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Lorentz-covariant quantum 4-potential and orbital angular momentum for the transverse confinement of matter waves

R. Ducharme[1] and I. G. da Paz[2]

[1]2112 Oakmeadow Pl., Bedford, Texas 76021, USA

[2]Departamento de Física, Universidade Federal do Piauí, Campus Ministro Petrônio Portela, CEP 64049-550, Teresina, Piauí, Brazil

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In two recent papers exact Hermite-Gaussian solutions to relativistic wave equations were obtained for both electromagnetic and particle beams. The solutions for particle beams correspond to those of the Schrödinger equation in the nonrelativistic limit. Here, it will be shown that each beam particle has additional 4-momentum resulting from transverse localization compared to a free particle traveling in the same direction as the beam with the same speed. This will be referred to as the quantum 4-potential term since it will be shown to play an analogous role in relativistic Hamiltonian quantum mechanics as the Bohm potential in the nonrelativistic quantum Hamilton-Jacobi equation. Low-order localization effects include orbital angular momentum, Gouy phase, and beam spreading. Toward a more systematic approach for calculating localization effects at all orders, it will be shown that both the electromagnetic and quantum 4-potentials couple into the canonical 4-momentum of a particle in a similar way. This offers the prospect that traditional methods used to calculate the affect of an electromagnetic field on a particle can now be adapted to take localization effects into account. The prospects for measuring higher order quantum 4-potential related effects experimentally are also discussed alongside some questions to challenge the quantum information and quantum field theorists.

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I. INTRODUCTION

Experiment shows that beams of particles still behave like beams even if only one particle is traveling through the apparatus at a time [1]. The converse of this argument is that isolated particles can behave like beams. Specifically, it is understood that the wave function Ψ for the particle must take account of a full compliment of wave beam features such as mode numbers [2], Gouy phase [3,4], and orbital angular momentum (OAM) [5]. The difference between free and beam particles is the transverse localization of the beam particles. It is therefore reasonable to assume that this localization is the cause of the special features the beam exhibits.

The purpose of this paper is to present a method for calculating the effects of transverse localization on beam particles starting from the observation that the 4-momentum of a beam particle can be split into the sum of constant and position-dependent parts. The position-dependent part will be called the quantum 4-potential term since it will be shown later to play an analogous role in relativistic Hamiltonian quantum mechanics as the Bohm potential [6–8,10] in the nonrelativistic quantum Hamilton-Jacobi equation.

Quantum potential is a unique form of potential for two reasons. One is that particles have quantum potential as a result of carrying it themselves and not as a result of their position with respect to an external carrier. The other is that the quantity of quantum potential a particle carries can be set using physical devices. It is the concept that quantum potential can be set that is the key to uniqueness here since other forms of potential can be carried.

The usefulness of quantum 4-potential in accounting for the affects of localization stems from the fact that both quantum and electromagnetic 4-potentials couple into the canonical (total) 4-momentum of particles in a similar way. This means that whenever there exists a calculation of the affect of an electromagnetic field on a particle there is also the possibility to adapt this calculation to include the effects of localization. Some areas of quantum physics that have these kinds of calculations include basic quantum mechanics, interpretations of quantum mechanics, quantum information theory, many-body quantum mechanics, and quantum field theory (QFT). It is of interest therefore to briefly consider how the concept of quantum 4-potential might intersect each of them.

In basic quantum mechanics, the fine-structure constant is used to classify the order of electromagnetic terms such that lowest order terms are the largest. The analogy between the electromagnetic and quantum 4-potentials implies the existence of a counterpart localization constant that will be identified here and used to assign an order to a range of localization effects specific to the particle beam application under development.

Quantum potential is a central concept of the de Broglie–Bohm interpretation [9] of quantum mechanics that differs from the standard Copenhagen interpretation. This raises the question, can the quantum potential energy of a particle be measured in an experiment? In this paper, a formula for the energy of isolated particles that includes a contribution from quantum potential energy will be derived and the details of an experiment that is, theoretically, sensitive enough to detect it will be outlined. The experiment would also have relevance to the field of quantum information theory [11] since the amount of quantum potential energy in a particle could be regarded as data. In particular, if the experiment were successful it would show data could be encoded in a particle through a confinement device, carried by the particle to a new location, and then retrieved at the receiving end.

Although the scope of this paper is limited to beams, it is known that quantum potential is also present in bound systems such as many-electron atoms. One possible research question here is, can quantum 4-potential be coupled into the electrons as a means of represent the confining effects of cavities on the atoms? The union of many-body theory and quantum 4-potential also has direct applicability to beams of charged particles owing to the electromagnetic interactions between the particles.

Introducing localization into QFT generates a couple of problems. One is that there are two different approaches to the second quantization of the beam. In particular there is the traditional method that would require the beam function to be Fourier decomposed into plane waves versus a direct approach [12] that does not utilize the Fourier decomposition. The other problem is how might a quantum 4-potential based self-interaction be represented on a Feynman diagram [13]? More specifically, how is the interaction mediated and what does its propagator look like?

It is clear from the foregoing paragraphs that quantum 4-potential has the capacity to generate much discussion. This will be continued in later sections. For now it is necessary to redirect our attention to developing the mathematical theory that is an essential foundation for such a discussion.

Quantum 4-potential like the Bohm potential is not an external potential but a concept extracted from Ψ. This requires a distinction to be made between the canonical $\hat{p}_\mu(\mu = 1,2,3,4)$ and kinetic $\hat{P}_\mu$ 4-momentum for the particle. The canonical 4-momentum is the sum of the kinetic 4-momentum and the quantum 4-potential term. The signature of a quantum potential is therefore the appearance of a term in a quantum mechanical equation that generates localization and has no association to an external source. It will be shown that the form of this term depends on the specific formulation of quantum mechanics under consideration but that all the variants interrelate and have two distinct common properties: they vanish in the free particle limit and have null expectation values.

External devices are responsible for collimating and focusing the particles in a beam. Once a particle has passed through these devices, it remains localized but is no longer subject to external confinement. Our solutions describe the localized state of the particles but not the passage of the particle through the devices responsible for confining the beam. It has been stated that particles are carriers of quantum 4-potential. From this perspective, the basic function of all confinement mechanisms is to induce the presence of quantum 4-potential in the particles that, in turn, generates localization alongside a host of localization related effects.

The basic structure of a wave beam can be understood using the Heisenberg uncertainty principle [14] that states uncertainty in momentum is inversely proportional to uncertainty in position. In a continuous-wave beam there is no localization of the particle along the axis of the beam, meaning that each particle can be assumed to have a precise axial momentum and therefore a precise axial velocity v_3. The uncertainty in the position of the particle along the transverse axis is smallest at the beam waist. It is therefore the size of the waist that determines the uncertainty in the transverse momentum of the particle. The presence of quantum 4-potential related transverse momentum explains the fact that beams spread. It also accounts for the existence of OAM in beams.

Linear wave equations have both plane-wave and localized solutions [15] often called wave packets [12]. The wave packet is smallest at the time of an event that localizes the particle then continuously grows in size afterward. One distinguishing characteristic of plane-wave and localized wave functions is the number of 4-position dependencies in them. Plane waves are local functions that only depend on the position $x_i(i = 1,2,3)$ of the particle at time t. By contrast, localized wave solutions are bilocal functions since they must depend on both the current 4-position of the particle as well as the 4-position $X_\mu = (X_i, cT)$ of the preceding confinement event where and when the size of the wave packet was at a minimum. It is the bilocal nature of wave packets that permits the probability density of finding free particles to have spatial extension as well as a 4-position. It is also the dependence of Ψ on X_μ as well as x_μ that will enable us to define distinct kinetic P_μ and canonical p_μ 4-momentum vectors.

Bateman-Hillion functions [16,17] are exact localized solutions of relativistic wave equations that trace back to early work of Bateman on conformal transformations [18]. In two recent papers, exact Bateman-Hillion solutions were obtained for the Hermite-Gaussian modes of both electromagnetic [19] and quantum particle [20] beams. These are detailed solutions for particle beams that include the Gouy phase [21–23]. The paraxial wave equation [2] for electromagnetic beams and the Schrödinger equation for nonrelativistic particle beams have both been demonstrated as limiting cases of the Bateman-Hillion method.

It has been remarked that there are alternate approaches to the second quantization of beam problems. Altaisky and Kaputkina [12] have specifically addressed the second quantization problem for Bateman-Hillion wave beam solutions. It is not known if local and bilocal [24] QFT will produce different results. If it turns out there are differences, then any experiment to detect those differences would be a test on a basic assumption of the standard model physics that is currently characterized as a local theory.

One method of obtaining Bateman-Hillion solutions to a wave equation is to start from an ansatz. In the case of the Klein-Gordon equation the ansatz eliminates the second-order time derivative reducing the wave equation to a parabolic form. This resolves problems of negative energies and negative probability densities [25] that afflict the unconstrained Klein-Gordon equation. It will be further shown in this paper that the probability density of finding a particle in a Bateman-Hillion beam is just $|\Psi|^2$ similar to the Schrödinger equation except that the probability density for Bateman-Hillion solutions is also form preserving under Lorentz transformations.

In this paper a unitary transformation will be made to the Bateman-Hillion solutions of the Klein-Gordon equation for particle beams to account for an earlier finding [20] that the components of the 4-momentum of the particles must have a shift in them related to the complex shift in the 4-position coordinates needed for the accurate description of any wave beam. This will be shown to facilitate a calculation for the total energy of each particle in terms of the rest mass of the particle, the kinetic energy of the propagation of the particle along the axis of the beam, and the quantum 4-potential related kinetic energy locked up in the transverse mass flows. Results will be presented for both Hermite-Gaussian and Laguerre-Gaussian beams. Laguerre-Gaussian beams are useful to describe the orbital angular momentum states of the particle.

After the seminal paper by Bliokh *et al.* introducing vortex beams carrying OAM for free quantum electrons [26], several experimental [27] and theoretical [28,29] results were obtained. Properties of the interaction of OAM with an electric field such as OAM Hall effect was studied in the nonrelativistic context [26]. Further the interaction of OAM with a magnetic field was also studied in the nonrelativistic context [30]. More recently the effect of the interaction of relativistic electron vortex beam with a laser field was studied showing that the beam center is shifted and that the shift in the paraxial beams is larger than that in the nonparaxial beams [31,32]. The results that we are obtaining in this paper could be useful to explore the relativistic effects in the properties such as OAM Hall and Zeeman effects resulting, respectively, from the interaction of a relativistic scalar (without spin) electron vortex beam with an electric and a magnetic field. Further we can similarly solve the Dirac equation to include the effects of the interaction of spin angular momentum (SAM) with a magnetic field.

It will be shown in this paper that the Schrödinger and Klein-Gordon equations give the same orbital angular momentum for each scalar mode of a Laguerre-Gaussian beam. To find relativistic corrections to orbital angular momentum it is therefore necessary to investigate solutions that mix multiple modes. For example, in the case of Bessel beam solutions to the Dirac equation it has been found [29] that the corrective amplitude coefficients take the form $a = \sqrt{1 - E_0/E}\,\sin\theta_0$ where E denotes the energy of each particle, E_0 is the rest energy, and θ_0 is the polar angle indicating the divergence of the beam. This results in a relativistic correction sa^2 to the total angular momentum of each particle with spin s. The correction clearly vanishes in both the nonrelativistic ($E \to E_0$) and paraxial ($\theta_0 \to 0$) limits but can otherwise affect the energies of beam particles in external electric and magnetic fields. Another source for relativistic corrections that may affect OAM is the repulsion between charged particles. This can be a stronger effect than the spin-orbit interaction that could be studied using either the Klein-Gordon or Dirac equations. The repulsion between charged particles is also known to have a greater affect on the beam for lower energy particles.

The fact that $\Psi(x_i,t,X_i,T)$ depends on two 4-position vectors requires the introduction of a constraint condition [33,34] to eliminate one of the independent time coordinates in the calculation of the physical properties for the beam. As in an earlier paper [20] the solution to be applied here is to use Dirac δ function notation to impose a relationship $\xi_3 - v_3\tau = 0$ between the relative position $\xi_i = x_i - X_i$ and relative time $\tau = t - T$. This relates back to the idea that particles in continuous-wave beams can be assigned a precise axial velocity v_3.

In Sec. II, we use the Bateman-Hillion ansatz to solve the Klein-Gordon equation for a particle that passes through a beam waist. In Sec. III, we determine the Lorentz invariant probability density of finding a particle in a Bateman-Hillion beam. In Sec. IV, we calculate the kinetic 4-momentum in terms of the canonical 4-momentum and the localization terms. In Sec. V, we calculate the quantum 4-potential. In Sec. VI, we discuss experimental considerations essential to the detection of quantum 4-potential related effects. In Sec. VII, we conclude our results in a summary.

II. BATEMAN-HILLION BEAMS

Consider a beam of particles each having a rest mass m_0, a 4-position $x_\mu = (x_i,ct)$, and a 4-momentum $p_\mu = (p_i,E/c)$. Let us assume each particle passes through a beam waist with a position X_i at the time T. The Klein-Gordon equation for the wave function $\Psi(x_i,t,X_i,T)$ representing each of the particles in Minkowski space can be expressed as

$$\hat{p}_\mu \hat{p}^\mu \Psi = \frac{1}{c^2}\left(\hat{E}^2 - c^2\hat{p}_i^2\right)\Psi = m_0^2c^2\Psi, \tag{1}$$

where

$$\hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}, \quad \hat{E} = -\frac{\hbar}{i}\frac{\partial}{\partial t} \tag{2}$$

are the canonical 4-momentum operators, $\hbar$ is Planck's constant divided by 2π, and c is the velocity of light.

One approach to solving Eq. (1) for a beam is to use a Bateman-inspired ansatz. In an earlier paper [20], the following trial form was taken as the starting point for the derivation of the positive-energy Hermite-Gaussian beam solutions:

$$\Psi_{mn}^O = \Phi_{mn}(\xi_1,\xi_2,\xi_3 + c\tau)\exp[i(k_3x_3 - k_4ct)], \tag{3}$$

where

$$\xi_i = x_i - X_i, \quad \tau = t - T, \tag{4}$$

gives the position of each point x_μ relative to the 4-position of the beam waist, $k_\mu = (0,0,k_3,k_4)$ is the wave vector, and Φ_{mn} are scalar functions. The positive integers m and n indicate the mode of the beam.

A curious feature of Eq. (3) derived in [20] is that it leads to the following expression for the particle current in a Gaussian beam:

$$\left\langle \Psi_{00}^O \middle| \hat{j}_\mu \middle| \Psi_{00}^O \right\rangle = \frac{\hbar}{m_0}\left(k_\mu - \kappa_\mu^{00}\right), \tag{5}$$

where

$$\Psi^*\hat{j}_\mu\Psi = \frac{1}{2m_0}(\Psi^*\hat{p}_\mu\Psi - \Psi\hat{p}_\mu\Psi^*), \tag{6}$$

and $\kappa_\mu^{mn} = (0,0,\kappa^{mn},-\kappa^{mn})$. Here, the axial parameter κ^{mn} takes the form

$$\kappa^{00} = \frac{1}{(k_3 + k_4)w_0^2}, \tag{7}$$

where w_0 is the radius of the beam at the waist.

Equation (5) suggests that k_μ is related to the expectation value of the axial current for a particle in a beam. In seeking an intuitive definition for k_μ we shall now make use of the unitary transformation

$$\Psi_{mn} = \Psi_{mn}^O \exp[i\kappa^{mn}(x_3 + ct)], \tag{8}$$

where

$$\kappa^{mn} = \frac{N^{mn}}{(k_3 + k_4)w_0^2}, \tag{9}$$

and N^{mn} is a constant. The general form of N^{mn} is to be determined but it can be seen from comparison of Eqs. (7) and (9) that $N^{00} = 1$. It is also readily verified that Eq. (8) is form invariant under the Lorentz transformation equations

$$x_3' = (x_3 - v_3\tau)\gamma, \quad \tau' = \left(\tau - \frac{v_3}{c^2}x_3\right)\gamma, \tag{10}$$

$$k_3' = \left(k_3 - \frac{v_3}{c}k_4\right)\gamma, \quad k_4' = \left(k_4 - \frac{v_3}{c}k_3\right)\gamma, \tag{11}$$

where $\gamma = 1/\sqrt{1 - v_3^2/c^2}$. Applying the transformation (8) to Eq. (3) gives

$$\Psi_{mn} = \Phi_{mn}(\xi_1,\xi_2,\xi_3 + c\tau) \times \exp[i(k_3 + \kappa^{mn})x_3 - ic(k_4 - \kappa^{mn})t], \tag{12}$$

equivalent to making the replacements $k_3 \to k_3 + \kappa^{mn}$ and $k_4 \to k_4 - \kappa^{mn}$. These replacements can be used, in turn, to reduce Eq. (5) to the simplified to the form

$$\langle\Psi_{00}|\hat{j}_\mu|\Psi_{00}\rangle = \frac{\hbar}{m_0}(0,0,k_3,k_4), \tag{13}$$

where it can be seen κ^{00} has been eliminated. One important goal of this paper will be to show that there exists N^{mn} such that the condition

$$\langle\Psi_{mn}|\hat{j}_\mu|\Psi_{mn}\rangle = \frac{\hbar}{m_0}(0,0,k_3,k_4) \tag{14}$$

is satisfied. If this hypothesis is true, it implies $\frac{\hbar}{m_0}k_\mu$ can be interpreted as the expectation value for the particle current in a relativistic beam thus giving a clear physical meaning to k_μ.

Inserting Eq. (12) into the Klein-Gordon equation (1) gives

$$\frac{\partial^2\Phi_{mn}}{\partial x_1^2} + \frac{\partial^2\Phi_{mn}}{\partial x_2^2} + 2i(k_3 + \kappa^{mn})\frac{\partial\Phi_{mn}}{\partial x_3} + \frac{2i}{c}(k_4 - \kappa^{mn})\frac{\partial\Phi_{mn}}{\partial t} = 0, \tag{15}$$

where

$$k_4^2 = k_3^2 + 2\kappa^{mn}(k_3 + k_4) + \frac{m_0^2c^2}{\hbar^2}. \tag{16}$$

It can be seen the unitary transformation (8) has introduced the term

$$K_T^{mn} = 2\kappa^{mn}(k_3 + k_4) \tag{17}$$

into this dispersion relationship. The physical interpretation of K_T^{mn} will be discussed later once the relativistic energy formula for each particle in the beam has been derived.

It is instructive to observe that

$$\frac{\partial}{\partial x_3}\Phi_{mn} = \frac{1}{c}\frac{\partial}{\partial t}\Phi_{mn}, \tag{18}$$

and equivalently

$$\frac{\partial}{\partial x_3}|\Psi_{mn}|^2 = \frac{1}{c}\frac{\partial}{\partial t}|\Psi_{mn}|^2, \tag{19}$$

owing to the fact Φ_{mn} only depends on ξ_3 and τ in the linear combination $\xi_3 + \tau$. Equations (15) and (18) can now be combined to obtain the operator relationships

$$\hat{p}_3\Phi_{mn} = -\hat{p}_4\Phi_{mn} = -\frac{\hat{p}_1^2 + \hat{p}_2^2}{2\hbar(k_3 + k_4)}\Phi_{mn}. \tag{20}$$

These results will prove useful later.

Equation (15) can be solved analogously to the paraxial equation [2] to give

$$\Phi_{mn} = \frac{C_{mn}^{\text{HG}}w_0}{w}H_m\left(\frac{\sqrt{2}\xi_1}{w}\right)H_n\left(\frac{\sqrt{2}\xi_2}{w}\right) \times \exp\left[\frac{i2b(\xi_1^2 + \xi_2^2)}{w_0^2(\xi_3 + c\tau - i2b)} - ig_{mn}\right], \tag{21}$$

where H_m and H_n are Hermite polynomials,

$$b = \frac{w_0^2}{4}(k_3 + k_4), \tag{22}$$

$$w(\xi_3,\tau) = w_0\sqrt{1 + \left(\frac{\xi_3 + c\tau}{2b}\right)^2}, \tag{23}$$

is the beam radius such that $w_0 = w(0,0)$, and

$$g_{mn}(\xi_3,\tau) = (1 + m + n)\arctan\left(\frac{\xi_3 + c\tau}{2b}\right) \tag{24}$$

is the Gouy phase of a relativistic quantum particle.

It is notable that the Klein-Gordon equation (1) can also be usefully solved in cylindrical coordinates starting from the expression

$$\Psi_{lp} = \Phi_{lp}(\xi_\rho,\xi_\phi,\xi_3 + c\tau) \times \exp[i(k_3 + \kappa^{lp})x_3 - ic(k_4 - \kappa^{lp})t] \tag{25}$$

equivalent to Eq. (12) where $\xi_\rho = \sqrt{\xi_1^2 + \xi_2^2}$ and $\xi_\phi = \text{atan2}(\xi_2,\xi_1)$. This gives

$$\Phi_{lp} = \frac{C_{lp}^{\text{LG}}w_0}{w}\left(\frac{\sqrt{2}\xi_\rho}{w}\right)^{|l|}L_p^{|l|}\left(\frac{2\xi_\rho^2}{w^2}\right) \times \exp\left[\frac{i2b\xi_\rho^2}{w_0^2(\xi_3 + c\tau - i2b)} + il\xi_\phi - ig_{lp}\right], \tag{26}$$

where $L_p^{|l|}$ are the generalized Laguerre polynomials and

$$g_{lp}(\xi_3,\tau) = (1 + |l| + 2p)\arctan\left(\frac{\xi_3 + c\tau}{2b}\right) \tag{27}$$

is the Gouy phase in terms of the radial Laguerre index p and the azimuthal index l, which may be positive or negative.

The operator for the axial component of canonical OAM can be expressed as

$$\hat{L}_3 = \xi_\rho \times \hat{p}_\phi = \frac{\hbar}{i}\frac{\partial}{\partial\xi_\phi}. \tag{28}$$

The Laguerre-Gaussian beam functions (25) can thus be seen to give

$$\hat{L}_3\Psi_{lp} = l\hbar\Psi_{lp}, \tag{29}$$

showing $L_3 = l\hbar$ are the possible eigenvalues of OAM for a Laguerre-Gaussian beam.

III. PROBABILISTIC INTERPRETATION

In this section, the correspondence between the particle current (6) for Bateman-Hillion beams and that of the Schrödinger equation for particle beams will be investigated as means of determining the probability density of finding a particle in a Bateman-Hillion beam. As a starting point it will be useful to evaluate each component of the Bateman-Hillion particle current

$$j_\mu^{mn} = \Psi_{mn}^* \hat{j}_\mu \Psi_{mn}. \tag{30}$$

This leads to

$$j_1^{mn} = \frac{4b(\xi_3 + c\tau)\xi_1}{w_0^2[(\xi_3+c\tau)^2+4b^2]}\frac{\hbar}{m_0}|\Psi_{mn}|^2, \tag{31}$$

$$j_2^{mn} = \frac{4b(\xi_3 + c\tau)\xi_2}{w_0^2[(\xi_3+c\tau)^2+4b^2]}\frac{\hbar}{m_0}|\Psi_{mn}|^2, \tag{32}$$

$$j_3^{mn} = \left[k_3 + \kappa^{mn} - \frac{2b(1+m+n)}{(\xi_3+c\tau)^2+4b^2}\right]\frac{\hbar}{m_0}|\Psi_{mn}|^2 - \frac{2b(\xi_1^2+\xi_2^2)[(\xi_3+c\tau)^2-4b^2]}{w_0^2[(\xi_3+c\tau)^2+4b^2]^2}\frac{\hbar}{m_0}|\Psi_{mn}|^2, \tag{33}$$

$$j_4^{mn} = \left[k_4 - \kappa^{mn} + \frac{2b(1+m+n)}{(\xi_3+c\tau)^2+4b^2}\right]\frac{\hbar}{m_0}|\Psi_{mn}|^2 + \frac{2b(\xi_1^2+\xi_2^2)[(\xi_3+c\tau)^2-4b^2]}{w_0^2[(\xi_3+c\tau)^2+4b^2]^2}\frac{\hbar}{m_0}|\Psi_{mn}|^2, \tag{34}$$

where

$$|\Psi_{mn}|^2 = \left(\frac{C_{mn}^{\rm HG} w_0}{w}\right)^2 H_m^2\left(\frac{\sqrt{2}\xi_1}{w}\right)H_n^2\left(\frac{\sqrt{2}\xi_2}{w}\right) \times \exp\left[-\frac{8b^2(\xi_1^2+\xi_2^2)}{w_0^2[(\xi_3+c\tau)^2+4b^2]}\right]. \tag{35}$$

The continuity equation for the Klein-Gordon equation (1) is

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{1}{c}\frac{\partial j_4}{\partial t} = 0. \tag{36}$$

Equations (33) and (34) enable this expression to be rewritten in the form

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{1}{m_0}\left(k_3\frac{\partial}{\partial x_3} + k_4\frac{\partial}{\partial t}\right)|\Psi_{mn}|^2 = 0, \tag{37}$$

or equivalently

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{1}{m_0}(k_3+k_4)\frac{\partial}{\partial t}|\Psi_{mn}|^2 = 0, \tag{38}$$

having used Eq. (19). This result reduces to the simplified expression

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial}{\partial t}\left|\Psi_{mn}^S\right|^2 = 0 \tag{39}$$

in the nonrelativistic limit where $k_3 \ll k_4$ and $m_0c^2 \simeq c\hbar k_4$.

In an earlier paper [20] it was shown that Eqs. (1) and (3) reduce to the Schrödinger equation

$$\frac{\partial^2\Psi_{mn}^S}{\partial x_1^2} + \frac{\partial^2\Psi_{mn}^S}{\partial x_2^2} + \frac{\partial^2\Psi_{mn}^S}{\partial x_3^2} + 2i\frac{m}{\hbar}\frac{\partial\Psi_{mn}^S}{\partial t} = 0, \tag{40}$$

and the nonrelativistic form of the Bateman-Hillion ansatz

$$\Psi_{mn}^{\rm OS} = \Phi_{mn}^S(\xi_1,\xi_2,\tau)\exp\left[\frac{i}{\hbar}(P_3x_3 - E_st)\right], \tag{41}$$

where E_S is the nonrelativistic energy of the particle and

$$\Phi_{mn}^S = \int \Phi_{mn}\delta(\xi_3 - v\tau)\,d\xi_3. \tag{42}$$

For comparison to results in the present context $\Psi_{mn}^{\rm OS}$ must be further subjected to the unitary transformation (8) that simplifies to

$$\Psi_{mn}^S = \Psi_{mn}^{\rm OS}\exp\left(\frac{iN^{mn}\hbar t}{m_0 w_0^2}\right) \tag{43}$$

in the the nonrelativistic limit $c \to \infty$.

It is readily shown that Eq. (39) is the continuity equation for the Schrödinger equation (40) since

$$\frac{\partial j_3}{\partial x_3} = \frac{P_s}{\hbar}\frac{\partial}{\partial x_3}\left|\Psi_{mn}^S\right|^2 = 0. \tag{44}$$

It is thus concluded from a direct comparison of Eqs. (38) and (39) that

$$P_{\rm BH} = m_0\frac{j_3+j_4}{k_3+k_4} = |\Psi_{mn}|^2 \tag{45}$$

is the relativistic probability density for finding a particle in a Bateman-Hillion beam. This differs from the widely cited [13] Klein-Gordon probability density

$$P_{\rm KG} = \frac{j_4}{c} \tag{46}$$

due to the fact Ψ_{mn} is further constrained under the parabolic equation (15). It is also of interest to notice that $P_{\rm BH}$ is form invariant under Lorentz transformations whereas $P_{\rm KG}$ is not as an isolated component of a 4-vector.

Bateman-Hillion functions can be normalized using the integral expression

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\Psi|^2\delta(\xi_3 - v_3\tau)d\xi_1 d\xi_2 d\tau = \frac{1}{L}, \tag{47}$$

having set the probability of finding the particle in a beam of length L as 1. This evaluates to

$$C_{mn}^{\rm HG} = \sqrt{\frac{2}{\pi w_0^2 L 2^{m+n} m!n!}} \tag{48}$$

for Hermite-Gaussian beams; and

$$C_{lp}^{\rm LG} = \sqrt{\frac{4p!}{w_0^2 L(p+|l|)!}} \tag{49}$$

for Laguerre-Gaussian beams.

Expectation values for the measurable properties of each particle in the beam can be calculated as

$$\langle\Psi|\hat{O}|\Psi\rangle_P = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\Psi^*\hat{O}\Psi)\delta(\xi_3 - v_3\tau)\,d\xi_1\,d\xi_2\,d\tau, \tag{50}$$

where $\hat{O}$ is the quantum mechanical operator for each observable quantity. Here, the subscript P has been included as a reminder that the integration is performed over a planar cross section perpendicular to the axis of the beam but not along the axis itself.

IV. CALCULATION OF 4-MOMENTUM

The canonical 4-momentum operator $\hat{p}_\mu$ is defined in Eq. (2) in terms of the 4-position vector x_μ. We next seek to use the fact that Ψ_{mn} depends on X_μ as well as x_μ to define a distinct kinetic 4-momentum operator $\hat{P}_\mu$ to satisfy the eigenvalue equation

$$\hat{P}_\mu \Psi_{mn} = \hbar k_\mu \Psi_{mn}. \tag{51}$$

The first step is to write

$$\Phi_{mn}(\xi_1,\xi_2,\xi_3 + c\tau) = \Phi_{mn}(x_1 - X_1, x_2 - X_2, x_3 - X_3 + ct - cT) \tag{52}$$

having used Eq. (4). This indicates

$$\frac{\partial \Phi_{mn}}{\partial x_\mu} = -\frac{\partial \Phi_{mn}}{\partial X_\mu}, \tag{53}$$

and therefore

$$i\hbar\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial X^\mu}\right)\Psi_{mn} = \hbar\left(k_\mu + \kappa_\mu^{mn}\right)\Psi_{mn}. \tag{54}$$

From comparison of this expression to Eq. (51) it can be seen that

$$\hat{P}_\mu \Psi_{mn} = \left(i\hbar\frac{\partial}{\partial x^\mu} + i\hbar\frac{\partial}{\partial X^\mu} - \hbar\kappa_\mu^{mn}\right)\Psi_{mn} = \hbar k_\mu \Psi_{mn} \tag{55}$$

or equivalently

$$\hat{P}_1\Psi_{mn} = \hat{P}_2\Psi_{mn} = 0, \quad \hat{P}_3\Psi_{mn} = \hbar k_3 \Psi_{mn}, \tag{56}$$

$$\hat{P}_4\Psi_{mn} = \sqrt{\hbar^2 k_3^2 + 2\kappa^{mn}(k_3 + k_4) + m_0^2 c^2}\,\Psi_{mn}, \tag{57}$$

having used Eq. (16). These results are the eigenvalue equations for the kinetic 4-momentum of each particle in a relativistic Hermite-Gaussian beam. In completing this argument, it is necessary to find the explicit form of N^{mn} from Eq. (14).

Inserting the Bateman-Hillion ansatz (12) into Eq. (14) gives

$$\langle\Psi_{mn}|m_0\hat{j}_\mu|\Psi_{mn}\rangle_P = \hbar\left(k_\mu + \kappa_\mu^{mn}\right) + \langle\Phi_{mn}|m_0\hat{j}_\mu|\Phi_{mn}\rangle_P. \tag{58}$$

Here, the term $\langle\Phi_{mn}|m_0\hat{j}_\mu|\Phi_{mn}\rangle_P$ can be evaluated using the integrals

$$\int_{-\infty}^{+\infty} x H_m^2(\sqrt{\alpha}x)e^{-\alpha x^2}dx = 0, \tag{59}$$

$$\int_{-\infty}^{+\infty} x^2 H_m^2(\sqrt{\alpha}x)e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha^3}}\left(\frac{1}{2} + m\right). \tag{60}$$

The result is

$$\langle\Phi_{mn}|m_0\hat{j}_\mu|\Phi_{mn}\rangle_P = -\hbar\kappa^{mn} \tag{61}$$

having set

$$N^{mn} = 1 + m + n. \tag{62}$$

Putting Eq. (61) into (58) gives

$$\langle\Psi_{mn}|m_0\hat{j}_\mu|\Psi_{mn}\rangle_P = \hbar k_\mu. \tag{63}$$

It is thus established that the eigenvalues of the kinetic 4-momentum operator $\hat{P}_\mu$ are equal to the expectations values for the mass current for all Hermite-Gaussian beam modes.

Equations (57) and (62) enable the total energy E_{HG}^{mn} for each particle in a Hermite-Gaussian mode to be written as

$$E_{\mathrm{HG}}^{mn} = c\sqrt{\hbar^2 k_3^2 + \frac{2\hbar^2}{w_0^2}(1 + m + n) + m_0^2 c^2}. \tag{64}$$

Comparing this result to the energy of a free particle

$$E_{\mathrm{FP}} = c\sqrt{\hbar^2 k_3^2 + m_0^2 c^2} \tag{65}$$

of identical mass m_0 and axial wave number k_3 shows that the beam particle picks up an additional energy contribution

$$\hbar^2 K_T^{mn} = \frac{2\hbar^2}{w_0^2}(1 + m + n), \tag{66}$$

where K_T^{mn} is defined in Eq. (17), as a result of being localized. The remaining task is therefore to assign a physical interpretation to this term.

It can be inferred from inspection of Eq. (6) that the expectation values of canonical 4-momentum and mass current must be related through the expression

$$\langle\Psi_{mn}|m_0\hat{j}_\mu|\Psi_{mn}\rangle_P = \mathrm{Re}\langle\Psi_{mn}|\hat{p}_\mu|\Psi_{mn}\rangle_P, \tag{67}$$

where the operator Re takes the real part of the argument. Equations (20), (58), (61), and (67) can therefore be used together to give

$$\mathrm{Re}\langle\Psi_{mn}|\hat{p}_1^2 + \hat{p}_2^2|\Psi_{mn}\rangle_P = \frac{2\hbar^2}{w_0^2}(1 + m + n). \tag{68}$$

This shows that the middle term under the square root sign in Eq. (64) represents the contribution of the fluctuating transverse components of momentum to the total energy of each particle.

V. QUANTUM POTENTIAL

The concept of distinguishing between canonical and kinetic 4-momentum has familiarity from the description [13] of a particle of charge e moving in an electromagnetic 4-potential A_μ. The kinetic 4-momentum for this problem is

$$\hat{\pi}_\mu = \hat{p}_\mu - eA_\mu. \tag{69}$$

For the purposes of comparison the relationship between the kinetic and the canonical 4-momentum of a beam particle given in Eq. (55) can be written as

$$\hat{P}_\mu = \hat{p}_\mu - m_0\hat{U}_\mu, \tag{70}$$

where

$$\hat{U}_\mu = \frac{\hbar}{m_0}\left(\frac{1}{i}\frac{\partial}{\partial X^\mu} + \kappa_\mu^{mn}\right). \tag{71}$$

Equations (69) and (70) are similar in form but A_μ is an external 4-potential whereas $\hat{U}_\mu$ is an operator. The understanding here is that wave equations are constructed using kinetic 4-momentum to take account of external potentials and canonical 4-momentum if no external potential is present. The Hermite-Gaussian function Ψ_{mn} was derived from a wave equation that contains only canonical 4-momentum operators but it is still possible to identify a 4-potential-like term $\hat{U}_\mu$ in the definition of the kinetic 4-momentum P_μ analogous to the role of the external 4-potential A_μ in π_μ. Equation (71) will be referred to as the 4-potential operator.

Kinetic 4-momentum was defined in Eq. (51) to be a real quantity. It follows from Eq. (70) that the particle current can be written in the form

$$j_\mu = \left(\frac{\hbar k_\mu}{m_0} + U_\mu\right)|\Psi|^2, \tag{72}$$

where

$$U_\mu = \frac{\hbar}{m_0}\mathrm{Re}\left(\frac{i}{\Psi}\frac{\partial\Psi}{\partial X^\mu} + \kappa_\mu^{mn}\right) \tag{73}$$

is a real quantum 4-potential field. Comparing Eq. (72) to the component Eqs. (31)–(34) gives

$$U_1^{mn} = \frac{\hbar}{m_0}\frac{4b(\xi_3 + c\tau)\xi_1}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]}, \tag{74}$$

$$U_2^{mn} = \frac{\hbar}{m_0}\frac{4b(\xi_3 + c\tau)\xi_2}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]}, \tag{75}$$

$$U_3^{mn} = \frac{\hbar}{m_0}\left[\kappa^{mn} - \frac{2b(1+m+n)}{(\xi_3 + c\tau)^2 + 4b^2}\right] - \frac{\hbar}{m_0}\frac{2b\left(\xi_1^2 + \xi_2^2\right)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]^2}, \tag{76}$$

$$U_4^{mn} = \frac{\hbar}{m_0}\left[-\kappa^{mn} + \frac{2b(1+m+n)}{(\xi_3 + c\tau)^2 + 4b^2}\right] + \frac{\hbar}{m_0}\frac{2b\left(\xi_1^2 + \xi_2^2\right)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]^2}, \tag{77}$$

to be the explicit form of the quantum 4-potential for a Hermite-Gaussian beam.

Expression (72) is a quantum mechanical equation describing a particle in a localized state. In the absence of localization ($w_0 \to \infty$) it reduces to the form

$$j_\mu = \frac{\hbar k_\mu}{m_0}|\Psi|^2 \tag{78}$$

showing that the quantum 4-potential term has vanished. Squaring Eq. (78) gives

$$j_M^2 - m_0^2 c^2|\Psi|^2 = 0, \tag{79}$$

where $j_M^2 = m_0^2 j_\mu j^\mu$. It is clear that if we now add back the quantum 4-potential into both Eqs. (78) and (79) then Eq. (79) must pick up an additional scalar term V^2 such that

$$j_M^2 - m_0^2 c^2|\Psi|^2 + V^2 = 0, \tag{80}$$

where

$$V^2 = -|\hbar k_\mu - m_0 U_\mu|^2 - m_0^2 c^2 |\Psi|^2. \tag{81}$$

Expanding this expression gives

$$V^2 = -m_0^2\left|U_\mu^{mn}\right|^2 - 2\hbar k^\mu m_0 U_\mu^{mn} - \hbar^2 K_T^{mn} = 4\hbar^2\left[\frac{1+m+n}{w^2} - \frac{\xi_1^2+\xi_2^2}{w^4}\right] \tag{82}$$

having used

$$\left|U_\mu^{mn}\right|^2 = -\frac{\hbar^2}{m_0^2}\frac{16b^2(\xi_3 + c\tau)^2\left(\xi_1^2+\xi_2^2\right)}{w_0^4[(\xi_3 + c\tau)^2 + 4b^2]^2}, \tag{83}$$

$$k^\mu U_\mu^{mn} = \frac{\hbar}{m_0}\left[-\frac{K_T^{mn}}{2} + \frac{8b^2(1+m+n)}{w_0^2[(\xi_3 + c\tau)^2 + 4b^2]}\right] + \frac{\hbar}{m_0}\frac{8b^2\left(\xi_1^2+\xi_2^2\right)[(\xi_3 + c\tau)^2 - 4b^2]}{w_0^4[(\xi_3 + c\tau)^2 + 4b^2]^2}, \tag{84}$$

alongside Eq. (16). It is concluded from this argument that V^2 is itself a quantum potential appearing in Eq. (81) as the scalar analog of the quantum 4-potential U_μ in Eq. (72).

The OAM operator (28) can be rewritten in Cartesian coordinates to give

$$\hat{L}_3 = \xi_1\hat{p}_2 - \xi_2\hat{p}_1 \tag{85}$$

or equivalently

$$\hat{L}_3 = \xi_1(\hat{P}_2 + m_0\hat{U}_2) - \xi_2(\hat{P}_1 + m_0\hat{U}_1) \tag{86}$$

having used Eq. (70). This last result simplifies to

$$\hat{L}_3 = \xi_1 m_0\hat{U}_2 - \xi_2 m_0\hat{U}_1, \tag{87}$$

since $P_\mu = (0,0,\hbar k_3,\hbar k_4)$. It is therefore concluded that the quantum 4-potential operator $\hat{U}_\mu$ and not the kinetic 4-momentum operator $\hat{P}_\mu$ is the source of the mass flow resulting in OAM.

Calculating the expectation value of each component $\hat{U}_\mu$ of the quantum 4-potential and the scalar analog V^2 we obtain

$$\langle\Psi_{mn}|U_\mu|\Psi_{mn}\rangle_P = \langle\Psi_{mn}|V_\mu^2|\Psi_{mn}\rangle_P = 0. \tag{88}$$

This result shows that quantum 4-potential is a fluctuating phenomenon. Specifically, the presence of quantum 4-potential can cause the canonical 4-momentum of a localized particle in a beam to instantaneously deviate from the kinetic 4-momentum, but it has no affect at all on the expected 4-momentum of the particle.

The original concept of a quantum potential was introduced by Bohm [6] who started from an ansatz to solve the Schrödinger equation. This takes the form

$$\Psi = R\exp\left(i\frac{S}{\hbar}\right), \tag{89}$$

where the amplitude R and $S/\hbar$ are real valued functions.

On inserting Eq. (89) into the Schrödinger equation (40), the imaginary part of the equation can be identified as the continuity equation (39) and the real part as the Hamilton-Jacobi equation

$$-\frac{\partial S_{mn}}{\partial t} = \frac{|\nabla S_{mn}|^2}{2m_0} + Q, \tag{90}$$

where

$$Q = -\frac{\hbar^2}{2m_0}\frac{\nabla^2 R_{mn}}{R_{mn}} \tag{91}$$

is the Bohm potential. It is of interest next to investigate how the quantum 4-potential and the Bohm potential are related to each other.

The solution to the Schrödinger equation for Hermite-Gaussian beams is given in Eqs. (41) and (42). On comparing Eq. (41) and (89) the explicit form of the amplitude R_{mn} and phase function S_{mn} can be read off to be

$$R_{mn} = \frac{C_{mn}^{\mathrm{HG}} w_0}{w_S} H_m\left(\frac{\sqrt{2}\xi_1}{w_S}\right) H_n\left(\frac{\sqrt{2}\xi_2}{w_S}\right) \times \exp\left(-\frac{\xi_1^2+\xi_2^2}{w_S^2}\right), \tag{92}$$

and

$$S_{mn} = P_3 x_3 - Et - (1+m+n)\hbar\omega_0 t + \frac{2(\xi_1^2+\xi_2^2)\hbar\omega_0\tau}{w_S^2} - \hbar(1+m+n)\arctan(2\omega_0\tau), \tag{93}$$

where

$$w_S = w_0\sqrt{1+4\omega_0\tau^2}, \quad \omega_0 = \frac{\hbar}{m_0 w_0^2}. \tag{94}$$

Inserting Eq. (92) into Eq. (91) shows the Bohm quantum potential for a nonrelativistic Hermite-Gaussian beam to be

$$Q = \frac{2\hbar^2}{m_0}\left[\frac{1+m+n}{w_S^2} + \frac{\xi_1^2+\xi_2^2}{w_S^4}\right]. \tag{95}$$

Equation (95) can in turn be inserted into the Hamilton-Jacobi equation (90) giving Eq. (93) as a solution.

It is clear from Eqs. (82) and (95) that

$$Q = \lim_{c\to\infty}\frac{V^2}{2m_0}. \tag{96}$$

This result shows that the Bohm potential for a Hermite-Gaussian beam is the nonrelativistic limit of the scalar form V^2 of the relativistic quantum potential defined in Eq. (82).

VI. EXPERIMENTAL CONSIDERATIONS

The range of quantum 4-potential effects treated in this paper is quite limited. The free-field Klein-Gordan equation was taken as the starting point to avoid adding additional layers of complexity to the mathematical sections that already had the burden of introducing quantum 4-potential and demonstrating its connection to the Bohm potential. More can be expected once external electromagnetic fields are incorporated into the calculations and Bateman-Hillion solutions are fully developed for the Dirac equation. Regardless of current limitations, it is still interesting to enumerate and estimate the size of the few measurable effects that have been identified.

It has been shown earlier that quantum 4-potential and electromagnetic 4-potential couple into the canonical 4-momentum of particles in a similar way. In estimating the size of electromagnetic terms it usual to make use of a dimensionless parameter called the fine-structure constant α such that any term containing α^n is said to be an nth-order effect. For quantum 4-potential related terms, it has been found useful to define an analogous constant

$$a_B = \frac{\lambdabar}{w_0} \tag{97}$$

representing the ratio of the reduced Compton wavelength

$$\lambdabar = \frac{\hbar}{m_0 c} \tag{98}$$

to the minimum beam radius w_0. We shall call a_B the beam particle constant.

Two of the most pronounced (zeroth order) localization effects are the Gouy phase and OAM. These can be called localization effects since it is clear that an isolated particle could exhibit neither of them unless it were localized. In the case of Gouy phase Eq. (24) is in good nonrelativistic correspondence to previous theoretical and experimental work as discussed in [20].

The OAM for a Laguerre-Gauss beam is calculated in Eq. (29). Equation (87) shows that the OAM of the beam is completely defined in terms of the quantum 4-potential the beam carries. It is clear that theory and experiment agree at the zeroth order. The next step is to consider higher order corrections to OAM that will result from intrinsic properties such as charge and spin. It is proposed to address these issues in a future paper once work on Bateman-Hillion solutions of the Dirac equation is completed.

Beam spreading provides a good illustration of a first-order localization effect. It can be seen, in particular, that

$$\frac{\partial w}{\partial x_3} \simeq \frac{w_0}{2b} \leqslant 2a_B, \tag{99}$$

providing $\xi_3 + c\tau \gg 2b$. This shows that $2a_B$ is an approximate upper bound for the constant rate of divergence of the beam at asymptotically large distances from the waist.

It is instructive to recall that a lot of early work on beams [2] was done in the paraxial limit where a_B is intentionally kept very small so that the divergence of the beam is imperceptible except over very large distances compared to the radius. It is clear now that the best hope of detecting higher order localization effects is to move to the other extreme where a_B is made as large as possible.

One higher order localization effect that perhaps could be measured in an experiment is the quantum potential energy contribution (66) in the total energy equation (64). It can be seen that the ratio of the quantum potential term to the rest energy squared term in Eq. (64) is

$$\frac{\hbar^2 K_T^{mn}}{m_0^2 c^4} = (1+m+n)a_B^2, \tag{100}$$

clearly showing that the quantum potential energy is of second-order importance compared to the rest energy.

The detection of quantum potential energy will require particles such as electrons to be strongly localized ahead of insertion into a measuring apparatus that is capable of determining the absolute energy of the beam. One estimate [35] for the best accuracy currently possible with such a detector is 10^{-4} indicating that a_B for a Gaussian ($m = n = 0$) beam must be larger than 10^{-2} for the experiment to be successful.

VII. SUMMARY

A relativistic solution for Hermite-Gaussian particle beams presented in an earlier paper [20] has been used to calculate the properties of a particle in the beam. It was found that the canonical 4-momentum for the particle can be separated into a constant and position-dependent part. The position-dependent part has been called the quantum 4-potential term since it has been shown to be the relativistic Hamiltonian counterpart of the Bohm potential in the nonrelativistic Hamilton-Jacobi equation.

A physical interpretation for quantum 4-potential has been proposed based on two observations. One is that free particles have no quantum 4-potential but beam particles do. It thus appears that quantum 4-potential is related to the transverse localization feature that distinguishes beam particles from free ones. The other is that quantum and electromagnetic 4-potentials couple into the canonical 4-momentum of particles in a similar way but quantum 4-potential has no external source. Quantum 4-potential is therefore proposed to be a field that isolated particles must carry to appear localized to outside observers.

Quantum 4-potential has no external source but the amount of quantum 4-potential a particle carries can be set using physical devices designed for collimating and focusing beam particles. The presence of the quantum 4-potential is then observable through the localization effects it produces including OAM, Gouy phase, and beam spreading.

In building on the analogy between quantum and electromagnetic 4-potentials, it has been recognized that the importance of localization effects can be classified using a dimensionless constant that has a comparable role to the fine-structure constant used to classify the order of electromagnetic effects. For example, OAM and Gouy phase are both zeroth-order localization effects but beam spreading is a first-order effect.

The fact that beam spreading is a first-order localization effect indicates that there may be higher order localization effects to observe but to do so it will be necessary to devise sensitive experiments that operate far outside the paraxial limit conditions that prevail in many beam experiments. One such second-order localization effect is the quantum potential energy contribution to the total energy of beam particles. An experiment to detect this small effect has been outlined.

In consideration of the next steps, it is recognized that one of the most significant limitations of the current paper is the fact it is based on the Klein-Gordon equation instead of the Dirac equation. One obvious next step is therefore to first obtain Bateman-Hillion solutions to the Dirac equation, then use them to catch Hermite-Gauss and Laguerre-Gauss beam models up to the high standard [29–32] that Bessel-type beam solutions of the Dirac equation have already reached. This should lead to the classification and deeper understanding of a broader range of localization effects than have been treated in this paper.

The existence of quantum 4-potential also raises some issues that may be interesting topics for future investigations. First, does quantum 4-potential have applicability to bound systems? Second, can the quantum potential energy stored in particles be used to transmit data? Finally, how might quantum 4-potential be represented in quantum field theory?

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