Nonlocality of three-qubit Greenberger-Horne-Zeilinger–symmetric states

Biswajit Paul,[1,*] Kaushiki Mukherjee,[2,†] and Debasis Sarkar[3,‡]

[1]*Department of Mathematics, South Malda College, Malda, West Bengal, India*

[2]*Department of Mathematics, Government Girls' General Degree College, Ekbalpore, Kolkata, India*

[3]*Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata-700009, India*

(Received 28 March 2016; published 2 September 2016)

Mixed states appear naturally in experiment over pure states. So for studying different notions of nonlocality and their relation with entanglement in realistic scenarios, one needs to consider mixed states. In a recent paper [Phys. Rev. Lett. **108**, 020502 (2012)], a complete characterization of entanglement of an entire class of mixed three-qubit states with the same symmetry as the Greenberger-Horne-Zeilinger state, known as GHZ-symmetric states, has been achieved. In this paper we investigate different notions of nonlocality of the same class of states. By finding the analytical expressions of maximum violation value of most efficient Bell inequalities we obtain the conditions of standard nonlocality and genuine nonlocality of this class of states. Also the relation between entanglement and nonlocality is discussed for this class of states. Interestingly, genuine entanglement of GHZ-symmetric states is necessary to reveal standard nonlocality. However, it is not sufficient to exploit the same.

DOI: 10.1103/PhysRevA.94.032101

I. INTRODUCTION

Quantum nonlocality is an inherent character of quantum theory where Bell inequalities [1] are used as witnesses to test the appearance of the same. Recently analysis of quantum nonlocality has become an interesting topic not only from a foundational viewpoint (see [2] and references therein) but also has been extensively used in various quantum information processing tasks, quantum communication complexity [3], randomness amplification [4], no-signaling [5], device-independent quantum key distribution [6], device-independent quantum state estimation [7,8], randomness extraction [9,10], etc. There exist several experimental evidences supporting the fact that presence of entanglement is necessary for nonlocality of quantum correlations. But determining which entangled state reveals nonlocality (i.e., violates Bell inequality) is difficult work. Any pure entangled state of two qubits violates Clauser-Horne-Shimony-Holt (CHSH) inequality [11,12], the amount of violation being proportional to the degree of bipartite entanglement [13]. However no such conclusion holds for mixed bipartite entangled states as there exists a class of mixed entangled states which admits a local hidden variable model and this class cannot violate any Bell inequality [14,15]. To date nonlocality in two-qubit systems has been explored in detail. However, the multiqubit case is much more difficult to analyze. There is an increasing complexity while shifting from bipartite to multipartite systems. This is mainly because of the fact that multipartite entanglement has comparatively much more complex and rich structure than that of bipartite entanglement [16,17]. So any study related to multipartite entanglement or dealing with multipartite nonlocality requires a deeper insight of the physics of many-particle systems which in general differ extensively from that of single or two party systems. However, study of many-particle systems gives rise to new interesting phenomena, such as phase transitions [18] and quantum computing. In this context, it is quite interesting to study the relationship between entanglement and nonlocality for multipartite system. To extend the two-qubit relationship between entanglement and quantum nonlocality, one needs to classify both entanglement and nonlocality in the multipartite system. In particular, entanglement of any tripartite state can either be biseparable or genuinely entangled [16,17]. Nonlocal character of a tripartite system can be categorized broadly in two categories of standard nonlocality and genuine nonlocality [2]. In the former case, nonlocality is revealed in at least one possible grouping of the parties whereas a state is said to be genuinely nonlocal if any possible grouping of parties reveals nonlocality. In [19], Śliwa gave the whole class of Bell inequalities which acts as a necessary and sufficient condition for detecting standard nonlocality. The relation between this notion of nonlocality and tripartite entanglement has been studied for three-qubit pure states [20–25] where it has been shown that entanglement (biseparable or genuine entanglement) of the pure state suffices to produce standard nonlocality. The notion of genuine tripartite nonlocality has been discussed in [26–28], which represents the strongest form of nonlocality for tripartite systems. There exists a relation between genuine tripartite nonlocality and three-tangle [29] (measure of genuine tripartite entanglement) which has been analyzed for some important classes of pure tripartite states [28,30–33]. Interestingly, Bancal *et al.* conjectured that all genuinely entangled pure quantum states can produce genuine nonlocal correlations [28]. While tripartite nonlocality turns out to be a generic feature of all entangled pure states, the situation becomes much more complex when we consider mixed states as there exists a genuine tripartite entangled state which admits a local hidden variable model [34,35]. In this context, it is interesting to characterize the state parameters for any class of tripartite mixed states on the basis of different notions of tripartite nonlocality and their relation with entanglement. Our paper goes in this direction. Recently, a new type of symmetry for the three-qubit quantum state was introduced [36], the so-called Greenberger-Horne-Zeilinger

*biswajitpaul4@gmail.com

†kaushiki_mukherjee@rediffmail.com

‡dsappmath@caluniv.ac.in

(GHZ) symmetry. In [36], they provided the whole class of states which has this type of symmetry. This class of states is referred to as GHZ-symmetric states. A complete classification of different types of entanglement of this class of tripartite mixed states is made in [36]. In this work we have classified the GHZ-symmetric states on the basis of different notions of tripartite nonlocality so that one can use this class of state in different information theoretic tasks. This helps us to establish the relationship between entanglement and nonlocality for this class of tripartite mixed states. The relation implies that genuine entanglement is necessary to reveal any type of nonlocality (standard nonlocality or genuine nonlocality) for this class of states.

The paper is organized as follows. In Sec. II, we give a brief introduction to some concepts and results which we will use in later sections. Subsequently, in Sec. III, we obtain the condition for which GHZ-symmetric states reveal standard nonlocality by deriving the analytical expressions of maximum violation value of the two most efficient facet inequalities. In Sec. IV, we deal with the classification of the class of states on the basis of genuine nonlocality. Section V shows how different types of entanglement are related with different notions of nonlocality for this class of mixed states. Finally we conclude with a summary of our results in Sec. VI.

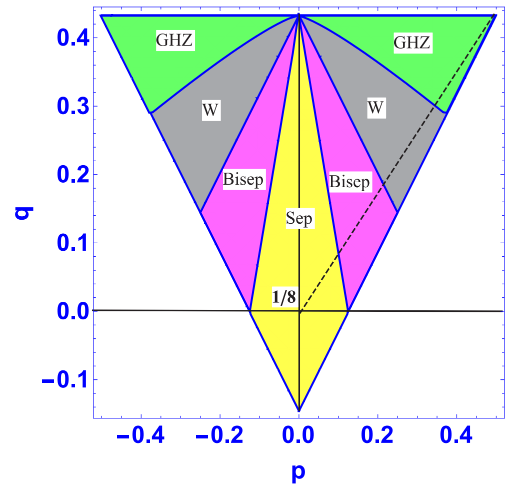

FIG. 1. The triangle of the GHZ-symmetric states for three qubits [36]. The upper corners of the triangle are the standard GHZ states $|\mathrm{GHZ}_+\rangle$ and $|\mathrm{GHZ}_-\rangle$. Mixed state $\frac{1}{8}$ is located at the origin. The black dashed line represents the generalized Werner state. We have indicated different types of three-qubit entanglement: GHZ (green), W (gray), biseparable (magenta), and separable (yellow).

II. BACKGROUND

A. GHZ-symmetric three-qubit states

As an important class of mixed states from a quantum theoretical perspective, GHZ-symmetric three-qubit states have been paid much attention [36–39]. In particular, in the eight-dimensional state space of three-qubit states, the set of GHZ-symmetric states defines a two-dimensional affine section, specifically a triangle of the full eight-dimensional set of states [40]. In this section, we review the properties of GHZ-symmetric three-qubit states [36]. GHZ-symmetric three-qubit states are defined to be invariant under the following transformations: (i) qubit permutations, (ii) simultaneous three-qubit flips (i.e., application of $\sigma_x \otimes \sigma_x \otimes \sigma_x$), and (iii) qubit rotation about the z axis of the form $U(\phi_1,\phi_2) = e^{i\phi_1\sigma_z} \otimes e^{i\phi_2\sigma_z} \otimes e^{-i(\phi_1+\phi_2)\sigma_z}$. Here σ_x and σ_z are the Pauli operators. The GHZ-symmetric states of three qubits can be written as

$$\rho(p,q) = \left(\frac{2q}{\sqrt{3}} + p\right)|\mathrm{GHZ}_+\rangle\langle\mathrm{GHZ}_+| + \left(\frac{2q}{\sqrt{3}} - p\right)|\mathrm{GHZ}_-\rangle\langle\mathrm{GHZ}_-| + \left(1 - \frac{4q}{\sqrt{3}}\right)\frac{\mathbf{1}}{8} \tag{1}$$

where $|\mathrm{GHZ}_\pm\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}$. The requirement $\rho(p,q) \geqslant 0$ gives the constraints: $-\frac{1}{4\sqrt{3}} \leqslant q \leqslant \frac{\sqrt{3}}{4}$ and

$$|p| \leqslant \frac{1}{8} + \frac{\sqrt{3}}{2}q. \tag{2}$$

This family of states forms a triangle in the state space and includes not only GHZ states, but also the maximally mixed state $\frac{1}{8}$ (located at the origin, see Fig. 1). Any point inside that triangle represents a GHZ-symmetric state. The generalized Werner states are found on the straight line $q = \frac{\sqrt{3}p}{2}$ connecting the origin with the $|\mathrm{GHZ}_+\rangle$ state. A GHZ-symmetric state is fully separable iff it is in the polygon defined by the four corner points $(0, -\frac{1}{4\sqrt{3}})$, $(\frac{1}{8},0)$, $(0,\frac{\sqrt{3}}{4})$, and $(-\frac{1}{8},0)$ (yellow area in Fig. 1). It is at most biseparable if and only if $|p| \leqslant \frac{3}{8} - \frac{\sqrt{3}}{2}q$(magenta area in Fig. 1). It is of W type (gray area in Fig. 1) if and only if $9216p^4 + p^2(-6768 + 17856\sqrt{3}q - 34560q^2 - 1024\sqrt{3}q^3) \leqslant 1521 - 5148\sqrt{3}q + 13536q^2 + 2432\sqrt{3}q^3 - 13056q^4 - 3072\sqrt{3}q^5$ and $|p| > \frac{3}{8} - \frac{\sqrt{3}}{2}q$.

B. Genuine multipartite concurrence (C_{GM})

In order to facilitate the discussion of our results, we briefly describe the genuine multipartite concurrence, a measure of genuine multipartite entanglement defined as [41] $C_{\mathrm{GM}}(|\psi\rangle) := \min_j \sqrt{2[1 - \Pi_j(|\psi\rangle)]}$ where $\Pi_j(|\psi\rangle)$ is the purity of the jth bipartition of $|\psi\rangle$. The genuine multipartite concurrence of three-qubit X states has been evaluated in [42]. It is given by

$$C_{\mathrm{GM}} = 2\max_i\{0,|z_i| - w_i\} \tag{3}$$

with $w_i = \sum_{j\neq i}\sqrt{a_j b_j}$ where a_j, b_j, and $z_j (j = 1,2,3,4)$ are the elements of the density matrix of tripartite X states:

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & z_1 \\ 0 & a_2 & 0 & 0 & 0 & 0 & z_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & z_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & {z_4}^* & b_4 & 0 & 0 & 0 \\ 0 & 0 & {z_3}^* & 0 & & b_3 & 0 & 0 \\ 0 & {z_2}^* & 0 & 0 & 0 & 0 & b_2 & 0 \\ {z_1}^* & 0 & 0 & 0 & 0 & 0 & 0 & b_1 \end{bmatrix}.$$

C. Tripartite nonlocality

In this section we provide a brief overview of the various notions of tripartite nonlocality and corresponding detectors of tripartite nonlocality for subsequent discussions. Consider a Bell-type experiment consisting of three spacelike separated parties, Alice, Bob, and Charlie. The measurement settings are denoted by $x, y, z \in \{0,1\}$ and their outputs are denoted by $a, b, c \in \{-1,1\}$ for Alice, Bob, and Charlie respectively. The experiment is thus characterized by the joint probability distribution (correlations) $p(abc|xyz)$. Now the correlations can exhibit different types of nonlocality. Any tripartite correlation $p(abc|xyz)$ is said to be local if it admits the following decomposition:

$$p(abc|xyz) = \sum_{\lambda} q_{\lambda} P_{\lambda}(a|x) P_{\lambda}(b|y) P_{\lambda}(c|z) \quad (4)$$

for all x, y, z, a, b, c, where $0 \leqslant q_{\lambda} \leqslant 1$ and $\sum_{\lambda} q_{\lambda} = 1$. $P_{\lambda}(a|x)$ is the conditional probability of getting outcome a when the measurement setting is x and λ is the hidden variable; $P_{\lambda}(b|y)$ and $P_{\lambda}(c|z)$ are similarly defined. Otherwise they are standard nonlocal. We denote L_3 as the set of local correlations that can be produced classically using shared randomness. The local set L_3 was fully characterized by Pitowsky and Svozil [43] and Śliwa [19]. It has 53 856 facets defining 46 different classes of inequalities that are inequivalent under relabeling of parties, inputs, and outputs [19]. Violation of any of these facet inequalities guarantees standard nonlocality. A tripartite correlation is local if it satisfies all the 46 facet inequalities. Inequality 2 (we follow Śliwa's numbering) is the Mermin inequality [44]:

$$M = |\langle A_1 B_0 C_0\rangle + \langle A_0 B_1 C_0\rangle + \langle A_0 B_0 C_1\rangle - \langle A_1 B_1 C_1\rangle| \leqslant 2. \quad (5)$$

Note that it is possible to violate Mermin inequality maximally (i.e., $M = 4$) using $|\mathrm{GHZ}_{\pm}\rangle$. However, in the tripartite scenario, Svetlichny [26] showed that there exist certain quantum correlations which can exhibit an even stronger form of nonlocality. Such type of correlations cannot be decomposed in the following form:

$$\begin{aligned} P(abc|xyz) = &\sum_{\lambda} q_{\lambda} P_{\lambda}(ab|xy) P_{\lambda}(c|z) \\ &+ \sum_{\mu} q_{\mu} P_{\mu}(ac|xz) P_{\mu}(b|y) \\ &+ \sum_{\nu} q_{\nu} P_{\nu}(bc|yz) P_{\nu}(a|x). \end{aligned} \quad (6)$$

Here $0 \leqslant q_{\lambda}, q_{\mu}, q_{\nu} \leqslant 1$ and $\sum_{\lambda} q_{\lambda} + \sum_{\mu} q_{\mu} + \sum_{\nu} q_{\nu} = 1$. The above form of correlations is not fully local as in Eq. (4); nonlocal correlations are present only between two particles (the two particles that are nonlocally correlated can change in different runs of the experiment) while they are only locally correlated with the third. If a correlation $P(abc|xyz)$ cannot be written in this form then such a correlation is said to exhibit genuine tripartite nonlocality. In [28], this type of nonlocality is referred to as Svetlichny nonlocality. Focusing on this form of correlations [Eq. (6)], Svetlichny designed a tripartite Bell type inequality (known as Svetlichny inequality):

$$S \leqslant 4 \quad (7)$$

where

$$\begin{aligned} S = |&\langle A_0 B_0 C_0\rangle + \langle A_1 B_0 C_0\rangle - \langle A_0 B_1 C_0\rangle \\ &+ \langle A_1 B_1 C_0\rangle + \langle A_0 B_0 C_1\rangle - \langle A_1 B_0 C_1\rangle \\ &+ \langle A_0 B_1 C_1\rangle + \langle A_1 B_1 C_1\rangle|. \end{aligned}$$

Thus violation of such inequality implies the presence of genuine tripartite nonlocality, implying in turn the presence of genuine tripartite entanglement. This inequalit y(7) is violated by GHZ and W states [30,31,45]. While Svetlichny's notion of genuine multipartite nonlocality is often referred to in the literature, it has certain drawbacks. As has been pointed out in [27,28], Svetlichny's notion of genuine tripartite nonlocality is so general that no restrictions were imposed on the bipartite correlations used in Eq. (6). They are allowed to display arbitrary correlations in the sense that there may be one-way or both-way signaling between a pair of parties or both the parties may perform simultaneous measurements. As a result, grandfather-type paradoxes arise [28] and inconsistency from an operational viewpoint appears [27]. Moreover it is found that there exist some genuine nonlocal correlations which satisfy this inequality [27,28,33]. In order to remove this sort of ambiguity, Bancal *et al.* [28], introduced a simpler definition of genuine tripartite nonlocality which is based on no-signaling principle, in which the correlations are no-signaling for all observers. Suppose $P(abc|xyz)$ is the tripartite correlation satisfying Eq. (6) with no-signaling criteria imposed on the bipartite correlations terms, i.e.,

$$P_{\lambda}(a|x) = \sum_{b} P_{\lambda}(ab|xy) \ \forall a, x, y, \quad (8)$$

$$P_{\lambda}(b|y) = \sum_{a} P_{\lambda}(ab|xy) \ \forall b, x, y \quad (9)$$

and similarly for the other bipartite correlation terms $P_{\mu}(ac|xz)$ and $P_{\nu}(bc|yz)$. The above form of correlations is called NS_2 local (where NS denotes nonsignaling). Otherwise, we say that they are genuinely three-way NS nonlocal (NS_2 nonlocal). In [28], 185 Bell-type inequalities are given which constitute the full class of facets of NS_2 local polytope. Violation of any of these facets (Bell-type inequalities) guarantees NS_2 nonlocality. Svetlichny inequality constitutes the 185th class. Throughout the paper, we use this notion of nonlocality as genuine tripartite nonlocality.

III. STANDARD NONLOCALITY OF GHZ-SYMMETRIC STATES

We have already discussed in the Introduction that all tripartite pure entangle states exhibit standard nonlocality but this relation does not hold for mixed states. From that point of view and also from experimental perspectives, characterization of mixed states on the basis of their ability to generate nonlocal correlations is far more interesting compared to that of pure states. As already discussed before, we aim to characterize the state parameters of the mixed class of GHZ-symmetric three-qubit states on the basis of their nonlocal nature. In this section we not only classify this class on the basis of standard nonlocality but also derived the necessary and sufficient condition of detecting standard nonlocality.

A. Maximum violation of Mermin inequality

We have already mentioned that standard nonlocality of correlations can be detected if the correlations violate at least one of the 46 inequivalent facet inequalities characterizing the local set(L_3). Among the 46 inequivalent facet inequalities, Mermin inequality is most frequently used. In [24], they gave a sufficient criterion to violate Mermin inequality for pure three-qubit states. Here we find a necessary and sufficient condition to obtain a violation of Mermin inequality for three-qubit GHZ-symmetric states. For this class of tripartite mixed entangled states the maximum value of M [Eq. (5)] with respect to projective measurement is given by $8|p|$ (see the Appendix). Then the Mermin inequality in Eq. (5) becomes

$$M_{\max} = 8|p| \leqslant 2. \tag{10}$$

Hence $\rho(p,q)$ violates Mermin inequality if and only if $|p| > \frac{1}{4}$. Due to this restriction on p, together with state constraints Eq. (2), the other state parameter q gets restricted: $q > \frac{1}{4\sqrt{3}}$. So standard nonlocality of any three-qubit GHZ-symmetric states with $|p| > \frac{1}{4}$ and $q > \frac{1}{4\sqrt{3}}$ is guaranteed via violation of Mermin inequality [see Fig. 2(a)].

B. Efficiency of 15th facet inequality over Mermin inequality

Mermin inequality, discussed in the last section, is not the most efficient detector of standard nonlocality. In this section it is argued that there exists another facet inequality which can be considered as a better tool for detecting standard nonlocality compared to use of Mermin inequality for doing the same. It is observed that the 15th facet can be considered as an inequality which is more efficient than Mermin inequality.

The 15th facet inequality is given by [19]

$$L \leqslant 4 \tag{11}$$

where $L=|2\langle A_0B_0\rangle+2\langle A_1B_0\rangle+\langle A_0C_0\rangle + \langle A_1C_0\rangle - 2\langle B_0C_0\rangle + \langle A_0B_1C_0\rangle - \langle A_1B_1C_0\rangle + \langle A_0C_1\rangle + \langle A_1C_1\rangle - 2\langle B_0C_1\rangle - \langle A_0B_1C_1\rangle + \langle A_1B_1C_1\rangle|$. The maximum value of L for three-qubit GHZ-symmetric states with respect to projective measurement is given by $\max[\frac{8(9|p^3|-8\sqrt{3}|q^3|)}{9p^2-12q^2}, -16\sqrt{3}q|]$ (see the Appendix). Using this, 15th facet inequality [Eq. (11)] gets modified as

$$L_{\max} = \max\left[\frac{8(9|p^3| - 8\sqrt{3}|q^3|)}{9p^2-12q^2}, -16\sqrt{3}q|\right] \leqslant 4. \tag{12}$$

As $-16\sqrt{3}q \leqslant 4$ for $-\frac{1}{4\sqrt{3}} \leqslant q \leqslant \frac{\sqrt{3}}{4}$, so 15th facet inequality is violated only when $\frac{8(9|p^3|-8\sqrt{3}|q^3|)}{9p^2-12q^2} > 4$. Using this relation and Eq. (2), we have $q > \frac{3}{148}(8+3\sqrt{3})$. It follows that for every $q > \frac{3}{148}(8+3\sqrt{3})$ there is at least one p for which the GHZ-symmetric states violate 15th facet inequality. In Fig. 2(b) we have plotted the range of the state parameters for which nonlocality is observed via the violation of 15th facet inequality. We have already discussed that a GHZ-symmetric state does not violate Mermin inequality if and only if $|p| \leqslant \frac{1}{4}$.

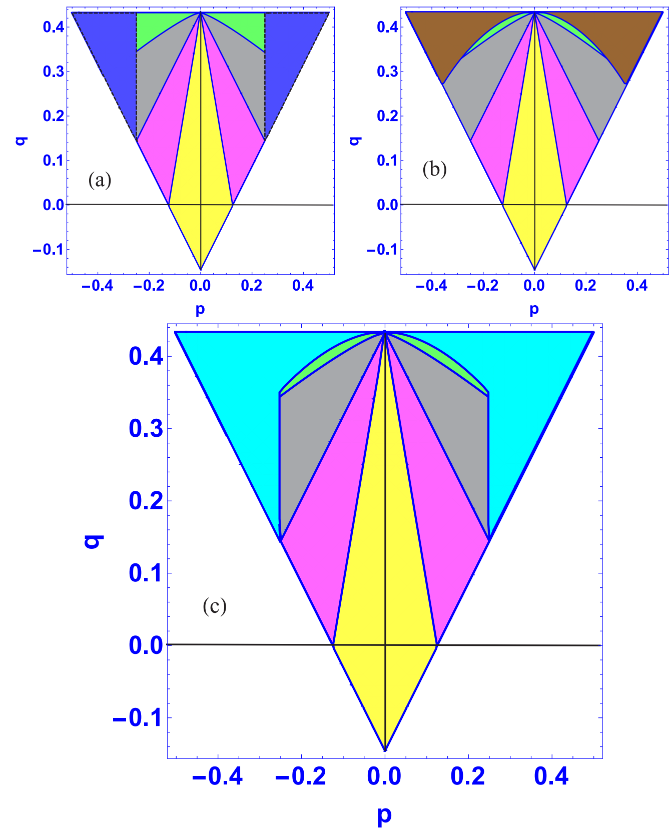

FIG. 2. (a) The blue areas represent the nonlocal region obtained via the violation of Mermin inequality [Eq. (5)]. (b) Regions of violation of 15th inequality are given by the brown regions. Clearly, the regions restricted by $p \leqslant \frac{1}{4}$ in the brown regions indicate the areas where 15th facet inequality [Eq. (11)] emerges as a more efficient tool over Mermin inequality [Eq. (5)] for revealing nonlocal nature of GHZ-symmetric states. (c) The cyan areas give the optimal region of standard nonlocality of GHZ-symmetric states. The states characterized by the state parameters lying in this region, when shared between Alice, Bob, and Charlie, do not admit any local hidden variable model.

Now this restriction, when imposed on $L_{\max} > 4$ gives at least one $q > \frac{1}{16}(\sqrt{15}+\sqrt{3})$ for all nonzero p. Hence there exists a region for $|p| \leqslant \frac{1}{4}$ where 15th facet inequality helps us to reveal standard nonlocality unlike Mermin inequality where the same is revealed only for $|p| > \frac{1}{4}$. For example, let us consider the GHZ-symmetric states with $p = 0.2$ and $q \in [-\frac{1}{4\sqrt{3}}, \frac{\sqrt{3}}{4}]$. This state does not violate Mermin inequality for any value of q but the same state violates 15th facet inequality for $q > 0.37861$. This in turn points out that 15th facet inequality is more efficient than the Mermin inequality over some restricted range of state parameters.

C. Necessary and sufficient detection criteria of standard nonlocality

By comparing the criteria necessary and sufficient for violation of each of the remaining 44 inequivalent facet inequalities (following procedure similar to that used for Mermin inequality, see the Appendix) with that of Mermin inequality and 15th facet inequality, we have observed that the region of standard nonlocality, as detected by any of the remaining 44 inequivalent facet inequalities, forms a subset of the region of standard nonlocality of Mermin and 15th facet inequality. So these two inequalities are the most efficient to detect standard nonlocality of this class of states. This in turn points out that the optimal region of standard nonlocality of GHZ-symmetric states is provided by the union of regions of standard nonlocality detected by Mermin and 15th facet inequality [see Fig. 2(c)]. So in totality the restricted state conditions for revealing standard nonlocality are given by

$$\text{(i)}\quad 4|p|>1,\quad q>\frac{1}{4\sqrt{3}}\quad \text{and}$$

$$\text{(ii)}\quad \frac{8(9|p^3|-8\sqrt{3}|q^3|)}{9p^2-12q^2}>4,\quad q>\frac{3}{148}(8+3\sqrt{3}). \tag{13}$$

A state exhibits standard nonlocality under projective measurements if and only if it satisfies at least one of the two sets of conditions [(i) or (ii) of Eq. (13)].

IV. GENUINE NONLOCALITY OF GHZ-SYMMETRIC STATES

Genuine nonlocality is the strongest form of nonlocality. So for a tripartite correlation it is natural to ask whether all three parties are nonlocally correlated. Such correlations play an important role in quantum information theory, phase transitions, and the study of many-body systems [18]. Also, the presence of genuine nonlocality implies the presence of genuine entanglement. So after discussing standard nonlocality, it becomes interesting to explore genuine nonlocality of this class of states. To be specific, in this section we have derived necessary and sufficient criteria for detecting genuine nonlocality.

A. Maximum violation of Svetlichny inequality

As we have discussed before, if we consider that all correlations between the observers are no-signalling, then the set of 185 facet inequalities acts as a necessary and sufficient condition for detecting genuine tripartite nonlocality. Among all of them, Svetlichny inequality is frequently used for the detection of genuine tripartite nonlocality. In [30], necessary and sufficient criteria for maximal violation of Svetlichny inequality are derived for some classes of tripartite pure states. Here we have derived the same for the class of GHZ-symmetric states. For this class of tripartite mixed entangled states, the maximum value of S [Eq. (7)] with respect to projective measurement is given by $8\sqrt{2}|p|$ (see the Appendix). Thus Eq. (7) gives

$$S_{\max}=8\sqrt{2}|p|\leqslant 4. \tag{14}$$

Hence $\rho(p,q)$ violates Svetlichny inequality if and only if $|p|>\frac{1}{2\sqrt{2}}$. Using this relation and Eq. (2), we have $q>\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}-\frac{1}{4})$. So Svetlichny nonlocality is revealed for three-qubit GHZ-symmetric states if and only if the relation $|p|>\frac{1}{2\sqrt{2}}$ and $q>\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}-\frac{1}{4})$ holds. In Fig. 3(a) we present the range of the state parameters of the three-qubit GHZ-symmetric states for which Svetlichny nonlocality is observed.

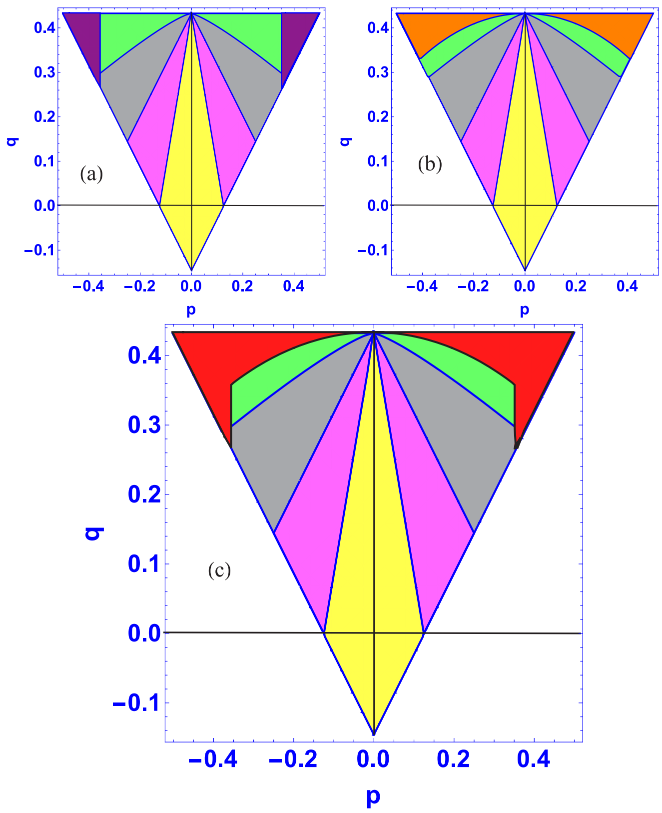

FIG. 3. (a) The purple areas give the restricted region of state parameters for which genuine nonlocality of the corresponding states is guaranteed by violation of Svetlichny inequality [Eq. (7)]. (b) Analogously, the orange regions represent the areas where genuine nonlocality is observed due to violation of 99th facet inequality [Eq. (15)]. Now the regions restricted by $|p|\leqslant\frac{1}{2\sqrt{2}}$ in the orange regions give the the areas where the 99th facet serves as a better tool to exploit genuine nonlocality of GHZ-symmetric states compared to Svetlichny inequality. (c) The optimal region of nonlocality of GHZ-symmetric states is given by the red regions.

B. Efficiency of 99th facet inequality over Svetlichny inequality

As we have mentioned in Sec. II, the newly introduced weaker definition of genuine nonlocality (genuine three-way NS nonlocality) gives advantage over Svetlichny's definition of genuine nonlocalty. So after completing the analysis of genuine nonlocality with respect to Svetlichny inequality, we search for an inequality which can be considered more efficient than Svetlichny inequality. In [33], we have shown that for detecting genuine nonlocality of some classes of tripartite pure entangled states, 99th facet inequality is more efficient compared to Svetlichny inequality. Here also, for the class of GHZ-symmetric states, 99th facet inequality emerges to be a more powerful tool for detecting genuine nonlocality for some subclasses. The 99th facet inequality is given by

$$\mathrm{NS}\leqslant 3 \tag{15}$$

where $\mathrm{NS}=|\langle A_1B_1\rangle+\langle A_0B_0C_0\rangle+\langle B_1C_0\rangle+\langle A_1C_1\rangle-\langle A_0B_0C_1\rangle|$. If projective measurement is considered, the maximum value of NS is given by $\frac{4q}{\sqrt{3}}+2\sqrt{\frac{16q^2}{3}+4p^2}$ (see the Appendix). Thus 99th facet inequality in Eq. (15) becomes

$$\mathrm{NS}_{\max} = \frac{4q}{\sqrt{3}} + 2\sqrt{\frac{16q^2}{3} + 4p^2} \leqslant 3. \tag{16}$$

Hence 99th facet inequality is violated if and only if $\frac{4q}{\sqrt{3}} + 2\sqrt{\frac{16q^2}{3} + 4p^2} > 3$. Using this along with the state constraints [Eq. (2)], we have $q > \frac{1}{28}(8\sqrt{5} - 5\sqrt{3})$. Thus NS_2 nonlocality is observed if $\frac{4q}{\sqrt{3}} + 2\sqrt{\frac{16q^2}{3} + 4p^2} > 3$ and $q > \frac{1}{28}(8\sqrt{5} - 5\sqrt{3})$. Hence for every $q > \frac{1}{28}(8\sqrt{5} - 5\sqrt{3})$ there exists at least one GHZ-symmetric state which is NS_2 nonlocal. We have already observed that any state restricted by $|p| \leqslant \frac{1}{2\sqrt{2}}$ fails to violate Svetlichny inequality. Now this restriction, when imposed on $\mathrm{NS}_{\max} > 3$ gives at least one $q > \frac{1}{4}(\sqrt{10} - \sqrt{3})$ for all nonzero p. Hence we get a subclass of GHZ-symmetric states restricted by $q > \frac{1}{4}(\sqrt{10} - \sqrt{3})$ and $\mathrm{NS}_{\max} > 3$ which is genuinely nonlocal even when $|p| \leqslant \frac{1}{2\sqrt{2}}$. This in turn points out efficiency of 99th facet inequality over Svetlichny inequality.

C. Necessary and sufficient criteria for detecting genuine nonlocality

A detailed comparison of the criteria required for violation of each of the remaining 183 facets (following the same procedure as that for Mermin inequality) with that of Svetlichny and 99th facet points out the fact that these two inequalities (99th facet inequality and Svetlichny inequality) are the most efficient detectors of genuine nonlocality. This in turn points out the fact that the optimal region of genuine nonlocality is given by

$$\text{(i)}\quad |p| > \frac{1}{2\sqrt{2}} \quad q > \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right) \quad \text{and}$$

$$\text{(ii)}\quad \frac{4q}{\sqrt{3}} + 2\sqrt{\frac{16q^2}{3} + 4p^2} > 3, \quad q > \frac{1}{28}(8\sqrt{5} - 5\sqrt{3}). \tag{17}$$

Genuine nonlocality of any state up to projective measurements is guaranteed if and only if it satisfies at least one of the two possible sets of conditions [(i) or (ii) of Eq. (17)].

V. RELATION BETWEEN ENTANGLEMENT AND NONLOCALITY

Entanglement of any state is necessary for nonlocality of the state. So after completing classification of GHZ-symmetric states with respect to different forms of nonlocality, we proceed to establish the relationship between nonlocality and entanglement of this class.

A. Relation between biseparable entanglement and standard nonlocality

Biseparable entanglement of a tripartite quantum state is necessary to produce standard nonlocality. Their relationship has been analyzed in [25] for three-qubit pure states where it is shown that biseparable entanglement of tripartite pure quantum states also turns out to be sufficient to exhibit standard nonlocality. Here we analyze whether it is sufficient for this class of tripartite mixed quantum states to obtain standard nonlocality. The criterion of biseparability of this class of states is [36,39] $|p| \leqslant \frac{3}{8} - \frac{\sqrt{3}}{2}q$. Interestingly, no biseparable GHZ-symmetric state can reveal standard nonlocality. We present our argument below.

We have already discussed that to detect standard nonlocality, Mermin and 15th facet inequality are the most efficient inequalities. In order to violate Mermin inequality the state parameters should satisfy $|p| > \frac{1}{4}$ and $q > \frac{1}{4\sqrt{3}}$. However, $|p| > \frac{1}{4}$, along with the biseparability criterion, gives $q \leqslant \frac{1}{4\sqrt{3}}$. This contradicts the required criterion for violation of Mermin: $q > \frac{1}{4\sqrt{3}}$. So violation of Mermin inequality is impossible. Now we consider the 15th facet inequality. Using the biseparability criterion, we get $L_{\max} \leqslant \frac{8(9(\frac{3}{8} - \frac{\sqrt{3}q}{2})^3 - 8\sqrt{3}|q^3|)}{9((\frac{3}{8} - \frac{\sqrt{3}q}{2})^2 - 12q^2)}$(say, f) where $f \leqslant 4$ and that makes violation of 15th inequality impossible by any biseparable state belonging to this class. Hence no biseparable state is capable of showing standard nonlocality.

B. Relation between genuine entanglement and standard nonlocality

In general for any tripartite state, genuine entanglement is necessary to reveal genuine nonlocality. So for GHZ-symmetric states, as argued in the last section, genuine entanglement is necessary to reveal even the weaker notion of standard nonlocality. However, one cannot claim it to be a sufficient criterion for revealing standard nonlocality for this class of states. We proceed with our argument below. For that we first consider genuinely entangled states. Such states are restricted by [36,39]

$$|p| > \frac{3}{8} - \frac{\sqrt{3}}{2}q. \tag{18}$$

As we have discussed earlier the locality criteria are

$$4|p| \leqslant 1$$

and

$$\frac{8(9|p^3| - 8\sqrt{3}|q^3|)}{9p^2 - 12q^2} \leqslant 4. \tag{19}$$

Clearly the conditions [Eqs. (18) and (19)] are feasible with the restricted range of the parameter q given by $\frac{1}{4\sqrt{3}} \leqslant q \leqslant \frac{\sqrt{3}}{4}$. This in turn proves the existence of genuinely entangled local states (see the pink region of Fig. 4). So any GHZ-symmetric state is genuinely entangled but local if it satisfies Eqs. (18) and (19). So strongest form of entanglement, i.e., genuine entanglement, turns out to be insufficient to generate even the weaker form of nonlocality, i.e., standard nonlocality. Hence we are able to present a class of genuinely entangled three-qubit states which does not violate a complete set of facet inequalities for standard nonlocality. Recently a similar type of result has been presented in [35], for some other class of states. In this context it will be interesting to study variation of standard nonlocality with the amount of genuine entanglement. Since Mermin and 15th facets are the most efficient Bell inequalities to detect standard nonlocality, now we deal with the variation of violation of these facet inequalities with the

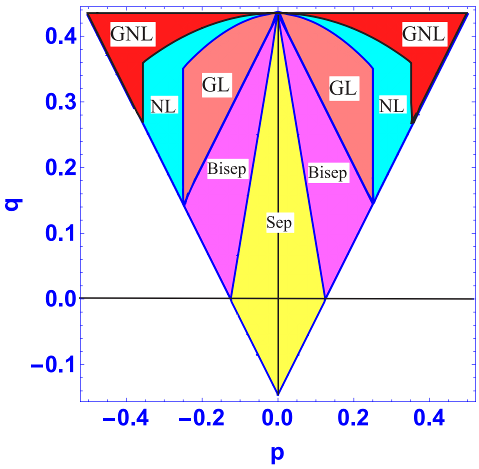

FIG. 4. The figure gives the nonlocality classification of three-qubit GHZ-symmetric states $\rho(p,q)$. The red regions give the optimal area where genuine nonlocality (GNL) is revealed with respect to projective measurement for GHZ-symmetric states. The cyan regions indicate the optimal area where standard nonlocality (NL) is revealed (except genuine nonlocality). As genuine nonlocality also implies standard nonlocality so red regions also give the region of standard nonlocality. Genuinely entangled but local states (GL) are represented by pink regions. Clearly, no nonlocal region lies within the biseparable region (magenta).

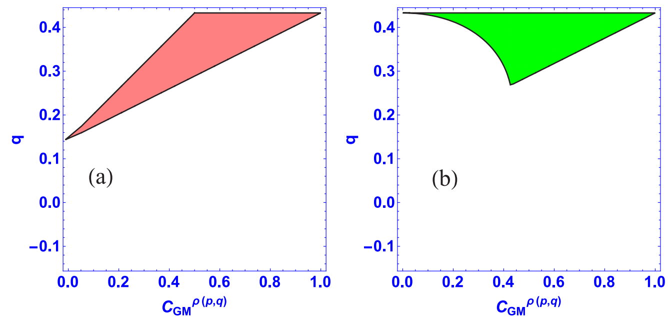

FIG. 5. Both of these figures depict variation of $C_{\rm GM}^{\rho(p,q)}$ with state parameter q for standard nonlocal states. Precisely, in (a) and (b), we have considered $C_{\rm GM}^{\rho(p,q)}$ of the states whose standard nonlocality is guaranteed by violation of Mermin and 15th facet inequality, respectively. Interestingly, for any arbitrarily small value of $C_{\rm GM}^{\rho(p,q)}$ there exists at least one GHZ-symmetric state which exhibits standard nonlocality.

amount genuine entanglement ($C_{\rm GM}^{\rho(p,q)}$). Since the three-qubit GHZ-symmetric states belong to the class of tripartite X states, their amount of entanglement can be measured by Eq. (3). So

$$C_{\rm GM}^{\rho(p,q)} = 2|p| - \tfrac{3}{4} + \sqrt{3}q. \tag{20}$$

For the state $\rho(p,q)$, one has $M_{\max} = 4(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q)$. Hence, GHZ-symmetric states violate Mermin inequality if $(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q) > \frac{1}{2}$. As we have proven in Sec. III, GHZ-symmetric states with $|p| > \frac{1}{4}$ and $q > \frac{1}{4\sqrt{3}}$ violate Mermin inequality. For this subclass of GHZ-symmetric states $C_{\rm GM}^{\rho(p,q)} > 0$ as $|p| > \frac{1}{4}$ and $q > \frac{1}{4\sqrt{3}}$. This subclass always violates Mermin inequality and the amount of violation (i.e., $M_{\max} - 2$) increases monotonically with $C_{\rm GM}^{\rho(p,q)}$ for any fixed value of q. Also for each $C_{\rm GM}^{\rho(p,q)} > 0$ there is a GHZ-symmetric state (i.e., a value of q) which violates Mermin inequality [see Fig. 5(a)]. Similarly, using Eq. (20), we have $L_{\max} = \frac{8(-8\sqrt{3}q^3 + \frac{9}{8}(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q)^3)}{-12q^2 + \frac{9}{4}(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q)^2)}$, which increases monotonically with $C_{\rm GM}^{\rho(p,q)}$ for any fixed value of q. GHZ-symmetric states violate 15th facet inequality if and only if $\frac{2(-8\sqrt{3}q^3 + \frac{9}{8}(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q)^3)}{-12q^2 + \frac{9}{4}(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q)^2)} > 1$. Clearly, for each value of $C_{\rm GM}^{\rho(p,q)}$ there is a GHZ-symmetric state with $q > \frac{3}{148}(8 + 3\sqrt{3})$ which violates 15th facet inequality. These are also shown in Fig. 5(b).

C. Relation between genuine entanglement and genuine nonlocality

To date no relationship between genuine entanglement and genuine nonlocality has been proved even for three-qubit pure quantum states. However, recently a conjecture has been reported in [28], which states that all three-qubit genuinely entangled pure states exhibit genuine nonlocality. More recently this conjecture has been proved for some important class of pure states [33]. But no such straightforward conclusion can be drawn for any class of three-qubit mixed states as there exist genuinely entangled mixed states which do not exhibit genuine nonlocality [34]. Here we obtain the relationship between the above two phenomena for three-qubit GHZ-symmetric states. In this context, we have presented a subclass of states of the GHZ-symmetric class of mixed tripartite states which is genuinely entangled yet fails to violate any of the 185 facet inequalities and thereby is not genuinely nonlocal (see Sec. V E). However, the discussion of genuine nonlocality (Sec. IV) points out that out of the 185 facet inequalities, Svetlichny inequality and the 99th facet inequality are the two most efficient detectors of genuine nonlocality for this class of states. In this section we have studied the variation of violation of these two efficient Bell inequalities with the amount of entanglement content $C_{\rm GM}^{\rho(p,q)}$. Using Eq. (20), the maximum violation value of Svetlichny inequality [Eq. (14)] becomes $S_{\max} = 4\sqrt{2}(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q)$. The algebraic expression clearly points out the relation between genuine nonlocality and entanglement [see Fig. 6(a)]. It is already argued in Sec. IV that for violation of Svetlichny inequality the state parameters get restricted as $|p| > \frac{1}{2\sqrt{2}}$ and $q > \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}} - \frac{1}{4})$. These restrictions, when imposed in Eq. (20), imply that for $C_{\rm GM}^{\rho(p,q)} > \sqrt{2} - 1$ Svetlichny inequality is violated. So any state having $C_{\rm GM}^{\rho(p,q)} \leqslant \sqrt{2} - 1$ cannot violate the Svetlichny inequality [see Fig. 6(a)]. Similarly by Eq. (20), the maximum violation value of the 99th facet becomes

$$\mathrm{NS}_{\max} = \frac{4q}{\sqrt{3}} + 2\sqrt{\frac{16q^2}{3} + \left(C_{\rm GM}^{\rho(p,q)} + \frac{3}{4} - \sqrt{3}q\right)^2} \leqslant 3. \tag{21}$$

Clearly for any arbitrary value of q, the amount of genuine nonlocality ($\mathrm{NS}_{\max} - 3$) increases monotonically with the amount of entanglement $C_{\rm GM}^{\rho(p,q)}$. Interestingly, for any positive value of $C_{\rm GM}^{\rho(p,q)}$, there exists a subclass which violates 99th

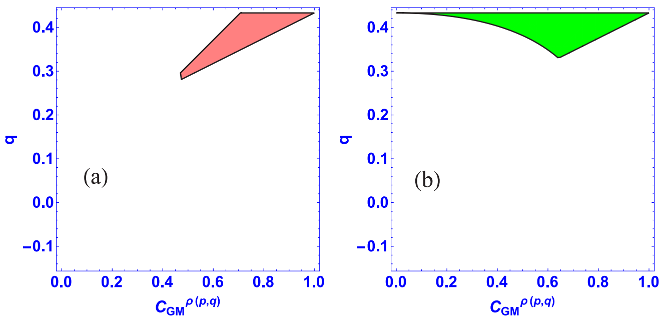

FIG. 6. Variations of state parameter q with that of $C_{\rm GM}^{\rho(p,q)}$ for genuinely nonlocal states are shown in these figures. The genuinely nonlocal states, as detected by Svetlichny inequality and 99th facet inequality, are considered separately in (a) and (b), respectively. It is interesting to note that for any positive value of $C_{\rm GM}^{\rho(p,q)}$, there exist some states whose genuine nonlocality is observed via violation of the 99th facet. However for the states whose genuine nonlocality is guaranteed by the violation of Svetlichny inequality, no such conclusion can be made. In fact for violation of Svetlichny inequality, the range of $C_{\rm GM}^{\rho(p,q)}$ gets restricted: $C_{\rm GM}^{\rho(p,q)} > \sqrt{2}-1$.

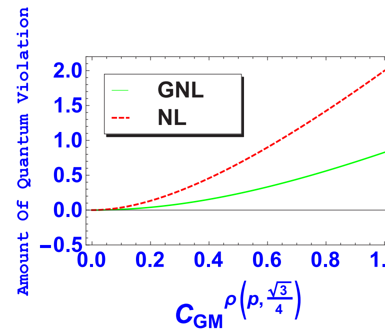

FIG. 7. The red dashed curve gives variation of amount of standard nonlocality ($L_{\max}-4$) with the amount of entanglement ($C_{\rm GM}^{\rho(p,q)}$) whereas the solid green curve represents the variation of amount of genuine nonlocality ($\mathrm{NS}_{\max}-3$) with the amount of entanglement ($C_{\rm GM}^{\rho(p,q)}$). The figure shows that standard nonlocality (NL) and genuine nonlocality (GNL) both are obtained for any positive value of $C_{\rm GM}^{\rho(p,q)}$. The curve showing variation of GNL with $C_{\rm GM}^{\rho(p,q)}$ for this mixed subclass of GHZ-symmetric states is the same as that of the curve showing variation of GNL with $C_{\rm GM}^{\rho(p,q)}$ for the pure generalized GHZ state [33].

facet inequality [see Fig. 6(b)]. To be precise, there exists a subclass of GHZ-symmetric states which is genuinely nonlocal for any amount of $C_{\rm GM}^{\rho(p,q)}$.

D. Genuinely nonlocal subclass

The genuinely nonlocal subclass is obtained for a fixed value of one of the two state parameters. Putting $q=\frac{\sqrt{3}}{4}$ in Eq. (1), we get

$$\rho\left(p,\frac{\sqrt{3}}{4}\right)=\left(\frac{1}{2}+p\right)|\mathrm{GHZ}_+\rangle\langle\mathrm{GHZ}_+| +\left(\frac{1}{2}-p\right)|\mathrm{GHZ}_-\rangle\langle\mathrm{GHZ}_-|. \quad (22)$$

This subclass of GHZ-symmetric states is genuinely entangled, the amount of entanglement given by [Eq. (20)]

$$C_{\rm GM}^{\rho(p,\frac{\sqrt{3}}{4})}=2|p|. \quad (23)$$

The optimal region of standard nonlocality of this subclass is detected by 15th facet inequality:

$$L_{\max}=4\frac{8p^3-1}{4p^2-1}>4. \quad (24)$$

Clearly for any nonzero value of p, $L_{\max}>4$. The relation between entanglement ($C_{\rm GM}^{\rho(p,q)}$) and standard nonlocality is given by

$$4\frac{\left(C_{\rm GM}^{\rho(p,\frac{\sqrt{3}}{4})}\right)^3-1}{\left(C_{\rm GM}^{\rho(p,\frac{\sqrt{3}}{4})}\right)^2-1}>4. \quad (25)$$

Equation (25) points out that the amount of standard nonlocality ($L_{\max}-4$) increases monotonically with amount of entanglement ($C_{\rm GM}^{\rho(p,\frac{\sqrt{3}}{4})}$). Clearly any arbitrary amount of entanglement is sufficient for violation of the 15th facet inequality (see Fig. 7). A similar sort of analysis can be made when we consider the stronger notion of genuine nonlocality. The 99th facet inequality is the most efficient detector of genuine nonlocality for this subclass:

$$\mathrm{NS}_{\max}=1+2\sqrt{1+4p^2}>3. \quad (26)$$

Using Eq. (23), the above inequality gets modified as

$$1+2\sqrt{1+\left(C_{\rm GM}^{\rho(p,\frac{\sqrt{3}}{4})}\right)^2}>3. \quad (27)$$

Clearly for any arbitrary amount of $C_{\rm GM}^{\rho(p,\frac{\sqrt{3}}{4})}$, this subclass can reveal genuine nonlocality (see Fig. 7). Interestingly, comparison between the $\mathrm{NS}_{\max}$ of this class $\rho(p,\frac{\sqrt{3}}{4})$ and that of the pure class of generalized Greenberger-Horne-Zeilinger (GGHZ) states [30,32,33] points out that for these two classes genuine nonlocality varies similarly with that of their corresponding entanglement content though one of these classes is pure (GGHZ) whereas the other one is mixed [33].

E. Genuinely entangled but not genuinely nonlocal subclass

In this subsection we present a subclass of GHZ-symmetric states which is genuinely entangled but satisfies all the 185 facet inequalities detecting genuine nonlocality. For that we first consider Eq. (18), which gives the criterion of genuine entanglement: $|p|>\frac{3}{8}-\frac{\sqrt{3}}{2}q$. Now any subclass having state parameters p and q restricted by this criterion cannot reveal genuine nonlocality if it cannot violate either Svetlichny inequality or 99th facet inequality, i.e., if $p\leqslant\frac{1}{2\sqrt{2}}$ and criteria for satisfying 99th facet inequality: $\frac{4q}{\sqrt{3}}+2\sqrt{\frac{16q^2}{3}+4p^2}\leqslant 3$, $q\leqslant\frac{1}{28}(8\sqrt{5}-5\sqrt{3})$. Clearly these three restrictions together give a feasible region in state parameter space (p,q) and that any GHZ-symmetric state having a state parameter lying in this feasible region fails to reveal genuine nonlocality in spite of being genuinely entangled.

VI. CONCLUSION

In summary, the above systematic study exploits the nature of different notions of nonlocality thereby giving the necessary and sufficient conditions for detecting nonlocality of an entire family of high-rank mixed three-qubit states with the same symmetry as the GHZ state. Generally Mermin inequality (which is a natural generalization of CHSH inequality) is used to detect standard nonlocality. However, we have showed that this inequality is not the most efficient Bell inequality for this class of three-qubit mixed states as for some restricted range of state parameters 15th facet inequality gives advantage over Mermin inequality. Our findings confirm that the nonlocality conditions given by 15th facet inequality and Mermin inequality are the best detector of standard nonlocality for this class of states. Analogously genuine nonlocality of the class is discussed. For detection of genuine nonlocality 99th facet inequality and Svetlichny inequality turn out to be the most effective tools. Further comparison between these two inequalities points out that 99th facet inequality is even far better than Svetlichny for some restricted subclasses of this class though the latter is extensively used for detection of genuine nonlocality. Also, our result illustrates the relationship between entanglement and nonlocality of this class of three-qubit mixed states. Interestingly no biseparable state is capable of revealing standard nonlocality. This in turn points out the necessity of genuine entanglement of this class for this purpose. However, for revelation of standard nonlocality existence of genuine entanglement is not sufficient. This fact becomes clear from the existence of a genuine entangled local subclass of GHZ-symmetric states. It will be interesting to explore the presence of hidden nonlocality (if any) [46–48] of this class of states. Also one may try to activate nonlocality of this class of states by using it in some suitable quantum network [49,50]. Also, the GHZ-symmetric class of states forms a two-dimensional affine subspace of the whole eight-dimensional space of three-qubit states [40]. So a study analyzing the relation between entanglement and nonlocality of tripartite states from other subspaces, or if possible characterization of the whole space itself, can be made in the future.

APPENDIX

In order to obtain the maximum value $M_{\max}$ [Eq. (10)] we consider the projective measurements: $A_0 = \vec{a}.\vec{\sigma_1}$ or $A_1 = \vec{\acute{a}}.\vec{\sigma_1}$ on the first qubit, $B_0 = \vec{b}.\vec{\sigma_2}$ or $B_1 = \vec{\acute{b}}.\vec{\sigma_2}$ on the second qubit, and $C_0 = \vec{c}.\vec{\sigma_3}$ or $C_1 = \vec{\acute{c}}.\vec{\sigma_3}$ on the third qubit, where $\vec{a},\vec{\acute{a}},\vec{b},\vec{\acute{b}}$ and $\vec{c},\vec{\acute{c}}$ are unit vectors and $\vec{\sigma_i}$ are the spin projection operators that can be written in terms of the Pauli matrices. Representing the unit vectors in spherical coordinates, we have $\vec{a} = (\sin\theta a_0 \cos\phi a_0, \sin\theta a_0 \sin\phi a_0, \cos\theta a_0)$, $\vec{b} = (\sin\alpha b_0 \cos\beta b_0, \sin\alpha b_0 \sin\beta b_0, \cos\alpha b_0)$, and $\vec{c} = (\sin\zeta c_0 \cos\eta c_0, \sin\zeta c_0 \sin\eta c_0, \cos\zeta c_0)$ and similarly we define $\vec{\acute{a}},\vec{\acute{b}}$, and $\vec{\acute{c}}$ by replacing zero in the indices by one. Then the value of M [Eq. (5)] for the state $\rho(p,q)$ can be written as

$$M[\rho(p,q)] = |2p[\cos(\beta b_0 + \eta c_0 + \phi a_0)\sin(\alpha b_0)\sin(\zeta c_0) \times \sin(\theta a_0) - \cos(\beta b_1 + \eta c_1 + \phi a_0)\sin(\alpha b_1) \times \sin(\zeta c_1)\sin(\theta a_0) + \cos(\beta b_1 + \eta c_0 + \phi a_1) \times \sin(\alpha b_1)\sin(\zeta c_0)\sin(\theta a_1) + \cos(\beta b_0 + \eta c_1 + \phi a_1)\sin(\alpha b_0)\sin(\zeta c_1)\sin(\theta a_1)]|. \tag{A1}$$

To obtain the maximum value of M we have to maximize the above function $M[\rho(p,q)]$ over all measurement angles. We first find the global maximum of $M[\rho(p,q)]$ with respect to θa_0 and θa_1. We begin by finding all critical points of $M[\rho(p,q)]$ inside the region $R = [0,2\pi] \times [0,2\pi]$ which are, namely, $(\frac{\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{2})$, and $(-\frac{\pi}{2}, -\frac{\pi}{2})$. The function gives maximum value with respect to θa_0 and θa_1 in all these critical points. In particular if we take $(\frac{\pi}{2}, \frac{\pi}{2})$ as the maximum critical point, then Eq. (A1) becomes

$$M[\rho(p,q)] \leqslant |2p[\cos(\beta b_0 + \eta c_0 + \phi a_0)\sin(\alpha b_0)\sin(\zeta c_0) - \cos(\beta b_1 + \eta c_1 + \phi a_0)\sin(\alpha b_1)\sin(\zeta c_1) + \cos(\beta b_1 + \eta c_0 + \phi a_1)\sin(\alpha b_1)\sin(\zeta c_0) + \cos(\beta b_0 + \eta c_1 + \phi a_1)\sin(\alpha b_0)\sin(\zeta c_1)]|. \tag{A2}$$

Now we carry out the same procedure over the pair of variables $(\beta b_0, \beta b_1)$ and $(\zeta c_0, \zeta c_1)$, one by one. Similar to the previous case, critical point $(\frac{\pi}{2}, \frac{\pi}{2})$ gives the maximum value for both of these pairs of variables. So the last inequality in Eq. (A2) takes the form

$$M[\rho(p,q)] \leqslant |2pG| \tag{A3}$$

where $G = \cos(\beta b_0 + \eta c_0 + \phi a_0) - \cos(\beta b_1 + \eta c_1 + \phi a_0) + \cos(\beta b_1 + \eta c_0 + \phi a_1) + \cos(\beta b_0 + \eta c_1 + \phi a_1)$. Now the algebraic maximum value of G is equal to 4 which can be obtained by taking $\beta b_0 = 0$, $\beta b_1 = -\frac{\pi}{2}$, $\phi a_0 = 0$, $\phi a_1 = \frac{\pi}{2}$, $\eta c_0 = 0$, and $\eta c_1 = -\frac{\pi}{2}$. Thus, $M_{\max} = 8|p|$ as obtained in Eq. (10). Similarly one can obtain $L_{\max}$ [Eq. (12)], $S_{\max}$ [Eq. (14)], and $NS_{\max}$ [Eq. (16)].

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