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Bell inequalities for quantum optical fields

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The commonly used "practical" Bell inequalities for quantum optical fields, which use intensities as the observables, are derivable only if specific additional assumptions hold. This limits the range of local hidden variable theories, which are invalidated by their violation. We present alternative Bell inequalities, which do not suffer from any (theoretical) loophole. The inequalities are for correlations of averaged products of local rates. By rates we mean ratios of the measured intensity in the given local output channel to the total local measured intensity, in the given run of the experiment. Bell inequalities of this type detect entanglement in situations in which the "practical" ones fail. Thus, we have full consistency with Bell's theorem, and better device-independent entanglement indicators. Strongly driven type-II parametric down conversion (bright squeezed vacuum) is our working example. The approach can be used to modify many types of standard Bell inequalities, to the case of undefined particle numbers. The rule is to replace the usual probabilities by rates.

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Introduction. With the experiments of the group of Hanson and other groups [1] we now know that natural phenomena, which do not satisfy Bell inequalities, can indeed be observed. Thus, assumptions behind the inequalities cannot be used to describe our world. All "loopholes"' were closed. Bell inequalities, as, e.g., the famous Clauser-Horne-Shimony-Holt (CHSH) ones [2], are based on assumptions of locality, of a form of realism (or *hidden* variables, causes, states, or counterfactual definiteness, etc.), and of the "freedom" of experimenters to choose their local measurement settings. In the decades after Bell's paper [3], many other inequalities were derived. They either apply to a different number of settings, or a different number of systems and observers, or different observables (for a recent review, see Ref. [4]). However, most of these inequalities were tailored for situations in which, in each run of the experiment, each observer receives just one particle. But, this is not so in the case of many quantum optical experiments, and in other physical situations. Local photon numbers may be undefined (before measurement), which leads to varying optical intensities. The trail-blazing inequalities of Reid and Walls [5] were specifically derived for such cases, which were the first ones not aimed at situations in which we have "one-run–one-particle" for each observer.

In the development of experimental quantum information one sees an ever-broadening palette of phenomena which are used to demonstrate various protocols. Still, quantum optical experiments dominate. As the security of many protocols rests in the possibility of putting them in a "device-independent" form, which usually leads to some form of Bell inequalities, or related concepts, we need such inequalities for new types of phenomena, and they must be loophole free. However, the inequalities of Ref. [5] cannot be derived solely by using the premises of Bell's theorem. An additional "reasonable" assumption is needed. Different inequalities without this deficiency exist. For example, see Ref. [6], in which photon number parity operators were used as observables (measured after a displacement). Such observables are fragile with respect to losses. A more robust approach simply using photon numbers is given in Ref. [7]. Still, the beauty of the natural approach of Ref. [5], which uses simply optical intensities, straightforwardly related to the usual quantum optical measurements, and its frequent use in the literature, calls for a basic reformulation, so that the resulting inequalities would require only the premises of Bell's theorem. Here, we report Bell inequalities which are for optical intensities, which do not rest on additional assumptions, and detect entanglement in (some) situations in which the "practical" ones of Ref. [5] fail (essentially, for stronger pumping). The crux of the approach is to use "normalized" intensities, or rates, as the variables entering the inequalities. They are defined as the intensity at the given detector divided by the total intensity of all (local) detectors (all for a given run of the experiment).

Note that the intention of our analysis is not an attempt to provide better Bell inequalities for a loophole-free experiment. This case is closed [1]. We aim at a loophole-free analysis of quantum optical experiments with states of undefined photon numbers, such as the (bright) four-mode squeezed vacuum, which will be our working example.

The approach of Ref. [5] set the standard for many years. Two spatially separated observers measure intensities of light at the outputs of their local devices. It is customary to consider them as the outputs of polarization analyzers. We shall follow this picture here. Reference [5] uses the following hidden variable description (for a minor modification, see Ref. [8]). The hidden values of intensities, potentially measurable by Alice, for the two outputs of her analyzer, are denoted here as $I_{A_\pm}(\theta,\lambda)$. For Bob we have $I_{B_\pm}(\phi,\lambda)$. They are non-negative and in principle unbounded. θ and ϕ symbolize the local settings of the polarization analyzers, controllable by the observers, and λ stands for (local) hidden variables, which are assumed to be distributed with some probability density $\rho(\lambda)$. Had the above been the only assumptions, one would not expect any (theoretical) loopholes in the resulting Bell inequalities. However, in Ref. [5], "total intensity through each polarized is written"

$$I_A(\lambda) = I_{A_+}(\theta,\lambda) + I_{A_-}(\theta,\lambda),$$
$$I_B(\lambda) = I_{B_+}(\phi,\lambda) + I_{B_-}(\phi,\lambda). \tag{1}$$

It is argued that $I_A(\lambda)$ and $I_B(\lambda)$ should be independent of θ and ϕ, as they (could) correspond to local intensities with polarizers removed. Conditions (1) are *additional* assumptions on the form of hidden variable theories, which do not have to hold. The polarizer could "enhance" the total intensity registered behind it for some values of λ, and lower it for some others (see the discussion in Refs. [9,10]). Conditions such as (1) perfectly hold for the classical theory of light, and thus seem "reasonable." Still, e.g., they do *not* hold in the case of stochastic electrodynamics [11].

Assumptions (1) allow one to construct local hidden variable correlation functions constrained by CHSH-like [2] inequalities,

$$E(\theta,\phi) = \frac{\langle [I_{A_+}(\theta) - I_{A_-}(\theta)][I_{B_+}(\phi) - I_{B_-}(\phi)]\rangle}{\langle I_A I_B\rangle}, \quad (2)$$

where $\langle\cdots\rangle$ denotes averaging over $\rho(\lambda)$. The denominator $\langle I_A I_B\rangle$ is *independent* of the local settings, as it reads $\int d\lambda \rho(\lambda) I_A(\lambda) I_B(\lambda)$. The inequality for these reads [5]

$$|E(\theta,\phi) + E(\theta,\phi') + E(\theta',\phi) - E(\theta',\phi')| \leqslant 2. \quad (3)$$

It holds *only* due to the fact that $\langle I_A I_B\rangle$ is the same in all E's. The additional assumption (1) is crucial.

In the case of CH-like inequalities [9] one proceeds similarly as in Ref. [5]. A version of a no-enhancement assumption is used: One requires that there exist variables, which are *independent* of the local settings (and the remote ones), $I_A(\lambda)$ and $I_B(\lambda)$, such that

$$I_J(\lambda) \geqslant I_{J_+}(\alpha,\lambda), \quad (4)$$

where $J = A$ or B, and $\alpha = \theta,\phi$, respectively. The variables $I_A(\lambda)$ and $I_B(\lambda)$ are (again) defined as hidden values of (total) local intensities which can be measured by the local observers "without the polarizers present." This allows one to introduce functions

$$G(\theta,\phi) = \int d\lambda \rho(\lambda) I_{A_+}(\theta,\lambda) I_{B_+}(\phi,\lambda), \quad (5)$$

and

$$r_A(\theta) = \int d\lambda \rho(\lambda) I_B(\lambda) I_{A_+}(\theta,\lambda),$$
$$r_B(\phi) = \int d\lambda \rho(\lambda) I_A(\lambda) I_{B_+}(\theta,\lambda). \quad (6)$$

As for real numbers, $0 \leqslant x,x' \leqslant X$ and $0 \leqslant y,y' \leqslant Y$, one has

$$-XY \leqslant xy + xy' + x'y - x'y' - xY - Xy \leqslant 0, \quad (7)$$

and one infers that

$$\begin{aligned} &I_{A_+}(\theta,\lambda) I_{B_+}(\phi,\lambda) + I_{A_+}(\theta,\lambda) I_{B_+}(\phi',\lambda) \\ &\quad + I_{A_+}(\theta',\lambda) I_{B_+}(\phi,\lambda) - I_{A_+}(\theta',\lambda) I_{B_+}(\phi',\lambda) \\ &\quad - I_{A_+}(\theta,\lambda) I_B(\lambda) - I_A(\lambda) I_{B_+}(\phi,\lambda) \leqslant 0. \end{aligned} \quad (8)$$

After averaging, this leads to a CH-like inequality,

$$\begin{aligned} &G(\theta,\phi,) + G(\theta,\phi') + G(\theta',\phi) \\ &\quad - G(\theta',\phi') - r_A(\theta) - r_B(\phi) \leqslant 0. \end{aligned} \quad (9)$$

CH-like inequalities free of the additional assumptions (1) and (4). The inequalities will use functions of intensities, which we call "rates," defined by $R_{A_\pm}(\theta) = \frac{I_{A_\pm}(\theta)}{I_A(\theta)}$ and $R_{B_\pm}(\phi) = \frac{I_{B_\pm}(\phi)}{I_B(\phi)}$, where $I_{J_\pm}(\alpha)$ stand for the measured intensities at detectors $J_\pm$, in the given run of the experiment, and

$$I_J(\alpha) = \sum_{i=\pm} I_{J_i}(\alpha). \quad (10)$$

Note that we do not assume that the total intensity is independent of the local setting. Whenever the denominator $I_J(\alpha)$ is zero, we put $R_{J_\pm} = 0$. The rates show the relative distribution of the measured intensities at the two outputs of the local polarizer. Quantum averages and operators for variables (10) do not depend on the local settings of the polarizers. However, we introduce the possible dependence on α having in mind the hidden variable theories for which "anything goes," provided it is local and realistic.

The local hidden variable model of the rates will be given by

$$R_{J_+}(\alpha,\lambda) = \frac{I_{J_+}(\alpha,\lambda)}{I_J(\alpha,\lambda)}, \quad (11)$$

with $I_J(\alpha,\lambda) = \sum_{i=\pm} I_{J_i}(\alpha,\lambda)$.

Since $0 \leqslant R_{J_+}(\alpha,\lambda) \leqslant 1$, the lemma (7) with $X = Y = 1$ gives

$$\begin{aligned} -1 \leqslant\ & R_{A_+}(\theta,\lambda) R_{B_+}(\phi,\lambda) + R_{A_+}(\theta,\lambda) R_{B_+}(\phi',\lambda) \\ & + R_{A_+}(\theta',\lambda) R_{B_+}(\phi,\lambda) - R_{A_+}(\theta',\lambda) R_{B_+}(\phi',\lambda) \\ & - R_{A_+}(\theta,\lambda) - R_{B_+}(\phi,\lambda) \leqslant 0. \end{aligned} \quad (12)$$

After averaging over $\rho(\lambda)$ we get a CH-like inequality,

$$\begin{aligned} -1 \leqslant\ & K(\theta,\phi) + K(\theta,\phi') + K(\theta',\phi) - K(\theta',\phi') \\ & - S_{A_+}(\theta,\lambda) - S_{B_+}(\phi,\lambda) \leqslant 0, \end{aligned} \quad (13)$$

where we have correlation functions for rates at distant detectors,

$$K(\theta,\phi) = \int d\lambda \rho(\lambda) R_{A_+}(\theta,\lambda) R_{B_+}(\phi,\lambda), \quad (14)$$

and averages of local rates,

$$S_{J_+}(\alpha) = \int d\lambda \rho(\lambda) R_{J_+}(\alpha,\lambda). \quad (15)$$

No additional assumptions are used. Loopholes are moved to experimental imperfections.

The quantum optical definition of the rates depends on the model of detection used, which defines the intensity operators at the exits of the polarizers. We shall use the simplest one, photon number operators, however, one can easily move to other quantum optical models of intensities or detector responses.

For our example involving polarizations, let us introduce annihilation (and creation) operators a_H, a_V, b_H, and b_V (creation operators have a dagger, $\dagger$). Here, H and V stand for horizontal and vertical polarizations, and a,b denote propagation beams towards Alice and Bob, respectively. The operators for arbitrarily oriented (linear) polarization analyzers are

$$\begin{aligned} a_\theta &= \cos\theta a_H + \sin\theta a_V, \\ a_{\theta^\perp} &= -\sin\theta a_H + \cos\theta a_V, \\ b_\phi &= \cos\phi b_H + \sin\phi b_V, \\ b_{\phi^\perp} &= -\sin\phi b_H + \cos\phi b_V. \end{aligned} \tag{16}$$

The operators with $\perp$ denote the "$-$" exit of the local polarizers, and the other ones, the "+" output. The chosen intensity model gives us

$$\hat{I}_{J_+}(\alpha) = c_\alpha^\dagger c_\alpha, \tag{17}$$

$$\hat{I}_{J_-}(\alpha) = c_{\alpha^\perp}^\dagger c_{\alpha^\perp}, \tag{18}$$

where $c = a,b$, $\alpha = \theta,\phi$ for $J = A,B$, respectively. Note that, most importantly, one has for *any* α,

$$\hat{I}_J = \sum_{i=\pm} \hat{I}_{J_\pm}(\alpha) = c_H^\dagger c_H + c_V^\dagger c_V. \tag{19}$$

That is, the total intensity *operator* $\hat{I}_J$ is independent of α. All this can be generalized in the known way to elliptic polarizations.

The quantum average values for correlation functions of the rates read

$$K(\theta,\phi)_Q = \mathrm{Tr}[\varrho \hat{R}_{A_+}(\theta)\hat{R}_{B_+}(\phi)], \tag{20}$$

and for the average local rate one has

$$S_{J_+}(\alpha)_Q = \mathrm{Tr}[\varrho \hat{R}_{J_+}(\alpha)], \tag{21}$$

where ϱ is the state, and $\hat{R}_{J_+}(\alpha)$ are given by

$$\hat{R}_{J_+}(\alpha) = \hat{\Pi}_J^{\perp 0} \frac{\hat{I}_{J_+}(\alpha)}{\hat{I}_J} \hat{\Pi}_J^{\perp 0}. \tag{22}$$

The operators $\hat{\Pi}_J^{\perp 0}$ are projectors into the subspace of the Fock space, of the modes described by annihilation operators c_H and c_V, which does not contain a vacuum. For example, the operator $\hat{R}_{a_+}(\theta)$ acts in the Fock space for modes a_θ and $a_{\theta^\perp}$, and reads

$$\hat{R}_{A_+}(\theta) = \hat{\Pi}_A^{\perp 0} \frac{a_\theta^\dagger a_\theta}{a_\theta^\dagger a_\theta + a_{\theta_\perp}^\dagger a_{\theta_\perp}} \hat{\Pi}_A^{\perp 0}, \tag{23}$$

with

$$\hat{\Pi}_A^{\perp 0} = \hat{\mathbb{1}}_a - |0,0\rangle_{aa}\langle 0,0|, \tag{24}$$

where $\mathbb{1}_a$ is the identity operator, and $|0,0\rangle_a$ is the vacuum satisfying $a_\theta|0,0\rangle_a = a_{\theta^\perp}|0,0\rangle_a = 0$.

Violations of the inequalities. In quantum optics, violations of Bell inequalities of the type presented in Ref. [5] usually decrease with the growing average numbers of photons. The inequalities introduced here are much less prone to this effect. The trivial reason for this is that their form warrants that terms with higher photon numbers contribute relatively less to $K(\theta,\phi)_Q$ and $S_{J_+}(\alpha)_Q$ than to $G(\theta,\phi)_Q$ and $r_{J_+}(\alpha)_Q$. A more profound one is that $K(\theta,\phi)_Q$ are averaged products of measured rates, which essentially give polarization readouts, whereas, for example, correlation functions $G(\theta,\phi)_Q$ effectively involve averaging of the product of rates weighted by the observed product of local total intensities. Thus, instead of a direct average over the statistical ensemble of a product of polarization readouts, we have in (5) an average with a weight proportional to the product of intensities. However, in classical electrodynamics the property of polarization of light waves has nothing to do with its intensity, and in quantum theory the energy of a photon and its polarization are unrelated variables. All this leads to a possibility of a new look at Stokes parameters, or Bloch vectors, for nonclassical light. The averages of the observables $\hat{R}_{A_+}(\theta) - \hat{R}_{A_-}(\theta)$, for three complementary elliptic polarization analyzer's settings, can be interpreted as such (redefined) parameters. In Ref. [12] we show that with such observables we get better entanglement witnesses for light with undefined photon numbers, e.g., better than the conditions given in Refs. [13–15].

Example. Let us consider the four-mode (bright) squeezed vacuum. The state reads

$$|\mathrm{BSV}\rangle = \frac{1}{\cosh^2\Gamma} \sum_{n=0}^{\infty} \sqrt{n+1}\tanh^n\Gamma|\psi_-^{(n)}\rangle, \tag{25}$$

where

$$|\psi_-^{(n)}\rangle = \frac{1}{\sqrt{n+1}} \sum_{m=0}^{n} (-1)^m |n-m\rangle_{a_H}|m\rangle_{a_V}|m\rangle_{b_H}|n-m\rangle_{b_V}, \tag{26}$$

where, in turn, a and b refer to the two directions along which the photon pairs are emitted, H (V) denotes horizontal (vertical) polarization, and Γ represents an amplification gain. The state is rotationally invariant, i.e., all probabilities are dependent only on $\theta - \psi$. A form of such an invariance holds also for measurements of all elliptic polarizations.

In Fig. 1 we compare the violation of the new CH-like Bell inequality (13) by the above family of states, with what happens for the standard inequality (9). We clearly see that the range of the pumping parameter Γ for which we observe violation is much wider than in the case of the standard

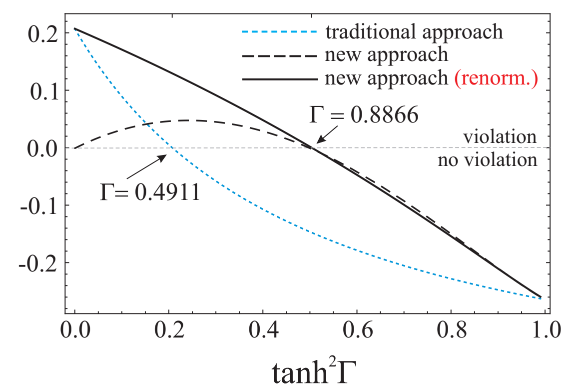

FIG. 1. Comparison of violations of the modified CH-like Bell inequality for rates (13) and the standard inequality (9). The threshold for violation of the CH inequalities is zero (indicated by the straight horizontal line). As vacuum events saturate the inequality (13), the curve pertaining to it, for renormalized averages (with vacuum terms removed), effectively shows violations conditioned on getting a nonvacuum event.

approach. Still, the new inequality seems to be less effective for lower pumping. However, this is only due to the fact that for low pumping the vacuum term in the state dominates, and the value of the left-hand side of inequality (13) for vacuum is zero. If one takes into account only nonvacuum events, the violation for all nonzero Γ's is higher than for the standard approach.

The stronger violation is a consequence of the fact that only nonvacuum events contribute to the visibility (interferometric contrast) of two detector correlations. For the bright squeezed vacuum state (25), the correlation function for the rates $K(\theta,\phi)_Q$ (20), with a removed vacuum term and renormalized by a factor $\frac{1}{1-p(\text{"vac"})}$, where $p(\text{"vac"}) = 1/\cosh^4\Gamma$ is the probability of the vacuum component of $|\text{BSV}\rangle$, is given by

$$K(\theta,\phi)_Q^{(\text{ren})} = \tfrac{1}{8}(\cosh 2\Gamma + 3)\sinh^2\Gamma - \tfrac{1}{96}\cos[2(\theta-\phi)] \times[12\cosh 2\Gamma + \cosh 4\Gamma - 16\ln(\cosh^{-2}\Gamma) - 13], \quad (27)$$

whereas the average (renormalized) rates at single detectors $S_{J_+}(\alpha)_Q^{(\text{ren})}$ are 1/2. The visibility of the correlation interference at pair of remote detectors, defined as

$$v = \frac{K_{\max}(\theta,\phi)_Q - K_{\min}(\theta,\phi)_Q}{K_{\max}(\theta,\phi)_Q + K_{\min}(\theta,\phi)_Q}, \quad (28)$$

where the minimum and maximum is taken over ϕ and θ, reads

$$v_{\text{new}} = \left(3 + \cosh^2\Gamma - 4\frac{\ln\cosh\Gamma}{\sinh^2\Gamma}\right) \Big/ (3 + 3\cosh^2\Gamma). \quad (29)$$

In the case of the approach of Ref. [5], the visibility is given by $v_{\text{standard}} = (1 + 2\tanh^2\Gamma)^{-1}$ (see, e.g., Ref. [16]), and is lower than v_{new} in the full range of the parameter Γ.

Note that the above removal of the vacuum contribution poses no problem when one uses (13) in analysis experimental data. In such a case one can rescale the observed correlation functions by a factor $\frac{1}{1-P(\text{"no"})}$, where is $P(\text{"no"})$ is the frequency of having no registrations in both output channels of the polarizers at both sides of the experiment. This is again due to the fact that no-counts-at-both-sides events saturate the inequality.

Extensions of the approach. The method can be used to rewrite various types of Bell inequalities, for one-local-particle-per-run scenario, to ones involving undefined numbers of particles. The usual Bell inequalities use hidden probabilities $P_J(i,\alpha,\lambda)$, where now α stands for some local parameter defining an observable to be measured on the local particle, on side J, and i numbers the outcomes. They can be replaced by hidden rates $R_J(i,\alpha,\lambda) = \frac{I(i,\alpha,\lambda)}{I_{\text{tot}}(\alpha,\lambda)}$, where $I_{\text{tot}}(\alpha,\lambda) = \sum_i I(i,\alpha,\lambda)$, where $I(i,\alpha,\lambda)$ is the (hidden) intensity at exit i of the local measuring station. If $I_{\text{tot}}(\alpha,\lambda) = 0$, one puts $R_J(i,\alpha,\lambda) = 0$. Note that both $P_J(i,\alpha,\lambda)$ and $R_J(i,\alpha,\lambda)$ are non-negative, and bounded by 1, and if $I_{\text{tot}}(\alpha,\lambda) \neq 0$, one has $\sum_i R_J(i,\alpha,\lambda) = 1$. Thus the bounds of the rewritten inequalities remain the same. In the Appendix we present how one can get loophole-free CHSH-like inequalities for rates.

Example of yet another extension. If one uses the wording of this Rapid Communication, the inequalities of Ref. [7] are based on averages of $|I_{A_+}(\theta) - I_{B_+}(\phi)|$, which can be shown, for hidden variable theories, to have the properties of geometric distance (the most important here is the triangle inequality, as it can be chained to give polygon inequalities). Let us replace these by $\langle|R_{A_+}(\theta) - R_{B_+}(\phi)|\rangle$, which obviously also has the geometric properties of a distance. The resulting chained (polygon) inequalities read

$$\sum_{i=1}^{L}\langle|R_{A_+}(\theta_i) - R_{B_+}(\phi_i)|\rangle + \sum_{i=1}^{L-1}\langle|R_{A_+}(\theta_{i+1}) - R_{B_+}(\phi_i)|\rangle \geqslant \langle|R_{A_+}(\theta_1) - R_{B_+}(\phi_L)|\rangle. \quad (30)$$

TABLE I. Critical values of gain $\Gamma^{\text{crit}}_{\text{chained}}$ and $\Gamma^{\text{crit}}_{\text{chained, ratios}}$ above which chained inequalities for moduli of intensity differences of [7] and similar ones for rates cannot be violated. Parameter L stands for the number of settings ("length") of the chained inequality.

L	$\Gamma^{\text{crit}}_{\text{chained}}$	$\Gamma^{\text{crit}}_{\text{chained, ratios}}$	L	$\Gamma^{\text{crit}}_{\text{chained}}$	$\Gamma^{\text{crit}}_{\text{chained, ratios}}$
2	0.915	1.123	5	1.260	1.586
3	1.053	1.367	6	1.345	1.687
4	1.165	1.482	7	1.427	1.795

We have numerically studied the violation of inequality (30). We cut off the expansion (25) at $n = 25$, and use the settings given in Ref. [7]. The values of the gain Γ, below which we still observe the violation of the inequalities, are given in Table I. Chained inequalities for rates are violated for higher gains than the original ones of Ref. [7]. Also, one needs fewer settings, i.e., lower L, to observe violations for higher gains. For example, to violate the inequality (30) at $\Gamma = 1.4$, one needs only four local settings, whereas it takes seven settings to violate the analogous inequality for the intensities of Ref. [7] [which has the form of inequality (30), with I's replacing R's].

The presented approach allows one to derive, without invoking additional assumptions, Bell inequalities with involve averaged correlations of the rates of local intensities (for the given run, the intensity at the given detector divided by the total local registered intensity) at pairs of spatially separated detectors. Especially for the case of stronger fields, the inequalities can be better entanglement witnesses than standard ones, which use additional assumptions, and not just those of local realism. One can generalize the approach to more observation stations, and to many parties, so that they can be used for situations with undefined particle numbers. Further generalizations of the approach, going beyond the problem of Bell inequalities, are presented in, e.g., Ref. [12].

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APPENDIX: DERIVATION OF CHSH-LIKE BELL INEQUALITIES FOR RATES

The CHSH-like inequalities can be derived using the same basic hidden variable objects $I_{J_\pm}(\alpha,\lambda)$, and their functions $R_{J_+}(\alpha,\lambda)$, plus the rates at the other outputs,

$$R_{J_-}(\alpha,\lambda) = \frac{I_{J_-}(\alpha,\lambda)}{I_J(\alpha,\lambda)}. \tag{A1}$$

One can introduce correlation functions

$$F(\theta,\phi) = \int d\lambda\rho(\lambda)[R_{A_+}(\theta,\lambda) - R_{A_-}(\theta,\lambda)] \times[R_{B_+}(\phi,\lambda) - R_{B_-}(\phi,\lambda)]. \tag{A2}$$

As $-1 \leqslant R_{J_+}(\alpha,\lambda) - R_{J_-}(\alpha,\lambda) \leqslant 1$, one gets

$$|F(\theta,\phi) + F(\theta,\phi') + F(\theta',\phi) - F(\theta',\phi')| \leqslant 2. \tag{A3}$$

While this is a correct CHSH-like inequality, its usefulness is highly limited. All R variables equal zero for zero total local intensity. In many experiments the produced states are such that the vacuum components dominate (see Ref. [17]). As a result, the values of correlation functions F are very low, and the left-hand side cannot breach the bound of 2. Still, such inequalities could be of use in the case of event-ready experiments [18].

To devise CHSH-like inequalities free of this deficiency, one can use the trick of Ref. [19] (used there to derive CHSH-type inequalities for two qubits which are optimal for less than perfect detection efficiency). We introduce rigged rate functions. We define, based on the ideas of Ref. [8], the value of $R_{J_+}(\alpha,\lambda)$ for zero total intensity as 1. No change is required for $R_{J_-}(\alpha,\lambda)$. We denote the rigged rate functions by adding a prime. We still have

$$-1 \leqslant R'_{J_+}(\alpha,\lambda) - R_{J_-}(\alpha,\lambda) \leqslant 1.$$

The correlation function can be defined as follows

$$C(\theta,\phi) = \int d\lambda\rho(\lambda)[R'_{A_+}(\theta,\lambda) - R_{A_-}(\theta,\lambda)] \times[R'_{B_+}(\phi,\lambda) - R_{B_-}(\phi,\lambda)]. \tag{A4}$$

This definition does not suffer from the mentioned deficiency, because in the case of vacuum, $C(\theta,\phi) = 1$. We get a modified inequality,

$$|C(\theta,\phi) + C(\theta,\phi') + C(\theta',\phi) - C(\theta',\phi')| \leqslant 2. \tag{A5}$$

To give a quantum formula for $C(\theta,\phi)_Q$, one puts

$$\hat{R}'_{A_+}(\theta) = \hat{R}_{A_+}(\theta) + |0,0\rangle_{aa}\langle 0,0| \tag{A6}$$

and

$$\hat{R}'_{B_+}(\phi) = \hat{R}_{B_+}(\phi) + |0,0\rangle_{bb}\langle 0,0|, \tag{A7}$$

where $|0,0\rangle_b$ is the vacuum for the b modes.

Let us see whether the new inequalities are violated by quantum predictions for a bright squeezed vacuum. First, let us compute the value of the CHSH expression in fashion of Ref. [5], according to the formula

$$E(\theta,\phi) = \frac{\langle[I_{A_+}(\theta) - I_{A_-}(\theta)][I_{B_+}(\phi) - I_{B_-}(\phi)]\rangle}{\langle I_A I_B\rangle}. \tag{A8}$$

For every Fock component of the state $|\mathrm{BSV}\rangle$, the correlation function is proportional to a cosine function with one of its minima at $\theta - \phi = 0$ and amplitude $\frac{1}{3}(2n + n^2)$. Thus, the numerator in Eq. (A8), summed over $n > 0$ with weights $w_n = (n+1)\tanh^{2n}\Gamma\cosh^{-4}\Gamma$, reads

$$\langle[I_{A_+}(\theta) - I_{A_-}(\theta)][I_{B_+}(\phi) - I_{B_-}(\phi)]\rangle = -2\cosh^2\Gamma\sinh^2\Gamma\cos[2(\theta-\phi)], \tag{A9}$$

while the denominator $\langle I_A I_B\rangle$ is equal to

$$(3\cosh 2\Gamma - 1)\sinh^2\Gamma. \tag{A10}$$

For the optimal settings $\theta = 0$, $\theta' = \pi/4$, $\phi = \pi/8$, and $\phi' = -\pi/8$, we get violations of the CHSH-Bell-like inequality [based on the additional assumption (1)] for the range $0 < \Gamma < 0.4911$.

In the loophole-free approach utilizing the ratios, for an individual $2n$-photon component we have

$$\langle\psi_-^{(n)}|[R'_{A_+}(\theta) - R_{A_-}(\theta)][R'_{B_+}(\phi,\lambda) - R_{B_-}(\phi,\lambda)]|\psi_-(n)\rangle = -\frac{n+2}{3n}\cos(\theta-\phi), \tag{A11}$$

and for the vacuum component we have 1. After the averaging components with weights w_n, which now includes $n = 0$, we get

$$C(\theta,\phi)_Q = \frac{1}{\cosh^4\Gamma}\left[1 - \frac{1}{3}\cos(\theta-\phi)[4\ln\cosh\Gamma + (3+\cosh^2\Gamma)\sinh^2\Gamma]\right]. \tag{A12}$$

This, put into the CHSH expression, gives a violation for the range $0 < \Gamma < 0.8866$.

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