Quantum trajectories under frequent measurements in a non-Markovian environment

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In this work we generalize the quantum trajectory (QT) theory from Markovian to non-Markovian environments. We model the non-Markovian environment by using a Lorentzian spectral density function with bandwidth (Λ), and find a perfect "scaling" property with the measurement frequency (τ^{-1}) in terms of the scaling variable $x = \Lambda\tau$. Our result bridges the gap between the existing QT theory and the Zeno effect, by rendering them as two extremes corresponding to $x \to \infty$ and $x \to 0$, respectively. This x-dependent criterion improves the idea of using τ alone and quantitatively identifies the validity condition of the conventional QT theory.

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I. INTRODUCTION

The quantum trajectory (QT) given by the stochastic Schrödinger equation (SSE) for an open system associated with *Markovian dynamics* can be interpreted as quantum state conditioned on continuous observation (monitoring) on the environment [1,2]. The QT theory of this type has been well demonstrated and broadly applied [3,4], including the recent experiments in superconducting solid-state circuits [5–13]. On the other hand, associated with the *non-Markovian dynamics* of open quantum systems, a similar non-Markovian stochastic Schrödinger equation (nMSSE) has been constructed [14,15]. However, the nMSSE is largely a working tool for unraveling the non-Markovian dynamics, which cannot be interpreted as a measurement-conditioned *physical* quantum trajectory [16–18]. After careful analysis by Wiseman *et al.*, the nMSSE might be at most interpreted as a certain "hidden variable" theory, i.e., taking the complex Wiener variable z_t involved in the nMSSE as an "objective property" which inherently exists in the environment, rather than as a consequence of continuous measurements [18].

In this work we consider the interesting problem of how to construct the *physical* QT associated with frequent monitoring on a non-Markovian environment. To be specific, we model the non-Markovian environment by using a Lorentzian spectral density function (SDF) with finite bandwidth. We show that the result is quite different from the nMSSE mentioned above. Elegantly, via slight modification by involving a "scaling" variable, the resultant QT formally resembles, but essentially generalizes, the conventional QT. Our result bridges the gap between the existing QT [1–4] and the quantum Zeno effect [19], by rendering them as two extremes which have quite different predictions [20,21].

Let us consider a two-level atom (qubit) prepared in a quantum superposition of the ground state ($|g\rangle$) and exited state ($|e\rangle$), $|\Psi(0)\rangle = \alpha_0|e\rangle + \beta_0|g\rangle$. Now consider its evolution under *continuous* (very frequent) measurements in the surrounding environment for the spontaneous emission of a photon. From the celebrated QT theory [1–4], conditioned on the *continuous* null-result (no-register of spontaneous emission) detection, the state would change, following the simple formula

$$|\Psi(t)\rangle = (\alpha_0 e^{-\Gamma t/2}|e\rangle + \beta_0|g\rangle)/\mathcal{N}, \tag{1}$$

where Γ is the spontaneous emission rate and $\mathcal{N}$ denotes the normalization factor. To interpret this result, reasoning based on *informational* evolution is sometimes put forward. That is, *no result* is a sort of *information*, so the state can change according to Bayesian inference, similar to classical probability theory.

On the other hand, the above continuous null-result quantum motion is prohibited by the quantum Zeno effect [19]. We may briefly summarize the treatment and result as follows. Starting with $|\Psi(0)\rangle$, let us expand the evolution operator up to second order in τ, $U(\tau) \simeq 1 - iH\tau - H^2\tau^2/2$, where τ is the time interval between the successive null-result measurements. Each null-result measurement would project the wave function onto the atomic subspace. Consider n subsequent null-result measurements during time t (with $n = t/\tau$). In the limit $\tau \to 0$ and $t =$ const, one obtains (see Appendix A for more details)

$$|\Psi_n\rangle \to \alpha_0|e\rangle + \beta_0|g\rangle \equiv |\Psi(0)\rangle. \tag{2}$$

So we find that the frequent null-result monitoring of the environment will prevent the change of the state, resulting thus in the quantum Zeno effect.

Actually, the QT theory leading to Eq. (1) is from unraveling the Markovian Lindblad master equation. In the Markovian approximation, one requires a wide-bandwidth environment (i.e., the bandwidth $\Lambda \to \infty$). Therefore, any τ is long compared to the environment's memory time Λ^{-1}, leading thus to the exponential decay of population which destroys the possibility of the Zeno effect. In the case of $\Lambda \to \infty$, the above expansion on $U(\tau)$ is invalid. In order to generate the Zeno effect, the physical condition is $\tau \ll \Lambda^{-1}$. In the remainder of this work, we develop a treatment to smoothly bridge these two extremes and construct the associated QT theory by introducing an external drive to the atom.

II. SPONTANEOUS DECAY

The two-level atom coupled to the electromagnetic vacuum (environment) is described by the Hamiltonian

$$H = \frac{\Delta_{eg}}{2}\sigma_z + \sum_r \left(b_r^\dagger b_r + \frac{1}{2}\right)\omega_r + \sum_r [V_r b_r^\dagger \sigma^- + \text{H.c.}]. \tag{3}$$

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Throughout this work we set $\hbar = 1$. Here we introduce the two-level energy difference $\Delta_{eg} = E_e - E_g$, and the atomic operators $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma^- = |g\rangle\langle e|$, and $\sigma^+ = |e\rangle\langle g|$. V_r is the coupling amplitude of the atom with the environment. Then, consider the evolution of the entire system, starting with an initial state $|\Psi(0)\rangle = (\alpha_0|e\rangle + \beta_0|g\rangle) \otimes |\mathrm{vac}\rangle$, where $|\mathrm{vac}\rangle$ stands for the environmental vacuum with no photon. Under the influence of the coupling, the entire state at time t can be written as

$$|\Psi(t)\rangle = \alpha(t)|e\rangle \otimes |\mathrm{vac}\rangle + \sum_r c_r(t)|g\rangle \otimes |1_r;0;\cdots\rangle + \beta_0|g\rangle \otimes |\mathrm{vac}\rangle, \tag{4}$$

where $|1_r;0;\cdots\rangle$ describes the environment with a photon excitation in the state "r" and no excitations of other states. The coefficients have initial conditions of $\alpha(0) = \alpha_0$ and $c_r(0) = 0$.

Substituting Eq. (4) into the Schrödinger equation and performing the Laplace transform, one can obtain the solution of $\alpha(t)$ in the frequency domain (see Appendix B for more details). That is, we replace $\sum_r |V_r|^2[\cdots] \to \int d\omega_r D(\omega_r)[\cdots]$, where $D(\omega_r) = \sum_{r'} |V_{r'}|^2 \delta(\omega_r - \omega_{r'}) \to D_0\Lambda^2/[(\omega_r - \omega_0)^2 + \Lambda^2]$ is the SDF, approximated here by a finite-band Lorentzian spectrum with ω_0 the spectral center and Λ the width [22]. We obtain then the time-dependent amplitude $\alpha(t) \equiv a(t)\alpha_0$ via the inverse Laplace transform as [21]

$$a(t) = \frac{1}{A_+ - A_-}(A_+ e^{-A_- t} - A_- e^{-A_+ t}), \tag{5}$$

with $A_\pm = [\Lambda - iE \pm \sqrt{(\Lambda - iE)^2 - 2\Gamma\Lambda}]/2$. Here we introduced the energy offset $E = (E_e - E_g) - \omega_0$ and the usual decay rate in the wideband limit, $\Gamma = 2\pi D_0$.

III. FREQUENT NULL-RESULT MEASUREMENTS

The null-result measurement in the environment, quantum mechanically, collapses the entire wave function onto the atomic subspace. After n such null-result measurements with subsequent time interval $\tau = t/n$, the final state of the atom is

$$|\widetilde{\Psi}(t)\rangle = [\bar{a}(t)\alpha_0|e\rangle + \beta_0|g\rangle]/\sqrt{\mathcal{N}_n(t)}, \tag{6}$$

where $\bar{a}(t) = a^n(\tau)$ and $\mathcal{N}_n(t) = |\bar{a}(t)\alpha_0|^2 + |\beta_0|^2$. Note that, unlike the case of the wideband-limit Markovian environment, $|\widetilde{\Psi}(t)\rangle$ differs from the single-null-measurement-collapsed state at the final moment from $|\Psi(t)\rangle$. It can be proved that the normalization factor $\mathcal{N}_n$ equals also the *joint* probability of getting *null* results in all the intermediate measurements, i.e., $(1 - \sum_r |c_r(\tau)|^2)^n$. Let us denote $\mathcal{N}_n(t) \equiv p_0^{(n)}(t)$. Accordingly, during time $(0,t)$, the probability of detecting a spontaneous photon is $p_1^{(n)}(t) = 1 - p_0^{(n)}(t)$.

Now let us consider the limit of "continuous" measurements, $n \to \infty$, by taking the measurement time interval $\tau \to 0$ and keeping $t = n\tau$ fixed. Supposing that we increase the bandwidth Λ so that the variable $x = \Lambda\tau$ remains constant, we can prove a "scaling" property that the final state becomes a function of x only. To reveal the full scaling behavior in the general case, we also assume the energy offset $E = c\Lambda$ (in usual treatment $c = 0$). One finds from Eq. (5) that $A_+ = \kappa\Lambda - \Gamma/(2\kappa)$ and $A_- = \Gamma/(2\kappa)$ [up to the order of $(\Gamma/\Lambda)^2$], where $\kappa = 1 - ic$. Using $(1 - \frac{z}{n})^n = e^{-z(1 + \frac{z}{2n} + \cdots)}$

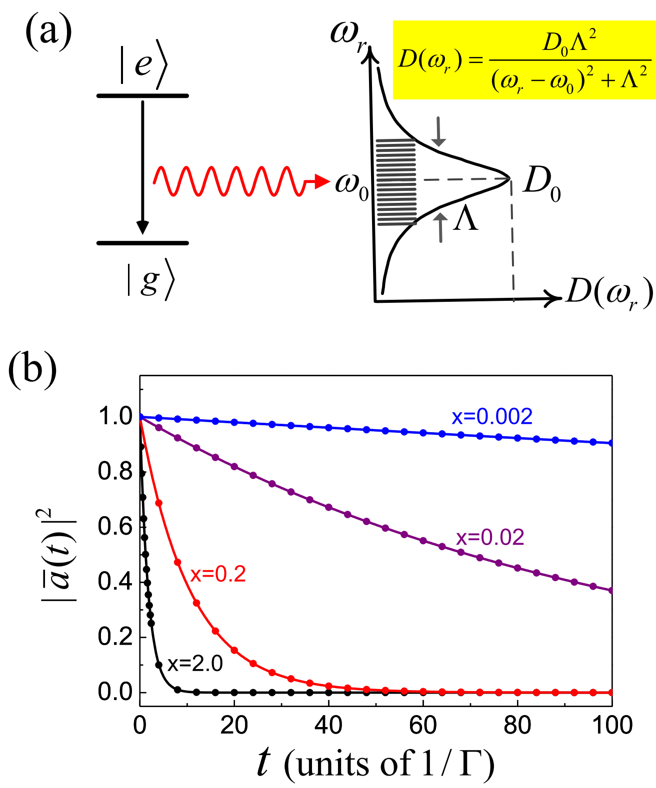

FIG. 1. (a) Spontaneous emission of a two-level atom coupled to a non-Markovian environment with finite-bandwidth Lorentzian spectrum. (b) Effective decay factor of the excited state started with a quantum superposition $\alpha_0|e\rangle + \beta_0|g\rangle$, under frequent null-result measurements in the environment. Scaling behavior is demonstrated by the remarkable agreement between Eq. (7) (continuous lines) and $a^n(\tau)$ (symbols) calculated using Eq. (5) with $\Lambda = 10\Gamma$ (as an example) and $E = \omega - \omega_0 = 0$. Recall that $t = n\tau$ and $x = \Lambda\tau$.

and neglecting small terms $\sim\Gamma/\Lambda$ in exponent, we arrive at [21]

$$\bar{a}(t) = a^n(\tau) = \exp\left\{-\left[\frac{1}{\kappa} - (1 - e^{-\kappa x})\frac{1}{\kappa^2 x}\right]\frac{\Gamma t}{2}\right\}. \tag{7}$$

Elegantly, this result reveals an explicit scaling property in the $x = \Lambda\tau$ variable. In Fig. 1(b), by relaxing the conditions ($n \to \infty$ and $\tau \to 0$) for obtaining this analytic formula, we illustrate the scaling behavior in broad parameter conditions.

Some remarks about Eq. (7)

(i) The numerical results in Fig. 1(b) for finite Λ and τ (e.g., the $x = 2.0$ curve for $\tau^{-1} = 0.5\Lambda$) show excellent agreement with Eq. (7), indicating that we can expect the scaling behavior beyond the limits $n \to \infty$ and $\tau \to 0$. This limiting procedure is only a mathematical technique leading us to obtain the analytic result of Eq. (7).

(ii) The scaling behavior can be understood via the time-energy uncertainty relation. Actually, the successive measurements with time interval τ in the reservoir will cause fluctuations of the atom's level (E_e) by an amount $\sim\tau^{-1}$, since the result of whether or not a spontaneous emission is detected in the reservoir allows us to know whether or not the atom is in the excited state. Then, if we (conceptually) expand the width of the reservoir's SDF by this same amount (i.e., by $\sim\tau^{-1}$), we can expect the same (identical) decay dynamics. This is the physical reason for the scaling behavior shown analytically by Eq. (7) and numerically in Fig. 1(b).

(iii) Note that the x dependence of the decay dynamics is the same as the τ-dependence for a given bandwidth Λ (usually it is difficult to change Λ in real setups). And, this τ dependence is the essential feature associated with measurements in a non-Markovian reservoir, which is in sharp contrast with the conventional τ*-independent* Markovian case.

(iv) From Eq. (7), in the wideband limit, $x \to \infty$ and $\kappa \to 1$, one recovers the result $\bar{a}(t) \to e^{-\Gamma t/2}$ predicted by the standard QT theory. On the other hand, in the limit of $x \to 0$, one finds from Eq. (7) that $\bar{a}(t) = 1$, so that the atom is frozen in its initial state, showing the Zeno effect.

(v) In the Zeno regime $\tau^{-1} \gg \Lambda$, one may encounter a "negative-frequency" problem if the central frequency ω_0 is not much larger than Λ. In this case (and for the transition energy $\Delta_{eg} > \omega_0$) the level E_e may fluctuate into the domain of negative frequency of the SDF, thus violating the condition of the symmetric Lorentzian SDF model and needing certain modification to Eq. (7). In this work, we assume a *symmetric* Lorentzian SDF model under the conditions $\Delta_{eg} > \omega_0 \gg \Lambda$, for the sake of showing a full transition from the Markovian behavior to the Zeno effect governed by the unified Eq. (7). In this case, there is no negative frequency difficulty to affect the validity of Eq. (7).

(vi) From Eq. (7), one can define an *effective* decay rate

$$\gamma_{\rm eff} = {\rm Re}\{[1 - (\kappa x)^{-1}(1 - e^{-\kappa x})]/\kappa\}\,\Gamma. \tag{8}$$

Note that for the wideband-limit Markovian environment the exponential decay process implies *no effect* of the intermediate null-result interruptions [21]. Equation (8), however, shows that the decay rate is influenced by the frequent null-result measurements. This x or τ dependence reflects the non-Markovian effect rooted in Eq. (5), despite that the frequent measurements cut off the usual non-Markovian correlation (memory) effect between different τ-period evolutions. It is just the accumulation of the "small" non-Markovian contributions over $t = n\tau$ that makes Eqs. (7) and (8) and the associated QT (to be constructed) generalize the usual Markovian results.

IV. QUANTUM TRAJECTORIES

Corresponding to direct photon detection, let us first construct the Monte Carlo wave-function (MCWF) approach, closely along the line proposed in Ref. [1]. Consider the state evolution under frequent null-result measurements between t and $t + \Delta t$, with thus $\Delta t = n\tau$. The probability with a photon register in the detector during Δt is $p_1^{(n)}(\Delta t) = |\alpha(t)|^2\gamma_{\rm eff}\Delta t$. Under the "scaling" consideration, the effective decay rate $\gamma_{\rm eff}$ is simply given by Eq. (8), or, alternatively, by

$$\gamma_{\rm eff} = [1 - |\bar{a}(\Delta t)|^2]/\Delta t \quad \text{ or } \gamma_{\rm eff} = -\ln[\,|\bar{a}(\Delta t)|^2]/\Delta t. \tag{9}$$

For small Δt, which implies $|\bar{a}(\Delta t)|^2 \simeq 1$, both definitions are equivalent and coincide with Eq. (8).

In practical simulations, we generate a random number ϵ between 0 and 1. If $\epsilon < p_1^{(n)}(\Delta t)$, which corresponds to the probability of having a photon register in the detector ($\Delta N_c = 1$), we update the state by a "jump" action

$$|\widetilde{\Psi}(t + \Delta t)\rangle = \sigma^-|\widetilde{\Psi}(t)\rangle/\parallel \bullet \parallel, \tag{10}$$

where $\parallel \bullet \parallel$ denotes the normalization factor. On the other hand, if $\epsilon > p_1^{(n)}(\Delta t)$, which corresponds to the null-result measurement (NRM) with $\Delta N_c = 0$, we update the state via the effective *smooth* evolution

$$|\widetilde{\Psi}(t + \Delta t)\rangle = \mathcal{U}(\Delta t)|\widetilde{\Psi}(t)\rangle/\parallel \bullet \parallel. \tag{11}$$

In terms of a matrix form defined by $\{\alpha(t + \Delta t),\beta(t + \Delta t)\}^T = \mathcal{U}(\Delta t)\{\alpha(t),\beta(t)\}^T$, the effective *nonunitary* evolution operator reads

$$\mathcal{U}(\Delta t) = \begin{pmatrix} \bar{a}(\Delta t) & 0 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

Noting that $\Delta t = n\tau$, as above, here we mention again that $\bar{a}(\Delta t) = [a(\tau)]^n$ which can be Eq. (7) in the limits $\tau \to 0$ and $n \to \infty$, or can be more generally determined using Eq. (5) for $a(\tau)$.

Based on the MCWF approach proposed above, one can simulate the (stochastic) quantum trajectories under frequent photon detections in the environment. The ensemble average over these trajectories of quantum (pure) state corresponds to the result given by the following master equation [1–4]:

$$\dot{\rho} = -i[H_S,\rho] + \gamma_{\rm eff}\mathcal{D}[\sigma^-]\rho, \tag{13}$$

where $\mathcal{D}[\bullet]\rho \equiv \bullet\rho\bullet^\dagger - \frac{1}{2}\{\bullet^\dagger\bullet,\rho\}$. Formally, this is an x- or τ-dependent Lindblad-type master equation. However, unlike its Markovian counterpart, a significant difference lies in the fact that this equation does *not* describe the *reduced* state $\varrho(t)$ of the (open) quantum system. It is well known that $\varrho(t)$ is defined by tracing the environment degrees of freedom from the entire (system-plus-environment) unitary wave function at time t. Here, "tracing" simply means performing projective measurements and making an average *only* at the last moment t, on the entire unitary wave function evolved from the same *initial* state. In contrast to $\varrho(t)$, the state $\rho(t)$ given by Eq. (13) is the ensemble-averaged state of the system under successive measurement interruptions. Remarkably, the successive measurements would destroy the correlation effect between *different* τ-period evolutions, resulting thus in the Markovian-Lindblad-type Eq. (13) with, however, an effective $\gamma_{\rm eff}$ rather than a certain "natural" decay rate.

Following Refs. [1–4], we now include external driving into Eq. (13), via $H_S = \frac{\Delta_{eg}}{2}\sigma_z + \Omega\sigma_x$. Note that the validity of this procedure is rooted in the *additivity of the state changes* over the very small time interval (τ). As a result, there are two contributions to the state change: one is *informational* owing to the continuous measurements, and the other is *physical* which is caused by the external driving. Note also that in general the dissipative two-level atom under driving is not exactly solvable. The underlying complexity can be imagined as follows: there are more and more photons emitted into the reservoir, and the emitted photon can reexcite the atom. However, in the presence of *frequent* measurements, the emitted photon will be destroyed by detectors. During each successive measurement interval (τ), it is reasonable to assume that there is *at most* one photon in the reservoir. Therefore, even

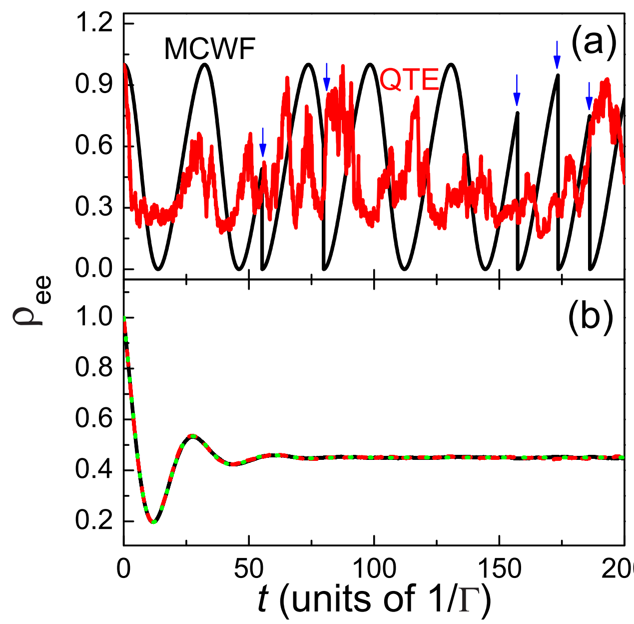

FIG. 2. (a) Two quantum trajectories from the MCWF (black) and QT equation (red) simulations. The blue arrows indicate quantum "jumps" owing to "direct" detection of the spontaneous emission of the atom. (b) Ensemble average of 2000 MCWF and QT equation trajectories and the result (green curve) from the master equation $\dot{\rho} = -i[H_S,\rho] + \gamma_{\rm eff}\mathcal{D}[\sigma^-]\rho$. Parameters used in the simulation: $\Omega = 0.1$, $\Gamma = 1.0$, $x = 0.2$, and $E = 0$.

in the presence of external driving, Eq. (13) is valid under the above considerations.

Instead of the *direct* detection of the spontaneous emission considered above, one can also adopt the so-called homodyne detection scheme by mixing the emitting photons with a classical field with modulating phase φ [2,3]. The measurement result (optical current) of this type can be expressed as [2,3] $I_\varphi(t) = \sqrt{\gamma_{\rm eff}}\langle\sigma^- e^{-i\varphi} + \sigma^+ e^{i\varphi}\rangle/2 + \xi(t)$, where $\langle\cdots\rangle = {\rm Tr}[(\cdots)\rho(t)]$ and $\xi(t)$ is the Gaussian white noise associated with quantum jumps. In this measurement scheme, the detection result is a sum of the classical reference field and the photon emitted by the atom. The "jump" (knowledge change of the atom state) associated with the photon register in the detector is relatively weak, developing thus a "diffusive" regime because of the mix of the reference field. Through a careful analysis [2,3], the difference of the detected result (in single realization) during $(t,t+dt)$ from the expected one using the earlier $\rho(t)$ is characterized by $\xi(t)dt$ in the expression of $I_\varphi(t)$. Conditioned on $I_\varphi(t)$, the state evolution is given by the *diffusive* QT equation [2,3]

$$\dot{\rho} = -i[H_S,\rho] + \gamma_{\rm eff}\mathcal{D}[\sigma^-]\rho + \sqrt{\gamma_{\rm eff}}\mathcal{H}[e^{-i\varphi}\sigma^-]\rho\xi(t), \tag{14}$$

where $\mathcal{H}[\bullet]\rho \equiv \bullet\rho + \rho\,\bullet^\dagger - \langle\bullet + \bullet^\dagger\rangle\rho$. Essentially, Eq. (14) generalizes the existing QT equation by accounting for the measurement frequency ($\nu = 1/\tau$) in the effective "spontaneous" emission rate $\gamma_{\rm eff}$.

In Fig. 2(a) we display two representative quantum trajectories from the MCWF and the diffusive QT equation (14). We see that the former type of quantum trajectory reveals a drastic "quantum jump" owing to the *direct* detection for the spontaneous emission, while the latter type has no such jump onto the ground state $|g\rangle$. However, as expected, an ensemble average of each type of quantum trajectories (over 2000) gives the same result of Eq. (13), as demonstrated in Fig. 2(b).

V. SUMMARY AND DISCUSSIONS

We have constructed a scheme to generalize the QT theory from Markovian to non-Markovian environments. Taking the specific model of Lorentzian SDF, we revealed a perfect scaling property between the spectral bandwidth and the measurement frequency. Our result bridges the gap between the existing QT and the quantum Zeno effect by rendering them as two extremes.

While leaving the possible existence of scaling behavior an open question for some non-Lorentzian SDFs, the main conclusion above is valid in general. Following the procedures in this work, one can develop similar generalized QT theory by numerically obtaining the $\bar{a}(\Delta t)$ in Eq. (9), rather than using the analytic Eqs. (7) and (8). In Appendix C, we outline the solution scheme for an arbitrary SDF.

Unlike the Markovian counterpart, the ensemble average of the proposed QTs does *not* describe the *reduced* state given by tracing the environment degrees of freedom from the entire (system-plus-environment) unitary wave function. Since the successive measurements in the QT destroy the correlation (memory) effect between different *free* evolutions, the ensemble-average state also differs from the one resulting from averaging the nMSSE discussed in literature [14–18]. For a non-Markovian environment, as pointed out by Wiseman *et al.* [16,18], the nMSSE is not consistent with any physical quantum trajectories (i.e., having *no* physical interpretations).

For the relevance of the present work to possible experiment, we may refer to the *partial collapse* quantum measurement of the solid-state superconducting qubit [23–25]. The changed state reported there is conditioned on a projective null-result at the final time t, but not on "continuous" or "frequent" null result over the interval $(0,t)$. For the Markovian environment, both results are identical; however, for the non-Markovian case, this is not true. Possible experiment may be guided by Eq. (7) or Eq. (8), via the scaling variable x. As an alternative demonstration, one may perform a large-derivation analysis on the emitted photons from driven atoms [26]. From the present work, we expect that if we alter the detection interval τ for the spontaneous emissions, the statistics of the emitted photons will be drastically different. We would like to leave this interesting problem for future investigation.

ACKNOWLEDGMENTS

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APPENDIX A: ZENO EFFECT FOR SUPERPOSITION STATE

Consider the superposition state $|\Psi(0)\rangle = (\alpha_0|e\rangle + \beta_0|g\rangle) \otimes |\text{vac}\rangle$. After small time τ the wave function becomes

$$|\Psi(\tau)\rangle = [\alpha_0(1 - iH\tau - H^2\tau^2/2 + \cdots)|e\rangle + \beta_0|g\rangle] \otimes |\text{vac}\rangle. \quad \text{(A1)}$$

The null-result measurement in the environment implies that the wave function is projected on the atomic subspace, $|\Psi(\tau)\rangle \to \hat{Q}|\Psi(\tau)\rangle$, where $\hat{Q} = (|e\rangle\langle e| + |g\rangle\langle g|)/\mathcal{N}$ and $\mathcal{N}$ is a normalization factor. Therefore,

$$|\Psi_1\rangle = \hat{Q}|\Psi(\tau)\rangle = [\alpha_0(1 - K\tau^2)|e\rangle + \beta_0|g\rangle]/\mathcal{N}_1, \quad \text{(A2)}$$

where $K = \sum_r V_r^2$ and $\mathcal{N}_1^2 = 1 - 2\alpha_0^2 K\tau^2$, with V_r the atom-environment (the rth mode) coupling amplitude. After n subsequent null-result measurements during time t, with $n = t/\tau$, we find

$$|\Psi_n\rangle = [\alpha_0(1 - K\tau^2)^n|e\rangle + \beta_0|g\rangle]/\mathcal{N}_n, \quad \text{(A3)}$$

where $\mathcal{N}_n = \sqrt{1 - 2n\alpha_0^2 K\tau^2}$. Thus in the limit $\tau \to 0$ and $t = \text{const}$, we obtain the result of Eq. (2) in the main text, $|\Psi_n\rangle \to |\Psi(0)\rangle$.

APPENDIX B: SOLUTION FOR SPONTANEOUS EMISSION

Substituting Eq. (4) in the main text into the Schrödinger equation, $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$, and performing the Laplace transformation, $\tilde{f}(\omega) = \int_0^\infty f(t)\exp(i\omega t)dt$, we obtain the following system of algebraic equations:

$$(\omega - E_e)\tilde{\alpha}(\omega) - \sum_r V_r\tilde{c}_r(\omega) = i\alpha_0, \quad \text{(B1a)}$$

$$[\omega - (E_g + \omega_r)]\tilde{c}_r(\omega) - V_r^*\tilde{\alpha}(\omega) = 0. \quad \text{(B1b)}$$

The right-hand side of the first equation reflects the initial condition. Substituting $\tilde{c}_r(\omega)$ from Eq. (B1b) into Eq. (B1a), we obtain

$$(\omega - E_e)\tilde{\alpha}(\omega) - \mathcal{F}(\omega)\tilde{\alpha}(\omega) = i\alpha_0, \quad \text{(B2)}$$

where

$$\mathcal{F}(\omega) = \int \frac{D(\omega_r)}{\omega - (E_g + \omega_r)} d\omega_r. \quad \text{(B3)}$$

Rather than the wideband limit for the "Markovian" reservoir, in this work we consider a finite-band spectrum by taking the SDF $D(\omega_r)$ in the Lorentzian form,

$$D(\omega_r) \equiv \sum_{r'} |V_{r'}|^2\delta(\omega_r - \omega_{r'}) \to D_0\Lambda^2/[(\omega_r - \omega_0)^2 + \Lambda^2], \quad \text{(B4)}$$

with ω_0 the spectral center, D_0 the spectral height, and Λ the spectral width. We obtain then

$$\mathcal{F}(\omega) = \frac{\Lambda\Gamma/2}{(\omega - \omega_0 - E_g) + i\Lambda}, \quad \text{where } \Gamma = 2\pi D_0. \quad \text{(B5)}$$

Substituting this result into Eq. (B2), we find the amplitude $\tilde{\alpha}(\omega)$. The time-dependent amplitude is obtained via the inverse Laplace transform, $\alpha(t) = \int_{-\infty}^{\infty} \tilde{\alpha}(\omega)e^{-i\omega t}d\omega/(2\pi)$. Then, we obtain $\alpha(t) = a(t)\alpha_0$, with an explicit expression of $a(t)$ given by Eq. (5) in the main text.

APPENDIX C: SOLUTION SCHEME FOR NON-LORENTZIAN SDF

For the Lorentzian SDF, as shown above, we can first solve Eq. (B2) in the frequency domain, then obtain the analytic solution of $\alpha(t)$ by means of an inverse-Laplace transformation. However, for arbitrary SDF $D(\omega_r)$, this strategy does not work. Instead, we can solve Eq. (B2) for $\alpha(t)$ numerically (and directly) in the time domain. For this purpose, an inverse-Laplace transformation to Eq. (B2) yields

$$i\dot{\alpha}(t) = E_e\alpha(t) + \int_0^t dt' F(t - t')\alpha(t'), \quad \text{(C1)}$$

where

$$F(t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\mathcal{F}(\omega) = -i\int d\omega_r D(\omega_r)e^{-i(\omega_r+E_g)(t-t')}. \quad \text{(C2)}$$

Here we have employed the well-known convolution formula in the Laplace transformation and the following result related to inverse-Laplace transformation:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}[\omega - (\omega_r + E_g)]^{-1} = -i\, e^{-i(\omega_r+E_g)t}. \quad \text{(C3)}$$

In practice, for a given SDF $D(\omega_r)$, one can first carry out $F(t - t')$ in advance, via Eq. (C2); then, one can numerically integrate Eq. (C1) to obtain $a(t)$. With this result at hand, it is straightforward to develop the generalized QT theory, by numerically generating the $\bar{a}(\Delta t)$ in Eq. (9), rather than using the analytic Eqs. (7) and (8). We have examined this numerical scheme on the Lorentzian SDF and found excellent agreement with the analytic solution. The same success can be anticipated when applying to arbitrary non-Lorentzian SDFs.

[1] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).
[2] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 642 (1993).
[3] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2009).
[4] K. Jacobs, *Quantum Measurement Theory and Its Applications* (Cambridge University Press, Cambridge, 2014).
[5] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. **6**, 442 (2010).
[6] J. P. Groen, D. Ristè, L. Tornberg, J. Cramer, P. C. de Groot, T. Picot, G. Johansson, and L. DiCarlo, Phys. Rev. Lett. **111**, 090506 (2013).
[7] A. J. Hoffman, S. J. Srinivasan, S. Schmidt, L. Spietz, J. Aumentado, H. E. Türeci, and A. A. Houck, Phys. Rev. Lett. **107**, 053602 (2011).

[8] M. Mariantoni *et al.*, Nat. Phys. **7**, 287 (2011).
[9] M. Hatridge, S. Shankar, M. Mirrahimi, F. Schackert, K. Geerlings, T. Brecht, K. M. Sliwa, B. Abdo, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Science **339**, 178 (2013).
[10] K. W. Murch, S. J. Weber, C. Macklin, and I. Siddiqi, Nature (London) **502**, 211 (2013).
[11] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, Nature (London) **490**, 77 (2012).
[12] D. Ristè, J. G. van Leeuwen, H. S. Ku, K. W. Lehnert, and L. DiCarlo, Phys. Rev. Lett. **109**, 050507 (2012).
[13] P. Campagne-Ibarcq, E. Flurin, N. Roch, D. Darson, P. Morfin, M. Mirrahimi, M. H. Devoret, F. Mallet, and B. Huard, Phys. Rev. X **3**, 021008 (2013).
[14] L. Diósi, N. Gisin, and W. T. Strunz, Phys. Rev. A **58**, 1699 (1998).
[15] W. T. Strunz, L. Diósi, and N. Gisin, Phys. Rev. Lett. **82**, 1801 (1999).
[16] H. M. Wiseman and J. M. Gambetta, Phys. Rev. Lett. **101**, 140401 (2008).
[17] L. Diósi, Phys. Rev. Lett. **100**, 080401 (2008).
[18] J. Gambetta and H. M. Wiseman, Phys. Rev. A. **68**, 062104 (2003).
[19] K. Koshino and A. Shimizu, Phys. Rep. **412**, 191 (2005).
[20] J. Ping, Y. Ye, X. Q. Li, Y. J. Yan, and S. Gurvitz, Phys. Lett. A **377**, 676 (2013).
[21] L. Xu, Y. Cao, X. Q. Li, Y. J. Yan, and S. Gurvitz, Phys. Rev. A **90**, 022108 (2014).
[22] A. G. Kofman, Phys. Rev. A **71**, 033806 (2005).
[23] N. Katz *et al.*, Science **312**, 1498 (2006).
[24] R. Ruskov, A. Mizel, and A. N. Korotkov, Phys. Rev. B **75**, 220501(R) (2007).
[25] L. P. Pryadko and A. N. Korotkov, Phys. Rev. B **76**, 100503(R) (2007).
[26] J. P. Garrahan and I. Lesanovsky, Phys. Rev. Lett. **104**, 160601 (2010).