Cancellation of the N-composite-boson correlation energy under a BCS-like potential: A dimensionality-dependent effect

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We use Richardson-Gaudin exact equations to derive the ground-state energy of N composite bosons (cobosons) interacting via a potential which acts between fermion pairs having zero center-of-mass momentum, that is, a potential similar to the reduced BCS potential used in conventional superconductivity. Through a density expansion, we show that while for two-dimensional (2D) systems, the N-coboson correlation energy undergoes a surprising cancellation which leaves the interaction part with an $N(N-1)$ dependence only, such a cancellation does not exist in 1D, 3D, or 4D systems (which correspond to 2D parabolic traps) nor when the cobosons interact via a similar short-range potential but between pairs having an arbitrary center-of-mass momentum. This shows that the previously found cancellation which exists for the Cooper-pair correlation energy results not only from the very peculiar form of the reduced BCS potential, but also from a quite mysterious dimensionality effect, the density of states for Cooper pairs feeling the BCS potential being essentially constant, as for 2D systems.

I. INTRODUCTION

The success of the BCS theory [1,2] for conventional superconductors remains fascinating today for its impressive agreement with experimental data, in spite of its extreme simplicity. One of its insightful ingredients is that the attraction between free electron pairs with opposite spins and opposite momenta is enough to capture the physics of superconductivity induced by Cooper pairing. This attractive potential reads

$$V_{\text{BCS}} = -V \sum_{\mathbf{p}\mathbf{p}'} w_{\mathbf{p}} w_{\mathbf{p}'} B_{\mathbf{0p}}^{\dagger} B_{\mathbf{0p}'}, \qquad (1)$$

where $B_{\mathbf{0p}}^{\dagger} \equiv a_{\mathbf{p}}^{\dagger} b_{-\mathbf{p}}^{\dagger}$ creates a zero-momentum pair made of an up-spin ($\mathbf{p}$) electron and a down-spin ($-\mathbf{p}$) electron. V denotes the potential amplitude, taken as positive and constant, and $w_{\mathbf{p}}$ designates the energy layer close to the normal electron Fermi energy ε_F, in which a phonon-mediated attraction acts; $w_{\mathbf{p}}$ is taken equal to unity within this energy layer and zero outside it.

This so-called reduced BCS potential leads to one of the very few exactly solvable many-body problems. Richardson [3–6] and Gaudin [7] have independently shown that the ground-state energy of N pairs reads as $E_N = \sum_{i=1}^{N} R_i$, where the R_i's are solutions of the N coupled equations

$$\frac{1}{V} = \sum_{\mathbf{p}} \frac{w_{\mathbf{p}}}{\varepsilon_{\mathbf{p}} - R_i} + \sum_{j=1}^{N}{}' \frac{2}{R_i - R_j}. \qquad (2)$$

In this set of equations, the prime ($'$) excludes the $j = i$ term of the j sum. $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2\mu_e$ denotes the free-pair kinetic energy, with $\mu_e = m_e/2$ being the electron pair reduced mass. Being valid for any N, these Richardson-Gaudin equations are quite appropriate for studying superconducting grains and nanostructures [8–15] outside the thermodynamical limit.

For conventional superconductors, the normal electron Fermi energy ε_F is much larger than the potential extension width Ω, on the order of a phonon energy; so, the density of states in this layer is essentially constant. It has been shown [16] that for a pair density of states ρ taken as constant, the ground-state energy of N Cooper pairs in the thermodynamic limit takes a surprisingly simple form, within under-extensive terms, as

$$E_N = N E_1 + \frac{N(N-1)}{2\rho} \frac{1+\sigma}{1-\sigma}, \qquad (3)$$

where $\sigma = e^{-1/\rho V}$. (The factor of 2 difference from the standard exponent comes from the fact that ρ here is the pair density of states.) E_1 is the single-pair energy found by Cooper [17], which can be obtained from Eq. (2) taken for $N = 1$, that is, without the second sum.

The exact cancellation in the N Cooper pair energy of all extensive terms beyond $N(N-1)$ suggests that under a BCS-like potential, a composite boson (coboson) only interacts with one among N cobosons, as if it is completely ignorant of the other surrounding cobosons. The physics behind this astonishing many-body effect remains obscure up to now.

Two reasons can lead to this strange result:

(i) The attractive potential given in Eq. (1) only acts between zero-momentum pairs. To prove this first point, we must show that for a short-range potential between pairs having an *arbitrary* center-of-mass momentum, the N-coboson ground-state energy has extensive terms beyond $N(N-1)$.

(ii) The density of states in the energy extension where the potential acts is constant. To prove this second point and ascribe the mysterious cancellation to the system dimensionality, we must show that exact cancellation of extensive terms beyond $N(N-1)$ does not occur for space dimensions other than two; i.e., it occurs only in two-dimensional (2D) systems.

The present work is devoted to these two possible reasons. The understanding of the underlying physics that leads to the

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N-pair energy given in Eq. (3) will shed light on the merits and limitations of the V_{BCS} potential when used in the study of realistic systems.

To this end, we slightly modify the potential given in Eq. (1) in order for the present work to be applicable for cold-atom (CA) systems [18,19] which today are of major interest: the operators $(a^\dagger, b^\dagger)$ now denote the creation operators of the two fermion species at hand with reduced mass $\mu^{-1} = m_a^{-1} + m_b^{-1}$; the range of $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2\mu$ energy in which the attractive potential acts, that is, $w_{\mathbf{p}} = 1$, now extends from 0 to Ω. Moreover, to study the first reason, we generalize the V_{BCS} potential given in Eq. (1) into a short-range separable potential V_{CA}, which acts between fermion pairs having an arbitrary center-of-mass momentum $\mathbf{K}$

$$V_{CA} = -V \sum_{\mathbf{Kpp}'} w_{\mathbf{p}} w_{\mathbf{p}'} B_{\mathbf{Kp}}^\dagger B_{\mathbf{Kp}'}, \qquad (4)$$

where $B_{\mathbf{Kp}}^\dagger \equiv a_{\mathbf{p}+\gamma_a \mathbf{K}}^\dagger b_{-\mathbf{p}+\gamma_b \mathbf{K}}^\dagger$ with $\gamma_a = 1 - \gamma_b = m_a/(m_a + m_b)$. In translationally invariant systems as here considered, the center-of-mass momentum $\mathbf{K}$ is conserved during particle scattering.

The most striking result of this work is that for N cobosons interacting via a BCS-like potential between zero-momentum pairs, the density expansion of their ground-state energy depends on space dimension D in a very compact form for $D = (1,2,3)$, namely,

$$\frac{E_N - N E_1}{E_1} = -(N-1)(4-D)\{\nu_1(N\lambda^2) + (2-D)D[\nu_2(N\lambda^2)^2$$

$$+ \nu_3(24-14D-D^2)(N\lambda^2)^3 + \cdots]\}. \qquad (5)$$

The ν_n's are numerical factors while λ^2 scales as $(a_B/L)^D$ with L being the sample size and a_B the single-pair Bohr radius; so $N\lambda^2$ scales as the pair density n. This result confirms that in 2D systems, the interaction part of the N-coboson energy is in $N(N-1)$ only. It further suggests that the whole interaction part should cancel for $D = 4$, which would be even stranger because the system energy would then reduce to that of N noninteracting cobosons. Four-dimensional systems actually are of physical interest because their densities of states have the same energy dependence as 2D parabolic traps [18]. We will show in Sec. VI that this seemingly pathological dimensionality-induced cancellation of all interaction terms does not occur in 4D.

II. SINGLE-PAIR BINDING ENERGY

Let us first consider the ground-state binding energy of a single pair in $D = (1,2,3,4)$ dimension. Since a single pair in its ground state has zero center-of-mass momentum, the ground-state energy obtained for V_{CA} coincides with that for V_{BCS}.

By turning the discrete sum into an integral, with a density of states in D dimension written as $\rho(\varepsilon/\Omega)^{(D-2)/2}$, where ρ is the density of states at the upper potential cutoff Ω, the single-pair binding energy follows from the $(-E_1) > 0$ solution of Eq. (2) taken for $N = 1$, namely,

$$\frac{1}{V} = \sum_{\mathbf{p}} \frac{w_{\mathbf{p}}}{\varepsilon_{\mathbf{p}} - E_1} \simeq \int_0^\Omega \rho d\varepsilon \frac{(\varepsilon/\Omega)^{(D-2)/2}}{\varepsilon - E_1}. \qquad (6)$$
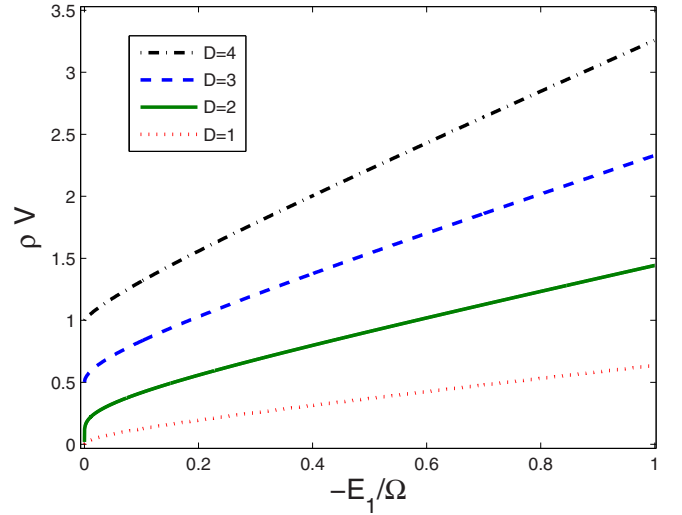

FIG. 1. ρV as a function of $(-E_1)/\Omega$ for $D = (1,2,3,4)$, according to Eqs. (7), (9), (11), and (13), respectively.

(i) For $D = 1$, the energy dependence of the density of states is in $1/\sqrt{\varepsilon}$; so

$$\frac{1}{\rho V} = \int_0^\Omega \frac{d\varepsilon}{\sqrt{\varepsilon/\Omega}} \frac{1}{\varepsilon - E_1^{(1)}} = 2\sqrt{\frac{\Omega}{-E_1^{(1)}}} \arctan\sqrt{\frac{\Omega}{-E_1^{(1)}}}. \qquad (7)$$

This equation has a $(-E_1^{(1)}) > 0$ solution for whatever V (see Fig. 1). In the regime of physical interest, that is, $|E_1^{(1)}| \ll \Omega$, it reads

$$E_1^{(1)} \simeq -\Omega(\rho V \pi)^2. \qquad (8)$$

(ii) For $D = 2$, the density of states is constant; so

$$\frac{1}{\rho V} = \int_0^\Omega d\varepsilon \frac{1}{\varepsilon - E_1^{(2)}} = \ln\left(\frac{\Omega}{-E_1^{(2)}} + 1\right), \qquad (9)$$

from which we analytically derive the single-pair energy as

$$E_1^{(2)} = -\Omega \frac{\sigma}{1-\sigma}, \qquad (10)$$

with $\sigma = e^{-1/\rho V}$ (see Fig. 1). This negative energy, which exists even for V vanishingly small, corresponds to the binding energy of a single pair obtained by Cooper [17]: indeed, the effect of a large normal electron Fermi energy ε_F is to make constant the density of states in the energy layer where the 3D potential acts.

(iii) For $D = 3$, the energy dependence of the density of states is in $\sqrt{\varepsilon}$; so

$$\frac{1}{\rho V} = \int_0^\Omega d\varepsilon \frac{\sqrt{\varepsilon/\Omega}}{\varepsilon - E_1^{(3)}} = 2 - 2\sqrt{\frac{-E_1^{(3)}}{\Omega}} \arctan\sqrt{\frac{\Omega}{-E_1^{(3)}}}. \qquad (11)$$

A solution $(-E_1^{(3)}) > 0$ exists for V larger than a threshold value $V_{th}^{(3)} = 1/2\rho$ (see Fig. 1). In the regime of physical

interest, that is, $|E_1^{(3)}| \ll \Omega$, this yields

$$E_1^{(3)} \simeq -\frac{4}{\pi^2}\Omega\left(1 - \frac{1}{2\rho V}\right)^2. \qquad (12)$$

(iv) For $D = 4$, the density of states is linear in ε; so

$$\frac{1}{\rho V} = \int_0^\Omega d\varepsilon \frac{\varepsilon/\Omega}{\varepsilon - E_1^{(4)}}$$

$$= 1 - \frac{-E_1^{(4)}}{\Omega}\ln\left(\frac{\Omega}{-E_1^{(4)}} + 1\right) \qquad (13)$$

has a $(-E_1^{(4)}) > 0$ solution provided that V is larger than the threshold value $V_{\text{th}}^{(4)} = 1/\rho$. The threshold for a single pair to form a bound state increases with space dimension, as seen in Fig. 1. In the physically relevant regime $|E_1^{(4)}| \ll \Omega$, the above equation yields

$$E_1^{(4)} \simeq \Omega\frac{1 - 1/\rho V}{\ln(1 - 1/\rho V) - \ln[-\ln(1 - 1/\rho V)]}. \qquad (14)$$

III. DENSITY EXPANSION PARAMETER

We are going to perform the density expansion of the energy of N cobosons interacting via the V_{CA} potential and the V_{BCS} potential. In this study, the natural dimensionless parameter appears as

$$\lambda^2 \equiv \frac{2}{\rho\Omega}\left(\frac{\Omega}{-E_1}\right)^{D/2}. \qquad (15)$$

Since the density of states ρ scales as the sample volume L^D, the product $N\lambda^2 \propto n$ can be used as a parameter to study the small density expansion of the N-coboson energy. More precisely, λ^2 scales as $(a_B/L)^D$, where a_B is the single-pair Bohr radius defined through $E_1 \equiv -1/2\mu a_B^2$. To show it explicitly, we first note that the density of states in D dimension is defined through

$$\Theta_D\left(\frac{L}{2\pi}\right)^D p^{D-1}dp = \rho\left(\frac{\varepsilon_{\mathbf{p}}}{\Omega}\right)^{D/2-1}d\varepsilon_{\mathbf{p}}, \qquad (16)$$

where $\Theta_D = 2\pi^{D/2}/\Gamma(D/2)$ is the solid angle in D dimension, with $\Gamma(x)$ being the gamma function. As $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2\mu$, this gives

$$\Theta_D\left(\frac{L}{2\pi}\right)^D = \frac{2\rho\Omega}{(2\mu\Omega)^{D/2}}. \qquad (17)$$

So for E_1 written in terms of a_B, Eq. (15) leads to

$$\lambda^2 = 2\Gamma(D/2)\left(\frac{2\sqrt{\pi}a_B}{L}\right)^D. \qquad (18)$$

This shows that the $N\lambda^2$ expansion we are going to perform, in fact, corresponds to an expansion in the usual dimensionless parameter that controls many-body effects in the coboson systems [20,21], namely, $\eta = N(a_B/L)^D$.

Another dimensionless coefficient that appears for cobosons interacting via the V_{BCS} potential is

$$\alpha_m = \frac{\lambda^2}{2}\sum_{\mathbf{p}} w_{\mathbf{p}}\frac{(-E_1)^{m+1}}{(\varepsilon_{\mathbf{p}} - E_1)^{m+1}} = \int_0^{\frac{\Omega}{-E_1}}dx\frac{x^{D/2-1}}{(x+1)^{m+1}}. \qquad (19)$$

Since the integrand of the above equation decreases as $x^{D/2-2-m}$, we can safely extend the upper integral limit to infinity for $m \geqslant 1$ and $D = (1,2,3)$ in the physically relevant regime $|E_1| \ll \Omega$. This gives

$$\alpha_1 = \frac{\pi}{2} \quad \text{for} \quad D = (1,3), \qquad (20a)$$

$$\alpha_1 = 1 \quad \text{for} \quad D = 2. \qquad (20b)$$

Higher α_m's are related through

$$2m\alpha_m = (2m - D)\alpha_{m-1}, \qquad (21)$$

as obtained from Eq. (19) by an integration by part. So, for $m \geqslant 2$,

$$\alpha_m = \frac{(2m - D)\cdots(4 - D)}{2^{m-1}m!}\alpha_1. \qquad (22)$$

As a result, all α_m's are finite. They form a geometrical series that depend on space dimension $D = (1,2,3)$, but not on the single-pair energy E_1 nor on the potential cutoff Ω, provided that $|E_1| \ll \Omega$.

IV. EFFECT OF ARBITRARY CENTER-OF-MASS MOMENTUM

We tackle the first question by considering cobosons having an arbitrary center-of-mass momentum $\mathbf{K}$ and interacting through the short-range potential V_{CA} given in Eq. (4). We want to study whether, in $D = (1,2,3)$ dimension, the density expansion of the N-coboson energy has extensive terms beyond $N(N - 1)$. For this purpose, we consider the Hamiltonian mean value in the N-coboson state, namely,

$$\langle H \rangle_N = \frac{\langle v|B_0^N H B_0^{\dagger N}|v\rangle}{\langle v|B_0^N B_0^{\dagger N}|v\rangle}, \qquad (23)$$

where $B_0^\dagger$ creates a ground-state coboson and $|v\rangle$ is the vacuum state. $\langle H \rangle_N$ corresponds to the Born value of the N-coboson ground-state energy.

Using the coboson many-body formalism [20,21], we have shown that $\langle H \rangle_N$ is given, within the present notation, by [see Eq. (86) in Ref. [22]]

$$\frac{\langle H \rangle_N - NE_1}{E_1}$$

$$\simeq -(N - 1)\left[\frac{\alpha_2}{2\alpha_1^2}(N\lambda^2) + \left(\frac{\alpha_2\alpha_3}{2\alpha_1^4} - \frac{\alpha_4}{4\alpha_1^3}\right)(N\lambda^2)^2 + \cdots\right], \qquad (24)$$

which, for α_n's given in Eq. (22), reduces to

$$\frac{\langle H \rangle_N - NE_1}{E_1} \simeq -(N - 1)(4 - D)\left[\frac{1}{8\alpha_1}(N\lambda^2)\right.$$

$$\left. + \frac{(6 - D)(8 - 3D)}{768\alpha_1^2}(N\lambda^2)^2 + \cdots\right]. \qquad (25)$$

The above result gives evidence that for a short-range potential which acts between fermion pairs having an arbitrary center-of-mass momentum, the N-coboson ground-state energy in the Born approximation has terms beyond $N(N-1)$ in $D = (1,2,3)$ dimension—but apparently not in 4D.

V. EFFECT OF SPACE DIMENSION

We now turn to the second question related to the effect of space dimension on the ground-state energy of N cobosons. To this end, we consider N pairs having zero center-of-mass momentum and interacting via the V_{BCS} potential given in Eq. (1) for $D = (1,2,3)$ dimension. Their energy follows from solving the Richardson-Gaudin equations given in Eq. (2) for a density of states not necessarily constant.

A. Resolution of Richardson-Gaudin equations

To solve these equations in an easy way, we rescale R_i as $E_1(1 - r_i)$ and expand $1/(\varepsilon_{\mathbf{p}} - R_i)$ in powers of r_i as

$$\frac{1}{\varepsilon_{\mathbf{p}} - R_i} = \sum_{m=0}^{\infty} \frac{(-E_1)^m}{(\varepsilon_{\mathbf{p}} - E_1)^{m+1}} r_i^m. \tag{26}$$

The Richardson-Gaudin equations (2) then read, with the help of Eq. (6),

$$0 = \sum_{m=1}^{\infty} r_i^m \sum_{\mathbf{p}} w_{\mathbf{p}} \frac{(-E_1)^{m+1}}{(\varepsilon_{\mathbf{p}} - E_1)^{m+1}} + \sum_{j=1}^{N} {}' \frac{2}{r_i - r_j}, \tag{27}$$

which, using Eq. (19), we rewrite as

$$0 = \sum_{m=1}^{\infty} \alpha_m r_i^m + \lambda^2 \sum_{j=1}^{N} {}' \frac{1}{r_i - r_j}. \tag{28}$$

To solve the above equation, we expand r_i as

$$r_i = a_i \lambda + b_i \lambda^2 + c_i \lambda^3 + d_i \lambda^4 + e_i \lambda^5 + f_i \lambda^6 + \cdots. \tag{29}$$

This expansion has been previously [23] worked out up to λ^4 to obtain the cubic term in the density expansion of the N-Cooper-pair ground-state energy, that is, for a constant density of states made possible by the existence of a large normal electron Fermi energy. We here wish not only to consider energy-dependent density of states in D dimension, but also to go up to λ^6 to obtain the quartic term in density, in order to better control the effect of space dimension on the N-coboson correlation energy.

To this end, we insert Eq. (29) into Eq. (28) and expand $1/(r_i - r_j)$ in powers of λ. By matching the coefficients of the λ and λ^2 terms, we get

$$0 = \alpha_1 a_i + \sum_j {}' \frac{1}{a_i - a_j}, \tag{30a}$$

$$0 = \alpha_1 b_i + \alpha_2 a_i^2 - \sum_j {}' \frac{b_i - b_j}{(a_i - a_j)^2}. \tag{30b}$$

Similar but heavier relations obtained from higher-order terms in λ are given in Appendix A.

The N-coboson ground-state energy

$$E_N = \sum_{i=1}^{N} R_i = E_1(N - \mathcal{R}_N) \tag{31}$$

follows from $\mathcal{R}_N = \sum_i r_i$. It is possible to obtain its λ expansion without having to determine the r_i's individually. To do it, we first sum Eq. (28) over i. As the second sum reduces to zero by symmetry, this yields

$$0 = \sum_{m=1}^{\infty} \alpha_m \sum_i r_i^m, \tag{32}$$

in which we replace the r_i's by their λ expansion.

(i) The λ term immediately gives

$$\sum_i a_i = 0, \tag{33}$$

from which we conclude that $\mathcal{R}_N$ has no term in λ.

(ii) The λ^2 term of Eq. (32) gives

$$0 = \alpha_1 \sum_i b_i + \alpha_2 \sum_i a_i^2. \tag{34}$$

To obtain $\sum_i a_i^2$, we multiply Eq. (30a) by a_i and we sum over i. This yields

$$0 = \alpha_1 \sum_i a_i^2 + \sum_i \sum_j {}' \frac{a_i}{a_i - a_j}. \tag{35}$$

As the double sum is equal to $(1/2) \sum_i \sum_j {}' (a_i - a_j)/(a_i - a_j)$, which readily gives $N(N-1)/2$, we get

$$\sum_i a_i^2 = -N(N-1) \frac{1}{2\alpha_1}. \tag{36}$$

Equation (34) then gives the λ^2 term of $\mathcal{R}_N$, with α_2 obtained from Eq. (22), as

$$\sum_i b_i = N(N-1) \frac{\alpha_2}{2\alpha_1^2} = \frac{N(N-1)}{8\alpha_1}(4 - D). \tag{37}$$

(iii) Similar calculations for the λ^3 and λ^5 terms of Eq. (32), shown in Appendix B, give

$$\sum_i c_i = 0 = \sum_i e_i, \tag{38}$$

which supports the fact that $\mathcal{R}_N$ has no odd term in λ. So $\mathcal{R}_N$, hence E_N, has an analytical expansion in the coboson density, which is rather reasonable.

(iv) The n^3 term of $\mathcal{R}_N$ follows from the λ^4 term of r_i, that is, the sum of d_i's. Using the α_m's given in Eq. (22), we find their sum as (see Appendix B)

$$\sum_i d_i \simeq \frac{N^3}{2\alpha_1^4} \left(3\alpha_2 \alpha_3 - 2\frac{\alpha_2^3}{\alpha_1} - \alpha_1 \alpha_4 \right)$$

$$= \frac{N^3}{384\alpha_1^2} D(2 - D)(4 - D). \tag{39}$$

(v) In the same way, the n^4 term of $\mathcal{R}_N$ follows from the λ^6 term of r_i, that is, the sum of f_i's. This sum is given (see Appendix B) by

$$\sum_i f_i \simeq \frac{N^4}{8\alpha_1^4}\left(32\frac{\alpha_2^5}{\alpha_1^4} - 92\frac{\alpha_2^3\alpha_3}{\alpha_1^3} + 48\frac{\alpha_2^2\alpha_4}{\alpha_1^2}\right.$$
$$\left. + 45\frac{\alpha_2\alpha_3^2}{\alpha_1^2} - 18\frac{\alpha_3\alpha_4}{\alpha_1} - 20\frac{\alpha_2\alpha_5}{\alpha_1} + 5\alpha_6\right), \quad (40)$$

which also reads, with the help of Eq. (22), in a surprisingly compact form as

$$\sum_i f_i \simeq \frac{N^4}{36864\alpha_1^3}D(2-D)(4-D)\left(24 - 14D - D^2\right). \tag{41}$$

B. N-coboson ground-state energy

Combining the above results, we find that in the thermodynamic limit, the expansion in density, $n \propto N\lambda^2$, of the N-coboson ground-state energy for $D = (1,2,3)$ dimension appears as given in Eq. (6), with $\nu_1 = 1/8\alpha_1$, $\nu_2 = 1/384\alpha_1^2$, and $\nu_3 = 1/36864\alpha_1^3$ where α_1 and λ^2 depend on space dimension D according to Eqs. (20) and (15). This result confirms that for $D = 2$, that is, for a constant density of states, the interaction part of the energy for N cobosons interacting via a BCS-like potential is in $N(N-1)$ only. By contrast, the exact cancellation of extensive terms beyond $N(N-1)$ does not occur for $D = (1,3)$.

Moreover, from the above result, the whole interaction part of the ground-state energy seems to cancel for $D = 4$, suggesting an even stranger many-body effect in 4D. Since the 4D density of states has the same energy dependence as the one of 2D parabolic trap [18], 4D systems have physical relevance. Actually, the previous calculations are not valid for 4D. On closer inspection, we found that the exact cancellation of all interaction terms does not occur in 4D, that is, in a 2D parabolic trap, as we will now show.

VI. 4D OR 2D PARABOLIC TRAP

Let us first calculate the α_m's given in Eq. (19) for 4D. To get α_1, we must keep the integral upper boundary $\Omega/(-E_1)$ in the integral to avoid logarithmic divergence. We then find

$$\alpha_1 = \ln\left(1 + \frac{\Omega}{-E_1}\right) - \frac{\Omega/(-E_1)}{1 + \Omega/(-E_1)}. \tag{42}$$

Contrary to $D = (1,2,3)$, the α_1 coefficient depends on the single-pair energy E_1 and the cutoff Ω through their ratio. By contrast, α_m's for $m \geqslant 2$ can be obtained by extending the integral upper boundary to infinity. This gives

$$\alpha_2 = \frac{1}{2}. \tag{43}$$

Higher α_m's again form a geometric series, as in Eq. (21), but the series now starts with α_2. The fact that α_1 does not

belong to this geometric series makes the D-dependence of the interaction energy for $D = 4$ different from lower dimensions. The density expansion of the N-coboson ground-state energy instead appears as

$$\frac{E_N - NE_1}{E_1}$$
$$\simeq -(N-1)\left[\frac{1}{4\alpha_1}(N\lambda^2) - \frac{\alpha_1^2 - 3\alpha_1 + 3}{24\alpha_1^4}(N\lambda^2)^2\right.$$
$$\left. + \frac{4\alpha_1^4 - 18\alpha_1^3 + 39\alpha_1^2 - 46\alpha_1 + 24}{192\alpha_1^8}(N\lambda^2)^3 + \cdots\right]. \tag{44}$$

So, in 4D or in a 2D parabolic trap, the exact cancellation of all interaction terms in the N-coboson energy does not occur.

Using these α_m's in Eq. (24), we also find that for the above same reason, exact cancellation does not occur in the interaction energy of the Hamiltonian mean value for N ground-state cobosons interacting via the V_{CA} potential.

VII. CONCLUSION

We study the ground-state energy of N cobosons interacting through a BCS-like potential between zero-momentum fermion pairs and through a short-range separable potential between fermion pairs having an arbitrary center-of-mass momentum. Our goal is to determine the effects of the potential characteristics and the space dimension on the interaction part of the N-coboson ground-state energy. We find that for 2D systems interacting via a BCS-like potential, the interaction part is in $N(N-1)$ only, in agreement with previous results obtained in the context of Cooper pairs. Such a striking exact cancellation of the correlation energy exists uniquely for 2D systems. This mysterious cancellation results from the marriage of the very peculiar form of the reduced BCS potential which acts between zero-momentum pairs only, and the constant density of states where the potential acts. Our analysis based on the density expansion of the N-coboson energy shows that this cancellation does not occur for systems with energy-dependent density of states nor for cobosons having an arbitrary center-of-mass momentum. This microscopic understanding allows us to better appreciate the beauty of the reduced BCS potential originally proposed to understand standard superconductivity, and to be cautious of its possible limitations when used in other fields such as nuclear [9,24,25] and cold-atom [19] physics.

APPENDIX A: RELATIONS FOR HIGHER-ORDER λ COEFFICIENTS

We here list the relations, obtained in the same way as Eq. (30), for the λ^3, λ^4, and λ^5 terms:

$$0 = \alpha_1 c_i + 2\alpha_2 a_i b_i + \alpha_3 a_i^3 + \sum_j{}' \left(\frac{(b_i - b_j)^2}{(a_i - a_j)^3} - \frac{c_i - c_j}{(a_i - a_j)^2} \right),$$
$$\text{(A1a)}$$

$$0 = \alpha_1 d_i + \alpha_2 (b_i^2 + 2a_i c_i) + 3\alpha_3 a_i^2 b_i + \alpha_4 a_i^4$$
$$+ \sum_j{}' \left(-\frac{(b_i - b_j)^3}{(a_i - a_j)^4} + 2\frac{(b_i - b_j)(c_i - c_j)}{(a_i - a_j)^3} - \frac{d_i - d_j}{(a_i - a_j)^2} \right),$$
$$\text{(A1b)}$$

$$0 = \alpha_1 e_i + 2\alpha_2 (a_i d_i + b_i c_i) + 3\alpha_3 \left(a_i^2 c_i + a_i b_i^2 \right) + 4\alpha_4 a_i^3 b_i$$
$$+ \alpha_5 a_i^5 + \sum_j{}' \left(\frac{(b_i - b_j)^4}{(a_i - a_j)^5} - 3\frac{(b_i - b_j)^2(c_i - c_j)}{(a_i - a_j)^4} \right.$$
$$\left. + 2\frac{(b_i - b_j)(d_i - d_j)}{(a_i - a_j)^3} + \frac{(c_i - c_j)^2}{(a_i - a_j)^3} - \frac{e_i - e_j}{(a_i - a_j)^2} \right).$$
$$\text{(A1c)}$$

These equations are necessary to obtain the λ expansion of r_i up to λ^6.

APPENDIX B: RESOLUTION FOR HIGHER-ORDER λ COEFFICIENTS

We here explicitly derive the λ^3, λ^4, λ^5, and λ^6 terms of $\mathcal{R}_N$. These straightforward but quite heavy calculations ultimately show that the $\mathcal{R}_N$ expansion only contains even powers of λ.

1. λ^3 term

The λ^3 term of $\mathcal{R}_N$ follows from $\sum_i c_i$. The λ^3 coefficient in Eq. (32) gives this sum through

$$0 = \alpha_1 \sum_i c_i + 2\alpha_2 \sum_i a_i b_i + \alpha_3 \sum_i a_i^3. \qquad \text{(B1)}$$

To calculate $\sum_i a_i^3$, we multiply Eq. (30a) by a_i^2 and we sum over i. This gives

$$0 = \alpha_1 \sum_i a_i^3 + \sum_i \sum_j{}' \frac{a_i^2}{a_i - a_j}. \qquad \text{(B2)}$$

The double sum gives zero since we can rewrite it as $(1/2) \sum_i \sum_j{}' (a_i^2 - a_j^2)/(a_i - a_j) = (1/2) \sum_i \sum_j{}'(a_i + a_j)$ while $\sum_i a_i = 0$, due to Eq. (33). So, $\sum_i a_i^3 = 0$.

To calculate $\sum_i a_i b_i$, we multiply Eq. (30b) by b_i and sum over i. This gives

$$0 = \alpha_1 \sum_i a_i b_i + \alpha_2 \sum_i a_i^3 - \sum_i \sum_j{}' a_i \frac{b_i - b_j}{(a_i - a_j)^2}. \quad \text{(B3)}$$

Because $\sum_i a_i^3 = 0$, while the double sum also reads

$$-\sum_i \sum_j{}' a_i \frac{b_i - b_j}{(a_i - a_j)^2}$$
$$= -\frac{1}{2} \sum_i \sum_j{}' \frac{b_i - b_j}{a_i - a_j}$$
$$= -\sum_i b_i \sum_j{}' \frac{1}{a_i - a_j} = \alpha_1 \sum_i a_i b_i, \qquad \text{(B4)}$$

as obtained with the help of Eq. (30a) for the j sum, we conclude from Eq. (B3) that $\sum_i a_i b_i = 0$. So, Eq. (B1) ultimately gives

$$\sum_i c_i = 0. \qquad \text{(B5)}$$

As a result, the λ^3 term of $\mathcal{R}_N$ is equal to zero.

2. λ^4 term

The λ^4 term of $\mathcal{R}_N$ follows from $\sum_i d_i$. The λ^4 coefficient of Eq. (32) gives this sum through

$$0 = \alpha_1 \sum_i d_i + \alpha_2 \sum_i \left(b_i^2 + 2a_i c_i \right) + 3\alpha_3 \sum_i a_i^2 b_i$$
$$+ \alpha_4 \sum_i a_i^4. \qquad \text{(B6)}$$

To calculate $\sum_i a_i^4$, we multiply Eq. (30a) by a_i^3 and we sum over i. This gives

$$0 = \alpha_1 \sum_i a_i^4 + \sum_i \sum_j{}' \frac{a_i^3}{a_i - a_j}. \qquad \text{(B7)}$$

We rewrite the double sum in the above equation as

$$\sum_i \sum_j{}' \frac{a_i^3}{a_i - a_j} = \frac{1}{2} \sum_i \sum_j{}' \frac{a_i^3 - a_j^3}{a_i - a_j}$$
$$= \frac{1}{2} \sum_i \sum_j{}' \left(a_i^2 + a_i a_j + a_j^2 \right), \quad \text{(B8)}$$

which gives $(N - 3/2) \sum_i a_i^2 + (1/2)(\sum_i a_i)^2$. Using Eq. (36) and $\sum_i a_i = 0$, we end up with

$$\sum_i a_i^4 = N(N - 1)(2N - 3) \frac{1}{4\alpha_1^2}. \qquad \text{(B9)}$$

To calculate $\sum_i a_i^2 b_i$, we multiply Eq. (30b) by a_i^2 and we sum over i. This gives

$$0 = \alpha_1 \sum_i a_i^2 b_i + \alpha_2 \sum_i a_i^4 - \sum_i \sum_j{}' \frac{a_i^2(b_i - b_j)}{(a_i - a_j)^2}. \quad \text{(B10)}$$

We rewrite the double sum in the above equation as

$$-\sum_i \sum_j {}' \frac{a_i^2(b_i - b_j)}{(a_i - a_j)^2}$$

$$= -\frac{1}{2} \sum_i \sum_j {}' \frac{(a_i^2 - a_j^2)(b_i - b_j)}{(a_i - a_j)^2}$$

$$= -\frac{1}{2} \sum_i \sum_j {}' \frac{(a_i + a_j)(b_i - b_j)}{a_i - a_j}. \quad (B11)$$

Since $\sum_i \sum_j {}'(b_i + b_j)(a_i + a_j)/(a_i - a_j) = 0$, while $(a_i + a_j)/(a_i - a_j) = (-1 + 2a_i)/(a_i - a_j)$, the right-hand side of the above equation also reads

$$-\sum_i b_i \sum_j {}' \left(-1 + \frac{2a_i}{a_i - a_j} \right), \quad (B12)$$

which, with the help of Eq. (30a), gives $(N - 1) \sum_i b_i + 2\alpha_1 \sum_i a_i^2 b_i$. Combining this result with Eqs. (37) and (B9), we end up with

$$\sum_i a_i^2 b_i = -N(N - 1)(4N - 5) \frac{\alpha_2}{12\alpha_1^3}. \quad (B13)$$

To calculate the second term of Eq. (B6), we multiply Eq. (A1a) by a_i and we sum over i. This gives

$$0 = \alpha_1 \sum_i a_i c_i + 2\alpha_2 \sum_i a_i^2 b_i + \alpha_3 \sum_i a_i^4$$

$$+ \sum_i \sum_j {}' a_i \left(\frac{(b_i - b_j)^2}{(a_i - a_j)^3} - \frac{c_i - c_j}{(a_i - a_j)^2} \right). \quad (B14)$$

The double sum in the above equation also reads

$$\frac{1}{2} \sum_i \sum_j {}'(a_i - a_j) \left(\frac{(b_i - b_j)^2}{(a_i - a_j)^3} - \frac{c_i - c_j}{(a_i - a_j)^2} \right), \quad (B15)$$

which reduces to

$$\sum_i b_i \sum_j {}' \frac{b_i - b_j}{(a_i - a_j)^2} - \sum_i c_i \sum_j {}' \frac{1}{a_i - a_j}. \quad (B16)$$

Using Eq. (30) for the two sums over j, we end up with

$$0 = \alpha_1 \sum_i \left(b_i^2 + 2a_i c_i \right) + 3\alpha_2 \sum_i a_i^2 b_i + \alpha_3 \sum_i a_i^4. \quad (B17)$$

By combining Eqs. (B6), (B17), (B9), and (B13), we get the prefactor of the λ^4 term in $\mathcal{R}_N$ as

$$\sum_i d_i = \frac{N(N - 1)}{4\alpha_1^4} \left[(6N - 8)\alpha_2 \alpha_3 - (4N - 5) \frac{\alpha_2^3}{\alpha_1} \right.$$

$$\left. -(2N - 3)\alpha_1 \alpha_4 \right]. \quad (B18)$$

3. λ^5 term

The λ^5 term of $\mathcal{R}_N$ follows from $\sum_i e_i$. The λ^5 coefficient of Eq. (32) gives this sum through

$$0 = \alpha_1 \sum_i e_i + 2\alpha_2 \sum_i (a_i d_i + b_i c_i)$$

$$+ 3\alpha_3 \sum_i \left(a_i^2 c_i + a_i b_i^2 \right) + 4\alpha_4 \sum_i a_i^3 b_i + \alpha_5 \sum_i a_i^5. \quad (B19)$$

To calculate $\sum_i a_i^5$, we multiply Eq. (30a) by a_i^4 and sum over i. This gives

$$0 = \alpha_1 \sum_i a_i^5 + \sum_i \sum_j {}' \frac{a_i^4}{a_i - a_j}. \quad (B20)$$

We rewrite the double sum in the above equation as

$$\sum_i \sum_j {}' \frac{a_i^4}{a_i - a_j} = \frac{1}{2} \sum_i \sum_j {}' \frac{a_i^4 - a_j^4}{a_i - a_j}$$

$$= \frac{1}{2} \sum_i \sum_j {}' \left(a_i^3 + a_i^2 a_j + a_i a_j^2 + a_j^3 \right), \quad (B21)$$

which leads to $(N - 2) \sum_i a_i^3 + \sum_i a_i^2 \sum_i a_i$; so $\sum_i a_i^5 = 0$, since $\sum_i a_i = 0 = \sum_i a_i^3$.

To calculate $\sum_i a_i^3 b_i$, we multiply Eq. (30b) by a_i^3 and we sum over i. This gives

$$0 = \alpha_1 \sum_i a_i^3 b_i + \sum_i a_i^5 - \sum_i \sum_j {}' \frac{a_i^3(b_i - b_j)}{(a_i - a_j)^2}. \quad (B22)$$

The double sum in the above equation also reads

$$-\frac{1}{2} \sum_i \sum_j {}' \frac{(a_i^3 - a_j^3)(b_i - b_j)}{(a_i - a_j)^2}$$

$$= -\sum_i b_i \sum_j {}' \frac{a_i^2 + a_i a_j + a_j^2}{a_i - a_j}. \quad (B23)$$

By writing $a_i^2 + a_i a_j + a_j^2$ as $(a_i - a_j)^2 + 3a_i a_j$ and $a_j/(a_i - a_j)$ as $-1 + a_i/(a_i - a_j)$, the above term is equal to

$$-\sum_i b_i \sum_j {}'(a_i - a_j) + 3 \sum_i a_i b_i \sum_j {}' 1$$

$$-3 \sum_i a_i^2 b_i \sum_j {}' \frac{1}{a_i - a_j}. \quad (B24)$$

The sum of the first two terms give $(2N - 3) \sum_i a_i b_i + \sum_i a_i \sum_i b_i = 0$, while using Eq. (30a), the last term gives $3\alpha_1 \sum_i a_i^3 b_i$. Since $\sum_i a_i^5 = 0$, we find from Eq. (B22) that $\sum_i a_i^3 b_i = 0$.

To calculate the third term of Eq. (B19), we multiply Eq. (A1a) by a_i^2 and we sum over i. This gives

$$0 = \alpha_1 \sum_i a_i^2 c_i + 2\alpha_2 \sum_i a_i^3 b_i + \alpha_3 \sum_i a_i^5$$
$$+ \sum_i \sum_j{}' a_i^2 \left(\frac{(b_i - b_j)^2}{(a_i - a_j)^3} - \frac{c_i - c_j}{(a_i - a_j)^2} \right). \quad \text{(B25)}$$

The double sum also reads

$$\frac{1}{2} \sum_i \sum_j{}' (a_i^2 - a_j^2) \left(\frac{(b_i - b_j)^2}{(a_i - a_j)^3} - \frac{c_i - c_j}{(a_i - a_j)^2} \right), \quad \text{(B26)}$$

which is equal to

$$\sum_i b_i \sum_j{}' (a_i + a_j) \frac{b_i - b_j}{(a_i - a_j)^2} - \sum_i c_i \sum_j{}' \frac{a_i + a_j}{a_i - a_j}.$$

B writing $(a_i + a_j)/(a_i - a_j)$ as $-1 + 2a_i/(a_i - a_j)$, we find from Eqs. (B25) and (30) that

$$0 = 3\alpha_1 \sum_i (a_i^2 c_i + a_i b_i^2) + 4\alpha_2 \sum_i a_i^3 b_i + \alpha_3 \sum_i a_i^5$$
$$+ (N - 1) \sum_i c_i. \quad \text{(B27)}$$

Since the last three terms are equal to zero, we end up with $\sum_i (a_i^2 c_i + a_i b_i^2) = 0$.

To calculate the second term, we multiply Eq. (A1b) by a_i and we sum over i. Similar calculation for the double sum leads to

$$0 = 2\alpha_1 \sum_i (a_i d_i + b_i c_i) + 3\alpha_2 \sum_i (a_i^2 c_i + a_i b_i^2)$$
$$+ 4\alpha_3 \sum_i a_i^3 b_i + \alpha_4 \sum_i a_i^5. \quad \text{(B28)}$$

Since the last three terms are equal to zero, we end up with $\sum_i (a_i d_i + b_i c_i) = 0$.

Equation (B19) then gives

$$\sum_i e_i = 0. \quad \text{(B29)}$$

So, $\mathcal{R}_N$ has no term in λ^5.

4. λ^6 term

The λ^6 term of $\mathcal{R}_N$ follows from $\sum_i f_i$. The λ^6 coefficient in Eq. (32) gives this sum through

$$0 = \alpha_1 \sum_i f_i + \alpha_2 \sum_i (c_i^2 + 2a_i e_i + 2b_i d_i)$$
$$+ \alpha_3 \sum_i (b_i^3 + 3a_i^2 d_i + 6a_i b_i c_i)$$
$$+ \alpha_4 \sum_i (4a_i^3 c_i + 6a_i^2 b_i^2) + 5\alpha_5 \sum_i a_i^4 b_i + \alpha_6 \sum_i a_i^6. \quad \text{(B30)}$$

To calculate $\sum_i a_i^6$, we multiply Eq. (30a) by a_i^5 and we sum over i. Since $\sum_i a_i = 0 = \sum_i a_i^3$, we get

$$0 = \alpha_1 \sum_i a_i^6 + \frac{2N - 5}{2} \sum_i a_i^4 + \frac{1}{2} \left(\sum_i a_i^2 \right)^2. \quad \text{(B31)}$$

To calculate $\sum_i a_i^4 b_i$, we multiply Eq. (30b) by a_i^4 and we sum over i. Similar algebras lead to

$$0 = 5\alpha_1 \sum_i a_i^4 b_i + \alpha_2 \sum_i a_i^6 + 3(N - 2) \sum_a a_i^2 b_i$$
$$+ \left(\sum_i b_i \right) \left(\sum_i a_i^2 \right). \quad \text{(B32)}$$

To calculate the fourth term of Eq. (B30), we multiply Eq. (A1a) by a_i^3 and we sum over i. This yields

$$0 = \alpha_1 \sum_i (4a_i^3 c_i + 6a_i^2 b_i^2) + 5\alpha_2 \sum_i a_i^4 b_i + \alpha_3 \sum_i a_i^6$$
$$+ \frac{2N - 3}{2} \sum_i (b_i^2 + 2a_i c_i) + \frac{1}{2} \left(\sum_i b_i \right)^2. \quad \text{(B33)}$$

To calculate the third term, we multiply Eq. (A1b) by a_i^2 and we sum over i. This yields

$$0 = \alpha_1 \sum_i (b_i^3 + 3a_i^2 d_i + 6a_i b_i c_i) + \alpha_2 \sum_i (4a_i^3 c_i + 6a_i^2 b_i^2)$$
$$+ 5\alpha_3 \sum_i a_i^4 b_i + \alpha_4 \sum_i a_i^6 + (N - 1) \sum_i d_i. \quad \text{(B34)}$$

To calculate the second term of Eq. (B30), we multiply Eq. (A1b) by b_i and Eq. (A1c) by a_i and we sum over i. By adding these two equations, we get

$$0 = \alpha_1 \sum_i (c_i^2 + 2a_i e_i + 2b_i d_i) + \alpha_2 \sum_i (b_i^3 + 3a_i^2 d_i + 6a_i b_i c_i)$$
$$+ \alpha_3 \sum_i (4a_i^3 c_i + 6a_i^2 b_i^2) + 5\alpha_4 \sum_i a_i^4 b_i + \alpha_5 \sum_i a_i^6. \quad \text{(B35)}$$

Equations (B30)–(B35) ultimately lead to the sum of f_i's given in Eq. (40).

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