Vector-soliton storage and three-pulse-area theorem

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In the present work, we present a high-speed method to control, manipulate, and retrieve an intense vector soliton stored in the ground-state coherences of a four-level atomic system. Additionally, we show the importance of the pulse area in determining the evolution of the system and present a constant in the evolution defined as the three-pulse area; a surprising extension to previously defined pulse areas.

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I. INTRODUCTION

Ever since its discovery, the adiabatic technique of electromagnetically induced transparency (EIT) [1] has been the crux for light storage and manipulation. The interaction of a Λ atomic system with a control field opens a transparency window for a resonant signal field to propagate through without absorption. As the group velocity is directly related to the properties of the optical susceptibility; this also gives a way to control the speed of propagation of the signal field all the way to where it can be stopped and thus mapped into a spin wave. This procedure (and the retrieval of the stored signal) is best described in terms of dark-state polaritons (light-matter excitations), as was done in Ref. [2].

This opened the door to numerous techniques for optical information storage where the main contenders are a far-off-resonant Raman scheme [3] and a photon-echo-based procedure [4]. For the first, the fields are highly detuned, thus allowing the adiabatic elimination of the higher atomic level. Thus, the field is mapped into the ground states via stimulated Raman transitions. The second is an extension of the well-known phenomenon of photon echo in inhomogeneously broadened two-level atoms [5]. A resonant signal-photon wave packet is absorbed and subsequentially mapped into the stable ground states by means of a short π-control pulse. The stored field can be recovered by a second counterpropagating π-control pulse. Furthermore, in a series of papers [6], Gorshkov *et al.* brought all of these techniques into a "universal approach to optimal photon storage" and devised a procedure to maximize the efficiency given any signal field.

The enormous success of the Λ configuration motivated the study of more complicated atomic systems, such as the double Λ system where enhancement of nonlinear effects can be achieved as well as the storage of two signal pulses [7]. An N-type system has also been proposed for the control of two-photon absorption via quantum interference which can lead to an improvement in EIT [8], as well as a giant enhancement of the Kerr nonlinearity [9].

One of the most prolific extensions has been the four-level system in a tripodal configuration (see Fig. 1) where many advances have been made for light control and storage (e.g., double EIT [10]). The formalism of dark-state polaritons [2] has been extended for this four-level system in different kinds of scenarios. Depending on the initial preparation of the medium, there can be either two signal pulses and one control, which leads to the possibility of storage of a photonic quantum bit [11], or there can be just one signal pulse and two control pulses that allow two-channel light storage [12]. The existence of temporal and spatial vector solitons in this four-level system has also been shown [13]. Furthermore, it has been demonstrated that, under the influence of a classical (intense) control field, a propagating probe field acquires a phase that affects its state of polarization. There are proposals for using this effect to enhance the sensibility of Faraday magnetometers [14] or to serve as a polarization phase gate [15]. In addition, some of these new phenomena have been demonstrated experimentally, such as dark resonance switching [16], the existence of two transparency windows, enhanced crossed-phase modulation [17], the propagation of matched slow pulses [18], and two-field storage [11,19] (we refer the interested reader to these manuscripts for examples of experimental realizations of the tripodal scheme).

The original work on pulse storage was tied to a requirement for adiabaticity (effectively near-constant intensity) of the control pulse, which limits the speed of the process. This point was carefully studied by Matsko *et al.* [20]. They showed that the storage and retrieval are still possible by "instantaneously" switching the control field off and on. This was further studied by Shakhmuratov *et al.* [21] where they added the effect of a rf field in an N configuration. Even though most proposals work with cw control fields that are turned off and on, this might not be the best strategy. When one considers the optimization problem, the resulting optimal field acquires a temporal structure that clearly deviates from the standard cw field [3,6]. Another usual assumption is that the signal field is of quantum nature (low intensity). In Ref. [22] they go beyond these assumptions by means of a series of numerical experiments, but they restrict themselves to cw control fields and use the adiabatic theory of Grobe *et al.* [23] to interpret their results because their signal pulses are long enough (the duration is about 100 times the lifetime in the excited state) that spontaneous emission needs to be included.

In the present work, we depart from the usual considerations for pulse storage because we consider the joint evolution of resonant-intense-broadband pulses (these are much shorter and about two orders of magnitude stronger that those considered in Ref. [22]). This is the realm of self-induced transparency

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(SIT), which was introduced by McCall and Hahn in their seminal papers [24]. They showed the crucial role that the total pulse area, defined as

$$\theta(x) = \int_{-\infty}^{\infty} \Omega(x,t)dt, \tag{1}$$

plays in this type of coherent interaction. Recent research has shown how the interaction of broadband pulses with matter opens the door to high-speed switching. In this framework, it has been shown that storage, manipulation, and retrieval of a signal pulse in a Λ system is possible [25] even in nonidealized conditions [26] and how the methodology presented there can be extended to accommodate the storage of multiple pulses and added control of the information stored [27]. A generalized two-pulse area was shown to play a role [28]: $\Theta_{12}(x) = [|\theta_1(x)|^2 + |\theta_2(x)|^2]^{1/2}$.

We now extend this exploration to the tripodal atom interacting with three fields in resonance. We show how a vector soliton can be stored in the coherences of the ground states and then retrieved. We also find that the competing process of stealing population from the common excited state leads to a constraint on the area of each field as determined by the three-pulse area defined in Eq. (10). This is in alignment with previous results for the Λ and double-Λ systems [28,29].

II. THEORETICAL MODEL

For the tripodal atom (see Fig. 1), each ground state is only connected to the excited state $|0\rangle$ by the dipole moment operator and the fields are given by

$$\vec{E}(x,t) = \sum_{j=1}^{3} \vec{\mathcal{E}}_j(x,t) e^{i(k_{j0}x - \omega_{j0}t)} + \text{c.c.} \tag{2}$$

Here, ω_{10}, ω_{20}, and ω_{30} are the field frequencies, k_{10}, k_{20}, and k_{30} are the vacuum wave numbers, and $\vec{\mathcal{E}}_1(x,t)$, $\vec{\mathcal{E}}_2(x,t)$, and $\vec{\mathcal{E}}_3(x,t)$ are the slowly-varying-field envelopes. For simplicity, the fields are taken to be at resonance, and we will neglect the effects of Doppler broadening by considering a gas of cold atoms (given some minor substitutions, the results presented here will remain valid even in the presence of Doppler broadening [30]). We will further assume that the fields are pulses

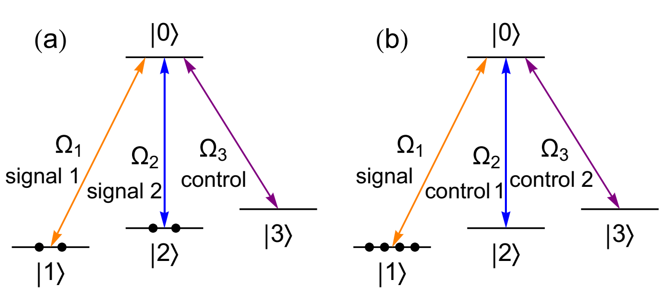

FIG. 1. Four-level atom in a tripodal configuration, interacting with three fields in resonance: (a) with the population initially distributed equally but incoherently among the ground states $|1\rangle$ and $|2\rangle$, which leads to the presence of two signal pulses and one control, and (b) with all the population in state $|1\rangle$, which leads to one signal pulse and two control pulses.

short enough to neglect spontaneous emission but long enough so that the variation of the envelopes is much slower than the oscillations given by the optical frequency. This justifies the use of the slowly-varying-envelope approximation (SVEA). Considering the dipole and rotating-wave approximations, the Hamiltonian takes the form

$$H = -\frac{\hbar}{2}(\Omega_1|0\rangle\langle 1| + \Omega_2|0\rangle\langle 2| + \Omega_3|0\rangle\langle 3|) + \text{c.c.}, \tag{3}$$

where we defined the Rabi frequencies $\Omega_j(x,t) = 2\vec{d}_{0j} \cdot \vec{\mathcal{E}}_j(x,t)/\hbar$, and the nonzero off-diagonal elements of H can then be written as $H_{0j} = -\hbar\Omega_j(x,t)/2$ for $j = 1, 2, 3$.

The dynamics of the field-matter system are described in terms of the von Neumann equation for the density matrix,

$$i\hbar\frac{\partial\rho}{\partial t} = [H,\rho], \tag{4}$$

and Maxwell's wave equations in the slowly-varying-envelope approximation:

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\Omega_1 = i\mu_{10}\rho_{01}, \tag{5a}$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\Omega_2 = i\mu_{20}\rho_{02}, \tag{5b}$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\Omega_3 = i\mu_{30}\rho_{03}. \tag{5c}$$

where we defined the atom-field coupling parameters as $\mu_{j0} = N\omega_{j0}|d_{j0}|^2/\hbar\epsilon_0 c$ with $j = 1, 2, 3$. These give a set of nonlinear partial differential equations that need to be solved simultaneously. Therefore, we cannot move forward, at least analytically, without making further assumptions. The key lies in considering equal atom-field coupling parameters for all three transitions, $\mu_{10} = \mu_{20} = \mu_{30} = \mu$ (this assumption might not be realistic but, given the stability of this type of solution in nonidealized conditions shown in Ref. [26], we expect the results to remain valid for the most part). Doing so, and introducing the constant matrix $W = i|0\rangle\langle 0|$, we can write the evolution equations as

$$i\hbar\frac{\partial\rho}{\partial T} = [H,\rho] \quad \text{and} \quad \frac{\partial H}{\partial Z} = -\frac{\hbar\mu}{2}[W,\rho], \tag{6}$$

in terms of the traveling-wave coordinates $T = t - x/c$ and $Z = x$. By computing the commutator, one can verify that the matrix equation for the field [Eqs. (6)] indeed reduces to Eqs. (5) plus some (irrelevant) trivial equations.

From the form in which the evolution equations were written in Eqs. (6), it is easy to show that the system is integrable (Appendix A) and therefore solvable by standard methods such as inverse scattering [31], the Bäcklund transformation [32], or the Darboux transformation [33], to name a few. This study contrasts with many previous results in that it considers the simultaneous evolution of the three fields instead of assuming one of them to be a strong constant field that induces some extra nonlinearities in the evolution of the other two [13–15]. In addition, we consider the full nonlinear interaction which contrasts with the first-order approximation in the fields done in Ref. [10].

III. TWO-PULSE STORAGE

We now proceed to solve the Maxwell–Bloch equations for a tripodal atomic system [Eqs. (6)] by using the single-soliton Darboux transformation and the nonlinear superposition rule. This method allows us to start from a trivial solution and obtain complicated pulse dynamics by some integration and algebraic manipulation. A review of this method can be found in Refs. [27,33] and is similar to the one presented by Clader and Eberly in Ref. [28]. For the interested reader, an outline of the principal steps is presented in Appendix A.

We start by taking the situation depicted in Fig. 1(a) as our trivial solution; that is, an incoherent preparation of the medium given by $\rho = 1/2(|1\rangle\langle 1| + |2\rangle\langle 2|)$ and no fields. By applying the Darboux transformation to this seed solution, we obtain a first-order solution for which the analytic expressions can be simplified in the limits of infinitely long negative and positive times (the expression of the involution matrix used to compute the solution in these limits is presented in Table I of Appendix B). This provides us with the state of the system before and after the interaction. An example of the pulse dynamics is represented in Fig. 2.

Initially ($T/\tau_a \ll -1$), we have two SIT pulses of duration τ_a propagating throughout the medium at a reduced group velocity for the two signal fields. Their shapes are given by

$$\Omega_1 = \frac{2}{\tau_a}\cos\left(\frac{\nu}{2}\right)\operatorname{sech}\left(\frac{T}{\tau_a} - \frac{\kappa_a}{4}Z + \eta\right), \tag{7a}$$

$$\Omega_2 = \frac{2}{\tau_a}e^{i\phi}\sin\left(\frac{\nu}{2}\right)\operatorname{sech}\left(\frac{T}{\tau_a} - \frac{\kappa_a}{4}Z + \eta\right), \tag{7b}$$

where we introduced the absorption coefficient $\kappa_a = \mu\tau_a/2$ and the constants of integration η, ϕ, and ν. The angles ϕ and ν define the area of each signal pulse, $\theta_1 = 2\pi\cos(\nu/2)$ and $\theta_2 = 2\pi e^{i\phi}\sin(\nu/2)$ (the area of the first pulse was taken to be real in order to fix the global phase). Note that the signal pulses

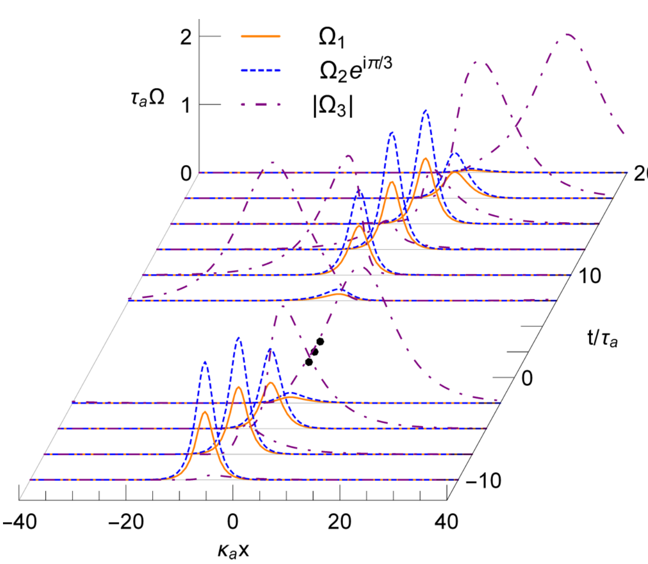

FIG. 2. Pulse evolution dictated by the second-order solution obtained from the medium preparation shown in Fig. 1(a). The encoding of the vector soliton is separated by the ellipses from its retrieval and displacement.

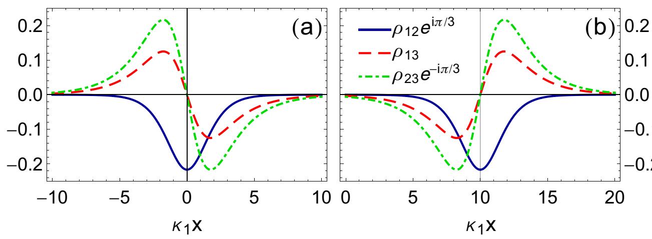

FIG. 3. Imprint as has been encoded by the pulse dynamics shown in Fig. 2 in the ground-state coherences (a) before and (b) after the displacement.

are "normalized" by the total two-pulse area (as defined by Clader and Eberly in Ref. [28]); that is, $(|\theta_1|^2 + |\theta_2|^2)^{1/2} = 2\pi$. In this limit, the control pulse tends towards zero. If we assume that the two fields address different transitions due to their polarization as in Ref. [13], then the stored field can be written as

$$\vec{E}_s(x,t) = \frac{\hbar}{d\tau_a}\left[\cos\left(\frac{\nu}{2}\right)\vec{p}_1 + e^{i\phi}\sin\left(\frac{\nu}{2}\right)\vec{p}_2\right] \times \operatorname{sech}\left(\frac{T}{\tau_a} - \frac{\kappa_a}{4}Z + \eta\right)e^{i(k_s x - \omega_s t)} + \text{c.c.} \tag{8}$$

to show that it can be seen as a vector soliton ($\vec{p}_1$ and $\vec{p}_2$ are two orthogonal polarization vectors). Given the free parameters ϕ and ν, the polarization can occupy any point on the surface of the Poincaré sphere.

As the signal pulses propagate, the control pulse starts to drive some of the population to the ground state $|3\rangle$, thus amplifying its asymptotically small amplitude while depleting the signal pulses. This transfer slowly takes over until we are left with a decoupled control pulse propagating away with the light's phase velocity. The full solution for the pulses is provided in Eqs. (B3). During the interaction, the information of the vector soliton is imprinted into the ground-state elements of the density matrix in the form of a spin-wave. This "imprint" is depicted in Fig. 3. Even if the coherences are nonzero everywhere (because of the exponentially weak but infinitely long tails of the sech function) it is clear that they are well localized around their center, which we identify as their location. Computing the total pulse area for each pulse [see Eqs. (B5)], we get

$$\frac{\sqrt{|\theta_1(x)|^2 + |\theta_2(x)|^2}}{|\theta_3(x)|} = e^{-\frac{\kappa_a}{2}(x - x_1)}, \tag{9}$$

where x_1 gives the location of the imprint. Therefore, the imprint is located at the position where the two-signal-pulse area is equal to the control pulse area.

This two-pulse storage, when taken to low intensities (quantum states of light), can be seen as a qubit-storage procedure because the phase between the two pulses is also encoded within the atomic medium. This was actually done in Ref. [11] but was based on the usual EIT procedure of slowing down the quantum signal by means of a classical control pulse.

Taking a closer look at the expressions for the area of each pulse in the first-order solution, we see that a role for a three-pulse area is implied:

$$\Theta_{123}(x) = \sqrt{|\theta_1(x)|^2 + |\theta_2(x)|^2 + |\theta_3(x)|^2} = 2\pi. \tag{10}$$

This result is a clear statement of the relationship between this solution and the original SIT solution. The pulse area is a key quantity, not only controlling the reshaping of pulses but also the storage and manipulation of the information stored in the medium. Its importance goes well beyond the two-level atom remaining a constant in pulse dynamics for multilevel systems [28,29]. For nonidealized input-pulse shapes, these multilevel systems follow a behavior similar to the predictions of the area theorem for a two-level system [28,29,34] even in the absence of Doppler broadening [26].

Other results referring to pulse-area theorems have been worked out for the Λ configuration. The first was done by Tan-no *et al.* [35] but is limited to highly detuned fields in two-photon resonance. Clader and Eberly presented a clear comparison of this stimulated Raman scattering (SRS) with the exact Maxwell–Bloch equations [28]. Another famous result is the dark-area theorem [36], where the dark area is not defined in the usual manner (it has time derivatives of the Rabi frequencies). Nevertheless, it provides a simple spatial evolution of a quantity involving both Rabi frequencies, which helps to understand the interaction of the two fields. This result is very much in line with the original area theorem [24]. A more recent result was presented by Shchedrin *et al.* in Ref. [37]. There they consider a pulse interaction beyond the rotating-wave approximation, extending the applicability of their results. However, the pulse-area theorem lacks any mention of spatial evolution and is in fact formulated more like an energy conservation equation which still involves the Rabi frequencies. To this we could add the conservation laws that can be deduced from the integrability of the Maxwell–Bloch equations (see, for example, Ref. [38]). As we already mentioned, our result is similar to those presented in Refs. [28,29], and thus we expect homologous results for the case of partially mixed states [34].

Another valid solution, which we can obtain from the same seed solution (by an appropriate choice of integration constants; see Appendix B), is that of a sech-shaped 2π-control pulse propagating through the medium at the speed of light in vacuum. Superimposing this solution with the one previously described, we obtain a second-order solution that (by an appropriate choice of parameters) describes a well-defined pulse sequence. Here, we first have the storage of the signal pulses and then the collision of another control pulse, of different duration τ_b, with the imprint. Upon interaction with the imprint, the control pulse retrieves the stored signal pulses along with the pulse area and relative phase information. The retrieved pulses then propagate farther into the medium and then are again stored, but in a displaced location (see Figs. 2 and 3). The displacement of the imprint is controlled by the duration of the pulses via

$$\delta = \kappa_a(x_2 - x_1) = 2\ln\left|\frac{\tau_a + \tau_b}{\tau_a - \tau_b}\right|, \tag{11}$$

where x_2 is the new location of the imprint.

In reality, we are not going to have an infinite medium, but this can be used to our advantage. Given a finite medium, Eq. (9) tells us how we should map this solution to initial conditions, because the ratio of the areas of the entering pulses will control the location of the imprint. Now, if the duration τ_b is tailored so that the displacement given by Eq. (11) is larger than the length of the medium, the control pulse will move the imprint outside. This would frustrate the re-encoding of the signal pulses and thus effectively retrieve the information stored in the atomic medium. Therefore, the finiteness of the medium provides us with the means to retrieve the stored pulses. Additionally, there could be some residual coherence between the ground states $|1\rangle$ and $|2\rangle$ from the initial preparation. However, by using the same formalism we can show that the storage-retrieval procedure is still viable (an extended discussion about the effects of partial coherence in a Λ system is presented in Ref. [34] and one would expect similar results for this case).

IV. TWO-CHANNEL MEMORY

The three-pulse area is, of course, not limited to the specific solution of the storage of a vector soliton. We can consider another possibility for the seed solution, such as the one depicted in Fig. 1(b). The medium is prepared in the ground state $|1\rangle$ and again with no fields. In this case, the first-order solution describes a signal pulse that is stored via the interaction of two control pulses with the atomic system as intermediary. The pulse dynamics start with an SIT signal pulse of duration τ_a propagating at a reduced velocity. As this pulse propagates, the asymptotically small control pulses start to drive some of the population to the ground states $|2\rangle$ and $|3\rangle$, thus amplifying them while depleting the signal pulse. The full solution for the pulses is provided in Eqs. (B8). During this interchange, the signal pulse is encoded in the coherences of the ground states. We can think of this system as having two channels represented by the coherences ρ_{12} (channel 1-2) and ρ_{13} (channel 1-3), connected by the coherence ρ_{23}. Computing the total pulse area for each pulse [see Eqs. (B10)], we get the following relation:

$$\frac{|\theta_1(x)|}{\sqrt{|\theta_2(x)|^2 + |\theta_3(x)|^2}} = e^{-\kappa_a(x-x_1)}, \tag{12}$$

where x_1 denotes the location of the imprint. In a similar fashion to the previous solution, the initial location of the imprint is determined by the ratio between the signal-pulse area and the two control-pulse areas. The amount of information stored in channel 1-2 is determined by the ratio between the area of the control pulse Ω_2 and the two-control-pulse area squared, $r_{1\text{-}2} = |\theta_2(x)|^2/[|\theta_2(x)|^2 + |\theta_3(x)|^2]$, and similarly for channel 1-3, $r_{1\text{-}3} = |\theta_3(x)|^2/[|\theta_2(x)|^2 + |\theta_3(x)|^2]$. The particular case of storing in just one channel can be seen in Fig. 4. The possibility for two-channel storage had already been mentioned in Ref. [12] but this study was based on an EIT-type interaction.

Here again, by studying the expressions for the individual pulse areas, we find that, by summing them as prescribed by Eq. (10), we obtain $\Theta_{123}(x) = 2\pi$. This goes to show that the three-pulse area is not limited to a specific type of preparation for the system. Now, the solution just presented is interesting by itself. Therefore, its interaction with subsequent pulses is well deserving of mention.

Let us consider the second-order solution born out of the superposition of the solution previously discussed and that of two control pulses of total two-pulse area equal to 2π and duration τ_b decoupled from the medium. We discover that

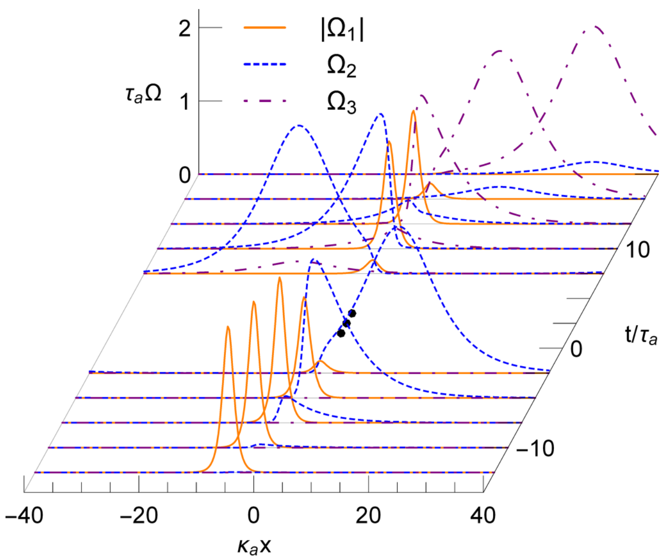

FIG. 4. Pulse evolution dictated by the second-order solution obtained from the medium preparation shown in Fig. 1(b). The encoding of the signal pulse is separated by the ellipses from the channel-switching and displacement.

the information can be displaced between channels by the subsequent interaction of the imprint with other control pulses. For simplicity, we consider the case in which the imprint was only encoded into channel 1-2 and we want to move all the information into the other channel (this case is depicted in Figs. 4 and 5). In this scenario, simple expressions can be worked out for the necessary pulse areas to achieve the channel switching of the information encoded in the medium and its displacement. These are given by

$$|\theta_2| = 2\pi\sqrt{\frac{1+\tau_b/\tau_a}{2}} \quad \text{and} \quad |\theta_3| = 2\pi\sqrt{\frac{1-\tau_b/\tau_a}{2}}, \tag{13}$$

and the corresponding displacement of the imprint is

$$\delta = \frac{1}{2}\ln\left(\frac{\tau_a+\tau_b}{\tau_a-\tau_b}\right). \tag{14}$$

We note that the two things are related and, depending on the duration of the control pulses, we have to choose the appropriate pulse area. Here, we assumed that $\tau_a > \tau_b$. The relative phase between the two control pulses determines the phase of the coherence which in turn dictates the phase of the retrieved pulse. The channel-switching of the imprint is shown in Fig. 5. Additionally, as long as the signal pulse is only stored in one channel, we can displace the imprint and retrieve it (if we consider a finite medium) by the results already obtained for a Λ system [25,26].

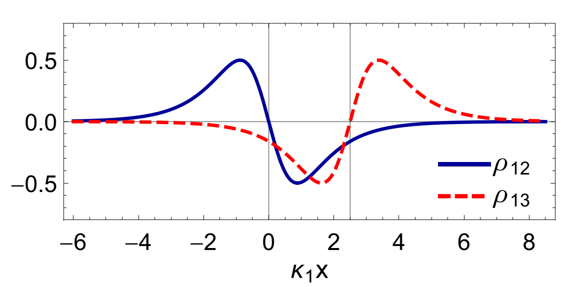

FIG. 5. Imprint as it has been encoded by the pulse dynamics shown in Fig. 4 in the ground-state coherences before (continuous lines) and after (dashed lines) the channel switching.

It is also interesting to note that, together with the displacement of the imprint, the intensity of the control pulses is inverted (see Fig. 4). If the two control pulses address different transitions due to their polarization, then this inversion of intensity is actually a rotation in the polarization state of the vector-control pulse. This change in the polarization in the tripodal configuration has already been studied in different regimes of light [14,15]. Of course the change of polarization is directly tied to the initial imprint; namely, its phase and the ratios $r_{1\text{-}2}$ and $r_{1\text{-}3}$.

It is also relevant to mention that, if we had considered an initial preparation of the medium analogous to the one in Ref. [39] (i.e., incoherent superposition of the ground states), we would have obtained a three-color switching analogous to the polarization switching mentioned in their work along with similar stochastic dynamics when dealing with disordered populations [40].

V. SUMMARY

In summary, we presented analytic solutions to the Maxwell–Bloch equations for a tripodal system, which suggest the possibility of high-speed storage and retrieval of a vectorial soliton as well as that of a two-channel memory. These results are similar to those derived in Refs. [11,12], which are based on the dark-state-polariton formalism which inevitably carries with it the adiabatic approximation limiting the speed of the processes involved. Still, it is interesting to note the similarities of our results to previous treatments dealing with quantum states of light entailing the free-space propagation of the control pulse and thus the omission of any change of shape during propagation. We also defined an extension for the area theorem in this three-pulse scenario, which is not tied to a specific preparation of the medium. This result raises new questions, such as: What is the appropriate way to combine the individual pulse areas in given multilevel and multipulse systems? Clearly, there must be a dependence on the connection between atomic levels and the number of ground and excited states. We expect these results to be of scientific importance for the continuing development of the pulse-storage and -manipulation field.

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APPENDIX A: DARBOUX TRANSFORMATION AND NONLINEAR SUPERPOSITION RULE

In what follows we present an outline of the steps to take to obtain the desired solution. The interested reader is referred to more complete treatments, such as those found in Refs. [27,33]. We start by defining the operators $U(\lambda) =$

$-(i/\hbar)H - \lambda W$ and $V(\lambda) = (i\mu/2\lambda)\rho$, where λ is a constant known as the spectral parameter. By using the Maxwell–Bloch equations [Eqs. (4) in the main text] it is easy to show that the equation

$$\partial_Z U - \partial_T V + [U,V] = 0 \tag{A1}$$

is satisfied. This is known as the Lax equation and is used as a criterion for the integrability of a system of equations. The operators U and V are called the Lax operators. Therefore, it is possible to obtain solutions by methods such as the Darboux transformation.

The idea behind this method is to build complicated (and interesting) solutions out of trivial ones. Hence, the first step is to find a trivial solution for our system of equations. An example is the one used for the two-channel-memory solution where the fields are taken equal to zero and the medium is prepared in the ground state $|1\rangle$. This trivial or zeroth-order solution is identified by the density matrix ρ^0 and the Hamiltonian H^0 which give the corresponding Lax operators U^0 and V^0. The next step is to solve the following linear equations:

$$[I\partial_T - U^0(-\lambda_a)]\,|\varphi\rangle = 0, \tag{A2a}$$

$$[I\partial_Z - V^0(-\lambda_a)]|\varphi\rangle = 0. \tag{A2b}$$

These equations are derived by requiring the new Lax pair to satisfy the Lax equation which is equivalent to the Maxwell–Bloch equations and demanding a number of additional properties, such as conservation of the spectral dependence and Hermiticity of the density matrix and Hamiltonian. By solving them we obtain the vector $|\varphi\rangle$ in terms of which a unitary involution is defined as

$$M^a = 2\frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle} - I. \tag{A3}$$

Finally, from this matrix we can construct a new first-order solution by using

$$H^a = H^0 - i\hbar\lambda_a[M^a,W], \tag{A4a}$$

$$\rho^a = M^a\rho^0 M^a. \tag{A4b}$$

In principle, the procedure just described could be used to compute higher-order solutions, but the reality is that Eqs. (A2) become harder to solve with each step. Fortunately, there is no need due to the existence of a nonlinear superposition rule. Given a seed (zeroth-order) solution we can derive a first-order solution with parameter λ_a and Lax operators U^a and V^a, as just described. From this solution, a second-order solution can be constructed by solving Eqs. (A2) with parameter λ_b and Lax operators U^a and V^a and thus obtain the second-order involution M^{ab} and Lax operators U^{ab} and V^{ab}. Alternatively, we could have first computed the first-order solution with parameter λ_b and from this obtained the second-order solution identified by Lax operators U^{ba} and V^{ba}. The theorem of permutability states that there is nothing special about the order in which we compute the solutions, thus both second-order solutions should be the same; that is, $U^{ab} = U^{ba}$ and $V^{ab} = V^{ba}$. From this condition we obtain the nonlinear superposition rule which gives us the involution matrix

$$M^{ab} = (\lambda_a M^a - \lambda_b M^b)(\lambda_a M^a M^b - \lambda_b I)^{-1}, \tag{A5}$$

and the second-order density matrix and Hamiltonian are

$$\rho^{ab} = M^{ab}M^a\rho^0 M^a M^{ab}, \tag{A6a}$$

$$H^{ab} = H^0 - i\hbar\big(\lambda_a^2 - \lambda_b^2\big)[(\lambda_a M^a - \lambda_b M^b),W]. \tag{A6b}$$

APPENDIX B: FIRST-ORDER SOLUTIONS AND LIMITING EXPRESSIONS

1. Two-pulse storage

Taking the seed to be the trivial solution of a medium in an incoherent superposition of the ground states $|1\rangle$ and $|2\rangle$

TABLE I. Elements of the involution matrix M^a for the two-pulse-storage first-order solution showing only the elements in the limits of infinite negative and positive times because these can be written in a simple form.[a]

	$T/\tau_a \ll -1$	$T/\tau_a \gg 1$
M^a_{00}	$-\tanh\left(\bar{T} - \frac{\bar{Z}}{2} + \eta_{+0}\right)$	-1
M^a_{11}	$\frac{\lvert c_1\rvert^2}{\lvert c_+\rvert^2}\left[1 + \tanh\left(\bar{T} - \frac{\bar{Z}}{2} + \eta_{+0}\right)\right] - 1$	$\frac{\lvert c_1\rvert^2}{\lvert c_+\rvert^2}\left[1 - \tanh\left(-\frac{\bar{Z}}{2} + \eta_{+3}\right)\right] - 1$
M^a_{22}	$\frac{\lvert c_2\rvert^2}{\lvert c_+\rvert^2}\left[1 + \tanh\left(\bar{T} - \frac{\bar{Z}}{2} + \eta_{+0}\right)\right] - 1$	$\frac{\lvert c_2\rvert^2}{\lvert c_+\rvert^2}\left[1 - \tanh\left(-\frac{\bar{Z}}{2} + \eta_{+3}\right)\right] - 1$
M^a_{33}	-1	$-\tanh\left(-\frac{\bar{Z}}{2} + \eta_{+3}\right)$
M_{01}	$\frac{c_0 c_1^*}{\lvert c_0 c_+\rvert}\operatorname{sech}\left(\bar{T} - \frac{\bar{Z}}{2} + \eta_{+0}\right)$	0
M_{02}	$\frac{c_0 c_2^*}{\lvert c_0 c_+\rvert}\operatorname{sech}\left(\bar{T} - \frac{\bar{Z}}{2} + \eta_{+0}\right)$	0
M_{03}	0	$\frac{c_0 c_3^*}{\lvert c_0 c_+\rvert}\operatorname{sech}\left(\bar{T} + \eta_{30}\right)$
M^a_{12}	0	$\frac{c_1 c_2^*}{\lvert c_+\rvert^2}\left[1 + \tanh\left(-\frac{\bar{Z}}{2} + \eta_{+3}\right)\right]$
M^a_{13}	0	$\frac{c_1 c_3^*}{\lvert c_3 c_+\rvert}\operatorname{sech}\left(-\frac{\bar{Z}}{2} + \eta_{+3}\right)$
M^a_{23}	0	$\frac{c_2 c_3^*}{\lvert c_3 c_+\rvert}\operatorname{sech}\left(-\frac{\bar{Z}}{2} + \eta_{+3}\right)$

[a]We defined the parameters $\bar{T} = T/\tau_a$, $\bar{Z} = \mu\tau_a Z/2$, $|c_+|^2 = |c_1|^2 + |c_2|^2$, $\eta_{jk} = \ln|c_j/c_k|$.

[$\rho^0 = 1/2(|1\rangle\langle 1| + |2\rangle\langle 2|)$] and fields off ($H^0 = 0$), Eqs. (A2) can be solved to obtain

$$|\varphi^a\rangle = \begin{pmatrix} c_0 e^{-T/\tau_a} \\ c_1 e^{-\mu\tau_a Z/4} \\ c_2 e^{-\mu\tau_a Z/4} \\ c_3 \end{pmatrix}. \tag{B1}$$

Here, the spectral parameter was written as $\lambda_a = i/\tau_a$, with $\tau_a \in \mathbb{R}$ and where c_0, c_1, c_2, c_3 are constants of integration. From this it is easy to compute the M matrix but the exact expression of its elements is cumbersome and not very illustrative. Therefore, we limit ourselves to write the simplified expressions for large positive and negative times in Table I.

From the constants of integration we can define other, more representative parameters:

$$\cos\left(\frac{\nu}{2}\right) = i\frac{c_0 c_1^*}{|c_0 c_+|}, \tag{B2a}$$

$$e^{i\phi}\sin\left(\frac{\nu}{2}\right) = i\frac{c_0 c_2^*}{|c_0 c_+|}, \tag{B2b}$$

$$\eta = \eta_{+0} = \ln\left|\frac{c_+}{c_0}\right|, \tag{B2c}$$

$$\kappa_a x_1 = 2\eta_{+3} = 2\ln\left|\frac{c_+}{c_3}\right|. \tag{B2d}$$

These are the parameters used in Eqs. (5)–(7). The most general solution (when all integration constants are not zero) gives us the solution that describes the storage of the vector soliton. This solution is then superimposed with another solution describing a control pulse propagating by itself decoupled from the medium. This second solution is obtained from setting $c_1 = c_2 = 0$.

TABLE II. Elements of the involution matrix M^a for the two-channel-memory first-order solution showing only the elements in the limits of infinite negative and positive times because these can be written in a simple form.[a]

	$T/\tau_a \ll -1$	$T/\tau_a \gg 1$
M^a_{00}	$-\tanh(\bar{T} - \bar{Z} + \xi_{10})$	-1
M^a_{11}	$\tanh(\bar{T} - \bar{Z} + \xi_{10})$	$\tanh(-\bar{Z} + \xi_{1+})$
M^a_{22}	-1	$\frac{\lvert d_2\rvert^2}{\lvert d_+\rvert^2}[1 - \tanh(-\bar{Z} + \xi_{1+})] - 1$
M^a_{33}	-1	$\frac{\lvert d_3\rvert^2}{\lvert d_+\rvert^2}[1 - \tanh(-\bar{Z} + \xi_{1+})] - 1$
M_{01}	$\frac{d_0 d_1^*}{\lvert d_0 d_1\rvert}\operatorname{sech}(\bar{T} - \bar{Z} + \xi_{10})$	0
M_{02}	0	$\frac{d_0 d_2^*}{\lvert d_0 d_+\rvert}\operatorname{sech}(\bar{T} + \xi_{+0})$
M_{03}	0	$\frac{d_0 d_3^*}{\lvert d_0 d_+\rvert}\operatorname{sech}(\bar{T} + \xi_{+0})$
M^a_{12}	0	$\frac{d_1 d_2^*}{\lvert d_1 d_+\rvert}\operatorname{sech}(-\bar{Z} + \xi_{1+})$
M^a_{13}	0	$\frac{d_1 d_3^*}{\lvert d_1 d_+\rvert}\operatorname{sech}(-\bar{Z} + \xi_{1+})$
M^a_{23}	0	$\frac{d_2 d_3^*}{\lvert d_+\rvert^2}[1 - \tanh(-\bar{Z} + \xi_{1+})]$

[a]We defined the parameters $\bar{T} = T/\tau_a$, $\bar{Z} = \mu\tau_a Z/2$, $|d_+|^2 = |d_2|^2 + |d_3|^2$, $\xi_{jk} = \ln|d_j/d_k|$.

For the first-order solution, the complete form for the fields is given by

$$\Omega_1 = \frac{4ic_0c_1^*}{\tau_a}e^{-\kappa_a Z/2}f(Z,T), \tag{B3a}$$

$$\Omega_2 = \frac{4ic_0c_2^*}{\tau_a}e^{-\kappa_a Z/2}f(Z,T), \tag{B3b}$$

$$\Omega_3 = \frac{4ic_0c_3^*}{\tau_a}f(Z,T), \tag{B3c}$$

with

$$f(Z,T) = \frac{e^{-T/\tau_a}}{|c_0|^2 e^{-2T/\tau_a} + |c_+|^2 e^{-\kappa_a Z} + |c_3|^2}. \tag{B3d}$$

It is clear that the integration with respect T for the pulse areas only involves the function f and

$$\int_{-\infty}^{\infty} f(Z,T)dT = \frac{\tau_a \pi}{2|c_0|}(|c_+|^2 e^{-\kappa_a Z} + |c_3|^2)^{-1/2}. \tag{B4}$$

The pulse area can be readily written as

$$\theta_1 = \frac{2\pi i c_0 c_1^*}{|c_0|}\frac{e^{-\kappa_a Z/2}}{\sqrt{|c_+|^2 e^{-\kappa_a Z} + |c_3|^2}}, \tag{B5a}$$

$$\theta_2 = \frac{2\pi i c_0 c_2^*}{|c_0|}\frac{e^{-\kappa_a Z/2}}{\sqrt{|c_+|^2 e^{-\kappa_a Z} + |c_3|^2}}, \tag{B5b}$$

$$\theta_3 = \frac{2\pi i c_0 c_3^*}{|c_0|}\frac{1}{\sqrt{|c_+|^2 e^{-\kappa_a Z} + |c_3|^2}}. \tag{B5c}$$

The results given in Eqs. (9) and (10) follow.

2. Two-channel memory

Now let us consider the other seed solution considered in the main text; the medium in the ground state $|1\rangle$ ($\rho^0 = |1\rangle\langle 1|$) and fields off ($H^0 = 0$). We solve Eqs. (A2) and obtain

$$|\varphi^a\rangle = \begin{pmatrix} d_0 e^{-T/\tau_a} \\ d_1 e^{-\mu\tau_a Z/2} \\ d_2 \\ d_3 \end{pmatrix}. \tag{B6}$$

Here again, the spectral parameter is identified with the pulse duration $\lambda_a = i/\tau_a$ and d_0, d_1, d_2, d_3 are the constants of integration. The simplified expressions of the elements of the involution matrix for large positive and negative times are shown in Table II. Similarly to the other solution, the location of the initial imprint is given by

$$\kappa_a x_1 = \xi_{1+} = \ln\left|\frac{d_1}{d_+}\right|. \tag{B7}$$

In this case, when all the coefficients are nonzero, the solution describes the storage of the signal pulse in the two channels. For the pulse evolution depicted in Fig. 4 we only considered the storage in one channel, for which we took

$d_3 = 0$. This was then superimposed with a solution consisting of just control pulses, in this case $d_1 = 0$.

For the first-order solution, the complete form for the fields is given by

$$\Omega_1 = \frac{4id_0d_1^*}{\tau_a}e^{-\kappa_a Z}g(Z,T), \tag{B8a}$$

$$\Omega_2 = \frac{4id_0d_2^*}{\tau_a}g(Z,T), \tag{B8b}$$

$$\Omega_3 = \frac{4id_0d_3^*}{\tau_a}g(Z,T), \tag{B8c}$$

with

$$g(Z,T) = \frac{e^{-T/\tau_a}}{|d_0|^2e^{-2T/\tau_a} + |d_1|^2e^{-2\kappa_a Z} + |d_+|^2}. \tag{B8d}$$

It is clear that the integration with respect T for the pulse areas only involves the function g and

$$\int_{-\infty}^{\infty} g(Z,T)dT = \frac{\tau_a\pi}{2|d_0|}(|d_1|^2e^{-2\kappa_a Z} + |d_+|^2)^{-1/2}. \tag{B9}$$

The pulse area can be readily written as

$$\theta_1 = \frac{2\pi id_0d_1^*}{|d_0|}\frac{e^{-\kappa_a Z}}{\sqrt{|d_1|^2e^{-2\kappa_a Z} + |d_+|^2}}, \tag{B10a}$$

$$\theta_2 = \frac{2\pi id_0d_2^*}{|d_0|}\frac{1}{\sqrt{|d_1|^2e^{-2\kappa_a Z} + |d_+|^2}}, \tag{B10b}$$

$$\theta_3 = \frac{2\pi id_0d_3^*}{|d_0|}\frac{1}{\sqrt{|d_1|^2e^{-2\kappa_a Z} + |d_+|^2}}. \tag{B10c}$$

From this, the result given in Eq. (12) and the three-pulse area being equal to 2π follow.

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