Spectrally dependent fluctuations of thermal photon sources

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Many current quantum optical systems, such as microcavities, interact with thermal light through a small number of widely separated modes. Previous theories for photon number fluctuations of thermal light have been primarily limited to special cases that are appropriate for large volumes or distances, such as single modes, many modes, or modes of uniform spectral distribution. Herein, a theory for the general case of spectrally dependent photon number fluctuations is developed for thermal light. The error in variance of prior art is quantitatively derived for an example cavity in the case where photon counting noise dominates. A method to reduce the spectral impact of this variance is described.

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I. INTRODUCTION

The development of microcavities of dimensions comparable to the wavelength of thermal light [1–5], as in coherent thermal emission [3,5–10], narrowband thermal detection [1,4,11], and cavity quantum electrodynamics [12–18], has opened an entire class of devices whose thermal statistics cannot be addressed by existing theory. Thermal light emitted into free space generally interacts with an enormous spectral density of modes. The photon number fluctuations of thermal emission into each mode have Poisson and Bose-Einstein contributions, but the latter average out when integrated over many modes, leaving only standard Poisson statistics. Historically, since almost all thermal emission occurred in systems with large numbers of modes, it has not been important to have a quantitative model of photon statistics for a small number; however, the aforementioned experimental and theoretical work in cavity micro- and nano-optics has greatly changed this situation.

A microcavity can define an enormous variety of mode distributions, and the strength of coupling between these modes and free space can vary from mode to mode. An analytical derivation of the thermal photon noise for the general case of an arbitrary number of modes with an arbitrary spectral distribution (determined by both the Planck distribution and the mode coupling) has eluded scientists since the late 1950s due to the complicated mathematics at hand [19–23]. We propose and demonstrate using an expansion of the probability density function to analytically find an exact general result for thermal photon population fluctuations for any average number of photons in any number of modes with any spectral dependence. This method sidesteps many of the mathematical complexities of previous treatments and produces a closed-form result.

Thermal photons will have number fluctuations given by the sum of Poisson and Bose-Einstein (BE) terms in single-mode systems. The BE contribution will be reduced to zero when integrated over many modes, resulting in only Poisson statistics for most thermal light. If there are multiple modes but the spectrum is completely uniform, then the variance in the number fluctuations can be given by the following equation [19,24]:

$$\langle(\Delta n)^2\rangle = \langle n\rangle + \frac{\langle n\rangle^2}{M}. \tag{1}$$

In Eq. (1) n is the number of photons, $\langle n\rangle$ is the expected number of photons, M is the number of modes, and $\langle(\Delta n)^2\rangle$ is the variance.

Following an algorithm developed previously [25] and in the Appendix, the general probability density for thermal photons is given by the following equation:

$$P(n) = \sum_{d=1}^{D}\prod_{m=1}^{M}\frac{1}{(1+\langle n_m\rangle)(1+\langle n_m\rangle^{-1})^{n_{m,d}}}. \tag{2}$$

In Eq. (2) and the rest of the paper the following variables are defined: n is the number of photons; m indicates the mode index; $\langle n_m\rangle$ is the average number of photons in mode *m; d* indicates the distribution index; $P(n)$ denotes the probability of having n photons given $\langle n_m\rangle$; M is the total number of modes; D is the total number of ways to distribute n photons in M modes; $n_{m,d}$ denotes the number of photons in mode m and distribution *d;* $\langle n\rangle$ is the average total number of photons; $\langle(\Delta n)^2\rangle$ is the variance in the total number of photons.

Note that n is a discrete random variable; in other words, for $n = 0$, Eq. (2) computes the probability of having 0 photons given the distribution $\langle n_m\rangle$. The number of possible photon distributions, D, is given by

$$D = \frac{(n+M-1)!}{n!(M-1)!}. \tag{3}$$

II. GENERAL SPECTRALLY DEPENDENT MODE DISTRIBUTIONS

To find the variance of general spectrally dependent thermal photon statistics, we first need to find an expression for the mode distribution scaled to the total average photon number. The scaled mode distribution in the general case can be visualized in Fig. 1. Naturally, the sum of the average photon number in each mode equals the total average photon number, but it is convenient to normalize a scaled distribution to the total average photon number such that the distribution is a function of the total average photon number.

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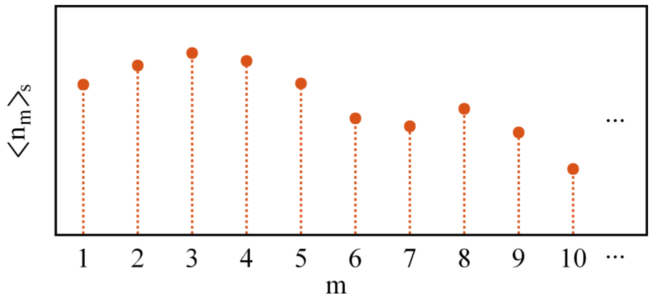

FIG. 1. General spectrally dependent mode distributions; the subscript s signifies that this distribution is scaled, and that the sum does not equal the total average photon number.

The sum of the average photon number in the modal distribution $\langle n_m \rangle$ is equal to the expected number of photons, $\langle n \rangle$ as in the following equations:

$$\langle n \rangle = \sum_{m=1}^{M} \langle n_m \rangle = \sum_{m=1}^{M} A \langle n_m \rangle_s, \tag{4}$$

$$A = \frac{\langle n \rangle}{\sum_{m=1}^{M} \langle n_m \rangle_s}, \tag{5}$$

$$\langle n_m \rangle = \frac{\langle n \rangle \langle n_m \rangle_s}{\sum_{m=1}^{M} \langle n_m \rangle_s}. \tag{6}$$

The scaling factor A in Eq. (4) scales the modal distribution to a normalized value as in Eq. (6). Now that there is an expression for the general mode distribution we can plug this into Eq. (2) and solve for the probability of having 0 or 1 photon in the system:

$$P(0) = \prod_{m=1}^{M} \frac{1}{\left(1 + \frac{\langle n \rangle \langle n_m \rangle_s}{\sum_{m=1}^{M} \langle n_m \rangle_s}\right)}, \tag{7}$$

$$P(1) = \sum_{d=1}^{M} \prod_{m=1}^{M} \frac{1}{\left(1 + \frac{\langle n \rangle \langle n_m \rangle_s}{\sum_{m=1}^{M} \langle n_m \rangle_s}\right)\left(1 + \frac{\sum_{m=1}^{M} \langle n_m \rangle_s}{\langle n \rangle \langle n_m \rangle_s}\right)^{\delta_{m,d}}}, \tag{8}$$

$$P(2) \cong 1 - P(1) - P(0), \tag{9}$$

$$P(n > 2) \cong 0. \tag{10}$$

These probabilities are accurate for average photon numbers much less than 1. However, notice that it is the scaling factor that has forced the photon number to this low value. We will later consider the limit as the average photon number approaches zero to recover the analytical result, and then adjust the scaling factor to show that it applies to all photon numbers, low and high.

With the above probabilities, the variance in the signal can be found for small expected photon numbers. Using standard statistical techniques the variance is defined by the following equation:

$$\langle (\Delta n)^2 \rangle = \sum_{n=0}^{2} (n - \langle n \rangle)^2 P(n). \tag{11}$$

From Eq. (1) it is reasoned that the variance must have a lower limit defined by Poissonian statistics in the case of infinite modes, and an upper limit defined by the sum of both Poissonian and BE terms in the case of a single mode. It follows then that the variance can be scaled and normalized by the following equation to force the variance between the limits of zero and 1:

$$\overline{\langle (\Delta n)^2 \rangle} = \frac{\langle (\Delta n)^2 \rangle - \langle n \rangle}{\langle n \rangle^2}. \tag{12}$$

From the normalized and scaled variance, finding the limit as the average photon number goes to zero can now be attempted. Solving the equation would be quite difficult; instead a limit-based approach is presented whereby the solution is found.

The first thing to notice is that in most cases where spectrally dependent thermal photon noise will be critical, the number of modes will be small. This is because in systems with large numbers of modes the statistics will approach Poissonian statistics, and the spectral dependence will become negligible. Therefore, the number of modes, M, will be set to 1 and the limit will be found. The number of modes will then be increased and a new limit will be found. This will continue until a fit is found for the limit as a function of the number of modes. The limit of the normalized and scaled variance as the average photon number goes to zero is found to be given by the following equation:

$$\lim_{\langle n \rangle \to 0} [\overline{\langle (\Delta n)^2 \rangle}] = \frac{\sum_{m=1}^{M} \langle n_m \rangle_s{}^2}{\left(\sum_{m=1}^{M} \langle n_m \rangle_s\right)^2}. \tag{13}$$

Equation (13) was verified and confirmed with MATHEMATICA to be exactly correct for $1 \leqslant M \leqslant 36$ modes for any photon distribution $\langle n_m \rangle_s$ with any photon occupancy greater than 0. Systems with more than 36 modes could not be solved exactly, but it is strongly implied that Eq. (13) is exactly correct for any arbitrarily large number of modes. To verify this assumption further, the limit can be solved numerically with some certain defined spectra with more than 36 modes, and no spectra were found that did not obey Eq. (13). More importantly Eq. (13) is exactly correct for any photon occupancy, even when the BE term dominates with an average photon occupancy greater than 1.

Denormalization of the result in Eq. (13) can be completed by substituting the result into Eq. (12) and solving for the variance. Doing so results in the following general theory of thermal photon statistics:

$$\langle (\Delta n)^2 \rangle = \langle n \rangle + \langle n \rangle^2 \frac{\sum_{m=1}^{M} \langle n_m \rangle_s{}^2}{\left(\sum_{m=1}^{M} \langle n_m \rangle_s\right)^2}. \tag{14}$$

It can be shown that Eq. (14) reduces to the standard estimation given in Eq. (1) for a uniform spectrum.

III. DISCUSSION

Equation (14) can be simplified further by noticing that a physical mode distribution is actually just the scaled mode distribution with a scaling factor equal to 1. In this case we can simplify the general theory of thermal photon statistics as in the following equation:

$$\langle(\Delta n)^2\rangle = \langle n\rangle + \langle n\rangle^2 \frac{\sum_{m=1}^{M}\langle n_m\rangle^2}{\left(\sum_{m=1}^{M}\langle n_m\rangle\right)^2}. \tag{15}$$

By substituting Eq. (4) into (15) the following simplifications can be made:

$$\begin{aligned}\langle(\Delta n)^2\rangle &= \sum_{m=1}^{M}\langle n_m\rangle + \left(\sum_{m=1}^{M}\langle n_m\rangle\right)^2 \frac{\sum_{m=1}^{M}\langle n_m\rangle^2}{\left(\sum_{m=1}^{M}\langle n_m\rangle\right)^2}\\ &= \sum_{m=1}^{M}\langle n_m\rangle + \sum_{m=1}^{M}\langle n_m\rangle^2\\ &= \sum_{m=1}^{M}\langle n_m\rangle + \langle n_m\rangle^2. \end{aligned}\tag{16}$$

What is proved by Eq. (16) is that spectrally dependent thermal photon statistics are very simple to compute with a general closed-form expression. The variance in photon number for a thermal source is given by the sum of the variances of each individual mode. This also means that the covariance between any two modes is zero for thermal photons.

As an example of how important this result can be, we calculate the thermal noise in the emission spectrum of an absorbing Fabry-Pérot cavity, as plotted in Fig. 2. When the cavity with this mode distribution is heated, it will emit thermal radiation defined by the spectral emissivity of the cavity multiplied by Planck's law of thermal radiation. The peaks generated by the cavity can be thought of as different thermal emission modes. Integration over each peak will produce the photon mode distribution to be modeled.

The number of modes can be estimated by the following equation [20]:

$$M \approx \frac{8\pi\nu^2\mu^3 V}{c^3}d\nu. \tag{17}$$

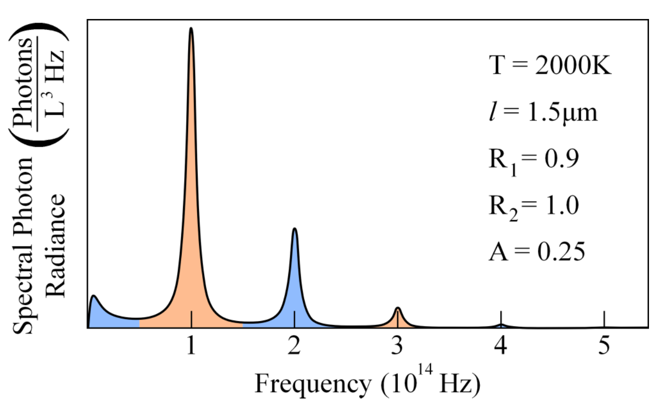

FIG. 2. Spectral photon radiance of a thermal source at 2000 K in a microcavity, calculated for normal incident light. R_1 and R_2 are the reflectivities of the top and bottom cavity mirrors, respectively, A is the single-pass absorption of the cavity, and l is the thickness of the cavity. The discrete modal spectral distribution can be calculated by integrating over each peak as highlighted, and is decidedly nonuniform.

In Eq. (17), μ is the index of refraction in the middle of the cavity, ν is the frequency of light, V is the volume of the cavity, and c is the speed of light. Using the cavity in Fig. 2, the weighted number of modes is approximated as 2.5.

Let us use this cavity to compare the variance predicted by assuming a uniform spectral distribution of photons in Eq. (1), and the exact results derived in Eq. (14). In Eq. (1) the variance is shown to be

$$\langle(\Delta n)^2\rangle \approx \langle n\rangle + (0.4)\langle n\rangle^2, \tag{18}$$

while in Eq. (14) the variance is shown to be (to three significant figures)

$$\langle(\Delta n)^2\rangle = \langle n\rangle + (0.459)\langle n\rangle^2 \tag{19}$$

The average number of photons in the cavity is about 0.173, found by integrating the spectrum in Fig. 2 and multiplying by the volume of the cavity. The standard uniform spectrum approximation results in about a 1% error in the total variance.

At this point is reasonable to ask if such errors would have a measurable impact on a practical microcavity. A cavity with the spectrum shown in Fig. 2 can be constructed of two distributed Bragg reflectors made from alternating SrF_2 and Ge layers, with a doped Ge absorbing layer in a central half-wave cavity layer. The finesse of the cavity is a function of the reflectivity of the mirrors and the absorptivity of the center layer.

Specifically, such a cavity might have a top mirror made of two pairs of 528-nm-thick SrF_2 and 185-nm-thick Ge layers, followed by an air cavity 571 nm thick with a 25-nm doped Ge absorbing layer in the center of the cavity, and finally a bottom mirror made from eight pairs of identical layers as the top mirror.

Two main noise sources, thermomechanical and photon counting noise, can cause the cavity dimensions to depart from their equilibrium positions. Photon pressure within the cavity is another source of noise, although the overall contribution to the total noise is negligible due to the extremely limited number of photons existing in the cavity at any one time. The photon counting noise is inversely proportional to the variance as seen in the following equation [26,27]:

$$\langle(\Delta z)^2\rangle \cong \left(\frac{\lambda}{4\pi F}\right)^2 \frac{1}{\langle(\Delta n)^2\rangle}. \tag{20}$$

In the previous equation λ is the wavelength of light, F is the finesse of the cavity, and z is the displacement of the mirrors and absorber within the cavity relative to their equilibrium positions. If the cavity contains very few photons, the photon counting noise will approach infinity and will dominate all other noise sources. The ambiguity in the cavity center frequency due to the apparent displacement degrades the finesse of the cavity proportionally to the variance in thermal photons. Given a spectrometer with a resolution of 1 cm^{-1}, this cavity could then be used to measure the thermal photon statistics accurately enough to measure a difference between the predictions of Eqs. (18) and (19).

An alternative cavity can be produced where the center frequency is almost independent of the noise, and therefore

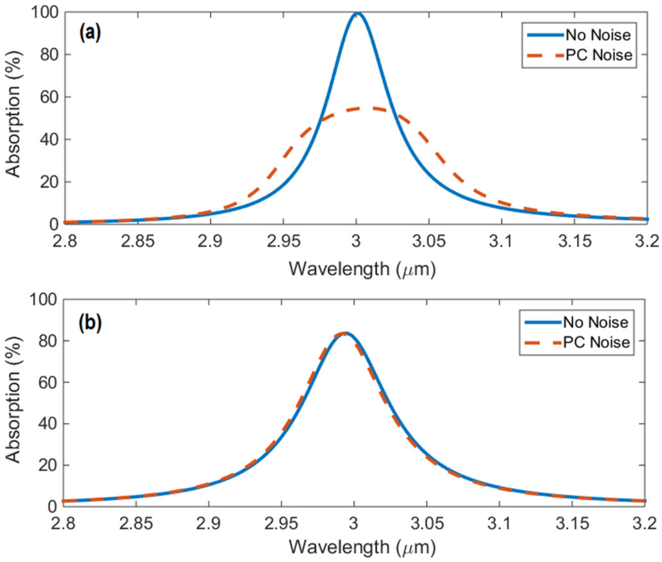

FIG. 3. Photon-counting-noise-limited peak broadening for two cavity designs. The solid lines are calculated for no noise in the system and the dashed lines are calculated for having 50 nm of displacement noise in the system. (a) The first cavity design showing a dramatic reduction in finesse as well as peak height. (b) The second design where the broadened peak is almost indistinguishable from the peak with no noise broadening.

little or no ambiguity in the spectrum occurs. In this case the bottom mirror starts with the Ge layer instead of the SrF_2 layer, and has a total of ten pairs, and also the cavity thickness is increased to 1135 nm. This design is more practical than the previous one where usually a higher finesse is desired. Figure 3 shows the spectral response of the two cavity designs with and without taking into account thermal photon noise.

IV. CONCLUSION

The modal and total variance of thermal photon populations in cavities with arbitrary mode distributions is described. Over the past 60 years, estimations have been used to find the thermal photon variance that work in situations where the number of modes is essentially infinite. With recent developments in optical micro- and nanocavities with small numbers of modes with different couplings to free space, these estimations could lead to significant quantitative inaccuracy. Examples of such error were described.

The analytical expressions derived in Eqs. (14) and (16) are marginally more complex than the standard noise expression of Eq. (1), yet they fully describes thermal photon noise for all systems, are derived from first principles, and make no assumptions.

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APPENDIX: DERIVATION OF THERMAL PHOTON PROBABILITY DENSITY

Photons are indistinguishable from one another, and can occupy the same energy state as each other, meaning they follow Bose-Einstein statistics. They also do not need to be number conserved, i.e., they can be created or destroyed within a system. This means the chemical potential is zero for photons obeying BE statistics.

Thermal photons then further obey the canonical ensemble whereby photons of higher energy are exponentially less likely to exist, given by the Boltzmann distribution. Taking the Boltzmann distribution and applying it to BE statistics one finds that the probability of finding n_m photons in a mode $h\nu$ is given by the following equation:

$$p(n_m) = e^{-n_m h\nu/k_B T}(1 - e^{-h\nu/k_B T}), \tag{A1}$$

where h is Planck's constant, ν is the frequency of the photon, k_B is the Boltzmann constant, and T is the temperature. The average number of photons, $\langle n_m \rangle$, is then given by the following:

$$\langle n_m \rangle = \sum_{n_m=0}^{\infty} n_m p(n_m) = \frac{1}{(e^{h\nu/k_B T} - 1)}. \tag{A2}$$

Multiplication of Eq. (A2) by the energy of a photon and the mode density results in Planck's law of thermal emission. It would be convenient if Eq. (A1) were given in terms of the average photon number as is calculated in (A2). After some algebraic manipulation of Eq. (A2), the following equations are derived [19,25]:

$$\begin{aligned} e^{-n_m h\nu/k_B T} &= \frac{1}{(1 + \langle n_m \rangle^{-1})^{n_m}}, \\ 1 - e^{-h\nu/k_B T} &= \frac{1}{(1 + \langle n_m \rangle)}. \end{aligned} \tag{A3}$$

Substituting the equalities from (A3) into Eq. (A1) results in the useful representation of the probability of finding n_m photons in a mode:

$$p(n_m) = \frac{1}{(1 + \langle n_m \rangle)(1 + \langle n_m \rangle^{-1})^{n_m}}. \tag{A4}$$

Although Eq. (A4) is mathematically nice and easy to work with it has a few limitations when working with thermal photon noise. The first limitation is that this is valid for a single mode. To incorporate systems with multiple or infinite modes (as is the case in most thermal light applications) the joint probability must be used as in Eq. (A5). To find the joint probability the probabilities of finding n_m photons in each mode must be multiplied together. However, there is an added difficulty in that there are multiple distributions possible, thus requiring a sum over all the distributions wherein each modal probability is multiplied [19,25]:

$$P(n) = \sum_{d=1}^{D} \prod_{m=1}^{M} \frac{1}{(1 + \langle n_m \rangle)(1 + \langle n_m \rangle^{-1})^{n_{m,d}}}. \tag{A5}$$

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