part of the spectrum is shifted to shorter wavelengths.

A pulse with the central frequency in anomalous group dispersion of nonlinear medium, as it is well known, can

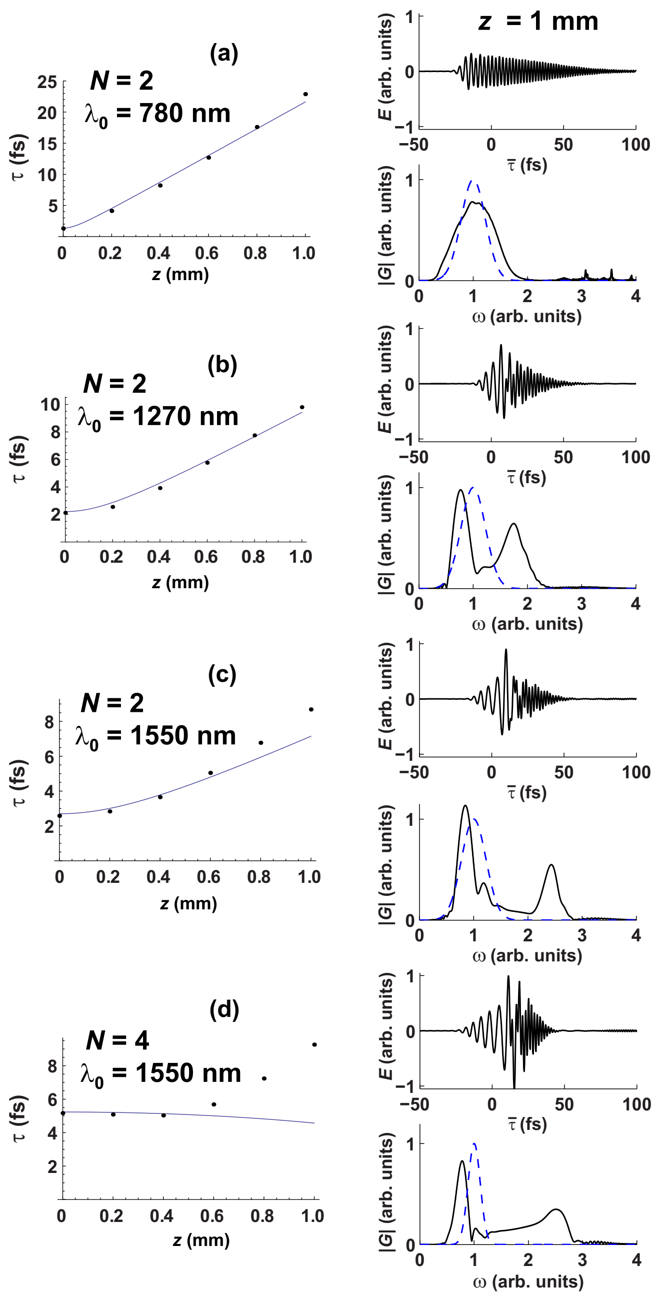

FIG. 8. Dynamics of the root-mean-square pulse duration τ with input intensity $I_0 = 5 \times 10^{12}$ W/cm^2 depending on the distance z in fused silica ($N_0 = 1.45$, $a = 2.74 \times 10^{-42}$ s^3/m, $b = 3.94 \times 10^{19}$ s^{-1}m^{-1}, $n_2' = 2.9 \times 10^{-16}$ cm^2/W) at different central wavelengths λ_0 and initial numbers of cycles N [(a) $N = 2.0$, $\lambda_0 = 780$ nm; (b) $N = 2.0$, $\lambda_0 = 1270$ nm; (c) $N = 2.0$, $\lambda_0 = 1550$ nm; (d) $N = 4.0$, $\lambda_0 = 1550$ nm] of radiation by the derived formula (27) (solid line) and results of numerical calculation by field equation (1) (dotted line). The electric field of a pulse E and its modulus of spectrum $|G|$ at $z = 1$ mm (solid line) are shown at the right side. The initial spectrum is also given by a dashed line.

compress. This is demonstrated in Fig. 8(d). In this case, high-frequency components appear in the spectrum of radiation up to tripled frequencies in relation to frequencies of the initial spectrum. The main part of this spectrum is shifted to longer wavelengths. It is shown from Fig. 8(c) that by decreasing the pulse duration until $N = 2$, a self-compression effect is no longer observed, giving way to effects of pulse broadening due to rapidly increasing contribution of dispersion of the change of pulse duration. Spectrum changes are also qualitatively similar to the ones demonstrated in Fig. 8(d).

Results of calculation of the change in pulse duration according to derived approximate formulas (12) and (27) are in agreement with more precise but time-consuming calculations utilizing field equation (1) [Figs. 8(a)–8(c)]. In the case of complication in pulse duration behavior (first compression and then broadening) derived in this work, formulas correctly display the initial character of radiation self-action [Fig. 8(d)].

V. CONCLUSION

In this article, analytical expressions describing the dynamics of average parameters of single-cycle, two-cycle, and other few-cycle waves in nonlinear media are obtained. Expressions take the form of elementary functions for the velocity of the mass center motion [Eqs. (15), (21), (25) and (29)] and the root-mean-square duration ([Eqs. (12), (17), (23), (27) and (30)] for such waves depending on the distance spent in a medium and its dispersive and nonlinear characteristics, as well as initial pulse parameters. This allows us to use these relations for the quick estimation of duration changes of few-cycle waves in various media and at different distances.

One can see that the obtained expressions for different pulse models look qualitatively alike. The reason for this is the proximity of the considered pulse models. Actually, Gaussian pulses and their second derivative allow us to describe similar relatively large-oscillation pulses. Further, single-cycle and two-cycles pulses may be considered as their individual cases. The analysis of these different pulse models was aimed at obtaining the simplest analytical expressions for different reasonable cases. Upon increase in the number of oscillations, these pulse models transform into quasimonochromatic pulses, removing any differences in dynamics of average parameters that can be seen when comparing the equations, for example, Eq. (15) vs Eq. (25) or Eq. (17) vs Eq. (27) with the assumption $N \to \infty$.

The obtained expressions are used for the calculation of the duration dynamics for THz and optical pulses in nonlinear media with dispersion. The comparison with numerical calculations by wave equation has shown that expressions extremely accurately describe the dynamics of high-intensity few-cycle pulses, even when their duration increases ten times during propagation in nonlinear media. The calculation of analytical expressions was performed with the help of Wolfram MATHEMATICA software.

According to the obtained analytical relations, it is proven that in media with normal group dispersion, the square of the few-cycle pulse duration increases with distance in inverse proportion to the fourth power of the number of input pulse cycles due to nonresonant dispersion. In media with normal group dispersion, it is shown that due to cubic nonlinearity, the square of the few-cycle pulse duration increases with distance in inverse proportion to the square power of the number of input pulse cycles. In media with anomalous group dispersion, it is shown that due to cubic nonlinearity, the effect of few-cycle pulse self-compression decreases with the reduction of the number of initial cycle pulses. The pulse self-compression effect has a threshold nature and terminates at a certain number of input pulses. This number of cycles is determined by the input intensity, the central frequency of the pulse, and the dispersive and nonlinear characteristics of the medium.

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