Entanglement as minimal discord over state extensions

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The characterization and quantification of quantum correlations, which play an instrumental role in exploring and exploiting the quantum world, have been extensively and intensively studied in the past few decades. Of special prominence and significance are the concepts of entanglement and discord, which are usually regarded as very distinctive quantum correlations, with the latter going beyond the former. In this work we establish a direct and natural link between entanglement and discord via state extensions and reveal that entanglement is actually the intrinsic discord, by which we mean that entanglement is the irreducible residue of discord viewed from ambient spaces. Our approach, taking into account the contextuality of a quantum state and being of a global nature, stands in sharp contrast to the local operations and classical communication paradigm of entanglement, which focuses on the state itself via a local approach. Furthermore, we introduce a figure of merit which, on the one hand, captures the essence of entanglement, i.e., nonlocality and quantumness of correlations, and, on the other hand, leads to a quantitative decomposition of total correlations into classical correlations, dissonance, and entanglement. This demystifies the meaning of entanglement from the perspective of quantum measurements and provides a unified framework for the interplay of various correlations in terms of quantum measurements and mutual information.

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I. INTRODUCTION

From both theoretical and experimental perspectives, quantum measurements lie at the heart of quantum mechanics [1] and remain a versatile and mysterious notion with profound applications and implications. The uncertainty principle and the complementarity principle are both manifestations of essential restrictions imposed by nature on quantum measurements [2–5]. The quantum-to-classical transition and related decoherence are consequences of quantum measurements [6–8]. In this work, we show that in a broad sense, the classification of correlations into separable and entangled can be put into the framework of an information-theoretic approach to quantum measurements, which has its early roots in Bohr's response to the Einstein-Podolsky-Rosen (EPR) argument [4,5], in Everett's work on universal wave function and relative states [9], and in Lindblad's investigations of entropy and quantum measurements [10,11]. The simple yet powerful idea we advocate here may be summarized as

$$\text{entanglement} = \text{minimal discord},$$

with the minimal being over all state extensions. Thus, entanglement may be interpreted as the intrinsic part of discord for enlarged states. Combining the above idea with the observation that discord = minimal MID (the minimal being over all von Neumann measurements and MID stands for measurement-induced disturbance, i.e., loss of correlations caused by quantum measurements), we get a more elementary picture: entanglement = Minimal MID. Here the capital Minimal indicates taking minimization over all state extensions and all von Neumann measurements.

The phenomenon of entanglement, with its inception in the seminal work of Einstein, Podolsky, and Rosen [12] and Schrödinger [13,14], dates back as early as the 1930s. It is deeply connected to quantum nonlocality [15–17], has gained prominence only in the last 30 years, and has now become a central character of quantum information theory [18–36]. Entanglement is the underpinning of many fundamental quantum tasks and is often regarded as a synonym of quantum correlations in early studies. However, since the explicit work of Ollivier and Zurek [37], Henderson and Vedral [38], Datta *et al.* [39], among others, the notion of discord has attracted increasing interest and is regarded as a more general type of quantum correlations than entanglement in the sense that separable states may possess nonvanishing discord, which is of nonclassical nature. Considerable efforts have been devoted to the calculation, variations, applications, and operational interpretations of discord in the last decade [39–72]. See Ref. [70] for a comprehensive review.

Entanglement and discord share intrinsic similarities as well as striking differences. Some remarkable relations and interplay between entanglement and discord have been uncovered in recent years [55–67]. For example, Cubitt *et al.* discovered that even separable states may help to distribute entanglement [55]. Koashi and Winter revealed a relation connecting entanglement (between two parties, a and b) with the discord (between party a and a third party, c, which serves to purify the state possessed by ab) [56]. The relation was further investigated by Cen *et al.* [57]. Adesso and Datta initiated the study of relations between entanglement and discord in the Gaussian case [58]. Madhok and Datta provided an operational interpretation of discord in terms of a quantum-state-merging protocol [61]. Cavalcanti *et al.* established a link between discord and entanglement consumption in quantum state distribution [62]. Streltsov *et al.* related discord between two parties to entanglement between a measurement apparatus and the system generated by quantum measurements [63]. Piani *et al.* devised a scheme to activate entanglement between a system and a local ancilla by use of discord [64], and

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Piani and Adesso defined quantum correlations in terms of entanglement [65]. Furthermore, Streltsov *et al.* [66], as well as Chuan *et al.* [67], obtained some intrinsic relations between discord and entanglement distribution.

Most of the above results are in the framework of comparing entanglement between two parties with the discord involving a third party (e.g., measurement apparatus). Here we incorporate entanglement directly into general measurement-induced disturbance and demonstrate that entanglement is actually the irreducible discord. This is achieved by a combination of state extensions and classification of the discord into two species: the reducible and extrinsic one, which can be eliminated via state extensions and is thus actually local quantum correlations, and the irreducible and intrinsic one, which cannot be eliminated by any state extension and thus captures really nonlocal quantum correlations. The latter is precisely entanglement, while the former is dissonance. A bona fide measure of entanglement follows naturally from this approach, and various desirable properties of this measure are established. This further leads to a quantitative decomposition of total correlations into three parts (classical correlations, dissonance, and entanglement) and provides a unified framework for their interplay in terms of quantum measurements and mutual information.

II. ENTANGLEMENT VERSUS DISCORD

We first recall two prototypical schemes for classifying correlations which lead to entanglement and discord: separable-entangled versus classical-quantum. A state ρ^{ab} shared by two parties a and b is called separable if it can be represented in a separable form [16]: There exist probabilities p_i and local states ρ_i^a and ρ_i^b for parties a and b, respectively, such that

$$\rho^{ab} = \sum_i p_i \rho_i^a \otimes \rho_i^b.$$

Otherwise, it is called entangled (nonseparable). Despite this formally clear and simple dichotomy, the detection and quantification of entanglement are extremely difficult and complicated in general cases. Some prominent measures of entanglement are the entanglement of formation, the entanglement cost, the distillable entanglement, the relative entropy of entanglement, the robustness of entanglement, and the squashed entanglement [18–32]. A fundamental property of an entanglement measure is the nonincreasing feature under local operations and classical communication (LOCC). The related equivalent characterizations for LOCC monotonicity are nicely discussed in Refs. [23,24,31].

In contrast to the separable-entangled paradigm, the classical-quantum scheme stipulates that a bipartite state ρ^{ab} is classically correlated (abbreviated as classical) if there exists a quantum measurement $\Pi = \{\Pi_i^a \otimes \Pi_j^b\}$ which does not disturb the state in the sense that

$$\rho^{ab} = \sum_{i,j} \left(\Pi_i^a \otimes \Pi_j^b\right) \rho^{ab} \left(\Pi_i^a \otimes \Pi_j^b\right),$$

where $\{\Pi_i^a\}$ and $\{\Pi_j^b\}$ are von Neumann measurements on parties a and b, respectively. Otherwise, the state ρ^{ab} is termed quantum correlated (abbreviated as quantum). Equivalently, ρ^{ab} is classical if and only if there exist orthonormal bases

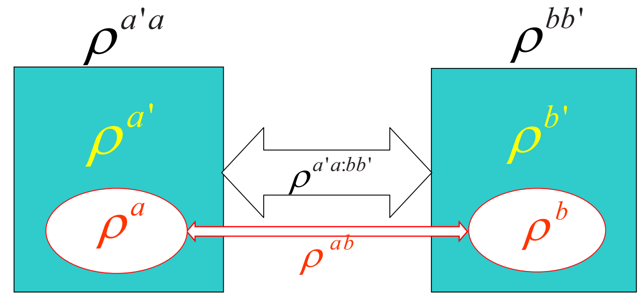

FIG. 1. Relation between the two schemes for classifying bipartite correlations: separable-entangled versus classical-quantum. The original bipartite state ρ^{ab} between two parties a and b may be formally extended to a four-partite state $\rho^{a'a:bb'}$ in an ambient space by appending ancillary systems a' and b' (may be correlated) to parties a and b, respectively, such that the reduced state on ab coincides with the original state: $\mathrm{tr}_{a'b'} \rho^{a'a:bb'} = \rho^{ab}$. The state ρ^{ab} is separable if and only if there exists an extension $\rho^{a'a:bb'}$ which is classical with respect to the bipartition $a'a : bb'$. Alternatively, ρ^{ab} is entangled if and only if it does not admit such a classical extension.

$\{|i\rangle_a\}$ and $\{|j\rangle_b\}$ of parties a and b, respectively, such that $\rho^{ab} = \sum_{i,j} p_{ij} |i\rangle_a\langle i| \otimes |j\rangle_b\langle j|$, $p_{ij} \geqslant 0$. Classical and quantum correlations can be neatly characterized by local broadcasting properties [45,50], as well as by monogamy [51]. This scheme for classifying correlations is more closely and intrinsically rooted in the fundamental and ubiquitous concept of quantum measurements. To put this into perspective, recall that ever since the beginning days of quantum mechanics, the most important and subtle issue is quantum measurement, and a delimiting line between classical and quantum is usually formulated in terms of measurement-induced disturbance [42]: In the classical world, we may perform measurements, at least in principle, to extract information without disturbing the measured system, while in the quantum world, a measurement usually disturbs the measured system except if the system turns out to be classical. The classical-quantum scheme for classifying correlations is a special incarnation of this general idea.

A natural question arises as to the relations between these two classifications of correlations: separable-entangled versus classical-quantum. On the one hand, the latter classification is more broad than the former in the sense that while a classical bipartite state is separable, the converse is not true: There are separable states which exhibit (unentangled) quantum correlations; that is, the notion of quantum correlations is more general than entanglement. On the other hand, by a result of Ref. [44], any separable state can be embedded into a larger classical state with a natural partition: For any separable state ρ^{ab}, there is a classical state $\rho^{a'a:bb'}$ (with the bipartition $a'a : bb'$) such that

$$\rho^{ab} = \mathrm{tr}_{a'b'} \rho^{a'a:bb'},$$

where a' and b' are two ancillary systems pertinent to parties a and b, respectively (see Fig. 1). Any entangled state does not admit such an extension. Consequently, a bipartite state is separable if and only if it is a reduced state of a classical state (with the natural partition), i.e., admits a classical extension.

This highlights the nature of entanglement as really *nonlocal* quantum correlations.

In the classical-quantum dichotomy, the quantum correlations in a bipartite state ρ^{ab} may be quantified by the (symmetric) discord as [46,47,60,70]

$$Q(\rho^{ab}) := I(\rho^{ab}) - C(\rho^{ab}),$$

where

$$I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab})$$

is the quantum mutual information [73–75], which is well established as a natural quantifier of the total correlations in ρ^{ab}, $S(\rho^a) := -\mathrm{tr}\rho^a \ln \rho^a$ is the von Neumann entropy,

$$C(\rho^{ab}) := \max_{\Pi} I(\Pi(\rho^{ab}))$$

is interpreted as the amount of classical correlation [10,11,45–47,60,70], i.e., the maximum amount of correlation extractable by local von Neumann measurements $\Pi = \{\Pi_i^a \otimes \Pi_j^b\}$ on ρ^{ab}, and $\Pi(\rho^{ab}) := \sum_{ij}(\Pi_i^a \otimes \Pi_j^b)\rho^{ab}(\Pi_i^a \otimes \Pi_j^b)$ is the postmeasurement state. In general, discord and classical correlations can be defined with respect to measurements on one party [37,38] or with respect to other distancelike measures [42,70], which yield the relative entropy of quantumness [48], the geometric discord [53], etc. Here we restrict ourselves to the symmetric case with the understanding that the nonsymmetric scenario can be treated similarly. Concerning the symmetric classical correlations, an intuitive conjecture of Lindblad [11], which states that the classical correlations account for at least half of the total correlations, or, equivalently, correlations are more classical than quantum, was disproved in Ref. [46]. Wu *et al.* studied extensively maximal extractable (i.e., classical) mutual information and related symmetric discord in the context of complementarity [47], and Lang *et al.* categorized and clarified various entropic measures of discord in detail [60].

III. ENTANGLEMENT VIA DISCORD

Based on the observation that entanglement is the irreducible discord, we introduce an entanglement measure as follows. For any bipartite state ρ^{ab}, consider any four-partite state $\rho^{a'a:bb'}$ which is an extension of ρ^{ab} in the sense that $\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'a:bb'}$. We include the cases when a' or b' is trivial (one-dimensional); that is, the state $\rho^{a'a:bb'}$ may reduce to $\rho^{a'a:b}$, $\rho^{a:bb'}$, or $\rho^{a:b}$. Now define a measure of entanglement as the minimal discord over all state extensions:

$$E(\rho^{ab}) := \min_{\mathrm{tr}_{a'b'}\rho^{a'a:bb'} = \rho^{ab}} Q(\rho^{a'a:bb'}),$$

with the discord $Q(\cdot)$ being taken with respect to the bipartition $a'a : bb'$. This entanglement measure, which may be termed the entanglement via discord for convenience, has the following desirable and remarkable properties:

(1) $E(\rho^{ab}) = 0$ if ρ^{ab} is separable.

(2) If $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ is a pure state, then $E(\rho^{ab}) = Q(\rho^{ab}) = S(\rho^a)$; that is, for any pure state, the entanglement $E(\rho^{ab})$ coincides with the discord $Q(\rho^{ab})$ and also coincides with the von Neumann entropy of the reduced state $\rho^a = \mathrm{tr}_b|\Psi^{ab}\rangle\langle\Psi^{ab}|$.

(3) $E(\cdot)$ is dominated by the discord; that is, for any ρ^{ab},

$$E(\rho^{ab}) \leqslant Q(\rho^{ab}).$$

(4) $E(\cdot)$ is convex in the sense that

$$E\left(\sum_i p_i \rho_i^{ab}\right) \leqslant \sum_i p_i E(\rho_i^{ab}),$$

where p_i are probabilities and ρ_i^{ab} are bipartite states shared by parties a and b.

(5) $E(\cdot)$ is dominated by the entanglement of formation $E_f(\cdot)$; that is, for any ρ^{ab},

$$E(\rho^{ab}) \leqslant E_f(\rho^{ab}).$$

(6) $E(\cdot)$ is locally unitary invariant in the sense that

$$E((U^a \otimes U^b)\rho^{ab}(U^a \otimes U^b)^\dagger) = E(\rho^{ab})$$

for any unitary operators U^a and U^b on parties a and b, respectively.

(7) $E(\cdot)$ is nonincreasing under local partial trace (state reduction) in the sense that $E(\rho^{ab}) \leqslant E(\rho^{a'a:bb'})$ for any state extension $\rho^{a'a:bb'}$ of ρ^{ab}.

(8) For any local channels (trace-preserving operations) Λ^a and Λ^b on parties a and b, respectively, it holds that

$$E((\Lambda^a \otimes \Lambda^b)(\rho^{ab})) \leqslant E(\rho^{ab}).$$

We now outline the reasoning leading to the above results.

(1) This property follows from the theorem in Ref. [44] concerning the relation between separable states and classical states, which establishes that a bipartite state ρ^{ab} is separable if and only if it can be extended to a certain classical state $\rho^{a'a:bb'}$ (with respect to the bipartition $a'a : bb'$).

(2) Since any pure state $|\Psi^{ab}\rangle\langle\Psi^{ab}|$ has only trivial extensions of the form $\rho^{a'b'} \otimes |\Psi^{ab}\rangle\langle\Psi^{ab}|$, the desired result follows from direct evaluation.

(3) This property follows from the definition since ρ^{ab} may be regarded as a (trivial) state extension of itself by regarding both the a' and b' systems to be one-dimensional.

(4) Taking two ancillary systems c and d of the same dimension, with orthonormal bases $\{|i\rangle_c\}$ and $\{|i\rangle_d\}$, respectively, noting that

$$\rho^{ca'a:bb'd} := \sum_i p_i |i\rangle_c\langle i| \otimes \rho_i^{a'a:bb'} \otimes |i\rangle_d\langle i|$$

is a state extension of $\rho^{ab} = \sum_i p_i \rho_i^{ab}$ whenever $\rho_i^{a'a:bb'}$ is a state extension of ρ_i^{ab} for all i (we may assume that all these extensions exist in the same large Hilbert space without loss of generality), we have

$$\begin{aligned} E(\rho^{ab}) &\leqslant Q(\rho^{ca'a:bb'd}) \\ &= Q\left(\sum_i p_i |i\rangle_c\langle i| \otimes \rho_i^{a'a:bb'} \otimes |i\rangle_d\langle i|\right) \\ &\leqslant \sum_i p_i Q(\rho_i^{a'a:bb'}). \end{aligned}$$

Now taking the minimum with respect to the state extensions $\rho_i^{a'a:bb'}$ of ρ_i^{ab} for all i, the desired result follows. The last inequality follows from direct evaluation and the fact that if $\{\Pi_{ij}^{a'a} \otimes \Pi_{ik}^{bb'}\}$ (with fixed i) is an optimal local von Neumann measurement for $\rho_i^{a'a:bb'}$ to achieve the corresponding discord, then $\{|i\rangle_c\langle i| \otimes \Pi_{ij}^{a'a} \otimes \Pi_{lk}^{bb'} \otimes |l\rangle_d\langle l|\}$ is a local von Neumann measurement on $\rho^{ca'a:bb'd}$.

(5) For any pure-state decomposition $\rho^{ab} = \sum_i p_i|\Psi_i^{ab}\rangle\langle\Psi_i^{ab}|$, by properties (1) and (4), we have

$$E(\rho^{ab}) \leqslant \sum_i p_i E\big(\big|\Psi_i^{ab}\big\rangle\big\langle\Psi_i^{ab}\big|\big) = \sum_i p_i E_f\big(\big|\Psi_i^{ab}\big\rangle\big\langle\Psi_i^{ab}\big|\big),$$

which implies the desired result. For the definition and properties of the entanglement of formation, see Refs. [18,21,31].

(6) This property follows from the fact that local unitary transforms do not alter the discord.

(7) Since any state extension $\rho^{a''a'a:bb'b''}$ of $\rho^{a'a:bb'}$ is necessarily a state extension of the reduced state $\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'a:bb'}$, we have

$$\begin{aligned} E(\rho^{ab}) &\leqslant \min_{\mathrm{tr}_{a''a'b'b''}\rho^{a''a'a:bb'b''}=\rho^{ab}} Q(\rho^{a''a'a:bb'b''}) \\ &\leqslant \min_{\mathrm{tr}_{a''b''}\rho^{a''a'a:bb'b''}=\rho^{a'a:bb'}} Q(\rho^{a''a'a:bb'b''}) \\ &= E(\rho^{a'a:bb'}). \end{aligned}$$

(8) For any local channels Λ^a and Λ^b on parties a and b, respectively, using the Stinespring dilation representation, we can write

$$\Lambda^a(\rho^a) = \mathrm{tr}_{a'}U^{a'a}(\sigma^{a'} \otimes \rho^a)(U^{a'a})^\dagger,$$
$$\Lambda^b(\rho^b) = \mathrm{tr}_{b'}U^{bb'}(\rho^b \otimes \sigma^{b'})(U^{bb'})^\dagger,$$

where $\sigma^{a'}$ and $\sigma^{b'}$ are states on the ancillary systems a' and b' and $U^{a'a}$ and $U^{bb'}$ are unitary operators on the composite systems $a'a$ and bb', respectively. From the above two equations and properties (6) and (7), we can show that $E(\cdot)$ is decreasing under local operations as follows:

$$\begin{aligned} &E((\Lambda^a \otimes \Lambda^b)(\rho^{ab})) \\ &= E(\mathrm{tr}_{a'b'}(U^{a'a} \otimes U^{bb'})(\sigma^{a'} \otimes \rho^{ab} \otimes \sigma^{b'})(U^{a'a} \otimes U^{bb'})^\dagger) \\ &\leqslant E((U^{a'a} \otimes U^{bb'})(\sigma^{a'} \otimes \rho^{ab} \otimes \sigma^{b'})(U^{a'a} \otimes U^{bb'})^\dagger) \\ &= E(\sigma^{a'} \otimes \rho^{ab} \otimes \sigma^{b'}) \\ &= E(\rho^{ab}). \end{aligned}$$

The entanglement measure $E(\cdot)$ is reminiscent of, but fundamentally different from, the squashed entanglement [26–28], which involves conditional quantum mutual information. Many important properties such as additivity, continuity, monogamy, operational meaning, asymptotic behavior, and relations to other entanglement measures, remain to be investigated.

With the above definition of discord and entanglement, we may interpret the difference

$$D(\rho^{ab}) := Q(\rho^{ab}) - E(\rho^{ab})$$

as a measure of dissonance as introduced by Kavan *et al.* [48], and thus, we come to a natural decomposition of quantum correlations, as quantified by the discord $Q(\cdot)$, into two parts involving entanglement $E(\cdot)$ and dissonance $D(\cdot)$:

$$Q(\rho^{ab}) = D(\rho^{ab}) + E(\rho^{ab}).$$

This in turn yields the separation of the total correlations, as quantified by the quantum mutual information $I(\rho^{ab})$, into classical correlations $C(\rho^{ab})$ plus dissonance $D(\rho^{ab})$ plus entanglement $E(\rho^{ab})$:

$$I(\rho^{ab}) = C(\rho^{ab}) + D(\rho^{ab}) + E(\rho^{ab}).$$

Clearly, for any pure state ρ^{ab}, we have $D(\rho^{ab}) = 0$; for any classical state ρ^{ab}, we have $Q(\rho^{ab}) = 0$, while for any separable state ρ^{ab}, we have $E(\rho^{ab}) = 0$. In this context, it is natural to investigate further the relations between the various correlations. For example, one may ask: Can quantum correlations exist without classical correlations? Can classical correlations exist without quantum correlations? While the answer to the first question in the present context is negative, that for the second is affirmative. It may happen that $C(\rho^{ab}) > 0$, while $D(\rho^{ab}) = 0$, $E(\rho^{ab}) = 0$, but whenever $E(\rho^{ab}) > 0$, it is necessary that $C(\rho^{ab}) > 0$. To summarize, classical correlations can exist without dissonance or entanglement, and dissonance can exist without entanglement, but entanglement and dissonance cannot exist without classical correlations. Symbolically,

$$C(\rho^{ab}) > 0 \nRightarrow D(\rho^{ab}) > 0,\ \ D(\rho^{ab}) > 0 \nRightarrow E(\rho^{ab}) > 0,$$
$$D(\rho^{ab}) > 0 \Rightarrow C(\rho^{ab}) > 0,\ \ E(\rho^{ab}) > 0 \Rightarrow C(\rho^{ab}) > 0,$$
$$E(\rho^{ab}) > 0 \nRightarrow D(\rho^{ab}) > 0,\quad I(\rho^{ab}) > 0 \Leftrightarrow C(\rho^{ab}) > 0.$$

One may wonder whether the entanglement $E(\rho^{ab})$ is dominated by the amount of classical correlation $C(\rho^{ab})$; that is, does it hold that $E(\rho^{ab}) \leqslant C(\rho^{ab})$? We leave this question open and suspect that it might not be true. See Ref. [46] for a similar problem and its resolution.

The discord measure $Q(\cdot)$ is usually defined via von Neumann measurements. However, one may also consider defining a discord measure $Q_{\mathrm{povm}}(\cdot)$ via general positive operator-valued measures (POVMs) [38], and there are subtle, albeit minor, differences between $Q(\cdot)$ and $Q_{\mathrm{povm}}(\cdot)$ [76–79]. Since any von Neumann measurement can be realized with the help of a POVM in an enlarged space consisting of the original system and an ancillary system, we conclude that it also holds that the entanglement measure is dominated by this modified discord measure, i.e., $E(\rho^{ab}) \leqslant Q_{\mathrm{povm}}(\rho^{ab})$, which refines property (3).

Although the entanglement measure is conceptually intuitive, its calculation is extremely difficult and at present seems intractable for general states. Actually, it is well known that the calculation of discord is a hard and complicated problem, and analytical formulas are rare. Even for the two-qubit X state, the original algorithm devised in Ref. [49] is only approximately correct [77]. It is evident that the evaluation of the entanglement measure via discord is even harder since now a further optimization apart from that in the discord enters the scenario: The optimization in the definition of $E(\cdot)$ is over both state extensions and von Neumann measurements. However, it is tempting to seek some explicit formulas of the entanglement measure for some symmetric states such as the Werner states, the isotropic states, and the Bell diagonal states. These seem highly nontrivial, and it is desirable to investigate various (lower or upper) bounds of the entanglement for general states. For this purpose, it will be helpful to first characterize and construct all state extensions, which is an important open issue of independent interest.

We have only worked in finite dimensions; it will also be interesting to consider the continuous-variable setup, such as the Gaussian scenario.

IV. DISCUSSION

The discord, as the minimal MID and a fundamental measure of quantum correlations, stems directly from the pivotal and ubiquitous notions of quantum measurements and mutual information. Here we have employed the discord to characterize entanglement as the irreducible discord and thus have subsumed entanglement into discord. A measure of entanglement was introduced via minimization of discord over state extensions. Some fundamental properties of this measure were demonstrated. This highlights the significance of interplay between quantum measurements and state extensions in quantum information science. Apart from further investigation of this connection and related properties of discord and entanglement, it will be highly desirable to relate and characterize the Bell nonlocality and quantum steering with discord and to pursue the role of state extensions in quantum foundations.

We remark that the method and results of the present approach can be applied to address the relations between entanglement and other measures of quantum correlations such as the relative entropy of quantumness and the geometric discord, as well as to the nonsymmetric case as in the original one-side discord.

State extensions are very powerful in addressing conceptual and fundamental issues of quantum theory and have various manifestations through operator dilations. We recall that POVMs can be represented as a reduction of von Neumann measurements through the Naimark dilation, and open dynamics (general channels) can be represented by the reduction of unitary dynamics via the Stinespring dilation [25,80,81]. Our result of viewing entanglement as a shadow of discord is also in this spirit and may be useful for providing deep insights into the nature and manipulation of entanglement. The philosophy is that whenever we face a mixed state, apart from concentrating on the state itself and working in the initial space, it is fruitful to keep an open mind by regarding the state as a shadow of larger states in ambient spaces. Then many features in the initial spaces can simply be regarded as shadows of some simpler structures in the ambient spaces. In this context, the intricate and subtle interplay between classicality, quantumness, nonlocality, contextuality, and entanglement may be recast in new forms, with quantum measurement explicitly or implicitly in the substratum. State extensions encode many quantum mysteries.

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