Approximations for the free evolution of self-gravitating quantum particles

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(Received 21 June 2016; published 1 August 2016)

The evolution of the center-of-mass wave function for a mesoscopic particle according to the Schrödinger-Newton equation can be approximated by a harmonic potential if the wave function is narrow compared to the size of the mesoscopic particle. It was noticed by Colin *et al.* [Phys. Rev. A **93**, 062102 (2016).] that, in the regime where self-gravitational effects are weak, intermediate and wider wave functions may be approximated by a harmonic potential as well but with a width-dependent coupling, leading to a time evolution that is determined only by a differential equation for the width of a Gaussian wave function as a single parameter. Such an approximation results in considerably less computational effort in order to predict the self-gravitational effects on the wave-function dynamics. Here, we provide an alternative approach to this kind of approximation, including a rigorous derivation of the equations of motion for an initially Gaussian wave packet under the assumption that its shape is conserved. Our result deviates to some degree from the result by Colin *et al.* [Phys. Rev. A **93**, 062102 (2016).], specifically in the limit of wide wave functions.

DOI: 10.1103/PhysRevA.94.022101

I. INTRODUCTION

As long as there is no agreement on what the correct theory is that unifies quantum physics and gravitation, there is no unambiguous prediction for the gravitational interaction of quantum matter. Although most physicists prefer the idea that the gravitational field itself must exhibit quantum properties at the microscopic regime, a fundamentally semiclassical coupling of quantum matter to classical space-time has been proposed as an alternative [1–3]. A natural way to source the curvature of classical space-time by quantum matter is provided by the semiclassical Einstein equations [1,2],

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \langle \Psi \mid \hat{T}_{\mu\nu} \mid \Psi \rangle, \tag{1}$$

where the stress-energy tensor of the classical Einstein equations is replaced by the expectation value of the corresponding quantum operator. Although some controversies exist about the relevance of this approach [4,5], the arguments against it are inconclusive [6–8]. The ultimate decision whether or not the gravitational field must be quantized, therefore, seems to be up to experiments [2,3,6].

Interestingly, the mass regime where such experiments become feasible is not that far from what is currently possible in the laboratory. This is because Eq. (1) yields a nonlinear Newtonian gravitational self-interaction in the nonrelativistic Schrödinger equation. Such a nonlinearity has observable consequences, despite the weakness of the gravitational interaction.

These self-gravitational effects were first modeled by the *one-particle* Schrödinger-Newton equation [3,9],

$$i\hbar \frac{\partial}{\partial t} \psi(t,\mathbf{r}) = \left(-\frac{\hbar^2}{2m} \nabla^2 - G m^2 \int d^3\mathbf{r}' \frac{|\psi(t,\mathbf{r}')|^2}{|\mathbf{r} - \mathbf{r}'|} \right) \psi(t,\mathbf{r}). \tag{2}$$

The self-gravitational interaction represented by the nonlinear term predicts inhibitions of the free spreading of a wave function for masses of about 10^{10} u and beyond [9]. Experiments aiming at testing this behavior could be realistic, at least, in space where a free spreading over longer time scales is feasible.

Evidently, the massive systems needed for such an experiment cannot be elementary particles. Rather will they consist of a large number of constituents. One may, therefore, ask in what sense Eq. (2) describes such a many-body system. The many-body dynamics following from the hypothesis of a fundamentally semiclassical gravity (1) will be discussed in some detail in Sec. II. One finds that—quite contrary to the situation where it is considered as a Hartree approximation [10]—Eq. (2) approximately describes the center of mass of a complex system, provided that its center of mass is sufficiently delocalized in order to be allowed to disregard its internal structure.

On the other hand, if the center-of-mass wave function is narrow compared to the size of the system, the resulting self-gravitational potential is a function of the first and second moments $\langle \mathbf{r} \rangle$ and $\langle r^2 \rangle$ and quadratic in $\mathbf{r}$. A potential of this form will leave the shape of a Gaussian wave function intact during its evolution. Contrary to Eq. (2) where for a numerical analysis the full wave function must be simulated, the center-of-mass dynamics can then be described by the evolution of the wave function width as a single parameter.

Colin *et al.* [11] make use of this, proposing to approximate the Schrödinger-Newton equation by a quadratic potential also in the case where the full dynamics are given by Eq. (2). Their approximation is valid for weak gravitational potentials and substantially simplifies numerical calculations. It has been employed in order to provide a crucial contribution towards an experimental test by comparing the self-gravitational inhibitions of dispersion to other decoherence sources.

The wave-function-dependent quadratic potential they use was, however, inferred from a comparison of energies, rather than derived directly from the Schrödinger-Newton equation. Here, after reviewing the many-body and center-of-mass equations in Sec. II, we will provide such a direct derivation in

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Sec. III. The result will be compared with the approximation used by Colin *et al.* [11,12] in Sec. IV where we find some deviations. We will conclude with a brief discussion.

II. THE MANY-BODY SCHRÖDINGER-NEWTON EQUATION

In a condensate of many constituents which collectively occupy the same state, Eq. (2) appears as a Hartree approximation and can be derived in a similar fashion as the Gross-Pitaevskii equation [13–15]; it is then, for example, a model for the self-gravitation of bosonic stars [16]. In this context, its origin is the *mutual* Newtonian interaction of the constituents, and it provides an effective description of the factors of a tensor product state of the condensate, regardless of whether Eq. (1) is fundamentally correct or not. This is *not* the point of view taken here.

On the other hand, if one starts with Eq. (1) as a hypothesis about the gravitational interaction of quantum matter, applies it to a Klein-Gordon or Dirac field, takes the nonrelativistic limit, and restricts oneself to the one-particle sector, then Eq. (2) follows immediately as the *fundamental* dynamical equation for a *single* nonrelativistic quantum particle [17,18]. Since in the nonrelativistic limit there is no interaction between the sectors of different particle numbers, one can choose a restriction to the N-particle sector instead for arbitrary N. In contrast to the Hartree approximation, which is already an effective description for N constituents, one then obtains a *different* equation for the dynamics of the full N-particle wave function [18,19],

$$i\hbar\frac{\partial}{\partial t}\Psi(t,\mathbf{r}_1,\ldots,\mathbf{r}_N) = \left(-\sum_{i=1}^{N}\frac{\hbar^2}{2m_i}\nabla_i^2 + V_{\text{nongrav.}} + V_g[\Psi]\right) \times \Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N), \tag{3a}$$

with the gravitational potential,

$$V_g[\Psi](t,\mathbf{r}_1,\ldots,\mathbf{r}_N) = -G\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j \times \int d^3r_1'\cdots d^3r_N'\frac{|\Psi(t;\mathbf{r}_1',\ldots,\mathbf{r}_N')|^2}{|\mathbf{r}_i-\mathbf{r}_j'|}. \tag{3b}$$

$\mathbf{r}_i$ and m_i are the coordinates and masses of the N constituents. The gravitational potential V_g contains both the self-gravitational interaction of each constituent and the mutual gravitational interactions between different constituents.

Due to the nonlinear gravitational term, Eq. (3a) does not strictly separate into center-of-mass and relative coordinates. Nonetheless, one can apply an appropriate Born-Oppenheimer-type approximation scheme, making use of the fact that the gravitational interaction is much weaker than intramolecular forces, which are making up the internal structure of the system modeled by a mass density distribution ρ_c. Precisely speaking, if the wave function separates into center-of-mass and relative coordinates $\Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N) = \psi(t,\mathbf{r})\chi(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})$, then ρ_c is given by the relative wave function [20],

$$\rho_c(\mathbf{r}) = \sum_{i=1}^{N-1} m_i \int d^3r_1\cdots d^3r_{i-1}d^3r_{i+1}\cdots d^3r_{N-1} \times|\chi(\mathbf{r}_1,\ldots,\mathbf{r}_{i-1},\mathbf{r},\mathbf{r}_{i+1},\ldots,\mathbf{r}_{N-1})|^2. \tag{4}$$

The center-of-mass wave function then satisfies [20]

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_g[\psi](t,\mathbf{r})\right)\psi(t,\mathbf{r}), \tag{5a}$$

$$V_g[\psi](t,\mathbf{r}) = -G\int d^3\mathbf{r}'|\psi(t,\mathbf{r}')|^2 I_{\rho_c}(|\mathbf{r}-\mathbf{r}'|), \tag{5b}$$

$$I_{\rho_c}(\mathbf{d}) = \int d^3\mathbf{u}\int d^3\mathbf{v}\frac{\rho_c(\mathbf{u})\rho_c(\mathbf{v}-\mathbf{d})}{|\mathbf{u}-\mathbf{v}|}. \tag{5c}$$

Here $\mathbf{r}$ is the center-of-mass coordinate, ψ is the center-of-mass wave function, and ρ_c is the mass density of the many-body system with respect to its center of mass. The function I_{ρ_c} is proportional to the gravitational energy between a mass distribution ρ_c and a copy of the same mass distribution shifted by a distance $\mathbf{d}$. Equation (5) is the Schrödinger-Newton equation for the center of mass of a complex system of many constituents. This is the equation we are interested in.

If the system is modeled by a homogeneous spherical mass distribution, the potential I_{ρ_c} takes the form [21]

$$I_{\rho_c}^{\text{sphere}}(d) = \frac{m^2}{R}\begin{cases}\frac{6}{5} - 2\left(\frac{d}{2R}\right)^2 + \frac{3}{2}\left(\frac{d}{2R}\right)^3 - \frac{1}{5}\left(\frac{d}{2R}\right)^5 & (d \leqslant 2R),\\ \frac{R}{d} & (d > 2R).\end{cases} \tag{6}$$

One can see immediately that in situations where the mass distribution is pointlike compared to the wave function, Eq. (5) is well approximated by Eq. (2).

In the opposite case where the spatial wave function can be considered narrow and the mass is distributed homogeneously over a sphere of radius R, a good approximation to V_g is given by [20,22]

$$V_{\text{narrow}}^{\text{sphere}}[\psi](t,\mathbf{r}) = \text{const} + \frac{m\omega_{\text{SN}}^2}{2}(\mathbf{r}^2 - 2\mathbf{r}\cdot\langle\psi\,|\,\mathbf{r}\,|\,\psi\rangle + \langle\psi\,|\,\mathbf{r}^2\,|\,\psi\rangle). \tag{7}$$

For a mesoscopic homogeneous sphere one finds $\omega_{\text{SN}}^2 = Gm/R^3$. The nonlinearity is still present, encoded in the expectation values. Based on this approximation, tests of the Schrödinger-Newton equation have been proposed also with optomechanical experiments [22–24].

Since the potential (7) depends only on the first and second moments and is quadratic in $\mathbf{r}$, it can be shown to leave the shape of an initially Gaussian wave packet unchanged [11,12,22]. Therefore, the full (as far as the narrow wave-function approximation is valid) dynamics for a Gaussian wave packet are given by the time evolution of the width $\sqrt{\langle r^2\rangle}$ of the wave packet.

III. EVOLUTION OF A SELF-GRAVITATING GAUSSIAN WAVE FUNCTION

For wider wave functions, the gravitational potential deviates from the quadratic form (7), and the wave-function shape will change over time. If the self-gravitational interaction is weak (i.e., the kinetic energy of the mesoscopic particle is much larger than the gravitational energy), one may, however, assume as a first-order approximation that the shape of the wave function stays a Gaussian, and self-gravitation only affects its width [11,12].

In the case of a spherically symmetric wave function, the expectation values of radial momentum and distance vanish exactly $\langle r\rangle = \langle p\rangle = 0$. The equations of motion for the second moment $u(t) = \langle r^2\rangle(t)$ follow straightforwardly from the Schrödinger-Newton equation (5). One finds

$$\frac{\partial}{\partial t}\langle r^2\rangle(t) = \frac{1}{m}\langle \mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}\rangle(t), \tag{8a}$$

$$\frac{\partial}{\partial t}\langle p^2\rangle(t) = -\langle \mathbf{p}\cdot(\nabla V_g) + (\nabla V_g)\cdot\mathbf{p}\rangle = -m\frac{\partial}{\partial t}\langle V_g\rangle(t), \tag{8b}$$

$$\frac{\partial}{\partial t}\langle \mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}\rangle(t) = \frac{2}{m}\langle p^2\rangle(t) - \langle \mathbf{r}\cdot\nabla V_g\rangle(t), \tag{8c}$$

where the second equality in Eq. (8b) is shown in Ref. [23]. If now one applies the aforementioned approximation, assuming that the gravitational potential V_g is simply a function of the width $u(t)$ of a Gaussian state, this yields

$$\dddot{u}(t) = -3\omega_{\mathrm{SN}}^2 f(u(t))\dot{u}(t), \tag{9a}$$

with (see the Appendix for a more detailed derivation)

$$f(u) = \frac{\partial}{\partial u}F(u), \tag{9b}$$

$$F(u) = \frac{R^3}{3Gm^2}\langle 2V_g + \mathbf{r}\cdot\nabla V_g\rangle = -\frac{144R^8}{\pi u^3}\int_0^\infty dp\int_0^p dq\,\exp\left[-\frac{3R^2}{u}(p^2+q^2)\right]\times(p^2-q^2)[\Pi(p)-\Pi(q)]. \tag{9c}$$

For a homogeneous sphere mass distribution, the function Π takes the form

$$\Pi(x) = \begin{cases} \frac{6}{5}x^2 - \frac{3}{2}x^4 + \frac{21}{20}x^5 - \frac{9}{70}x^7 & (x \leqslant 1),\\ \frac{3}{4}x & (x > 1). \end{cases} \tag{9d}$$

Note that the dependence on mass and the gravitational constant are completely encoded in ω_{SN}, and the function f depends only on the wave function width u as well as the radius R of the mesoscopic particle. For the homogeneous sphere, the integral can be solved analytically, resulting in

$$f(u) = \mathrm{erf}\left(\sqrt{\frac{3}{u}}\right) + \sqrt{\frac{u}{3\pi}}\left(u - \frac{7}{2} - \frac{324 - 162u - 35u^4 + 70u^5}{70u^4}e^{-3/u}\right), \tag{10}$$

where erf is the Gauss error function and u is expressed in units of R^2. One can easily check that this function has the appropriate limits,

$$\lim_{u\to 0} f(u) = 1 \quad\text{and}\quad \lim_{u\to\infty} f(u) = 0. \tag{11}$$

A. Initial conditions

For an initial real-valued Gaussian of width $\langle r^2\rangle_0 = u_0$,

$$\psi(t=0,r) = \left(\frac{3}{2\pi u_0}\right)^{3/4}\exp\left(-\frac{3r^2}{4u_0}\right), \tag{12}$$

the differential equation (9a) must be solved with initial conditions,

$$u(0) = u_0, \tag{13a}$$

$$\dot{u}(0) = 0, \tag{13b}$$

$$\ddot{u}(0) = \frac{9\hbar^2}{2m^2u_0} - \omega_{\mathrm{SN}}^2 g(u_0)u_0, \tag{13c}$$

with the function,

$$g(u) = \frac{R^3}{Gm^2u}\langle \mathbf{r}\cdot\nabla V_g\rangle_0 = -\frac{432R^8}{\pi u^4}\int_0^\infty dp\int_0^p dq\,\exp\left[-\frac{3R^2}{u}(p^2+q^2)\right]\times(p^2-q^2)[K(p)-K(q)]. \tag{14}$$

For the homogeneous sphere, K is

$$K(x) = \begin{cases} -\frac{1}{2}x^4 + \frac{9}{20}x^5 - \frac{1}{14}x^7 & (x \leqslant 1),\\ -\frac{1}{4}x & (x > 1), \end{cases} \tag{15}$$

which yields (again for units with $R = 1$)

$$g(u) = \mathrm{erf}\left(\sqrt{\frac{3}{u}}\right) + \sqrt{\frac{u}{3\pi}}\times\left(\frac{2}{3}u - 3 + \frac{486 + 105u^3 - 70u^4}{105u^3}e^{-3/u}\right). \tag{16}$$

The dynamics depend, in general, on the radius R of the mesoscopic sphere, its mass m, and the initial width u_0 of the center-of-mass wave function. However, the dependence on the latter is only a dependence on the ratio u_0/R for the functions f and g, whereas ω_{SN} is a function of mass density only. For a given mass density, if units are chosen such that $R = 1$, the only true dependence on mass comes through the initial condition (13c).

Although for a precise numerical analysis of Eqs. (2) and (5) the full wave function must be simulated on a grid of many points, and for this approximation it is sufficient to determine the evolution of $u(t) = \langle r^2\rangle$. This significantly simplifies numerical estimations of the magnitude of effects and may even allow for some analytical results which cannot be seen from the full equation.

IV. COMPARISON WITH PREVIOUS RESULTS

Equation (9a) also follows if one starts with a harmonic-oscillator ansatz with width-dependent coupling,

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{k(u)}{2}r^2\right)\psi(t,\mathbf{r}). \tag{17}$$

The authors of Refs. [11,12] employ an energy argument in order to determine an approximation for $k(u)$. To follow this argument, first note that Eq. (17) possesses the conserved energy,

$$E = \frac{\hbar^2}{2m}\int d^3\mathbf{x}|\nabla\psi(t,r)|^2 + U(u(t)), \tag{18}$$

if the potential U satisfies

$$\frac{dU}{du} = \frac{k(u)}{2}. \tag{19}$$

For the standard harmonic-oscillator potential, where k is a constant, U is simply the potential energy of the oscillator. The potential U is then chosen such that

$U(u) = -GI_{\rho_c}^{\text{sphere}}(\sqrt{u})$ with $I_{\rho_c}^{\text{sphere}}$ defined in Eq. (6), which leads to the functions,

$$\tilde{f}(u) = \begin{cases} 1 - \frac{21}{32}\sqrt{u} + \frac{3}{64}\sqrt{u^3} & (u \leqslant 4), \\ \frac{1}{2\sqrt{u^3}} & (u > 4), \end{cases} \tag{20a}$$

$$\tilde{g}(u) = \begin{cases} 1 - \frac{9}{16}\sqrt{u} + \frac{1}{32}\sqrt{u^3} & (u \leqslant 4), \\ \frac{1}{\sqrt{u^3}} & (u > 4), \end{cases} \tag{20b}$$

rather than $f(u)$ and $g(u)$ in Eqs. (9a) and (13c).

Intuitively speaking, the mesoscopic particle moves in a time-dependent harmonic potential, whose coupling constant is tuned in such a way that the potential energy of the particle in the trap matches the gravitational potential energy between two particles, one sitting at the center of the harmonic potential and the other one sitting at $\sqrt{\langle r^2\rangle}$.

From this intuitive picture it is already evident what could go wrong with such an approximation if applied to the self-gravitational potential in the Schrödinger-Newton equation. For the case of a narrow wave function, the gravitational potential within the mesoscopic particle is, in fact, a harmonic potential as is also indicated by the approximation (7). However, in the wide wave-function case where the mass density is well approximated by a δ distribution, the gravitational potential for a Gaussian wave function with $\langle r^2\rangle = u$ is

$$V_g(r) = -\frac{Gm^2}{r}\text{erf}\left(\sqrt{\frac{3r^2}{2u}}\right). \tag{21}$$

Although this can be approximated by a quadratic function for small r, one obtains $V_g \sim 1/r$ for large r. Therefore, for the full Schrödinger-Newton equation the part of the wave function at positions $r^2 \gg u$ will not see a quadratic potential, not even as an approximation, but a Coulomb-like potential. Hence, from the evolution according to the full Schrödinger-Newton equation one expects a weaker response to gravity than the approximation by a harmonic potential according to Eqs. (17) and (20) suggests.

In Fig. 1 the functions $f(u)$ and $\tilde{f}(u)$ are plotted. There is a clear deviation in the intermediate regime, where $u \approx R^2$. For wide wave functions, the deviation decreases in magnitude. Nevertheless, the functions also differ in their limiting behavior for $u \to \infty$. Although $f(u)$ approaches zero like $\sqrt{3/(16\pi\, u^3)}$, $\tilde{f}(u)$ falls off like $1/\sqrt{4\,u^3}$. Figure 2 shows how this can lead to a qualitatively different behavior in the equations of motion. The plotted solution corresponds to an osmium sphere of about 100 nm diameter. For a wide wave function both solutions show at least a similar tendency to slightly decrease in width and oscillate, even though with a significantly lower frequency for the approximation with $f(u), g(u)$ compared to $\tilde{f}(u), \tilde{g}(u)$. In the intermediate regime, on the other hand, the solution according to Eqs. (9a) spreads only a little slower than the free solution, whereas the solution according to Eqs. (17) and (20) is bound close to its initial width.

One can use Eq. (13c) for the initial acceleration in order to define a critical mass [by the condition $\ddot{u}(0) = 0$], which can serve as a rough estimate of whether the initial state will decrease or increase in width upon free evolution. Comparison of the functions $g(u)$ and $\tilde{g}(u)$ then shows that, for wider wave functions with $u \gtrsim (10R)^2$, the approximation chosen in Refs. [11,12] underestimates this mass value by an approximate 20%.

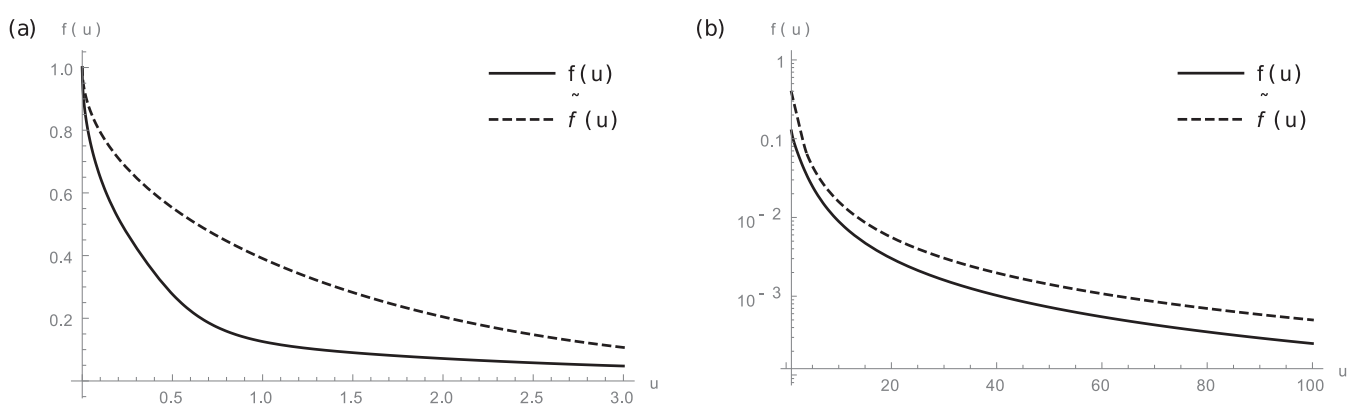

FIG. 1. Comparison of the function $f(u)$ defined in Eq. (9a) (solid curve) and the function $\tilde{f}(u)$ defined in Eq. (20) (dashed curve). u is in units of R^2 (a) for small u and (b) for large u (logarithmic scale).

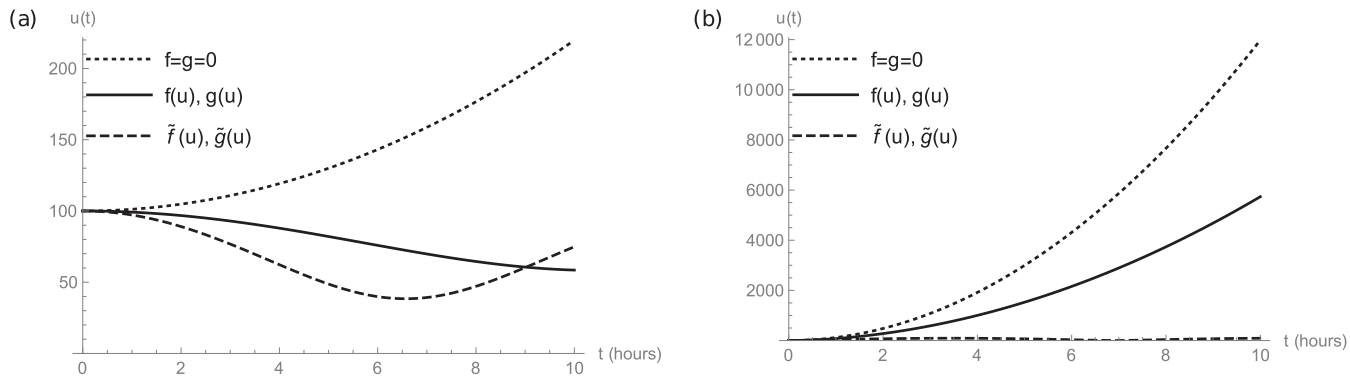

FIG. 2. Solution of the equations of motion for a wide and intermediate wave function. The dotted curve shows the free solution in the absence of self-gravity, the solid curve is the solution using $f(u)$ and $g(u)$ as defined in Eqs. (9a) and (13c), and the dashed curve uses the approximation according to Eq. (20). The mass is chosen to be $m = 10^{10}$ u for an osmium sphere ($\rho = 22.6$ g/cm^3), corresponding to a diameter of approximately 100 nm. u is in units of R^2. (a) Wide wave-function $u_0 = (10R)^2$. (b) Intermediate wave-function $u_0 = R^2$.

V. DISCUSSION

We have shown how, under the assumption of a weak self-gravitational effect that leaves the shape of an initially Gaussian wave packet intact, the free quantum evolution according to the Schrödinger-Newton equation can be approximated by a single parameter differential equation for the square of the wave-function width $\langle r^2\rangle$. The comparison with the previous approximation scheme used by Colin *et al.* [11,12] for the very same purpose revealed deviations for wider wave functions. Although their results made use of an approximation of the gravitational potential by a quasiharmonic interaction, which was determined by a rather *ad hoc* energy argument, here we provided a systematic derivation, based on the aforementioned approximation.

For the purpose of Ref. [11] where self-gravitating effects were compared to decoherence sources, both approximation schemes can serve as sufficiently good order of magnitude estimates. The behavior in Fig. 2(b) shows, however, that large deviations from the real evolution according to the full equation are possible.

The alternative approximation scheme presented in this paper is in no event inferior to the previously proposed one by Colin *et al.* as far as numerical efforts are concerned since the functions $f(u)$ and $g(u)$ are known analytically. This scheme can also be easily adapted to mass density distributions other than the homogeneous sphere.

ACKNOWLEDGMENTS

I gratefully acknowledge funding by the John Templeton Foundation (Grant No. 39530) and the German Research Foundation (DFG). I would like to thank T. Durt for helpful discussions.

APPENDIX: DERIVATION OF $\langle V_g\rangle$ AND $\langle \mathbf{r}\cdot\nabla V_g\rangle(u)$

We want to derive $\langle V_g\rangle$ with V_g as defined in Eq. (5) for the homogeneous sphere potential (6). We write $u = \langle r^2\rangle$. The absolute value squared of the wave function is

$$|\psi|^2(u) = \left(\frac{3}{2\pi u}\right)^{3/2} \exp\left(-\frac{3r^2}{2u}\right). \tag{A1}$$

First define $\alpha = 3R^2/(2u)$ and

$$i(x) = \begin{cases} \frac{6}{5} - 2x^2 + \frac{3}{2}x^3 - \frac{1}{5}x^5 & (x \leqslant 1), \\ \frac{1}{2x} & (x > 1). \end{cases} \tag{A2}$$

Then with appropriate substitutions we have

$$\langle V_g\rangle(\alpha) = -\frac{512Gm^2\alpha^3}{\pi R}\int_0^\infty da \int_0^\infty db \int_{-1}^1 dz \times e^{-4\alpha(a^2+b^2)} a^2 b^2 i(x), \tag{A3}$$

$$x = \sqrt{a^2 + b^2 - 2abz}. \tag{A4}$$

Further substituting z by x, and finally $p = a + b$, $q = |a - b|$ yields

$$\begin{aligned}\langle V_g\rangle(\alpha) &= -\frac{512Gm^2\alpha^3}{\pi R}\int_0^\infty da \int_0^\infty db \times e^{-4\alpha(a^2+b^2)} ab \int_{|a-b|}^{a+b} dx\, x i(x) \\ &= -\frac{128Gm^2\alpha^3}{\pi R}\int_0^\infty dp \int_0^p dq \times e^{-2\alpha(p^2+q^2)}(p^2 - q^2)[J(p) - J(q)],\end{aligned} \tag{A5}$$

with

$$J(x) = \begin{cases} \frac{3}{5}x^2 - \frac{1}{2}x^4 + \frac{3}{10}x^5 - \frac{1}{35}x^7 & (x \leqslant 1), \\ \frac{1}{2}x & (x > 1). \end{cases} \tag{A6}$$

In order to obtain $\langle \mathbf{r}\cdot\nabla V_g\rangle(u)$ we have to repeat essentially the same calculation with the replacement,

$$i(x) \to (a^2 - b^2 + x^2)\frac{i'(x)}{2x}, \tag{A7}$$

where the a^2 and b^2 terms cancel for symmetry reasons. This yields the same final relation,

$$\langle \mathbf{r}\cdot\nabla V_g\rangle(\alpha) = -\frac{128Gm^2\alpha^3}{\pi R}\int_0^\infty dp\int_0^p dq e^{-2\alpha(p^2+q^2)}(p^2-q^2)[K(p)-K(q)], \tag{A8}$$

but with

$$K(x) = \begin{cases} -\frac{1}{2}x^4 + \frac{9}{20}x^5 - \frac{1}{14}x^7 & (x \leqslant 1), \\ -\frac{1}{4}x & (x > 1). \end{cases} \tag{A9}$$

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