Exactly solvable time-dependent models of two interacting two-level systems

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(Received 18 April 2016; published 9 August 2016)

Two coupled two-level systems placed under external time-dependent magnetic fields are modeled by a general Hamiltonian endowed with a symmetry that enables us to reduce the total dynamics into two independent two-dimensional subdynamics. Each of the subdynamics is shown to be brought into an exactly solvable form by appropriately engineering the magnetic fields and thus we obtain an exact time evolution of the compound system. Several physically relevant and interesting quantities are evaluated exactly to disclose intriguing phenomena in such a system.

DOI: 10.1103/PhysRevA.94.022108

I. INTRODUCTION

A rigid and localized dimeric structure (simply dimer) consists of a pair of independent distinguishable quantum subsystems living, by definition, in finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and, therefore, hereafter referred to as spins $\hat{\mathbf{S}}_1 \equiv (\hat{S}_1^x, \hat{S}_1^y, \hat{S}_1^z)$ and $\hat{\mathbf{S}}_2 \equiv (\hat{S}_2^x, \hat{S}_2^y, \hat{S}_2^z)$, respectively, with $\hat{S}_i^a$ ($i = 1,2$; $a = x,y,z$) being the operator for the a-Cartesian component of $\hat{\mathbf{S}}_i$ in the laboratory reference frame. The dimension of the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of the dimer is $(2S_1 + 1)(2S_2 + 1)$, indeed postulating the absence in the two subsystems as well as in the compound system of classical degrees of freedom (a situation previously described using the adjectives "rigid" and "localized"). The physical nature of $\hat{\mathbf{S}}_i$ depends on the particular scenario under scrutiny: it may be the spin of an electron or a nucleus, the angular momentum of an atom in its ground state, or an effective representation of a few-level system dynamical variable. The Hamiltonian H of the dimer is then a true or effective spin Hamiltonian where the terms linear in $\hat{S}_i^a$ may (even fictitiously) be interpreted as Zeeman coupling of each of the two spins with classical, external, generally different, and time-dependent effective magnetic fields $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$, while the bilinear contributions may be thought of as stemming from the spin-spin interaction [1].

Over the last two decades, a great deal of theoretical, experimental, and applicative attention has been devoted to the field of molecular magnetic materials, in particular after the discovery of the so-called single magnet molecule (SMM), that is a single molecule behaving like a nanosized magnet associated to an unusual high value (even $S = 10$ [2]) of the spin in the ground state of the molecule. It is a matter of fact that as a result of a successful, extraordinary, and synergically interdisciplinary effort aimed at searching and producing SMM in the laboratory, in the last few years we have witnessed a very fast growth of efficient protocols for synthesizing a variety of such molecular magnets with the added value of possessing a number of constituent paramagnetic ions embodied in the molecule, running from 2 to 10 in different samples [3]. Such important technological advances, on the one hand, open very good applicative perspectives in many directions, from the realization of an experimental setup for testing theoretical prediction concerning qudits-based single-purpose quantum computers to the availability of new materials with magnetic properties tailored on demand to meet specific tasks. On the other hand, the production of crystalline or powder samples made up of molecular magnetic units provides an ideal platform to investigate and reveal the emergence of nonclassical signatures in the quantum dynamics of two or few interacting spins.

The simplest coupled spin system we may conceive consists, of course, of two interacting spin 1/2's only in a dimer, isolated from its environment (the rest of the sample) degrees of freedom. Some binuclear copper(II) compounds (e.g., Refs. [4,5]) provide a possible scenario of this kind and, in the previous references, the values of the parameters characterizing the spin-spin interaction in such a molecule have been experimentally determined by exploiting electron-paramagnetic resonance techniques. Motivations to investigate the emergence of quantum signatures in the behavior of two coupled spins ($\geqslant 1/2$) go beyond the area of magnetic materials. Two spin-1/2 Hamiltonians provide indeed experimentally implementable, powerful, effective models to capture quantum properties of such systems as two coupled semiconductor quantum dots [6] or a pair of two neutral cold atoms each nested into two adjacent sites of an optical lattice made up of isolated double wells [7]. Spin models provide a successful language to investigate possible manipulations of the qubits aimed at quantum computing purposes and quantum information transfer between two-spin qubits [8], encompassing rather different physical contents such as, for example, cavity QED [9,10], superconductors [11,12], and trapped ions [13,14].

The most general Hamiltonian model of an isolated dimer hosting two spin 1/2's may be written as a bilinear form involving the two sets of operators $\{\hat{S}_1^x, \hat{S}_1^y, \hat{S}_1^z, \hat{S}_1^0\}$ and $\{\hat{S}_2^x, \hat{S}_2^y, \hat{S}_2^z, \hat{S}_2^0\}$, that is,

$$H = \sum_{(i,j)\neq(0,0)} \gamma_{ij} \hat{S}_1^i \otimes \hat{S}_2^j, \tag{1}$$

where i and j run in the set $(x,y,z,0)$ and the operator $\hat{S}_i^0$ ($i = 1,2$) is the identity operator $\mathbb{1}_i$ in $\mathcal{H}_i$. The six real parameters γ_{i0} and γ_{0j} ($i, j \neq 0$) are assumed to be generally time dependent, while all the other parameters characterizing the spin-spin coupling are real and time independent. Without further specific constraints on the 15 parameters γ_{ij}

($i,j = x,y,z,0$), the Hamiltonian possesses no symmetries and, in particular, it does not commute with the collective angular momentum operators $\hat{\mathbf{S}}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2$ and/or $\hat{S}^z = \hat{S}_1^z + \hat{S}_2^z$. In such a case, even if H is time independent, the four roots of the relative secular equation, albeit determinable, are rather involved functions of all the 15 parameters and then are practically not exploitable for extracting physical prediction on the physical system under scrutiny. Thus, either legitimated by investigations on specific physical situations or motivated by the interest in studying models possessing, by construction, constants of motion, some constraints on the parameters γ_{ij} have been introduced in the literature, making the Hamiltonian (1) less general and, at the same time, nontrivial and of physical interest. It is enough to quote the main declinations of the three-dimensional quantum Heisenberg models or the Dzyaloshinskii-Moriya (DM) models [15,16] in conjugation or not with simplified contributions to terms describing anisotropy effects in the Hamiltonian.

In this paper, we too investigate a Hamiltonian model included in (1), still general enough to remain not commutating with $\hat{\mathbf{S}}^2$ and $\hat{S}^z$, but such to possess a symmetry property at the origin of significant properties characterizing its quantum dynamics. A peculiar aspect of such a symmetry property is that it displays its usefulness even when we wish to study our physical system in a time-dependent scenario. Exploiting, indeed, the symmetry-induced existence of two dynamically invariant subspaces of $\mathcal{H}$, we are able to successfully apply a recently reported [17] systematic approach for generating exactly solvable quantum dynamics of a single spin 1/2 subjected to a time-dependent magnetic field. Thus the main result of this paper is twofold. First we report the exact explicit solution of the time-dependent Schrödinger equation of a system of two coupled spin 1/2's described by a time-dependent generalized Heisenberg model. Second, we demonstrate that the method reported in Ref. [17], even as it stands, proves to be a useful tool to treat more complex time-dependent scenarios.

The paper is organized as follows. The Hamiltonian model and the decoupling procedure are discussed in Sec. II, where, in addition, the structure of the time-evolution operator is also constructed with the help reported in Ref. [17]. In the subsequent Sec. III, some exactly solvable time-dependent Hamiltonian models of the two coupled qubits are singled out and analyzed. Sections IV and V are, respectively, dedicated to a systematic study of the time behavior of exemplary collective spin operators and of the concurrence. Some conclusive remarks are finally reported in the last Sec. VI.

II. THE HAMILTONIAN MODEL

The Hamiltonian model (1) includes all contributions stemming from internal or external couplings of our two spin-1/2 system. It may be cast in the following form:

$$H' = \mu_B(\mathbf{B}_1 \cdot \mathbf{g}_1 \cdot \mathbf{S}_1 + \mathbf{B}_2 \cdot \mathbf{g}_2 \cdot \mathbf{S}_2) + \mathbf{S}_1 \cdot \Gamma_{12} \cdot \mathbf{S}_2, \quad (2)$$

where $\mathbf{g}_1$, $\mathbf{g}_2$, and Γ_{12} are appropriate second-order Cartesian tensors whose entries are related to the 15 parameters appearing in Eq. (1) and μ_B denotes the Bohr magneton. Equation (2) mimics the usual way of representing the Hamiltonian used in a molecular or nuclear context to describe the coupling of two true spin 1/2's. In general, we may claim that $\mathbf{g}_1$ and $\mathbf{g}_2$ include possible corrections to the coupling terms between each spin and its local time-dependent external magnetic field, while the other term includes contact termlike couplings as well as anisotropiclike spin-spin couplings.

The model we are going to propose assumes in the laboratory frame that $\mathbf{B}_i(t) \equiv [0,0,B_i^z(t)]$, and

$$\Gamma_{12} = \begin{pmatrix} \gamma_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \gamma_{yy} & 0 \\ 0 & 0 & \gamma_{zz} \end{pmatrix}, \quad \mathbf{g}_i = \begin{pmatrix} g_i^{xx} & g_i^{xy} & 0 \\ g_i^{yx} & g_i^{yy} & 0 \\ 0 & 0 & g_i^{zz} \end{pmatrix}, \quad (3)$$

with ($i = 1,2$). The structure of $\mathbf{g}_i$ is, for example, appropriate when the dimer is a binuclear unit characterized by a C_2 symmetry with respect to the $\hat{z}$ axis [5].

In accordance with our previous assumptions, in this paper we investigate the quantum dynamics of the following time-dependent two-spin Hamiltonian model:

$$H = \hbar\omega_1\hat{\sigma}_1^z + \hbar\omega_2\hat{\sigma}_2^z + \gamma_{xx}\hat{\sigma}_1^x\hat{\sigma}_2^x + \gamma_{yy}\hat{\sigma}_1^y\hat{\sigma}_2^y + \gamma_{zz}\hat{\sigma}_1^z\hat{\sigma}_2^z + \gamma_{xy}\hat{\sigma}_1^x\hat{\sigma}_2^y + \gamma_{yx}\hat{\sigma}_1^y\hat{\sigma}_2^x, \quad (4)$$

where $\hat{\sigma}_i^x$, $\hat{\sigma}_i^y$, and $\hat{\sigma}_i^z$ ($i = 1,2$) are the Pauli matrices related to the respective components of the spin operator $\hat{\mathbf{S}}_i$ as

$$\hat{\mathbf{S}}_i = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}_i, \quad (5)$$

with $\hat{\boldsymbol{\sigma}}_i \equiv (\hat{\sigma}_i^x,\hat{\sigma}_i^y,\hat{\sigma}_i^z)$, while

$$\omega_i(t) = \frac{\mu_B g_i^{zz} B_i^z(t)}{2}. \quad (6)$$

Note that the identity operators $\mathbb{1}_i$ are and will mostly be suppressed for notational simplicity.

A. Symmetry-based decoupling of the two spins

Our Hamiltonian does not commute with $\hat{\mathbf{S}}^2$ and $\hat{S}^z$ but, by construction, it exhibits the following canonical and symmetry transformation:

$$\hat{\sigma}_i^x \to -\hat{\sigma}_i^x, \quad \hat{\sigma}_i^y \to -\hat{\sigma}_i^y, \quad \hat{\sigma}_i^z \to \hat{\sigma}_i^z, \quad i = 1,2. \quad (7)$$

This fact implies the existence of a unitary time-independent operator accomplishing the transformation (7), which is by construction a constant of motion. This unitary operator is given by $\pm\hat{\sigma}_1^z\hat{\sigma}_2^z$, being the transformation (7) is nothing but the rotations of π around the $\hat{z}$ axis with respect to each spin. The unitary operator accomplishing this transformation is

$$e^{i\pi\hat{S}_1^z/\hbar} \otimes e^{i\pi\hat{S}_2^z/\hbar} = -\hat{\sigma}_1^z\hat{\sigma}_2^z = \cos\left(\frac{\pi}{2}\hat{\Sigma}_z\right), \quad (8)$$

where $\hat{\Sigma}_z \equiv \hat{\sigma}_1^z + \sigma_2^z$. Equation (8) shows that the constant of motion $\hat{\sigma}_1^z\hat{\sigma}_2^z$ is indeed a $\hat{\Sigma}_z$-based parity operator since in correspondence to its integer eigenvalues $M = 0, \pm 2$, $\hat{\sigma}_1^z\hat{\sigma}_2^z$ has eigenvalues +1 and −1, respectively.

The existence of this constant of motion implies the existence of two subdynamics related to the two eigenvalues of $\hat{\sigma}_1^z\hat{\sigma}_2^z$. We can extract these two subdynamics by considering that the operator $\hat{\sigma}_1^z\hat{\sigma}_2^z$ has the same spectrum of $\hat{\sigma}_2^z$, i.e., the same eigenvalues (± 1) with the same twofold degeneracy. Therefore, there exists a unitary time-independent operator $\mathbb{U}$ transforming $\hat{\sigma}_1^z\hat{\sigma}_2^z$ in $\hat{\sigma}_2^z$. It can be easily seen that the unitary and Hermitian operator

$$\mathbb{U} = \frac{1}{2}\left[\mathbb{1} + \hat{\sigma}_1^z + \hat{\sigma}_2^x - \hat{\sigma}_1^z\hat{\sigma}_2^x\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{9}$$

in the standard ordered basis

$$\mathcal{B} = \{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\} \tag{10}$$

accomplishes the desired transformation:

$$\mathbb{U}^\dagger\hat{\sigma}_1^z\hat{\sigma}_2^z\mathbb{U} = \mathbb{U}\hat{\sigma}_1^z\hat{\sigma}_2^z\mathbb{U} = \hat{\sigma}_2^z. \tag{11}$$

Transforming H into $\tilde{H} = \mathbb{U}^\dagger H \mathbb{U}$, we get

$$\begin{aligned}\tilde{H} &= \hbar\omega_1\hat{\sigma}_1^z + \hbar\omega_2\hat{\sigma}_1^z\hat{\sigma}_2^z + \gamma_{zz}\hat{\sigma}_2^z + \gamma_{xx}\hat{\sigma}_1^x - \gamma_{yy}\hat{\sigma}_1^x\hat{\sigma}_2^z \\ &\quad + \gamma_{xy}\hat{\sigma}_1^y\hat{\sigma}_2^z + \gamma_{yx}\hat{\sigma}_1^y.\end{aligned} \tag{12}$$

It is easy to check that $\hat{\sigma}_2^z$ is a constant of motion of $\tilde{H}$ and that, consequently, $\tilde{H}$ may be represented as

$$\tilde{H} = \sum_{\sigma_2^z} \tilde{H}_{\sigma_2^z}|\sigma_2^z\rangle\langle\sigma_2^z| = \tilde{H}_+ \otimes |+\rangle\langle+| + \tilde{H}_- \otimes |-\rangle\langle-|, \tag{13}$$

where

$$\begin{aligned}\tilde{H}_{\sigma_2^z} &= \gamma_{zz}\sigma_2^z + \hbar(\omega_1 + \omega_2\sigma_2^z)\hat{\sigma}_1^z + (\gamma_{xx} - \gamma_{yy}\sigma_2^z)\hat{\sigma}_1^x \\ &\quad + (\gamma_{xy}\sigma_2^z + \gamma_{yx})\hat{\sigma}_1^y.\end{aligned} \tag{14}$$

This implies the existence of two ($\sigma_2^z = \pm 1$) subdynamics relative to a fictitious spin 1/2 immersed in different magnetic fields, each one possessing three components with the z one only depending on time.

B. Evolution operator in the presence of inhomogeneous time-dependent magnetic field

If ω_1 and ω_2 were time independent, it would be straightforward to find the eigenstates of $\tilde{H}$ as

$$|\tilde{\psi}\rangle = |\phi_{1i}\rangle_{\sigma_2^z} \otimes |\sigma_2^z\rangle \tag{15}$$

($i = 1,2$), where $|\phi_{1i}\rangle_{\pm 1}$ are the two eigenvectors of $\tilde{H}_\pm$, that is the two eigenvectors related to the subdynamics with $\sigma_2^z = \pm 1$. Through the relation

$$|\psi\rangle = \mathbb{U}|\tilde{\psi}\rangle, \tag{16}$$

we could in turn find the eigenvectors of H and the time evolution of an arbitrary state of the two spins.

When ω_1 and ω_2 depend on time, thanks to the fact that the unitary and Hermitian operator $\mathbb{U}$ is time independent, we succeed, in view of the structure possessed by $\tilde{H}$ as given by Eq. (12), in decoupling the time-dependent Schrödinger equation into two time-dependent Schrödinger equations of single spin 1/2. Therefore, we can construct the time-evolution operator of the whole dynamics of the two interacting spin 1/2's, starting from the construction of the two time-evolution operators of the two subdynamics of single spin 1/2. Indeed, the Cauchy problem for the evolution operator $\mathcal{U}$ generated by H: $i\hbar\dot{\mathcal{U}} = H\,\mathcal{U}$ with $\mathcal{U}(0) = \mathbb{1}$ is easily converted into the following two Cauchy problems related to the subdynamics associated to $\tilde{H}_+$ and $\tilde{H}_-$,

$$i\hbar\dot{\tilde{\mathcal{U}}}_\pm = \tilde{H}_\pm\tilde{\mathcal{U}}_\pm, \quad \tilde{\mathcal{U}}_\pm(0) = \mathbb{1}, \tag{17}$$

where $\tilde{\mathcal{U}} \equiv \mathbb{U}^\dagger\mathcal{U}\mathbb{U} \equiv \tilde{\mathcal{U}}_+ \otimes |+\rangle\langle+| + \tilde{\mathcal{U}}_- \otimes |-\rangle\langle-|$. If we are able to solve these two single spin-1/2 time-dependent Schrödinger equations, we then can construct

$$\mathcal{U} = \mathbb{U}\tilde{\mathcal{U}}\,\mathbb{U}^\dagger = \mathbb{U}\tilde{\mathcal{U}}\mathbb{U}. \tag{18}$$

The importance of this result consists in the possibility of applying the *Messina-Nakazato* approach [17] to each of the two subdynamics of single spin 1/2 to generate a class of time-dependent exactly solvable models whose Hamiltonian could be generally written as in (4). It is also important to stress that the procedure and the results are valid also if all the coupling constants γ were time dependent too, besides ω_1 and ω_2. To illustrate such a possibility, we solve in detail the quantum dynamics of the two coupled spins, taking advantage of some results reported in Ref. [17].

The two-dimensional matrix of each subdynamics we derived before can be written as follows:

$$\tilde{H}_\pm = \tilde{H}'_\pm \pm \gamma_{zz}\mathbb{1} = \begin{pmatrix} \Omega_\pm(t) & \Gamma_\pm \\ \Gamma_\pm^* & -\Omega_\pm(t) \end{pmatrix} \pm \gamma_{zz}\mathbb{1}, \tag{19}$$

where we have put

$$\begin{aligned}\Omega_\pm(t) &= \hbar[\omega_1(t) \pm \omega_2(t)], \\ \Gamma_\pm &= (\gamma_{xx} \mp \gamma_{yy}) - i(\pm\gamma_{xy} + \gamma_{yx}).\end{aligned} \tag{20}$$

Denoting by $\mathcal{E}_\pm$ the time-evolution operator generated by $\tilde{H}'_\pm$, the evolution operator generated by $\tilde{H}_\pm$ is simply given by

$$\tilde{\mathcal{U}}_\pm = e^{\mp i\gamma_{zz}t/\hbar}\mathcal{E}_\pm, \tag{21}$$

and therefore we will directly search the time-evolution operator $\mathcal{E}_\pm$ in accordance with Ref. [17]. Following the example given in section 3.3 of [17] and considering the case when the transverse component of the magnetic field is constant (because in our context all of the internal coupling coefficients are time independent), the time-evolution operator for each subdynamics of single fictitious spin 1/2 may be cast in the form

$$\mathcal{E}_\pm = \begin{pmatrix} |a_\pm|e^{i\phi_a^\pm} & |b_\pm|e^{i\phi_b^\pm} \\ -|b_\pm|e^{-i\phi_b^\pm} & |a_\pm|e^{-i\phi_a^\pm} \end{pmatrix}, \tag{22}$$

where

$$|a_\pm| = \cos\left\{\frac{|\Gamma_\pm|}{\hbar}\int_0^t \cos[\Theta_\pm(t')]dt'\right\}, \tag{23}$$

and, since $|a_\pm|^2 + |b_\pm|^2 = 1$,

$$|b_\pm| = \sin\left\{\frac{|\Gamma_\pm|}{\hbar}\int_0^t \cos[\Theta_\pm(t')]dt'\right\} \tag{24}$$

and

$$\phi_a^\pm = -\left(\frac{\Theta_\pm}{2} + \mathcal{R}_\pm\right), \quad \phi_b^\pm = -\frac{\Theta_\pm}{2} + \mathcal{R}_\pm - \frac{\pi}{2}, \tag{25}$$

with

$$\mathcal{R}_\pm = \frac{|\Gamma_\pm|}{\hbar}\int_0^t \frac{\sin\Theta_\pm}{\sin\left[\frac{2|\Gamma_\pm|}{\hbar}\int_0^{t'}\cos\Theta_\pm dt''\right]}dt'. \tag{26}$$

Here, $\Theta_\pm(t)$ are arbitrary, well-behaved mathematical functions fulfilling the condition $\Theta_\pm(0)=0$. Consistently, the longitudinal components of the effective magnetic fields of the two subdynamics vary over time so that

$$\Omega_\pm = \frac{\hbar}{2}\dot{\Theta}_\pm + |\Gamma_\pm|\sin\Theta_\pm \cot\left[\frac{2|\Gamma_\pm|}{\hbar}\int_0^t \cos\Theta_\pm dt'\right], \tag{27}$$

and we easily get the time dependence of ω_1 and ω_2 [and that of B_1^z and B_2^z through the relation (6)], resulting in

$$\omega_1 = \frac{\Omega_+ + \Omega_-}{2\hbar}, \quad \omega_2 = \frac{\Omega_+ - \Omega_-}{2\hbar}. \tag{28}$$

These equations practically single out the class of time-dependent Hamiltonians exactly treatable when the realistic assumption is made that the tensors Γ_{12}, $\mathbf{g}_1$, and $\mathbf{g}_2$ are time independent. Deriving the time dependence of $\mathcal{E}_\pm$, we finally get

$$\mathcal{U} = \begin{pmatrix} |a_+|e^{i\Phi_a^+} & 0 & 0 & |b_+|e^{i\Phi_b^+} \\ 0 & |a_-|e^{i\Phi_a^-} & |b_-|e^{i\Phi_b^-} & 0 \\ 0 & -|b_-|e^{-i\Phi_b^{'-}} & |a_-|e^{-i\Phi_a^{'-}} & 0 \\ -|b_+|e^{-i\Phi_b^{'+}} & 0 & 0 & |a_+|e^{-i\Phi_a^{'+}} \end{pmatrix}, \tag{29}$$

where we have put

$$\Phi_{a/b}^{\pm} = \phi_{a/b}^{\pm} \mp \frac{\gamma_{zz}}{\hbar}t, \tag{30a}$$

$$\Phi_{a/b}^{'\pm} = \phi_{a/b}^{\pm} \pm \frac{\gamma_{zz}}{\hbar}t. \tag{30b}$$

It is important to point out that if $\omega_1(t)=\omega_2(t)$,

$$\hat{\tilde{H}}_- = \begin{pmatrix} 0 & \Gamma_- \\ \Gamma_-^* & 0 \end{pmatrix}, \tag{31}$$

and then the relative evolution operator reads

$$\tilde{\mathcal{U}}_- = e^{+i\frac{\gamma_{zz}}{\hbar}t}\begin{pmatrix} \cos\left(\frac{|\Gamma_-|}{\hbar}t\right) & e^{i\Phi}\sin\left(\frac{|\Gamma_-|}{\hbar}t\right) \\ e^{+i\Phi}\sin\left(\frac{|\Gamma_-|}{\hbar}t\right) & \cos\left(\frac{|\Gamma_-|}{\hbar}t\right) \end{pmatrix}, \tag{32}$$

where $\Phi = \arctan(\frac{\gamma_{xx}+\gamma_{yy}}{\gamma_{yx}-\gamma_{xy}})$. In this instance, hence, the whole evolution operator of the initial dynamics becomes

$$\mathcal{U} = \begin{pmatrix} |a_+|e^{i\Phi_a^+} & 0 & 0 & |b_+|e^{i\Phi_b^+} \\ 0 & e^{+i\frac{\gamma_{zz}}{\hbar}t}\cos\left(\frac{|\Gamma_-|}{\hbar}t\right) & e^{i(\Phi+\frac{\gamma_{zz}}{\hbar}t)}\sin\left(\frac{|\Gamma_-|}{\hbar}t\right) & 0 \\ 0 & e^{-i(\Phi-\frac{\gamma_{zz}}{\hbar}t)}\sin\left(\frac{|\Gamma_-|}{\hbar}t\right) & e^{+i\frac{\gamma_{zz}}{\hbar}t}\cos\left(\frac{|\Gamma_-|}{\hbar}t\right) & 0 \\ -|b_+|e^{-i\Phi_b^{'+}} & 0 & 0 & |a_+|e^{-i\Phi_a^{'+}} \end{pmatrix}. \tag{33}$$

Of course, the evolution operator has the same form as that given by Eq. (29), where the two-by-two internal block is now completely determined regardless of the way H depends on time. This means that when $\omega_1(t)=\omega_2(t)=\omega(t)$ in the Hamiltonian model given in Eq. (4), the time evolutions of $|+-\rangle$ and $|-+\rangle$ (and so, of every linear combination of these states) are independent of $\omega(t)$ and are characterized by Bohr frequencies related to the coupling constants appearing in H.

It is useful to underline that the condition $\omega_1(t)=\omega_2(t)$ is not implied simply by the condition $\mathbf{B}_1(t)=\mathbf{B}_2(t)$ because, in general, we may have different $\mathbf{g}$ tensors (or factors) for the two spins which "rule" the coupling with the magnetic field and are responsible for the different effective local magnetic fields in the two sites, even when $\mathbf{B}_1(t)=\mathbf{B}_2(t)$. So, the more general condition implying $\omega_1(t)=\omega_2(t)$ is

$$\mathbf{B}_1(t)\cdot\mathbf{g}_1 = \mathbf{B}_2(t)\cdot\mathbf{g}_2. \tag{34}$$

III. EXACTLY SOLVABLE TIME-DEPENDENT SCENARIOS FOR THE TWO SPIN-1/2 MODEL

In this section, we report and discuss some particular time-dependent physical scenarios leading to exact analytical solutions for the time-evolution operator $\mathcal{U}$ by taking advantage of the approach described in the previous section. We notice that for such a determination, the knowledge of $\tilde{\mathcal{U}}_+$ and $\tilde{\mathcal{U}}_-$ is sufficient. This means that in practice our task is the resolution of two dynamical problems, each formally referred to as a spin 1/2. This points to the relevance of the method in Ref. [17].

The following two sections report two exact solutions of the quantum dynamics of a spin 1/2 based on such a method. In practice, to single out a treatable scenario amounts to engineering the time-dependent magnetic field acting upon the spin 1/2. Both scenarios are useful in our problem meaning that each of them allows the selection of appropriate time-dependent exactly solvable models for $\tilde{H}_+$ and $\tilde{H}_-$. The last section is dedicated to the explicit construction of the two-spin Hamiltonian models emerging from the intermediate steps leading to $\tilde{H}_+$ and $\tilde{H}_-$.

A. First exactly solvable time-dependent scenario for one spin 1/2

We may put

$$\frac{|\Gamma|}{\hbar}\int_0^t \cos\Theta dt' = \frac{1}{2}\arcsin[\tanh(\gamma t)], \quad \gamma = \frac{2|\Gamma|}{\hbar}. \tag{35}$$

With this choice, $|a|(|b|)$ goes from 1(0), at $t=0$, to $1/\sqrt{2}(1/\sqrt{2})$, as $t\to\infty$. Indeed, we have

$$|a(t)| = \sqrt{\frac{\cosh(\gamma t)+1}{2\cosh(\gamma t)}}, \quad |b(t)| = \sqrt{\frac{\cosh(\gamma t)-1}{2\cosh(\gamma t)}}. \tag{36}$$

Moreover, we have

$$\cos\Theta(t) = \frac{1}{\cosh(\gamma t)}, \quad \sin\Theta(t) = \tanh(\gamma t), \tag{37}$$

and the integral $\mathcal{R}$ is trivially integrated to yield

$$\mathcal{R} = \frac{\gamma}{2}t. \tag{38}$$

From (37), we derive

$$\dot{\Theta} = \frac{\gamma}{\cosh(\gamma t)}, \quad \Theta = 2\arctan\left[\tanh\left(\frac{\gamma}{2}t\right)\right], \tag{39}$$

so that we get

$$\phi_a = -\arctan\left[\tanh\left(\frac{\gamma}{2}t\right)\right] - \frac{\gamma}{2}t = \phi_b - \gamma t + \frac{\pi}{2}, \tag{40}$$

and the longitudinal component of the magnetic field,

$$\Omega = 2|\Gamma|\frac{1}{\cosh(\gamma t)}. \tag{41}$$

The plot of $\frac{\Omega}{|\Gamma|}$ is shown in Fig. 1 as a function of $\tau_1 = \frac{2|\Gamma|}{\hbar}t$.

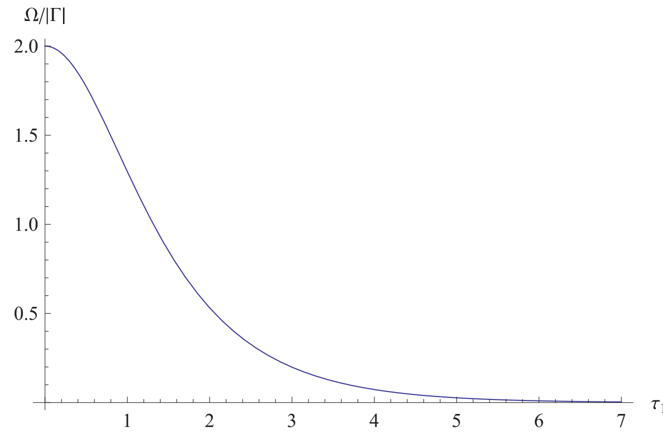

FIG. 1. Plot of $\frac{\Omega(\tau_1)}{|\Gamma|}$ according to Eq. (41).

B. Second exactly solvable time-dependent scenario for one spin 1/2

We have a monotonically decreasing trend of the function $|a(t)|$ also by putting

$$\frac{|\Gamma|}{\hbar}\int_0^t \cos\Theta dt' = \arcsin[\tanh(\gamma t)], \quad \gamma = \frac{|\Gamma|}{\hbar}, \tag{42}$$

which, in view of Eqs. (23) and (24), implies

$$|a(t)| = \frac{1}{\cosh(\gamma t)}, \quad |b(t)| = \tanh(\gamma t). \tag{43}$$

In this case, thus, $|a|$ ($|b|$) varies from 1(0), at $t=0$, to 0(1) when $t\to\infty$, realizing a perfect inversion of the spin. The expressions of $\cos\Theta(t)$ and $\sin\Theta(t)$ are the same as those in (37) of the previous case and so also $\dot{\Theta}$ and Θ have the same expressions as those given in (39) (though the definition of γ is different in the two cases). What is different is the value of integral $\mathcal{R}$ which, in this case, results in

$$\mathcal{R} = \tfrac{1}{2}\sinh(\gamma t), \tag{44}$$

and for the phases of a and b, we have

$$\phi_a = -\arctan\left[\tanh\left(\frac{\gamma}{2}t\right)\right] - \frac{1}{2}\sinh(\gamma t),$$
$$\phi_b = \phi_a + \sinh(\gamma t) - \frac{\pi}{2}. \tag{45}$$

With this choice, the longitudinal component of the magnetic field must be engineered as

$$\Omega = \frac{|\Gamma|}{2}\left[\frac{3}{\cosh(\gamma t)} - \cosh(\gamma t)\right]. \tag{46}$$

Figure 2 shows the behavior of $\frac{\Omega}{|\Gamma|}$ in this case against $\tau_2 = \frac{|\Gamma|}{\hbar}t$.

It is important to point out that the factor 1/2 (1) multiplying the function $\arcsin[\tanh(\gamma t)]$ in Eq. (35) [(42)] is crucial for the possibility of exactly getting the integral $\mathcal{R}$. Furthermore, such a factor has a remarkable consequence in the time dependence of $|a|$, $|b|$, and Ω in the first and second scenarios. We saw, indeed, that the asymptotic ($t\to\infty$) values of $|a|$ and $|b|$ are very different in the two cases determining a completely different dynamical evolution in time. Finally, as we can see from Fig. 2, the multiplying factor significantly determines the time trend of the longitudinal component of the magnetic field which must be engineered appropriately to have the exact dynamics we are studying.

C. Time-dependent scenarios for the two-spin model

In closing this section, we emphasize the significance of our results by explicitly giving all the time dependences

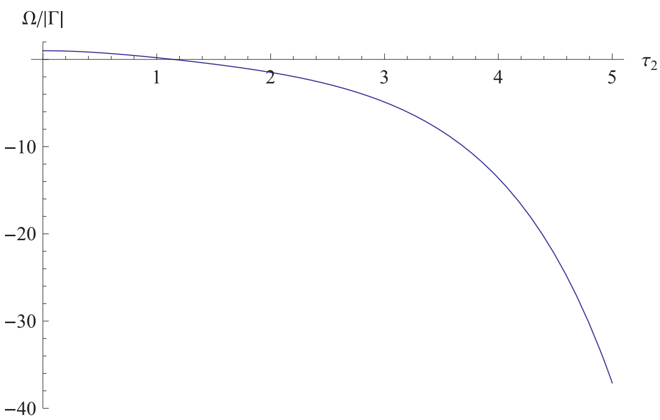

FIG. 2. Plot of $\frac{\Omega(\tau_2)}{|\Gamma|}$ according to Eq. (46).

[constructed on the basis of Eq. (28)] of ω_1 and ω_2 [and so of the two magnetic fields B_1^z ans B_2^z in view of Eq. (6)] in the two-spin Hamiltonian model (4) leading to exactly solvable and solved models. If we are interested in studying the time evolution of an initial state that belongs to one of the two dynamically invariant subspaces of H, wherein the dynamics is described by $\tilde{H}_+$ or $\tilde{H}_-$, we get classes of time-dependent scenarios which can be treated and solved exactly. Precisely, if we consider, e.g., the subdynamics characterized by $\sigma_1^z\sigma_2^z = 1$ and described by $\tilde{H}_+$, the two classes of time-dependent exactly solvable problems of two spins interacting according to our model in (4) are given by

$$\hbar[\omega_1(t)+\omega_2(t)] = \frac{2|\Gamma_+|}{\cosh\left(\frac{2|\Gamma_+|}{\hbar}t\right)}, \tag{47a}$$

$$\hbar[\omega_1(t)+\omega_2(t)] = \frac{|\Gamma_+|}{2}\left[\frac{3}{\cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)} - \cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)\right]. \tag{47b}$$

Equations (47) make clear the reason why we are talking of classes of time-dependent exactly solvable models. Indeed, we see that we have different possible choices of the two magnetic fields B_1^z and B_2^z such that their combination, in accordance with Eq. (6), satisfies one of the previous conditions, i.e., getting different time-dependent scenarios in which we are able to know the dynamics exactly. Obviously, we also have the analogous situation for the other subdynamics characterized by $\sigma_1^z\sigma_2^z = -1$ and described by $\tilde{H}_-$. In this case, the classes of exactly solvable models are due by the conditions

$$\hbar[\omega_1(t)-\omega_2(t)] = \frac{2|\Gamma_-|}{\cosh\left(\frac{2|\Gamma_-|}{\hbar}t\right)}, \tag{48a}$$

$$\hbar[\omega_1(t)-\omega_2(t)] = \frac{|\Gamma_-|}{2}\left[\frac{3}{\cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)} - \cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)\right]. \tag{48b}$$

We stress that Eqs. (48) are not compatible with the situation corresponding to $\omega_1(t) = \omega_2(t)$ for which, on the other hand, the quantum dynamics in the subspace under scrutiny has been completely solved as explicitly given by Eq. (33). In other words, Eqs. (48) display their usefulness, generating exactly solvable time-dependent Hamiltonian models of the two spins, only when $\omega_1(t) \neq \omega_2(t)$. If we look, instead, at the entire dynamics of the two interacting spin 1/2's, considering a general initial condition belonging to the total four-dimensional Hilbert space $\mathcal{H}$, we have the following four exactly solvable time-dependent cases [+(−) in ± corresponds to 1(2)]:

$$\hbar\omega_{1/2}(t) = \frac{|\Gamma_+|}{\cosh\left(\frac{2|\Gamma_+|}{\hbar}t\right)} \pm \frac{|\Gamma_-|}{\cosh\left(\frac{2|\Gamma_-|}{\hbar}t\right)}, \tag{49a}$$

$$\hbar\omega_{1/2}(t) = \frac{|\Gamma_+|}{\cosh\left(\frac{2|\Gamma_+|}{\hbar}t\right)} \pm \frac{|\Gamma_-|}{4}\left[\frac{3}{\cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)} - \cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)\right], \tag{49b}$$

$$\hbar\omega_{1/2}(t) = \frac{|\Gamma_+|}{4}\left[\frac{3}{\cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)} - \cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)\right] \pm \frac{|\Gamma_-|}{\cosh\left(\frac{2|\Gamma_-|}{\hbar}t\right)}, \tag{49c}$$

$$\hbar\omega_{1/2}(t) = \frac{|\Gamma_+|}{4}\left[\frac{3}{\cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)} - \cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)\right] \pm \frac{|\Gamma_-|}{4}\left[\frac{3}{\cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)} - \cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)\right]. \tag{49d}$$

For example, if we consider the time-dependent scenario given by Eq. (49a) with a particular choice,

$$\gamma_x = \gamma_y = 2\gamma_{xy} = 2\gamma_{yx} = c, \tag{50}$$

we have, in view of Eqs. (20) and (35),

$$|\Gamma_+| = c, \quad |\Gamma_-| = 2c, \quad \gamma_+ = \frac{2c}{\hbar}, \quad \gamma_- = \frac{4c}{\hbar}, \tag{51}$$

and the time behavior of $\frac{\hbar\omega_1}{c}$ and $\frac{\hbar\omega_2}{c}$ in terms of $\tau_c = \frac{ct}{\hbar}$ is seen in Fig. 3 ($\hbar = 1$).

The above four cases, therefore, provide $\omega_1(t)$ and $\omega_2(t)$ [and, consequently, the magnetic fields $B_1^z(t)$ and $B_2^z(t)$] such that our corresponding time-dependent Hamiltonian model given by Eq. (4) turns out to be exactly solvable and the related global time-evolution operator $\mathcal{U}$, given by Eq. (29), can be derived by plugging Eqs. (36) and (40), or (43) and (45), in place of $\{|a_+|,|b_+|,\phi_a^+,\phi_b^+\}$ or of $\{|a_-|,|b_-|,\phi_a^-,\phi_b^-\}$ at will, depending on which of the four time-dependent scenarios, given in Eqs. (49), we choose.

We notice that when $g_1^{zz} \neq g_2^{zz}$, Eqs. (49), by specializing Eq. (4), generate time-dependent Hamiltonian models of the two spins which cannot exactly be solved when the same external homogeneous magnetic field $B_1^z(t) = B_2^z(t)$ is applied on the two spins. In such a case, indeed, the time dependence of the magnetic field determined from Eq. (47) is incompatible with that of the same magnetic field derivable from Eq. (48). However, it is of relevance to point out that in this case, if we choose the time dependence of the unique magnetic field derived from either Eqs. (47a) or (47b) [(48a) or (48b)], we get

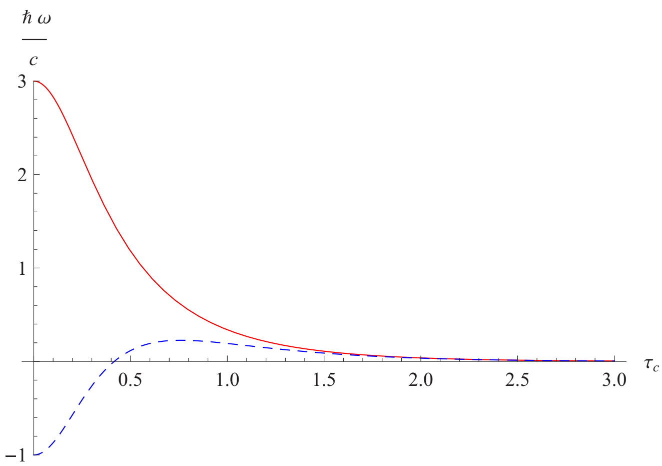

FIG. 3. Plots of $\frac{\hbar\omega_1}{c}$ (red solid line) and $\frac{\hbar\omega_2}{c}$ (blue dashed line), in terms of $\tau = \frac{ct}{\hbar}$, according to the time-dependent model satisfying Eqs. (49a) and (51).

particular time-dependent Hamiltonian models for which we are able to solve exactly the subdynamics in the subspace singled out by the condition $\sigma_1^z\sigma_2^z = 1$ ($\sigma_1^z\sigma_2^z = -1$). It is finally useful to underline that when the physical system may be described assuming $g_1^{zz} = g_2^{zz}$, a homogeneous time-dependent magnetic field, as derivable from either Eq. (47a) or (47b), leads to $\omega_1(t) = \omega_2(t)$, whose implications on the two-spin quantum dynamics have already been discussed after Eq. (30).

IV. DYNAMICAL PROPERTIES OF THE TWO SPIN-1/2 MODEL

In the previous sections, we have built exactly solvable models for two coupled spin 1/2's and solved them as well. This result is important for two reasons. The first one is that it shows that the systematic route reported in Ref. [17] may be successfully applied to physical systems living in an f-dimensional Hilbert space with $f > 2$. The second one is related to the construction of new time-dependent exactly solvable Hamiltonian models on its own, since solutions of such problems, generally speaking, are very rare. Thus we are going to exploit the knowledge of the solutions we have found, in Sec. III A for the first time-dependent scenario, and in Sec. III B for the second time-dependent scenario, to investigate physical properties exhibited by our two-spin system under the corresponding engineered magnetic fields.

A. Quantum evolution in the dynamically invariant subspace with parity + and −

In the subspace where the constant of motion $\hat{S}_1^z\hat{S}_2^z$ assumes the value $\frac{\hbar^2}{4}$ with certainty, in view of Eq. (29), the initial states [as a rule, the upper (lower) sign corresponds to α (β) here and in what follows]

$$|\psi_{\alpha/\beta}^+(0)\rangle \equiv |\pm\pm\rangle \tag{52}$$

at time t, respectively, become

$$|\psi_\alpha^+(t)\rangle = |a_+|e^{i\Phi_a^+}|++\rangle - |b_+|e^{-i\Phi_b^{\prime+}}|--\rangle, \tag{53a}$$

$$|\psi_\beta^+(t)\rangle = |b_+|e^{i\Phi_b^+}|++\rangle + |a_+|e^{-i\Phi_a^{\prime+}}|--\rangle. \tag{53b}$$

Since both $|++\rangle$ and $|--\rangle$ are eigenvectors of $\hat{\mathbf{S}}^2$, the subspace they span pertains to the quantum number $S = 1$ of such collective observable. It is worthwhile to observe that the magnetization $\langle\hat{S}^z(t)\rangle_{\alpha/\beta}$ of the system in this subspace is not known with certainty. Indeed, the mean value of $\hat{S}^z \equiv \hat{S}_1^z + \hat{S}_2^z$ on the states $|\psi_\alpha^+(t)\rangle$ and $|\psi_\beta^+(t)\rangle$ may be respectively cast as follows:

$$\langle\hat{S}^z(t)\rangle_\alpha \equiv \langle\psi_\alpha^+(t)|\hat{S}^z|\psi_\alpha^+(t)\rangle = \hbar(|a_+|^2 - |b_+|^2), \tag{54a}$$

$$\langle\hat{S}^z(t)\rangle_\beta \equiv \langle\psi_\beta^+(t)|\hat{S}^z|\psi_\beta^+(t)\rangle = -\hbar(|a_+|^2 - |b_+|^2), \tag{54b}$$

where $|a_+(t)|$ and $|b_+(t)|$ appear as entries of the matrix for $\mathcal{U}$.

1. First time-dependent scenario

As soon as $\omega_1(t)$ and $\omega_2(t)$ satisfy Eq. (47a), we get the following magnetization time dependences:

$$\langle\hat{S}^z(t)\rangle_{\alpha/\beta} = \pm\frac{\hbar}{\cosh\left(\frac{2|\Gamma_+|}{\hbar}t\right)}. \tag{55}$$

It should be appreciated that the recipe provided by Eq. (47a) enables, in principle, the construction of infinitely many time-dependent Hamiltonian models, all of them exactly predicting such an evolution of the magnetization of the coupled two-spin system. In Fig. 4, we plot such time dependences (for α and β, in units of $\hbar$) against the dimensionless time $\frac{2|\Gamma_+|}{\hbar}t$.

The common asymptotic value of $\langle\hat{S}^z(t)\rangle_\alpha$ and $\langle\hat{S}^z(t)\rangle_\beta$ can be understood by noticing that $|\psi_\alpha^+(t)\rangle$ and $|\psi_\beta^+(t)\rangle$, up to inessential phase factors, for large t evolve into the following entangled Bell states of the two spins:

$$|\psi_{\alpha/\beta}^+(t)\rangle \to e^{\mp i\left(\frac{\pm\gamma_{zz}+|\Gamma_+|}{\hbar}t+\frac{(2\mp1)\pi}{4}\right)}\frac{|++\rangle \pm |--\rangle}{\sqrt{2}}, \tag{56}$$

with both having vanishing average magnetizations for $t \to \infty$. Equations (56) clearly evidence that engineering the

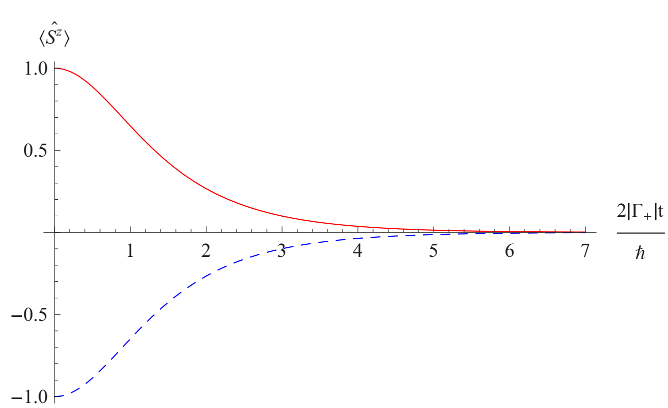

FIG. 4. Time dependence of $\langle\hat{S}^z(t)\rangle_\alpha$ (red solid line) and $\langle\hat{S}^z(t)\rangle_\beta$ (blue dashed line), in units of $\hbar$, in the subdynamics with parity + for the class of time-dependent models characterized by Eq. (47a).

magnetic fields on the two spins, in accordance with the constraint imposed by Eq. (47a), might be, in principle, at the heart of many possible experimental schemes successfully exploitable for generating such fully entangled states, whatever the internal coupling coefficients appearing in the Hamiltonian model, given by Eq. (4), are.

2. *Second time-dependent scenario*

The time dependences of $\langle \hat{S}^z(t)\rangle_{\alpha/\beta}$ in the subdynamics with parity + stemming from the class of time-dependent models characterized by Eq. (47b) takes, instead, the following forms:

$$\langle \hat{S}^z(t)\rangle_{\alpha/\beta} = \pm\hbar\left[\frac{2}{\cosh^2\left(\frac{|\Gamma_+|}{\hbar}t\right)} - 1\right], \tag{57}$$

which are plotted (in units of $\hbar$) in Fig. 5 as functions of $\frac{|\Gamma_+|}{\hbar}t$.

In this case, a gradual inversion of the magnetization of the system occurs due to the fact that as $t \to \infty$, we have a perfect inversion of the probability of finding the two spins in the state $|++\rangle$ ($|--\rangle$) when its initial state is $|--\rangle$ ($|++\rangle$). The asymptotic states for $t \to \infty$, in this scenario, are, indeed

$$|\psi_\alpha^+(t)\rangle \to e^{-i(\Phi_b^{'+}+\pi)}|--\rangle, \quad |\psi_\beta^+(t)\rangle \to e^{i\Phi_b^+}|++\rangle, \tag{58}$$

implying immediately that

$$|\langle\psi^+_{\alpha/\beta}(\infty)|\psi^+_{\alpha/\beta}(0)\rangle|^2 = 0. \tag{59}$$

The subspace with parity − is invariant for $\hat{S}^z$, while the mean value of $\hat{\mathbf{S}}^2$ evolves in time running in the interval $[0,2\hbar^2]$. It is easy to convince oneself that preparing the two-spin system in the state $|\psi^-_{\alpha/\beta}(0)\rangle \equiv |\pm\mp\rangle$, we get

$$\langle\hat{\mathbf{S}}^2(t)\rangle_{\alpha/\beta} = \hbar^2\left[1 \pm \tanh^2\left(\frac{2|\Gamma_-|}{\hbar}t\right)\right] \tag{60}$$

in the first time-dependent scenario (48a), and

$$\langle\hat{\mathbf{S}}^2(t)\rangle_{\alpha/\beta} = \hbar^2\left[1 \pm \frac{2\tanh^2\left(\frac{|\Gamma_-|}{\hbar}t\right)}{\cosh\left(\frac{|\Gamma_-|}{\hbar}t\right)}\right] \tag{61}$$

in the second time-dependent scenario (48b). These two time evolutions are graphically represented (in units of $\hbar^2$) in Figs. 6 and 7 against the dimensionless times $\frac{2|\Gamma_-|}{\hbar}t$ and $\frac{|\Gamma_-|}{\hbar}t$, respectively.

Figure 6 suggests that the two spins tend toward states identifiable as eigenstates of $\hat{\mathbf{S}}^2$ of eigenvalue $S = 1$, $|S = 1, M = 0\rangle$, and $S = 0$, $|S = 0, M = 0\rangle$, in correspondence to α and β, respectively. Thus, whatever $\omega_1(t) \neq \omega_2(t)$, fulfilling Eq. (48a), are, the envisioned time-dependent scenario under scrutiny leads to the generation of the following Bell states:

$$|\psi^-_{\alpha/\beta}(t)\rangle \to e^{\mp i\left(\frac{\mp\gamma_{zz}+|\Gamma_-|}{\hbar}t+\frac{(2\mp1)\pi}{4}\right)}\frac{|+-\rangle \pm |-+\rangle}{\sqrt{2}}. \tag{62}$$

Even in this case, then, our results, as expressed by Eq. (62), might provide implementable experimental strategies for the generation of two maximally entangled states $|S = 1, M = 0\rangle$ and $|S = 0, M = 0\rangle$.

On the other hand, the asymptotic trends of the two curves in Fig. 7 reflect the $t \to \infty$ asymptotic states,

$$|\psi_\alpha^-(t)\rangle \to e^{-i(\Phi_b^{'-}+\pi)}|-+\rangle, \quad |\psi_\beta^-(t)\rangle \to e^{i\Phi_b^-}|+-\rangle, \tag{63}$$

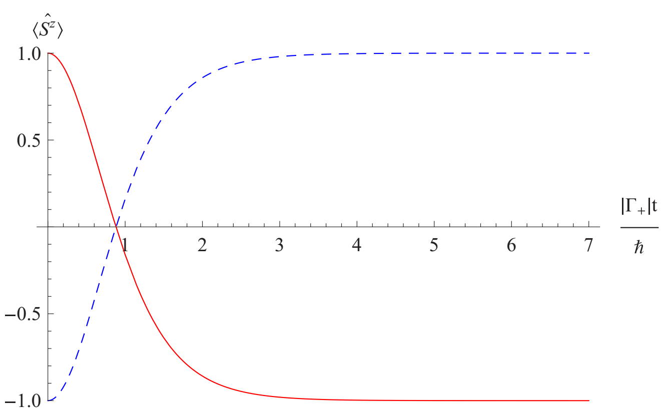

FIG. 5. Time dependence of $\langle\hat{S}^z(t)\rangle_\alpha$ (red solid line) and $\langle\hat{S}^z(t)\rangle_\beta$ (blue dashed line), in units of $\hbar$, in the subdynamics with parity + for the class of time-dependent models characterized by Eq. (47b).

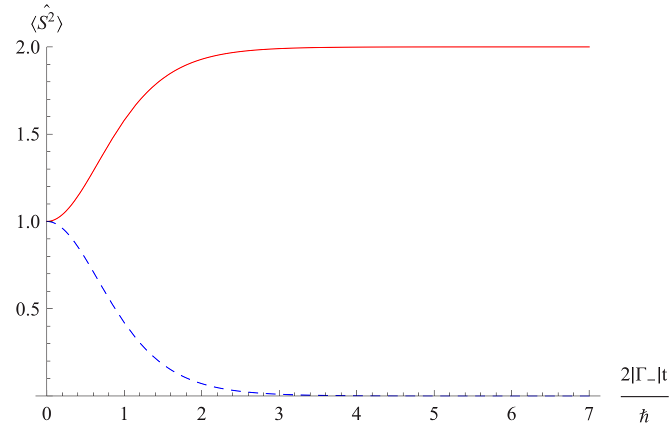

FIG. 6. Time dependences of $\langle\hat{\mathbf{S}}^2(t)\rangle_\alpha$ (red solid line) and $\langle\hat{\mathbf{S}}^2(t)\rangle_\beta$ (blue dashed line), in units of $\hbar^2$, in the subdynamics with parity − for the class of time-dependent models characterized by Eq. (48a).

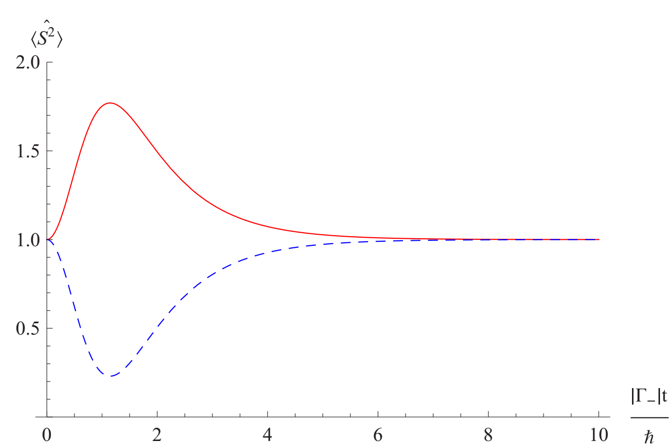

FIG. 7. Time dependences of $\langle\hat{\mathbf{S}}^2(t)\rangle_\alpha$ (red solid line) and $\langle\hat{\mathbf{S}}^2(t)\rangle_\beta$ (blue dashed line), in units of $\hbar^2$, in the subdynamics with parity − for the class of time-dependent models characterized by Eq. (48b).

which means that under the strategy dictated by Eq. (48b), the initial state $|+-\rangle$ is converted into the state $|-+\rangle$, while $|-+\rangle$ undergoes the analogous complete inversion.

B. Quantum dynamics from an arbitrary initial condition

In this last section, we report the time evolution of a generic initial state in $\mathcal{H}$,

$$|\psi(0)\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle, \tag{64}$$

generated by one of the Hamiltonian models given in Eqs. (49). It is easy to show that the expression of $|\psi(t)\rangle \equiv \mathcal{U}|\psi(0)\rangle$ can be cast as follows:

$$|\psi(t)\rangle = c_{++}(t)|++\rangle + c_{+-}(t)|+-\rangle + c_{-+}(t)|-+\rangle + c_{--}(t)|--\rangle, \tag{65}$$

with

$$c_{\pm\pm}(t) = e^{-i\frac{\gamma_{zz}}{\hbar}t}(|a_+(t)|e^{\pm i\phi_a^+(t)}c_{\pm\pm} \pm |b_+(t)|e^{\pm i\phi_b^+(t)}c_{\mp\mp}), \tag{66a}$$

$$c_{\pm\mp}(t) = e^{+i\frac{\gamma_{zz}}{\hbar}t}(|a_-(t)|e^{\pm i\phi_a^-(t)}c_{\pm\mp} \pm |b_-(t)|e^{\pm i\phi_b^-(t)}c_{\mp\pm}), \tag{66b}$$

where $|a_\pm(t)|$, $\phi_a^\pm(t)$, $|b_\pm(t)|$, and $\phi_b^\pm(t)$ appear as entries in the matrix representation of the evolution operator $\mathcal{U}$ (29) generated by the specific two-spin Hamiltonian model under scrutiny [that is, determined by one of Eqs. (49)].

The time evolutions of $\hat{\mathbf{S}}^2(t)$ as well as of $\hat{S}^z$ exhibit no interference terms stemming from the presence of states of different parity in (65). For example, in view of Eqs. (54), we have

$$\langle\psi(t)|\hat{S}^z|\psi(t)\rangle = \hbar(|c_{++}(t)|^2 - |c_{--}(t)|^2) \tag{67}$$

because the mean value of $\hat{S}^z$ in any state of negative parity identically vanishes. It is thus interesting to evaluate the time evolution of the mean value of an observable which has nonvanishing matrix elements between states of different parities, for example $\hat{S}^x = \hat{S}_1^x + \hat{S}_2^x$. We limit ourselves to an exemplary case, namely, the one obtained by choosing H with $\omega_1(t)$ and $\omega_2(t)$ as prescribed in Eq. (49a) and the amplitudes of $|\psi(0)\rangle$ real and such to make the initial state a common eigenstate of $\hat{\mathbf{S}}^2$ and $\hat{S}^x$ with maximum eigenvalues, that is,

$$c_{++} = c_{+-} = c_{-+} = c_{--} = \tfrac{1}{2}. \tag{68}$$

Equation (65) immediately yields

$$\begin{aligned}\langle\hat{S}^x(t)\rangle &\equiv \langle\psi(t)|\hat{S}^x|\psi(t)\rangle\\ &= \hbar\Big\{\cos\left(\frac{2\gamma_{zz}}{\hbar}t\right)[|a_+||a_-|\cos(\phi_a^+)\cos(\phi_a^-)\\ &\quad + |b_+||b_-|\sin(\phi_b^+)\sin(\phi_b^-)] + \sin\left(\frac{2\gamma_{zz}}{\hbar}t\right)\\ &\quad \times[|a_+||b_-|\cos(\phi_a^+)\sin(\phi_b^-)\\ &\quad - |a_-||b_+|\cos(\phi_a^-)\sin(\phi_b^+)]\Big\},\end{aligned} \tag{69}$$

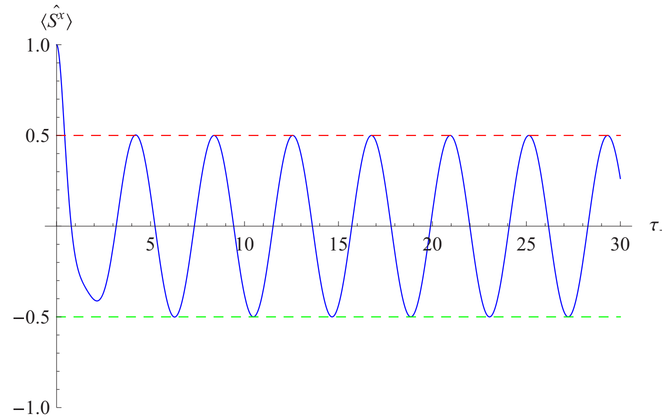

FIG. 8. Plot of $\langle\hat{S}^x(t)\rangle$ (blue solid line), in units of $\hbar$, starting from $c_{++} = c_{+-} = c_{-+} = c_{--} = \frac{1}{2}$ according to the time-dependent scenario (49a) and the special choice in Eqs. (50) and (51); the red upper (green lower) dashed straight line represents $\langle\hat{S}^x(t)\rangle = \frac{1}{2}$ ($\langle\hat{S}^x(t)\rangle = -\frac{1}{2}$).

with $[|a_\pm(\tau_\pm)| = X_\pm^{(1)}(\tau_\pm), |b_\pm(\tau_\pm)| = X_\pm^{(2)}(\tau_\pm)]$, and

$$\begin{aligned}|X_\pm^{(i)}(\tau_\pm)| &= \sqrt{\frac{\cosh(\tau_\pm) - (-1)^i}{2\cosh(\tau_\pm)}},\\ \phi_a^\pm(\tau_\pm) &= -\arctan\left[\tanh\left(\frac{\tau_\pm}{2}\right)\right] - \frac{\tau_\pm}{2}\\ &= \phi_b^\pm(\tau_\pm) - \tau_\pm + \frac{\pi}{2},\end{aligned} \tag{70}$$

where $\tau_\pm = \gamma_\pm t$ and $\gamma_\pm = \frac{2|\Gamma_\pm|}{\hbar}$, according to the time-dependent scenario (49a) (that is, the "first time-dependent scenario" for each subdynamics). The plot of $\langle\hat{S}^x(t)\rangle$, in this instance, and considering the special case characterized by Eqs. (50) and (51), is given (in units of $\hbar$) in Fig. 8 against τ_+. It is possible to understand the peculiar behavior for large t, characterized evidently by one frequency, by deriving analytically the asymptotic expression of $\langle\hat{S}^x(t)\rangle$, which, indeed, acquires the following clear form:

$$\langle\hat{S}^x(t)\rangle = \frac{\hbar}{2}\cos\left[\left(\frac{\gamma_{zz}}{|\Gamma_+|} + \frac{|\Gamma_-|}{2|\Gamma_+|} - \frac{1}{2}\right)\tau_+\right]. \tag{71}$$

V. CONCURRENCE IN THE TWO SUBDYNAMICS

Spurred by the results of the previous section, which, in particular cases, allow a direct and first-glance comparison between the initial level of entanglement with that stored in the asymptotic states, in this section we are going to derive and analyze the exact time-evolution law of the entanglement established in the two-spin system when it is initially prepared in the generic state given by Eq. (64). For a pair of qubits, a good measure of entanglement is the concurrence C introduced by Wootters [18] as well as the negativity, introduced by Vidal and Werner [19], which in a generic state coincides with C [20]. At a generic time instant t, it may be expressed as

$$C(t) = 2|c_{++}(t)c_{--}(t) - c_{+-}(t)c_{-+}(t)|, \tag{72}$$

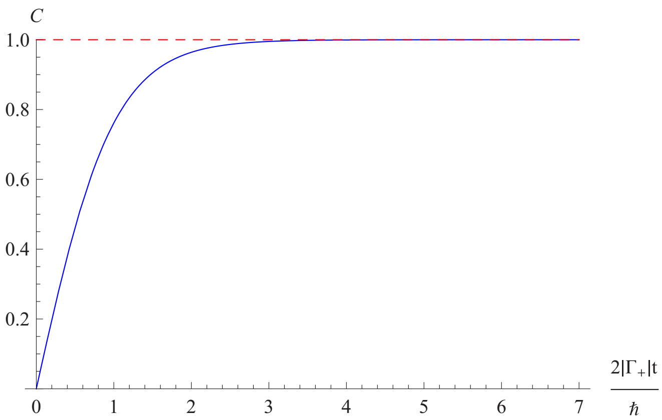

FIG. 9. Plot of $C(t)$ starting from $|{+}{+}\rangle$ when the time-dependent scenario (47a) is adopted.

where the four time-dependent coefficients are the complex amplitudes of the normalized state $|\psi(t)\rangle$, into which $|\psi(0)\rangle$ evolves, and are given in Eqs. (66). As expected, when the system starts in a state of definite parity, Eq. (72) yields $C(t) = 2|c_{++}(t)c_{--}(t)|$ $(= 2|c_{+-}(t)c_{-+}(t)|)$ for parity $+(-)$. When $c_{++}(0) = 1$ or $c_{--}(0) = 1$ and the first time-dependent scenario for this subdynamics is assumed, that is (47a), the concurrence results in

$$C(t) = 2|a(t)||b(t)| = \tanh\left(\frac{2|\Gamma_+|}{\hbar}t\right), \tag{73}$$

whose plot is reported in Fig. 9 against $\tau_+ = \frac{2|\Gamma_+|}{\hbar}t$. The asymptotic behavior of $C(t)$ in this case is easily understood in view of Eqs. (56).

Considering, instead, $c_{++}(0) = c_{--}(0) = \frac{1}{\sqrt{2}}$, still together with (47a), we obtain

$$C(t) = \sqrt{1 - \tanh^2\left(\frac{2|\Gamma_+|}{\hbar}t\right)\sin^2\left(\frac{2|\Gamma_+|}{\hbar}t\right)}, \tag{74}$$

which is plotted in Fig. 10 against $\tau_+ = \frac{2|\Gamma_+|}{\hbar}t$.

It shows that after a transient regime ($\tau_+ < 2\pi$), the concurrence oscillates between 0 and 1 as $|\cos(\tau_+)|$, which is immediately deduced from Eq. (74) for large t. The meaning of this behavior is that the system periodically evolves alternating factorized states and the Bell states. To understand and better appreciate this statement quantitatively, we exploit Eqs. (56) to recover the asymptotic expression of $|\psi(t)\rangle$ $(\gamma_+ = \frac{2|\Gamma_+|}{\hbar})$,

$$|\psi(t)\rangle = e^{-i\frac{\gamma_{zz}}{\hbar}t}\left[e^{-i\frac{\pi}{2}}\sin\left(\frac{\gamma_+}{2}t + \frac{\pi}{4}\right)|{+}{+}\rangle + \cos\left(\frac{\gamma_+}{2}t + \frac{\pi}{4}\right)|{-}{-}\rangle\right], \tag{75}$$

which clearly exhibits oscillations of period $T_+ = \frac{4\pi}{\gamma_+} = \frac{2\pi\hbar}{|\Gamma_+|}$ in accordance with the asymptotic expression of $C(t)$. Equation (75) easily explains the oscillations exhibited by $C(t)$ since it predicts that the two-spin system, up to a global phase factor, comes back to its initial condition and, after a time $\frac{T_+}{4}$ $(\frac{T_+}{2})$, it reaches the factorized (Bell-like) state $|{+}{+}\rangle$ $(\frac{-i|{+}{+}\rangle - |{-}{-}\rangle}{\sqrt{2}})$, whereas in the last semiperiod, it reaches first the factorized state $|{-}{-}\rangle$ and eventually its initial state. The reason why the concurrence does not vanish in this case in the transient region stems from the fact that $C(t) = 0$ necessarily implies $c_{++}(t) = 0$ or $c_{--}(t) = 0$. In view of the structure of the two amplitudes $c_{++}(t)$ and $c_{--}(t)$ [see Eqs. (66a)], we deduce that $|a_+(t)| = |b_+(t)|$ is a necessary condition in order for $c_{++}(t)$ or $c_{--}(t)$ to vanish. Since such a condition is only asymptotically reached by the system, the concurrence cannot vanish during the transient regime.

We now study $C(t)$ when the system is initially prepared in one of the same three states considered above, adopting this time as the Hamiltonian model the one stemming from Eq. (47b) (called "second time-dependent scenario" in the previous section). When $c_{++}(0) = 1$ or $c_{--}(0) = 1$ and the second time-dependent scenario (47b) is assumed, the concurrence becomes

$$C(t) = 2\frac{\tanh\left(\frac{|\Gamma_+|}{\hbar}t\right)}{\cosh\left(\frac{|\Gamma_+|}{\hbar}t\right)}, \tag{76}$$

and this is plotted in Fig. 11 as a function of $\tau_+^{'} = \frac{|\Gamma_+|}{\hbar}t$.

We note that at time instant $(\tau_+)_0 = \text{arcsinh}(1) \approx 0.88$, the system of the two spins reaches a maximally entangled state from which it asymptotically evolves toward a factorized

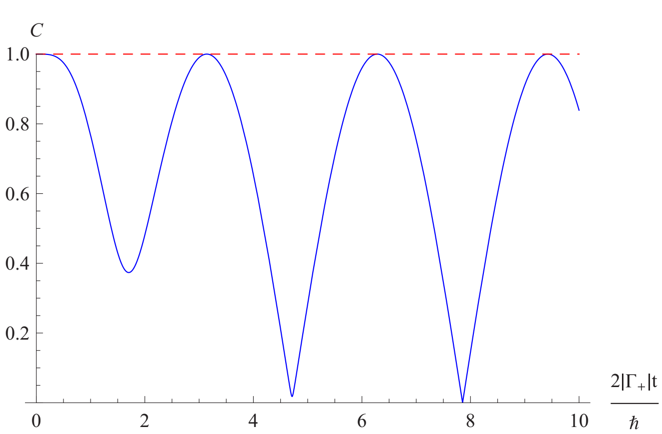

FIG. 10. Plot of $C(t)$ starting from $\frac{|{+}{+}\rangle + |{-}{-}\rangle}{\sqrt{2}}$ under the time-dependent scenario (47a).

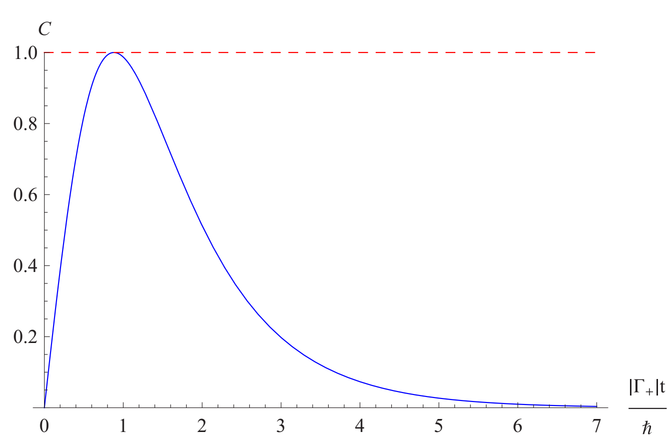

FIG. 11. Plot of $C(t)$ starting from $|{+}{+}\rangle$ according to the time-dependent scenario (47b).

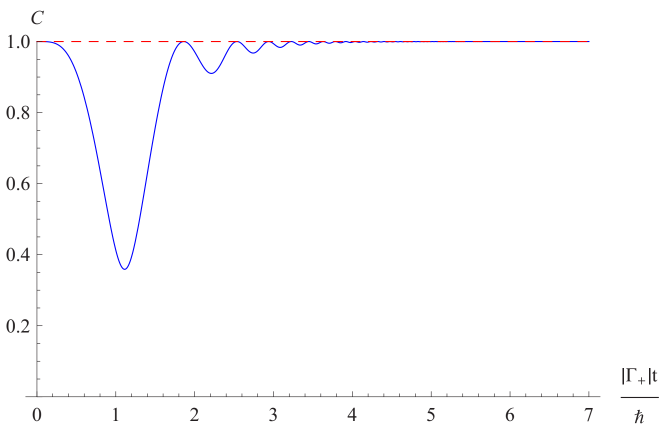

FIG. 12. Plot of $C(t)$ starting from $\frac{|{+}{+}\rangle+|{-}{-}\rangle}{\sqrt{2}}$ according to the time-dependent scenario (47b).

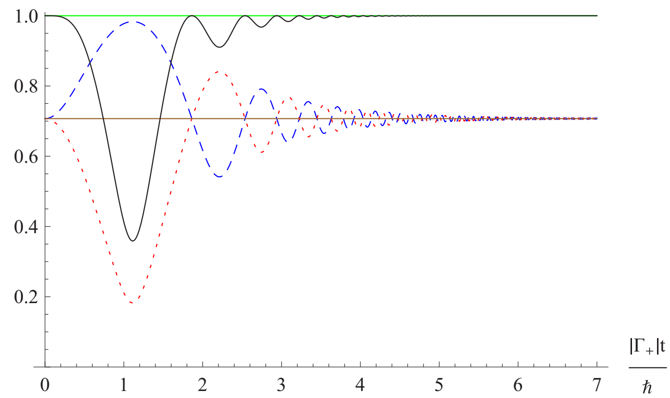

FIG. 13. Plots of $|c_{++}(t)|$ (blue dashed line) and $|c_{--}(t)|$ (red dotted line) and the resulting concurrence $C(t)$ (black line) starting from $c_{++}(0)=\frac{1}{\sqrt{2}}$ and $c_{--}(0)=\frac{1}{\sqrt{2}}$ according to the time-dependent scenario (48b); the upper green (middle brown) line represents the $y=1$ ($y=\frac{1}{\sqrt{2}}$) level.

state. Even in this case, it is useful to exploit Eqs. (53a) and (53b) together with Eqs. (43) and (45), predicting that at the particular time instant $(\tau_+)_0$, the evolved states, respectively, become

$$|\psi_\alpha^+[(\tau_+)_0]\rangle = e^{i\left(\frac{-\gamma_{zz}(\tau_+)_0}{\hbar}+\phi_a[(\tau_+)_0]\right)}\frac{|{+}{+}\rangle - e^{i(\frac{\pi}{2}-1)}|{-}{-}\rangle}{\sqrt{2}}, \tag{77a}$$

$$|\psi_\beta^+[(\tau_+)_0]\rangle = e^{i\left(\frac{-\gamma_{zz}(\tau_+)_0}{\hbar}+\phi_b[(\tau_+)_0]\right)}\frac{|{+}{+}\rangle + e^{-i(\frac{\pi}{2}-1)}|{-}{-}\rangle}{\sqrt{2}}, \tag{77b}$$

which is in accordance with the result on the concurrence. Equations (58) provide the asymptotic form of the two evolutions under scrutiny, confirming the expectation of vanishing concurrence for large t.

If the two-spin system is initially prepared in the Bell state $|\psi(0)\rangle=\frac{|{+}{+}\rangle+|{-}{-}\rangle}{\sqrt{2}}$, the concurrence may be expressed as

$$C(t)=\sqrt{1-4\frac{\tanh^2\left(\frac{|\Gamma_+|}{\hbar}t\right)}{\cosh^2\left(\frac{|\Gamma_+|}{\hbar}t\right)}\sin^2\left[\sinh\left(\frac{|\Gamma_+|}{\hbar}t\right)\right]}, \tag{78}$$

which is graphically represented in Fig. 12 as a function of $\tau_+^{'}=\frac{|\Gamma_+|}{\hbar}t$. Since on the basis of Eqs. (58) the state $|{+}{+}\rangle$ ($|{-}{-}\rangle$) asymptotically evolves into $|{-}{-}\rangle$ ($|{+}{+}\rangle$), up to an initial state-dependent global phase factor, the concurrence in the case under scrutiny must asymptotically comes back to its maximum value. The peculiar oscillatory behavior as time goes on is due to the fact that the time evolution of $|c_{++}(t)|$ and $|c_{--}(t)|$ is dominated by progressive oscillations of decreasing amplitudes around $1/\sqrt{2}$ until they asymptotically stabilize at such values as we can transparently appreciate in Fig. 13. The normalization of $|\psi(t)\rangle$ justifies the coincidence of the time instants where $|c_{++}(t)|$ and $|c_{--}(t)|$ assume the value $1/\sqrt{2}$. On the other hand, the independence of $C(t)$ on both phases of $c_{++}(t)$ and $c_{--}(t)$ explains why these time instants are exactly those at which $C(t)=1$. Moreover, Fig. 12 makes evident that all of the infinitely many minima of the concurrence occur at those time instants where $||c_{++}(t)|-|c_{--}(t)||$ reaches local maxima in time (maximally unbalanced condition).

It is possible to find exactly the infinite sequence of states at which the concurrence assumes its maximum value. To this end, we first calculate the time instants at which $C(t)=1$ and $|c_{++}(t)|=|c_{--}(t)|=1/\sqrt{2}$ simultaneously. They are given by

$$(\tau_+^{'})_n=\text{arcsinh}(n\pi), \tag{79}$$

with $n=0,1,2,\ldots$. Plugging $(\tau_+^{'})_n$ into the state given in Eq. (65) after making explicit its time dependence with the help of Eqs. (66a) and (70) yields the following sequence of maximally entangled states progressively emerging in the time evolution of the initial Bell state:

$$|\psi[(\tau_+^{'})_n]\rangle = e^{i\left(\frac{-\gamma_{zz}(\tau_+)_n}{\hbar}+\phi_n+\theta_n\right)}\frac{|{+}{+}\rangle+e^{-2i\phi_n}|{-}{-}\rangle}{\sqrt{2}}, \tag{80}$$

where

$$\phi_n\equiv\phi_a^+[(\tau_+^{'})_n]=\sqrt{\frac{\sqrt{1+(n\pi)^2}-1}{\sqrt{1+(n\pi)^2}+1}} \tag{81}$$

and

$$\theta_n=\arctan[(-1)^{n+1}n\pi], \tag{82}$$

with n being an arbitrary non-negative integer.

We emphasize that if we start from the initial condition $|{+}{-}\rangle$ adopting the time-dependent scenario (48a) [(48b)], we might again go through the arguments previously used to discuss the case $|{+}{+}\rangle$ in conjunction with (47a) [(47b)], getting results that coincide with those expressed by Eqs. (73) [(76)] provided that γ_+ is substituted by γ_- and γ_{zz} with $-\gamma_{zz}$. Analogously, had we started from the Bell state $(|{+}{-}\rangle+|{-}{+}\rangle)/\sqrt{2}$, Eq. (74) [(78)] represents a valid result in this case too, provided the same substitution of γ_+ and γ_{zz} is made and Eq. (48a) [(48b)] is adopted.

It is worth noticing that in the parity-constrained dynamical evolution under scrutiny, the concurrence $C(t)$ at a generic time instant may be expressed as [21]

$$C(t) = \sqrt{C_{xx}^2 + C_{yy}^2}, \tag{83}$$

where

$$C_{xx}(t) = \langle\psi(t)|\hat{\sigma}_1^x\hat{\sigma}_2^x|\psi(t)\rangle - \langle\psi(t)|\hat{\sigma}_1^x|\psi(t)\rangle\langle\psi(t)|\hat{\sigma}_2^x|\psi(t)\rangle \tag{84}$$

is the covariance of $\hat{\sigma}_1^x$ and $\hat{\sigma}_2^x$ and, analogously, $C_{yy}(t)$ is the covariance of $\hat{\sigma}_1^y$ and $\hat{\sigma}_2^y$. It is simple to show that since [21] $C_{xx}(t) = 2\mathrm{Re}[c_{++}(t)c_{--}^*(t)]$ and $C_{yy}(t) = 2\mathrm{Im}[c_{++}(t)c_{--}^*(t)]$,

$$C_{xx}[(\tau_+^{'})_n] = \cos(2\phi_n), \tag{85a}$$

$$C_{yy}[(\tau_+^{'})_n] = \sin(2\phi_n), \tag{85b}$$

in accordance with the property $C[(\tau_+^{'})_n] = 1$ for any $n = 0,1,2,\ldots$. Since, in view of Eq. (81), $2\phi_n$ spans an infinite countable number set between 0 and 2 made up of irrationally related elements, $C_{xx}[(\tau_+^{'})_n]$ is a decreasing function of n, changing its sign as soon as $2(\tau_+^{'})_n > \pi$ and asymptotically tending to cos(2), whereas $C_{yy}[(\tau_+^{'})_n]$ is an increasing (decreasing) function of n for $2(\tau_+^{'})_n < \pi$ $[2(\tau_+^{'})_n > \pi]$, asymptotically tending to sin(2). It is remarkable that the quantitative link expressed by Eq. (83) enables a direct measurement of the level of the entanglement established in the system at any time instant.

VI. CONCLUSIVE REMARKS

The Hamiltonian model given by Eq. (4) adopted in this paper contains seven parameters and then is potentially useful to describe a huge variety of physical systems and/or physical situations in its parameter space. The key guidance leading us to extract this model from the general one given in Eq. (1) is the idea of assuring to our model the existence of a constant of motion with two eigenvalues only, holding at the same time the noncommutativity with $\hat{\mathbf{S}}^2$ and/or $\hat{S}^z$. Such a constant of motion, by construction, subdivides $\mathcal{H}$ into two dynamically invariant and orthogonal subspaces sharing the same dimension 2. The merit of such a decomposition is that it paves the way for extending our Hamiltonian model to a time-dependent scenario, namely that wherein the two spins are subjected to an appropriate inhomogeneous time-dependent magnetic field.

In this paper, we report the exact time evolutions generated by such a time-dependent Hamiltonian. This result is first of all important on its own since exactly solvable problems involving two coupled bodies driven by time-dependent external fields are rare. In connection with the last consideration, we point out that our exact treatment holds its validity even when the spin-spin coupling constants are time dependent, as for example happens when two neutral atoms located in the left and right sites of a double well are induced to merge in a single well by carefully adjusting (that is, time controlling) the trapping potential [7]. Our treatment possesses an additional merit of providing not a lucky trick confined to the problem under scrutiny only, but indeed an exportable route. This claim, on the one hand, stems from the circumstance that the symmetry condition imposed to our Hamiltonian may be easily attributed to other Hamiltonian models representing dimers hosting two spins higher than 1/2 and even of different values. On the other hand, the consequent emergence of invariant subdynamics is traceable back to such a symmetry leading indeed to the possibility of taking advantage of the method reported in Ref. [17]. Our treatment is illustrated finding the time behavior of the two spins in correspondence to different choices of the inhomogeneous magnetic field. In particular, the time evolution of the mean value of some physically transparent observables as well as of the entanglement exhibited by the two-qubit system during its time evolution is carefully reported and discussed. Summing up, we wish to remark that providing exact solutions of a class of rather general Hamiltonian models describing two coupled qubits, although of relevance, is not the only result reported in this paper. We emphasize indeed that the strategic double exploitation of the decoupling treatment and of the systematic approach of Ref. [17] demonstrates, in a very transparent way, the usefulness of such an approach beyond the original application to the quantum dynamics of a spin 1/2 subjected to a time-dependent magnetic field.

ACKNOWLEDGMENTS

A.M. and R.G. warmly thank G. Buscarino and M. Todaro for useful comments on possible experimental realizations of the Hamiltonian model and for suggesting Refs. [4] and [5]. Moreover, the same authors acknowledge stimulating conversation on the subject with B. Militello.

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