μ-symmetry breaking: An algebraic approach to finding mean fields of quantum many-body systems

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One of the most fundamental problems in quantum many-body systems is the identification of a mean field in spontaneous symmetry breaking which is usually made in a heuristic manner. We propose a systematic method of finding a mean field based on the Lie algebra and the dynamical symmetry by introducing a class of symmetry-broken phases which we call μ-symmetry breaking. We show that for μ-symmetry breaking the quadratic part of an effective Lagrangian of Nambu-Goldstone modes can be block-diagonalized and that homotopy groups of topological excitations can be calculated systematically.

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I. INTRODUCTION

Spontaneous symmetry breaking (SSB) has long played a pivotal role in our understanding of Nature [1]. Examples include ferromagnetism [2], superconductivity [3], Bose-Einstein condensation [4,5], chiral symmetry breaking [6,7], and unification of the fundamental forces [8]. Both static and dynamic properties of a symmetry-broken phase can be described by the corresponding mean field, which is usually found in a heuristic manner. The identification of the mean field amounts to that of an order parameter or that of an operator that supports a long-range order (LRO) in quantum field theory [9].

In this paper, we propose a systematic method of finding mean fields of quantum many-body systems based on the Lie algebra and the dynamical symmetry. The dynamical symmetry has achieved a remarkable success in few-body systems for finding, e.g., the atomic spectrum of hydrogen [10,11] and collective excitation spectra of nuclei [12]. Here we apply the dynamical symmetry to a particular class of broken symmetry systems in which the mean fields are described in terms of the weight vector in the representation of the Lie algebra. Since the weight of the Lie algebra is often labeled by the Greek letter μ, we refer to such symmetry breaking as μ-symmetry breaking. We show that for μ-symmetry breaking the quadratic part of an effective Lagrangian of Nambu-Goldstone modes (NG modes) can be block-diagonalized and that homotopy groups of topological excitations can be calculated systematically. By applying this method to a U(N)-symmetric system which has recently been realized in an ultracold atomic gas [13,14], we show that a large class of symmetry-broken phases can be described in terms of μ-symmetry breaking.

This paper is organized as follows. In Sec. II, we introduce the concept of μ-symmetry breaking and identify mean fields by combining it with the dynamical symmetry. In Sec. III, mean fields of μ-SB are derived through minimization of energy functionals constructed from the underlying Lie algebra. In Sec. IV, we show that the quadratic part of an effective Lagrangian of Nambu-Goldstone modes can be block-diagonalized for μ-symmetry breaking. In Sec. V, we show how to systematically calculate homotopy groups of topological excitations for μ-symmetry breaking. In Sec. VI, we apply our method to a U(N)-symmetric system. In Sec. VII, the cases of higher-dimensional representations are discussed from the standpoint of μ-symmetry breaking by using the examples of spin-2 Bose-Einstein condensates (BECs) [15–17] and spin-1 color superconductors [18,19]. In Sec. VIII, we conclude this paper. In Appendix, we prove some formulas on homotopy groups used in Sec. V.

II. μ-SYMMETRY BREAKING

We consider a quantum field theory whose symmetry group G is described by a finite-dimensional unitary representation $\boldsymbol{R}$:

$$\phi_i \mapsto \sum_j \left[\exp\left(i\sum_{a=1}^{d} T_a t_a\right)\right]_{ij} \phi_j, \tag{1}$$

where $\{\phi_i\}_i$ is a set of fields of particles, T_a is an element of the Lie algebra $\mathfrak{g} = \{T_a\}_{a=1}^d$ constituted from finite-dimensional Hermitian matrices of the representation $\boldsymbol{R}$, respectively. Here, d is the dimension of G and t_a's are real parameters. We denote the Noether charge associated with the generator T_a by $\hat{Q}_{T_a}$. For the present discussion, we do not need to specify quantum statistics of particles and the system can be defined either on a lattice or in continuous space. We assume that the Lie algebra $\mathfrak{g}$ of the symmetry group is the direct product of a simple compact Lie algebra $\bar{\mathfrak{g}}$ and $\mathfrak{u}(1) = \{xI | x \in \mathbb{R}\}$:

$$\mathfrak{g} = \bar{\mathfrak{g}} \oplus \mathfrak{u}(1), \tag{2}$$

where I is the identity matrix. The particle-number operator is the Noether charge associated with the generator I.

The key ingredient in the following analysis is the quadratic Casimir invariant defined by

$$C_2^{\bar{\mathfrak{g}}}(\boldsymbol{v}) := \sum_{a=1}^{\bar{d}} (\boldsymbol{v}|T_a|\boldsymbol{v})^2, \tag{3}$$

where $|\boldsymbol{v})$ and $\bar{d} = d - 1$ are a vector in the representation $\boldsymbol{R}$ and the dimension of $\bar{\mathfrak{g}}$, respectively. Let $\{H_b\}_{b=1}^{\bar{r}}$ be the Cartan subalgebra of $\bar{\mathfrak{g}}$, i.e., the maximal commutative subalgebra of $\bar{\mathfrak{g}}$, where $\bar{r} = \mathrm{rank}\,\bar{\mathfrak{g}}$ is the rank of the Cartan subalgebra. Let $|\boldsymbol{\mu})$ be a weight vector which is a simultaneous eigenstate of $\{H_b\}_{b=1}^{\bar{r}}$:

$$H_b|\boldsymbol{\mu}) = \mu_b|\boldsymbol{\mu})\ (b = 1,2,\ldots,\bar{r}), \tag{4}$$

$$\boldsymbol{\mu} = {}^t(\mu_1,\mu_2,\ldots,\mu_{\bar{r}}), \tag{5}$$

where t denotes the transpose. The highest weight $|\boldsymbol{\mu}_H)$ is the weight vector that maximizes the expectation value of the quadratic Casimir invariant $C_2^{\bar{\mathfrak{g}}}$:

$$C_2^{\bar{\mathfrak{g}}}(\boldsymbol{\mu}_H) = \sum_{a=1}^{\bar{d}} (\boldsymbol{\mu}_H|T_a|\boldsymbol{\mu}_H)^2 = \max_{\phi,(\phi|\phi)=1} \sum_{a=1}^{\bar{d}} (\phi|T_a|\phi)^2. \tag{6}$$

The zero weight $|\boldsymbol{\mu}_0)$ is the weight vector that has the zero eigenvalue of Eq. (4) and therefore minimizes the expectation value of $C_2^{\bar{\mathfrak{g}}}$ to 0:

$$C_2^{\bar{\mathfrak{g}}}(\boldsymbol{\mu}_0) = 0. \tag{7}$$

An irreducible representation of $\bar{\mathfrak{g}}$ is uniquely determined by its highest weight $\boldsymbol{\mu}_H$ [20], and we denote a complete set of weight vectors belonging to $\boldsymbol{\mu}_H$ as $W[\boldsymbol{\mu}_H]$.

We now introduce the concept of μ-symmetry breaking. Let $\langle\boldsymbol{\phi}\rangle$ be an order parameter. If $\langle\boldsymbol{\phi}\rangle$ transforms in a low-dimensional representation of the symmetry group G in the presence of off-diagonal long-range order (ODLRO), $\langle\boldsymbol{\phi}\rangle$ will be shown to take either of the following forms:

$$\langle\boldsymbol{\phi}\rangle = |\boldsymbol{\mu}_H), \tag{8}$$

$$\langle\boldsymbol{\phi}\rangle = |\boldsymbol{\mu}_0). \tag{9}$$

If the lattice of space, which we denote by L, is free from frustration in the presence of diagonal long-range order (DLRO), the mean-field ground state $|\mathrm{GS}\rangle$ will be shown to take either of the following forms:

$$|\mathrm{GS}\rangle = \bigotimes_{i\in L} |\boldsymbol{\mu}_H)_i, \tag{10}$$

$$|\mathrm{GS}\rangle = \bigotimes_{u\in\mathcal{U}} \bigotimes_{i\in u} |\boldsymbol{\mu}_i)_i, \tag{11}$$

where $|\boldsymbol{\mu}_i)_i$, u, and $\mathcal{U}$ denote a simultaneous eigenstate of $\{H_b\}_{b=1}^{\bar{r}}$ at lattice site i, a unit cell of an ordered state, and the lattice constituted from the entire set of the unit cells, respectively. In Eq. (11), the set $\{\boldsymbol{\mu}_i\}_i$ is chosen so that the expectation value of $\widehat{C}_2^{\mathfrak{g}}$ within each unit cell vanishes (see Eq. (32) and the following explanation for detail):

$$\langle\widehat{C}_2^{\mathfrak{g}}\rangle = \sum_{a=1}^{\bar{d}} \left[\sum_{i\in u} {}_i(\boldsymbol{\mu}_i|T_a|\boldsymbol{\mu}_i)_i\right]^2 = 0. \tag{12}$$

As is shown in Eq. (32), Eq. (12) implies that the sum of the weight vectors within each unit cell vanishes.

The derivations of Eqs. (8)–(11) will be shown in the following section. We call these four types of symmetry breaking μ-symmetry breaking (μ-SB), which is characterized by the combination of the highest or zero weight and ODLRO or DLRO. Prototypical examples of μ-SB include the ferromagnetic phase and the polar (antiferromagnetic) phase of a spin-1 BEC [21,22], classical ferromagnets (FMs), and classical antiferromagnets (AFMs) (see Table I).

TABLE I. Examples of four types of μ-symmetry broken phases. Here $\boldsymbol{\mu}_H$ and $\boldsymbol{\mu}_0$ represent the highest-weight and zero-weight vectors, respectively. The order parameters for ODLRO are given by Eqs. (8) and (9) and the ground states for DLRO are given by Eqs. (10) and (11).

	ODLRO	DLRO
$\boldsymbol{\mu}_H$	Ferromagnetic spin-1 BEC	Ferromagnet
$\boldsymbol{\mu}_0$	Polar spin-1 BEC	Antiferromagnet

The order parameter of a spin-1 BEC is described by a three-dimensional complex vector

$$\langle\boldsymbol{\phi}\rangle = \begin{pmatrix} \langle\phi_1\rangle \\ \langle\phi_0\rangle \\ \langle\phi_{-1}\rangle \end{pmatrix}, \tag{13}$$

and the symmetry group G is $\mathrm{U}(1)\times\mathrm{SO}(3)$. The Cartan generator of SO(3) is the S_z-operator defined by

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{14}$$

The quadratic Casimir invariant of this system is given by

$$C_2^{\mathfrak{so}(3)} = \boldsymbol{S}\cdot\boldsymbol{S}, \tag{15}$$

where $\boldsymbol{S} = (S_x,S_y,S_z)$ is the vector of spin operators in the Cartesian representation. In the ferromagnetic phase, the order parameter has the form

$$\langle\boldsymbol{\phi}\rangle_{\mathrm{FM}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

This is the eigenstate of S_z with eigenvalue 1 and maximizes the expectation value of $\widehat{C}_2^{\mathfrak{so}(3)} = \widehat{\boldsymbol{S}}\cdot\widehat{\boldsymbol{S}}$:

$$\langle\widehat{C}_2^{\mathfrak{so}(3)}\rangle = \langle\widehat{\boldsymbol{S}}\rangle\cdot\langle\widehat{\boldsymbol{S}}\rangle = 1. \tag{17}$$

Thus, the ferromagnetic phase of a spin-1 BEC is characterized by μ-SB with ODLRO and the highest weight $\boldsymbol{\mu}_H$. In the polar phase, the order parameter has the form

$$\langle\boldsymbol{\phi}\rangle_{\mathrm{polar}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{18}$$

This is the eigenstate of S_z with eigenvalue 0 and minimizes the expectation value of $\widehat{C}_2^{\mathfrak{so}(3)}$:

$$\langle\widehat{C}_2^{\mathfrak{so}(3)}\rangle = 0. \tag{19}$$

Thus, the polar phase of a spin-1 BEC is characterized by μ-SB with ODLRO and the zero weight $\boldsymbol{\mu}_0$.

For both the classical FM and the classical AFM, the quadratic Casimir invariant is again the square of a spin $\boldsymbol{S}$:

$$C_2^{\mathfrak{so}(3)} = \boldsymbol{S}\cdot\boldsymbol{S}. \tag{20}$$

Let S be the spin quantum number. The highest-weight state $|\boldsymbol{\mu}_H)$ corresponds to the eigenstate of S_z with the highest magnetic quantum number m_z, i.e., $|\boldsymbol{\mu}_H) = |m_z = S)$. The mean-field ground state $|\mathrm{GS}\rangle_{\mathrm{FM}}$ of the classical FM represents a uniform alignment of the highest magnetic quantum-number state $|m_z = S)$ on every site of the lattice L:

$$|\mathrm{GS}\rangle_{\mathrm{FM}} = \bigotimes_{i\in L} |m_z = S)_i. \tag{21}$$

Therefore, the classical FM is characterized by μ-SB with DLRO and $\boldsymbol{\mu}_H$. On the other hand, the mean-field ground state $|\mathrm{GS}\rangle_{\mathrm{AFM}}$ of the classical AFM is described by a uniform alignment of the highest magnetic quantum-number state $|m_z = S)$ on the sites of one sublattice L_A and that of the lowest magnetic quantum-number state $|m_z = -S)$ on the other sublattice L_B:

$$|\mathrm{GS}\rangle_{\mathrm{AFM}} = \bigotimes_{u\in\mathcal{U}} \big(|m_z = S)_{u_A} \otimes |m_z = -S)_{u_B}\big), \tag{22}$$

where u_A and u_B indicate the sites in each unit cell u belonging to L_A and L_B, respectively. Within each unit cell, the total magnetization vanishes. Therefore, this phase is characterized by μ-SB with DLRO and $\boldsymbol{\mu}_0$.

III. DERIVATIONS OF MEAN FIELDS OF μ-SB

In this section, we show that mean fields described by the highest or zero-weight states are obtained through the minimization of an energy functional constructed from Casimir invariants of the underlying Lie algebra $\bar{\mathfrak{g}}$.

We first discuss the case of ODLRO. In this case, the energy functional can be obtained in a manner similar to the case of the dynamical symmetry [11,12,23]. However, as shown later, for μ-SB we can block-diagonalize the quadratic part of an effective Lagrangian of NG modes and calculate homotopy groups of topological excitations systematically, neither of which can be done from the dynamical symmetry alone. For the case of the lowest-dimensional representation such as a BEC in degenerate N-component bosons the mean-field energy functional $V(\langle\boldsymbol{\phi}\rangle)$ can be expressed up to the fourth order in $\langle\boldsymbol{\phi}\rangle$ as

$$V(\langle\boldsymbol{\phi}\rangle) = -c|\langle\boldsymbol{\phi}\rangle|^2 + c_0|\langle\boldsymbol{\phi}\rangle|^4, \tag{23}$$

where c and c_0 are real constants. Since any minimizer $\langle\boldsymbol{\phi}\rangle$ of $V(\langle\boldsymbol{\phi}\rangle)$ in the lowest-dimensional representation can be transformed into $|\boldsymbol{\mu}_H)$ by an appropriate element of G, the energy functional is minimized for

$$\langle\boldsymbol{\phi}\rangle = |\boldsymbol{\mu}_H). \tag{24}$$

For the case of the next lowest-dimensional representation such as a spin-1 BEC [21,22] and an s-wave superfluid in degenerate N-component fermions [24–27] $V(\langle\boldsymbol{\phi}\rangle)$ can be expressed in terms of $|\langle\boldsymbol{\phi}\rangle|^2$ and the quadratic Casimir invariant as

$$V(\langle\boldsymbol{\phi}\rangle) = -c|\langle\boldsymbol{\phi}\rangle|^2 + c_0|\langle\boldsymbol{\phi}\rangle|^4 + c_1 C_2^{\bar{\mathfrak{g}}}(\langle\boldsymbol{\phi}\rangle), \tag{25}$$

where c, c_0, and c_1 are real constants. For the case of a ferromagnetic interaction between condensate particles with $c_1 < 0$, the energy functional is minimized for

$$\langle\boldsymbol{\phi}\rangle = |\boldsymbol{\mu}_H). \tag{26}$$

For the case of an antiferromagnetic interaction with $c_1 > 0$, the energy functional is minimized for

$$\langle\boldsymbol{\phi}\rangle = |\boldsymbol{\mu}_0), \tag{27}$$

if the representation includes the zero-weight state $|\boldsymbol{\mu}_0)$. This condition is satisfied for low-dimensional Lie algebras such as $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$. The general case of $\bar{\mathfrak{g}} = \mathfrak{su}(N)$ will be discussed in Sec. VI. Since the energy functional can be written in neither form of Eq. (23) nor Eq. (25) in higher-dimensional representations, the ground states are no longer described by μ-SB. Such a case will be discussed in Sec. VII.

We next discuss the case of DLRO. We assume that particles are placed on a lattice L whose geometry is free from frustration. A prototypical example of DLRO is described by the Heisenberg-type Hamiltonian

$$H = -J\sum_{\langle i,j\rangle}\sum_{a=1}^{\bar{d}} T_{a,i}T_{a,j}, \tag{28}$$

where $\langle i,j\rangle$ represents a pair of nearest-neighbor sites i and j, and $\{T_{a,i}\}_{a=1}^{\bar{d}}$ is a set of generators of a simple compact Lie algebra $\bar{\mathfrak{g}}$ on site i. For the case of a ferromagnetic interaction with $J > 0$, the ground state $|\mathrm{GS}\rangle$ is written as

$$|\mathrm{GS}\rangle = \bigotimes_{i\in L} |\boldsymbol{\mu}_H)_i, \tag{29}$$

which agrees with Eq. (10). For the case of an antiferromagnetic interaction with $J < 0$, it follows from the frustration-free assumption that the mean-field classical ground state $|\mathrm{GS}\rangle$ is obtained by a tensor product of the state on site i that minimizes the interaction energy with its neighboring sites [28]. Therefore, the mean-field ground state can be written in terms of a site-factorized wave function as

$$|\mathrm{GS}\rangle = \bigotimes_{u\in\mathcal{U}}\bigotimes_{i\in u} |\boldsymbol{\mu}_i)_i, \tag{30}$$

where $|\boldsymbol{\mu}_i)_i$, u, and $\mathcal{U}$ denote a simultaneous eigenstate of $\{H_b\}_{b=1}^{\bar{r}}$ at lattice site i, a unit cell of an ordered state, and the lattice constituted from the entire set of the unit cells, respectively. In Eq. (30), two weight vectors $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_j$ at nearest-neighbor sites i and j should satisfy

$$(\boldsymbol{\mu}_i|\boldsymbol{\mu}_j) = \min\{(\boldsymbol{\mu}_k|\boldsymbol{\mu}_l)|\boldsymbol{\mu}_k,\boldsymbol{\mu}_l \in W[\boldsymbol{\mu}_H]\} \tag{31}$$

to minimize the nearest-neighbor interaction energy $\sum_{a=1}^{\bar{d}} T_{a,i}T_{a,j}$. We note that we do not impose Eq. (12) on Eq. (30) so far. Since the expectation value $(\boldsymbol{\mu}_i|T_a|\boldsymbol{\mu}_i)$ vanishes for any off-diagonal matrix T_a, which is nothing but the raising or lowering operator of the Cartan canonical form (see Sec. IV A), the expectation value of $\widehat{C}_2^{\bar{\mathfrak{g}}}$ within each unit cell u can be calculated from Eqs. (4) and (30) by considering the contribution from the Cartan generators $\{H_b\}_{b=1}^{\bar{r}}$ alone. We thus obtain

$$\begin{aligned}\langle\widehat{C}_2^{\bar{\mathfrak{g}}}\rangle &= \sum_{b=1}^{\bar{r}}\left[\sum_{i\in u} {}_i(\boldsymbol{\mu}_i|H_b|\boldsymbol{\mu}_i)_i\right]^2\\ &= \sum_{b=1}^{\bar{r}}\left[\sum_{i\in u}(\boldsymbol{\mu}_i)_b\right]^2 = \left\|\sum_{i\in u}\boldsymbol{\mu}_i\right\|^2,\end{aligned} \tag{32}$$

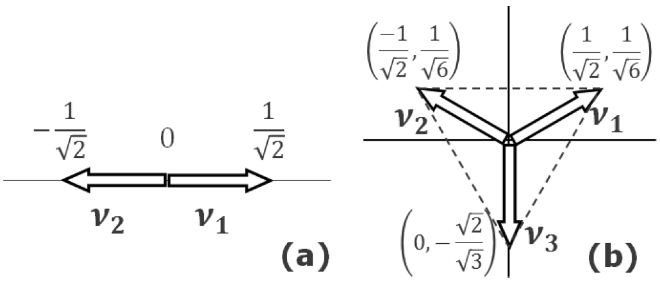

FIG. 1. (a) Weight vectors of the two-dimensional representation of the $\mathfrak{su}(2)$-Lie algebra. The weight vectors are two one-dimensional vectors $\boldsymbol{\nu}_1 = 1/\sqrt{2}$ and $\boldsymbol{\nu}_2 = -1/\sqrt{2}$, which have opposite directions with the same magnitude $1/\sqrt{2}$. In this representation, a pair of the weight vectors that satisfy Eq. (31) is uniquely determined to be $(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)$. (b) Weight vectors of the three-dimensional representation of the $\mathfrak{su}(3)$-Lie algebra. The weight vectors are three two-dimensional vectors $\boldsymbol{\nu}_1 = (1/\sqrt{2},1/\sqrt{6})$, $\boldsymbol{\nu}_2 = (-1/\sqrt{2},1/\sqrt{6})$, and $\boldsymbol{\nu}_3 = (0,-\sqrt{2/3})$ from the origin to each of the three apexes of the equilateral triangle. The sum of these three weight vectors vanishes. Three pairs of the weight vectors, $(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)$, $(\boldsymbol{\nu}_2,\boldsymbol{\nu}_3)$, and $(\boldsymbol{\nu}_3,\boldsymbol{\nu}_1)$, satisfy Eq. (31). In both (a) and (b), the weight vectors are normalized so as to satisfy Eq. (108).

where $\|\boldsymbol{x}\|$ denotes the magnitude of the vector $\boldsymbol{x}$. If there exist a pair of weight vectors with opposite directions such as the case of SU(2) [see Fig. 1(a)], Eq. (31) is satisfied and the sum of the weight vectors in Eq. (32) vanishes within each unit cell. Thus, the ground-state wave function satisfies Eq. (12). However, in a larger group such as SU(3), it is known that there do not exist two weight vectors with opposite directions and that there is more than one pair of weight vectors that satisfy Eq. (31) [see Fig. 1(b)] [20]. For such cases, the mean-field ground states represented by Eq. (30) are degenerate [29]. Therefore, we have to consider higher-order contributions arising from quantum fluctuations to determine the ground state. This can be done by using the flavor-wave theory [29–32]. The Hamiltonian $H_{\rm fw}$ of quantum fluctuations is given as [31,32]

$$H_{\rm fw} = \sum_{\langle i,j\rangle} A_{ij}^\dagger A_{ij}, \tag{33}$$

where

$$A_{ij} = b_{\boldsymbol{\mu}_i,j} + b_{\boldsymbol{\mu}_j,i}^\dagger, \tag{34}$$

and $b_{\boldsymbol{\mu}_i,j}$ is the annihilation operator of a boson with with weight vector $\boldsymbol{\mu}_i$ at site j. The expectation value of $H_{\rm fw}$ is minimized when

$$\langle{\rm GS}|A_{ij}^\dagger A_{ij}|{\rm GS}\rangle = 0 \Leftrightarrow A_{ij}|{\rm GS}\rangle = 0 \quad \text{for } \forall\, i,j. \tag{35}$$

Let i,j and j,k be pairs of nearest-neighbor sites. Then, i and k share the common nearest-neighbor site j. It follows from Eq. (35) that

$$\begin{aligned} 0 = [A_{ij},A_{jk}]|{\rm GS}\rangle &= \left[b_{\boldsymbol{\mu}_i,j},b_{\boldsymbol{\mu}_k,j}^\dagger\right]|{\rm GS}\rangle \\ &= \delta_{\boldsymbol{\mu}_i,\boldsymbol{\mu}_k}|{\rm GS}\rangle, \end{aligned} \tag{36}$$

and hence we have

$$\boldsymbol{\mu}_i \neq \boldsymbol{\mu}_k \tag{37}$$

for any pair of sites i and k that have a common nearest-neighbor site j. Thus, the ground state must satisfy both Eqs. (31) and (37). Equation (31) implies that two weight vectors, $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_j$, must give the minimum value of the inner product $(\boldsymbol{\mu}_i|\boldsymbol{\mu}_j)$ for any pair of nearest-neighbor sites i and j. Equation (37) implies that two weight vectors, $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_k$, must be different for any pair of sites i and k that share a common nearest-neighbor site. As a consequence, the sum of the weight vectors within each unit cell in Eq. (32) tend to cancel with each other and an ordered state with the vanishing expectation value of $\widehat{C}_2^{\bar{\mathfrak{g}}}$ within each unit cell is favored. Thus, the ordered state is characterized by μ-SB with $\boldsymbol{\mu}_0$.

IV. EFFECTIVE LAGRANGIAN OF NG MODES FOR μ-SB

A. Cartan canonical form and the generalized magnetization

The analyses of NG modes and topological excitations in μ-SB can be done conveniently in terms of a special basis of the Lie algebra $\bar{\mathfrak{g}}$ known as the Cartan canonical form [20],

$$\bar{\mathfrak{g}} = \left\{\{H_b\}_{b=1}^{\bar{r}},\left\{E_{\boldsymbol{\alpha}}^R,E_{\boldsymbol{\alpha}}^I\right\}_{\boldsymbol{\alpha}\in R_+}\right\}, \tag{38}$$

where R and I indicate the real and imaginary parts of the raising operators $E_{\pm\boldsymbol{\alpha}}$ of the Cartan canonical form:

$$E_{\boldsymbol{\alpha}}^R := \frac{E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}}{\sqrt{2}}, \quad E_{\boldsymbol{\alpha}}^I := \frac{E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}}{\sqrt{2}i}. \tag{39}$$

The Cartan canonical form (38) is a generalization of the basis of the $\mathfrak{su}(2)$ Lie algebra $\{S^z,\{S^x,S^y\}\}$, which decomposes the generators into the diagonal matrices $\{H_b\}_{b=1}^{\bar{r}}$ and the off-diagonal ones $\{E_{\boldsymbol{\alpha}}^R,E_{\boldsymbol{\alpha}}^I\}_{\boldsymbol{\alpha}\in R_+}$, where $\boldsymbol{\alpha}$ is an $\bar{r}$-dimensional real vector known as a positive root and R_+ denotes the set of all the positive roots. Here the positive roots distinguish different $\mathfrak{su}(2)$-subalgebras of $\bar{\mathfrak{g}}$. Defining $H_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \mathbb{R}^{\bar{r}}$ by

$$H_{\boldsymbol{\alpha}} := \sum_{b=1}^{\bar{r}} \alpha_b H_b, \tag{40}$$

where

$$\boldsymbol{\alpha} = {}^t(\alpha_1,\alpha_2,\ldots,\alpha_{\bar{r}}), \tag{41}$$

we obtain a triad $\boldsymbol{S}_{\boldsymbol{\alpha}}$ for a positive root $\boldsymbol{\alpha}$ as

$$\boldsymbol{S}_{\boldsymbol{\alpha}} = \left(E_{\boldsymbol{\alpha}}^R,E_{\boldsymbol{\alpha}}^I,H_{\boldsymbol{\alpha}}\right). \tag{42}$$

The triad $\boldsymbol{S}_{\boldsymbol{\alpha}}$ forms an $\mathfrak{su}(2)$ subalgebra of $\bar{\mathfrak{g}}$ analogous to the $\mathfrak{su}(2)$ spin algebra. In fact, it satisfies the following commutation relations:

$$\left[E_{\boldsymbol{\alpha}}^R,E_{\boldsymbol{\alpha}}^I\right] = i(\boldsymbol{\alpha},\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} \quad \text{for } \forall\, \boldsymbol{\alpha} \in R_+, \tag{43}$$

$$\left[E_{\boldsymbol{\alpha}}^I,H_{\boldsymbol{\alpha}}\right] = i(\boldsymbol{\alpha},\boldsymbol{\alpha})E_{\boldsymbol{\alpha}}^R \quad \text{for } \forall\, \boldsymbol{\alpha} \in R_+, \tag{44}$$

$$\left[H_{\boldsymbol{\alpha}},E_{\boldsymbol{\alpha}}^R\right] = i(\boldsymbol{\alpha},\boldsymbol{\alpha})E_{\boldsymbol{\alpha}}^I \quad \text{for } \forall\, \boldsymbol{\alpha} \in R_+, \tag{45}$$

$$\left[H_{\boldsymbol{\alpha}},H_{\boldsymbol{\alpha}'}\right] = 0 \quad \text{for } \forall\, \boldsymbol{\alpha},\forall\, \boldsymbol{\alpha}' \in R_+. \tag{46}$$

We refer to $\boldsymbol{S}_{\boldsymbol{\alpha}}$ as a generalized magnetization in analogy with the $\mathfrak{su}(2)$ spin $\boldsymbol{S}$. As shown later, the textures of NG modes and topological excitations are described in terms of the generalized magnetization $\boldsymbol{S}_{\boldsymbol{\alpha}}$.

B. Broken generators for μ-SB

To determine the quadratic part of an effective Lagrangian of NG modes, we first prove the following theorem on the spaces of broken generators $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of the remaining (unbroken) symmetry of the state.

Theorem 1. Consider μ-SB phases and define the sets of positive root vectors R_H, R_0^{OD}, and R_0^{D} by

$$R_H := \{\boldsymbol{\alpha} \in R_+ | (\boldsymbol{\alpha},\boldsymbol{\mu}_H) \neq 0\}, \tag{47}$$

$$R_0^{\mathrm{D}} := \bigcup_{i\in u} \{\boldsymbol{\alpha} \in R_+ | |\boldsymbol{\mu}_i - \boldsymbol{\alpha}) \in W[\boldsymbol{\mu}_H]\}, \tag{48}$$

$$R_0^{\mathrm{OD}} := \{\boldsymbol{\alpha} \in R_+ | |\boldsymbol{\mu}_0 - \boldsymbol{\alpha}) \in W[\boldsymbol{\mu}_H]\}. \tag{49}$$

Then the bases of the space of broken generators $\mathfrak{g}/\mathfrak{h}$ are given by

$$\mathrm{basis}(\mathfrak{g}/\mathfrak{h}) = \begin{cases} \{E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I | \boldsymbol{\alpha} \in R_H\} & \text{for DLRO and } \boldsymbol{\mu}_H, \\ \{E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I | \boldsymbol{\alpha} \in R_0^{\mathrm{D}}\} & \text{for DLRO and } \boldsymbol{\mu}_0, \\ \{E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I | \boldsymbol{\alpha} \in R_H\} \cup \{I\} & \text{for ODLRO and } \boldsymbol{\mu}_H, \\ \{E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I | \boldsymbol{\alpha} \in R_0^{\mathrm{OD}}\} \cup \{I\} & \text{for ODLRO and } \boldsymbol{\mu}_0. \end{cases} \tag{50}$$

Proof. First, we consider the DLRO with $\boldsymbol{\mu}_H$. From the mean-field ground state in Eq. (10), unitary transformations generated by $E_{\boldsymbol{\alpha}}^{R,I}$ leave the ground state unchanged up to a global phase if and only if

$$E_{\boldsymbol{\alpha}}^{R,I} |\boldsymbol{\mu}_H) = 0 \\ \Leftrightarrow E_{+\boldsymbol{\alpha}} |\boldsymbol{\mu}_H) = E_{-\boldsymbol{\alpha}} |\boldsymbol{\mu}_H) = 0. \tag{51}$$

Since $\boldsymbol{\mu}_H$ is the highest weight, $E_{+\boldsymbol{\alpha}} |\boldsymbol{\mu}_H)$ always vanishes [20], so that Eq. (51) is equivalent to

$$E_{-\boldsymbol{\alpha}} |\boldsymbol{\mu}_H) = 0 \Leftrightarrow |\boldsymbol{\mu}_H - \boldsymbol{\alpha}) \notin W[\boldsymbol{\mu}_H], \tag{52}$$

which, in turn, is equivalent to

$$\boldsymbol{\alpha} \notin R_H. \tag{53}$$

This equivalence can be shown as follows. From Eq. (52), we obtain

$$0 = [E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I] |\boldsymbol{\mu}_H) = i(\boldsymbol{\alpha},\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}} |\boldsymbol{\mu}_H) \\ = i(\boldsymbol{\alpha},\boldsymbol{\alpha})(\boldsymbol{\mu}_H,\boldsymbol{\alpha}) |\boldsymbol{\mu}_H). \tag{54}$$

Since $(\boldsymbol{\alpha},\boldsymbol{\alpha}) \neq 0$ and $|\boldsymbol{\mu}_H) \neq 0$, we obtain $(\boldsymbol{\mu}_H,\boldsymbol{\alpha}) = 0$ and hence Eq. (53) from the definition of R_H. Conversely, we can derive Eq. (52) by assuming Eq. (53). The Weyl reflection [20] of $\boldsymbol{\mu}_H + \boldsymbol{\alpha}$ with respect to $\boldsymbol{\alpha}$ is

$$(\boldsymbol{\mu}_H + \boldsymbol{\alpha}) - 2\frac{(\boldsymbol{\mu}_H + \boldsymbol{\alpha},\boldsymbol{\alpha})}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\boldsymbol{\alpha} = \boldsymbol{\mu}_H - \boldsymbol{\alpha}. \tag{55}$$

Since $|\boldsymbol{\mu}_H)$ is the highest weight, $|\boldsymbol{\mu}_H + \boldsymbol{\alpha})$ is not a weight vector, nor is $|\boldsymbol{\mu}_H - \boldsymbol{\alpha})$. Thus, Eq. (52) is obtained. Unitary transformations generated by $\{H_b\}_{b=1}^{\bar{r}}$ and I change the ground state $|\mathrm{GS}\rangle$ only by a global phase factor. Therefore, these generators are not broken ones, which completes the proof of the first row of Eq. (50).

Second, we consider the DLRO with $\boldsymbol{\mu}_0$. From the mean-field ground state in Eq. (11), unitary transformations generated by $E_{\boldsymbol{\alpha}}^{R,I}$ leave $|\mathrm{GS}\rangle$ invariant if and only if

$$E_{\boldsymbol{\alpha}}^{R,I} |\boldsymbol{\mu}_i) = 0 \quad \text{for } \forall\, i \in u. \tag{56}$$

This condition is equivalent to $\boldsymbol{\alpha} \notin R_0^{\mathrm{OD}}$. From the discussions similar to the case of DLRO and $\boldsymbol{\mu}_H$, all of the Cartan generators are not broken ones, which completes the proof of the second row of Eq. (50).

Third, we consider the ODLRO with the highest weight, i.e., $\langle\boldsymbol{\phi}\rangle = |\boldsymbol{\mu}_H)$. Unitary transformations generated by $E_{\boldsymbol{\alpha}}^{R,I}$ leave $|\mathrm{GS}\rangle$ invariant if and only if Eq. (52) is satisfied. From above discussion, this condition is equivalent to $\boldsymbol{\alpha} \notin R_H$. Unitary transformations generated by $\{H_b\}_{b=1}^{\bar{r}}$ and I change $|\boldsymbol{\mu}_H)$ only by a phase factor. This phase shift can be eliminated by taking a linear combination of $\{H_b\}_{b=1}^{\bar{r}}$ and I except for the direction of I, which completes the proof of the third row of Eq. (50). The proof of the fourth row can be given similarly by replacing $\boldsymbol{\mu}_H$ by $\boldsymbol{\mu}_0$ and using Eq. (52). Thus, the proof of Theorem 1 is completed.

C. Effective Lagrangian of NG modes

We now derive the quadratic part of an effective Lagrangian of NG modes. We note that in nonrelativistic systems the type-2 NG mode with a quadratic dispersion is allowed, in contrast to relativistic systems where only the type-1 NG mode with a linear dispersion is allowed [33]. Let $N_B := \dim(G/H)$ be the number of the broken generators. It has been shown that the numbers of type-1 and type-2 NG modes, n_1 and n_2, can be determined only from the ground state $|\mathrm{GS}\rangle$ and the set of the Noether charges associated with the broken generators of the symmetry group $\{\hat{T}_{a'}\}_{a'=1}^{N_B}$ as follows (the prime indicates that the generators are broken ones) [34,35]:

$$\begin{cases} n_1 + 2n_2 = \dim(G/H), \\ n_2 = \frac{1}{2}\mathrm{rank}\rho, \end{cases} \tag{57}$$

$$\rho_{a'b'} = -i\langle \mathrm{GS}| [\hat{Q}_{T_{a'}}, \hat{Q}_{T_{b'}}] |\mathrm{GS}\rangle, \tag{58}$$

where $\rho = \{\rho_{a'b'}\}_{a',b'=1}^{N_B}$ is a Kostant-Kirillov symplectic form (K-K form) [36–38] which is employed in Ref. [39] in the context of NG modes. The basis of the Lie algebra that block-diagonalizes Eq. (58) is constituted from a set of canonically conjugate pairs and generates type-2 NG modes [34,35]. While the types and the numbers of NG modes can be found from Eqs. (57) and (58), the dynamics and the corresponding broken generator of the NG mode cannot be determined from them since the quadratic part of an effective Lagrangian is not diagonalized in Refs. [34,35]. Here, we show that for μ-SB, the K-K form and the quadratic part of an effective Lagrangian in the fields of NG modes can be simultaneously block-diagonalized in terms of the Cartan canonical form. Moreover, NG modes are classified into three categories according to their dynamics. In fact, we can prove the following theorem.

Theorem 2. Consider a μ-SB in a nonrelativistic system. Let $\pi_{\boldsymbol{\alpha}}^R$, $\pi_{\boldsymbol{\alpha}}^I$ and π_I be the fields of the NG mode generated by the broken generators $E_{\boldsymbol{\alpha}}^R$, $E_{\boldsymbol{\alpha}}^I$, and the identity operator I and define $\Pi_{\boldsymbol{\alpha}}$ by $\Pi_{\boldsymbol{\alpha}} := \pi_{\boldsymbol{\alpha}}^R + i\pi_{\boldsymbol{\alpha}}^I$. Then, the quadratic parts of effective Lagrangians $\mathcal{L}_{\rm eff}$ can be block-diagonalized in terms of $\Pi_{\boldsymbol\alpha}$ and π_I as follows:

$$\mathcal{L}_{\rm eff}=\begin{cases}\sum_{\boldsymbol\alpha\in R_H}\mathcal{L}^{\boldsymbol\alpha}_{\rm pre} & \text{for DLRO and } \boldsymbol\mu_H,\\ \sum_{\boldsymbol\alpha\in R_0^{\rm D}}\mathcal{L}^{\boldsymbol\alpha}_{\rm osc} & \text{for DLRO and } \boldsymbol\mu_0,\\ \mathcal{L}_{\rm pha}+\sum_{\boldsymbol\alpha\in R_H}\mathcal{L}^{\boldsymbol\alpha}_{\rm pre} & \text{for ODLRO and } \boldsymbol\mu_H,\\ \mathcal{L}_{\rm pha}+\sum_{\boldsymbol\alpha\in R_0^{\rm OD}}\mathcal{L}^{\boldsymbol\alpha}_{\rm osc} & \text{for ODLRO and } \boldsymbol\mu_0,\end{cases} \tag{59}$$

$$\mathcal{L}^{\boldsymbol\alpha}_{\rm pre}=\rho_{\boldsymbol\alpha}(\Pi_{\boldsymbol\alpha}\partial_t\Pi^*_{\boldsymbol\alpha}-\Pi^*_{\boldsymbol\alpha}\partial_t\Pi_{\boldsymbol\alpha})+b_{\boldsymbol\alpha}|\nabla\Pi_{\boldsymbol\alpha}|^2, \tag{60}$$

$$\mathcal{L}^{\boldsymbol\alpha}_{\rm osc}=b_{\boldsymbol\alpha}|\nabla\Pi_{\boldsymbol\alpha}|^2+\bar b_{\boldsymbol\alpha}|\partial_t\Pi_{\boldsymbol\alpha}|^2, \tag{61}$$

$$\mathcal{L}_{\rm pha}=g_I(\nabla\pi_I)^2+\bar g_I(\partial_t\pi_I)^2, \tag{62}$$

where $\rho_{\boldsymbol\alpha},b_{\boldsymbol\alpha},\bar b_{\boldsymbol\alpha},g_I,\bar g_I$ are real constants.

Proof. Consider the quadratic part of an effective Lagrangian in the fields of NG modes in a nonrelativistic system. Then, the most general form of it can be written as [34]

$$\mathcal{L}_{\rm eff}=\sum_{a',b'=1}^{N_B}\left(\rho_{a'b'}\pi_{a'}\partial_t\pi_{b'}+\frac{1}{2}\bar g_{a'b'}\partial_t\pi_{a'}\partial_t\pi_{b'}+\frac{1}{2}g_{a'b'}\nabla\pi_{a'}\cdot\nabla\pi_{b'}\right), \tag{63}$$

where ρ is the K-K form defined in Eq. (58), $\{\pi_{a'}\}_{a'=1}^{N_B}$ is the set of the fields of NG modes associated with the set of the broken generators $\{T_{a'}\}_{a'=1}^{N_B}$, and $\bar g_{a'b'}$ and $g_{a'b'}$ are real constants.

We first block-diagonalize the quadratic part of an effective Lagrangian. Here we consider the case of ODLRO and $\boldsymbol\mu_H$. The proof for the other cases can be made in a similar manner. The quadratic forms constructed from π_I, $\Pi^*_{\boldsymbol\alpha}$, and $\Pi_{\boldsymbol\alpha}$ are the following six terms:

$$(\pi_I)^2,\ \pi_I\Pi^*_{\boldsymbol\alpha},\ \pi_I\Pi_{\boldsymbol\alpha},\quad \Pi_{\boldsymbol\alpha}\Pi_{\boldsymbol\beta},\ \Pi^*_{\boldsymbol\alpha}\Pi^*_{\boldsymbol\beta},\ \Pi_{\boldsymbol\alpha}\Pi_{\boldsymbol\beta}. \tag{64}$$

Let $\{H'_b\}_{b=1}^{\bar r-1}$ be a basis of $(\bar r-1)$-dimensional subspace that is orthogonal to $\boldsymbol\mu_H$. Under the unitary transformation generated by

$$H_s:=\sum_{b=1}^{\bar r-1}s_aH'_b+\frac{s_{\bar r}}{|\boldsymbol\mu_H|}\left(H_{\boldsymbol\mu_H}-I|\boldsymbol\mu_H|^2\right)\in\mathfrak{h}, \tag{65}$$

π_I is invariant since the generator I commutes with H_s. Using the commutation relations of the Cartan canonical form

$$\left[E^R_{\boldsymbol\alpha},H_s\right]=-i(\boldsymbol\alpha,\boldsymbol s)E^I_{\boldsymbol\alpha}, \tag{66}$$

$$\left[E^I_{\boldsymbol\alpha},H_s\right]=i(\boldsymbol\alpha,\boldsymbol s)E^R_{\boldsymbol\alpha}, \tag{67}$$

we have

$$e^{-iH_s}\left(E^R_{\boldsymbol\alpha}+iE^I_{\boldsymbol\alpha}\right)e^{iH_s}=e^{-i(\boldsymbol s,\boldsymbol\alpha)}\left(E^R_{\boldsymbol\alpha}+iE^I_{\boldsymbol\alpha}\right), \tag{68}$$

$$e^{-iH_s}\left(E^R_{\boldsymbol\alpha}-iE^I_{\boldsymbol\alpha}\right)e^{iH_s}=e^{i(\boldsymbol s,\boldsymbol\alpha)}\left(E^R_{\boldsymbol\alpha}-iE^I_{\boldsymbol\alpha}\right). \tag{69}$$

Therefore, the corresponding fields $\Pi_{\boldsymbol\alpha}$ and $\Pi^*_{\boldsymbol\alpha}$ transform under the same unitary transformation into $\Pi_{\boldsymbol\alpha}e^{i(\boldsymbol s,\boldsymbol\alpha)}$ and $\Pi^*_{\boldsymbol\alpha}e^{-i(\boldsymbol s,\boldsymbol\alpha)}$, respectively. Among the quadratic forms in Eq. (64), only $(\pi_I)^2$ and $\Pi^*_{\boldsymbol\alpha}\Pi_{\boldsymbol\alpha}$ are invariant under the transformations generated by H_s for any $\boldsymbol s\in\mathbb{R}^{\bar r}$. Thus, $\mathcal{L}_{\rm eff}$ in Eq. (63) can be written as

$$\mathcal{L}_{\rm eff}=\rho_I\pi_I\partial_t\pi_I+g_I(\nabla\pi_I)^2+\bar g_I(\partial_t\pi_I)^2+\sum_{\boldsymbol\alpha\in R_H}(\rho_{\boldsymbol\alpha}\Pi_{\boldsymbol\alpha}\partial_t\Pi^*_{\boldsymbol\alpha}+\rho'_{\boldsymbol\alpha}\Pi^*_{\boldsymbol\alpha}\partial_t\Pi_{\boldsymbol\alpha}+b_{\boldsymbol\alpha}|\nabla\Pi_{\boldsymbol\alpha}|^2+\bar b_{\boldsymbol\alpha}|\partial_t\Pi_{\boldsymbol\alpha}|^2). \tag{70}$$

where $\rho_{\boldsymbol\alpha}$, $\rho'_{\boldsymbol\alpha}$, $b_{\boldsymbol\alpha}$, and $\bar b_{\boldsymbol\alpha}$ are real constants. The first, fourth, and fifth terms correspond to the first term in Eq. (63) and hence to the K-K form in Eq. (58). From Eq. (58), we have

$$\rho_I=-i\langle[\hat Q_I,\hat Q_I]\rangle=0, \tag{71}$$

$$\rho_{\boldsymbol\alpha}=-i\left\langle\left[\hat Q_{E^R_{\boldsymbol\alpha}}+i\hat Q_{E^I_{\boldsymbol\alpha}},\hat Q_{E^R_{\boldsymbol\alpha}}-i\hat Q_{E^I_{\boldsymbol\alpha}}\right]\right\rangle=-2i\left\langle\hat Q_{H_{\boldsymbol\alpha}}\right\rangle, \tag{72}$$

$$\rho'_{\boldsymbol\alpha}=-i\left\langle\left[\hat Q_{E^R_{\boldsymbol\alpha}}-i\hat Q_{E^I_{\boldsymbol\alpha}},\hat Q_{E^R_{\boldsymbol\alpha}}+i\hat Q_{E^I_{\boldsymbol\alpha}}\right]\right\rangle=-\rho_{\boldsymbol\alpha}. \tag{73}$$

Thus, we obtain the block-diagonalized form of the quadratic part of an effective Lagrangian in $\Pi^*_{\boldsymbol\alpha},\pi_I$:

$$\mathcal{L}_{\rm eff}=g_I(\nabla\pi_I)^2+\bar g_I(\partial_t\pi_I)^2+\sum_{\boldsymbol\alpha\in R_H}[\rho_{\boldsymbol\alpha}(\Pi_{\boldsymbol\alpha}\partial_t\Pi^*_{\boldsymbol\alpha}-\Pi^*_{\boldsymbol\alpha}\partial_t\Pi_{\boldsymbol\alpha})+b_{\boldsymbol\alpha}|\nabla\Pi_{\boldsymbol\alpha}|^2+\bar b_{\boldsymbol\alpha}|\partial_t\Pi_{\boldsymbol\alpha}|^2], \tag{74}$$

$$\rho_{\boldsymbol\alpha}=-2i\left\langle\hat Q_{H_{\boldsymbol\alpha}}\right\rangle. \tag{75}$$

Here R_H represents the set of the positive root $\boldsymbol\alpha$ vectors associated with the broken generators. For the ODLRO (DLRO) with $\boldsymbol\mu_0$, the quadratic part of an effective Lagrangian is obtained by replacing R_H into $R_0^{\rm OD}$ ($R_0^{\rm D}$).

Next, we calculate $\langle\hat Q_{H_{\boldsymbol\alpha}}\rangle$ in Eq. (75). For μ-SB with $\boldsymbol\mu_0$, $\langle\hat Q_{H_{\boldsymbol\alpha}}\rangle$ vanishes for any positive root vector $\boldsymbol\alpha$ because the expectation value of any generator also vanishes. Therefore, the term $\rho_{\boldsymbol\alpha}(\Pi_{\boldsymbol\alpha}\partial_t\Pi^*_{\boldsymbol\alpha}-\Pi^*_{\boldsymbol\alpha}\partial_t\Pi_{\boldsymbol\alpha})$ in Eq. (74) vanishes and hence we obtain the fourth row of Eq. (59). On the other hand, it follows from the definition of μ-SB for $\boldsymbol\mu_H$ in Eqs. (8) and (9) that

$$\left\langle\hat Q_{H_{\boldsymbol\alpha}}\right\rangle=N_C(\boldsymbol\mu_H,\boldsymbol\alpha)\neq0\quad\text{for }\boldsymbol\alpha\in R_H, \tag{76}$$

where N_C is the number of particles that contribute to the LRO. The term $\bar b_{\boldsymbol\alpha}|\partial_t\Pi_{\boldsymbol\alpha}|^2$ in Eq. (74) is much smaller than the term $\rho_{\boldsymbol\alpha}(\Pi_{\boldsymbol\alpha}\partial_t\Pi^*_{\boldsymbol\alpha}-\Pi^*_{\boldsymbol\alpha}\partial_t\Pi_{\boldsymbol\alpha})$ at low energy because the former is the second order in ∂_t while the latter is the first order in ∂_t. Therefore, the term $\bar b_{\boldsymbol\alpha}|\partial_t\Pi_{\boldsymbol\alpha}|^2$ can be ignored and hence we obtain the third row of Eq. (59). For the case of DLRO, the terms involving π_I are absent since I is not a broken generator from Theorem 1. Therefore, we obtain the first and second rows of Eq. (59), which completes the proof of Theorem 2.

D. Three types of NG modes in μ-SB

From Theorem 2, we find that three types of NG modes arise in μ-SB described by π_I and $\Pi_{\boldsymbol\alpha}$ for $\boldsymbol\mu_0$, and $\Pi_{\boldsymbol\alpha}$ for $\boldsymbol\mu_H$, which we call the phason, the oscillaton, and the precesson, respectively (see Table II). The dynamics of these three NG modes are similar to those of the phonon in the scalar BEC, the magnon in the AFM, and the magnon in the FM, respectively.

TABLE II. Three types of NG modes in μ-SB in a nonrelativistic system. The second column shows the classification of μ-SB. The third, fourth, fifth, and sixth columns show the generators, fields, dynamics, and dispersion relations of the NG modes, respectively, where π_I is the field of the NG mode generated by the broken generators I (the identity operator), $\Pi_{\boldsymbol{\alpha}}$ is defined as $\Pi_{\boldsymbol{\alpha}} := \pi^R_{\boldsymbol{\alpha}} + i\pi^I_{\boldsymbol{\alpha}}$ with $\pi^R_{\boldsymbol{\alpha}}$ and $\pi^R_{\boldsymbol{\alpha}}$ being the fields of NG modes generated by the broken generators $E^R_{\boldsymbol{\alpha}}$ and $E^I_{\boldsymbol{\alpha}}$, respectively, and $\boldsymbol{S}_{\boldsymbol{\alpha}}$ is the generalized magnetization defined in Eq. (42). The last column shows a pair of the NG modes into which each of three types of NG modes, π_I, $\Pi_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} \in R_0)$ and $\Pi_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} \in R_H)$ decay through the interaction Lagrangians in Eqs. (82) and (83).

Name	Classification	Generator	Field	Dynamics	Dispersion	Decay processes
Phason	ODLRO with $\boldsymbol{\mu}_H$ ODLRO with $\boldsymbol{\mu}_0$	I	π_I	Density fluctuation	Linear	Two phasons π_I Two oscillatons $\Pi_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} \in R_0)$ Two precessons $\Pi_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} \in R_H)$
Oscillaton	DLRO with $\boldsymbol{\mu}_0$ ODLRO with $\boldsymbol{\mu}_0$	$E^R_{\boldsymbol{\alpha}}$ and $E^I_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in R_0)$	$\Pi_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in R_0)$	Oscillation of $\boldsymbol{S}_{\boldsymbol{\alpha}}$ [Eq. (78)]	Linear	One phason π_I and one oscillaton $\Pi_{\boldsymbol{\alpha}}$ Two oscillatons $\Pi_{\boldsymbol{\beta}}$ and $\Pi_{\boldsymbol{\gamma}}(\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma})$
Precesson	DLRO with $\boldsymbol{\mu}_H$ ODLRO with $\boldsymbol{\mu}_H$	$E^R_{\boldsymbol{\alpha}}$ and $E^I_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in R_H)$	$\Pi_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in R_H)$	Precession of $\boldsymbol{S}_{\boldsymbol{\alpha}}$ [Eq. (80)]	Quadratic	One phason π_I and one precesson $\Pi_{\boldsymbol{\alpha}}$ Two precessons $\Pi_{\boldsymbol{\beta}}$ and $\Pi_{\boldsymbol{\gamma}}(\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma})$

The phason is the type-1 NG mode which arises from the generator I (the identity operator) and similar to the phonon in the scalar BEC which describes density fluctuations. Although both the oscillaton and the precesson arise from the real and imaginary parts, $E^R_{\boldsymbol{\alpha}}$ and $E^I_{\boldsymbol{\alpha}}$, of the raising operators $E_{\pm\boldsymbol{\alpha}}$, they are different in their dispersion relations and the dynamics. These differences arise from the expectation value of the Noether charge $\langle \hat{Q}_{H_{\boldsymbol{\alpha}}} \rangle$ associated with the generator $H_{\boldsymbol{\alpha}}$. For $\boldsymbol{\mu}_0$, both the expectation value $\langle \hat{Q}_{H_{\boldsymbol{\alpha}}} \rangle$ and the first-order term in ∂_t in the quadratic part of an effective Lagrangian in Eq. (63) vanish. The quadratic part of the effective Lagrangian of the oscillaton associated with the generator $E^{R,I}_{\boldsymbol{\alpha}}$ can be given from Eq. (61) as

$$\begin{aligned}\mathcal{L}^{\boldsymbol{\alpha}}_{\text{osc}} &= b_{\boldsymbol{\alpha}}|\nabla\Pi_{\boldsymbol{\alpha}}|^2 + \bar{b}_{\boldsymbol{\alpha}}|\partial_t\Pi_{\boldsymbol{\alpha}}|^2 \\ &= b_{\boldsymbol{\alpha}}\left(\nabla\pi^R_{\boldsymbol{\alpha}}\right)^2 + \bar{b}_{\boldsymbol{\alpha}}\left(\partial_t\pi^R_{\boldsymbol{\alpha}}\right)^2 + b_{\boldsymbol{\alpha}}\left(\nabla\pi^I_{\boldsymbol{\alpha}}\right)^2 + \bar{b}_{\boldsymbol{\alpha}}\left(\partial_t\pi^I_{\boldsymbol{\alpha}}\right)^2.\end{aligned} \tag{77}$$

From the effective Lagrangian, two fields $\pi^R_{\boldsymbol{\alpha}}$ and $\pi^I_{\boldsymbol{\alpha}}$ associated with the generators $E^R_{\boldsymbol{\alpha}}$ and $E^I_{\boldsymbol{\alpha}}$ are decoupled, producing two independent harmonic oscillations of the generalized magnetization $\boldsymbol{S}_{\boldsymbol{\alpha}}$,

$$\boldsymbol{S}_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \begin{pmatrix} \sin[\Delta\sin(\boldsymbol{k}\cdot\boldsymbol{x})]\cos\phi \\ \sin[\Delta\sin(\boldsymbol{k}\cdot\boldsymbol{x})]\sin\phi \\ \cos[\Delta\sin(\boldsymbol{k}\cdot\boldsymbol{x})] \end{pmatrix} \quad \text{for } \phi = 0 \text{ or } \frac{\pi}{2}, \tag{78}$$

where Δ ($\ll$1) represents the amplitude of the oscillatons, $\boldsymbol{k}$ is the wave number of the oscillatons, and $\boldsymbol{x}$ is the coordinate in space. These modes are reminiscent of magnons in the AFM, where two magnons representing harmonic oscillations of the magnetization appear. On the other hand, for $\boldsymbol{\mu}_H$, neither the expectation value $\langle \hat{Q}_{H_{\boldsymbol{\alpha}}} \rangle$ nor the first-order term in ∂_t in the quadratic part of the effective Lagrangian in Eq. (63) vanish. The quadratic part of effective Lagrangian of the precesson associated with the generator $E^{R,I}_{\boldsymbol{\alpha}}$ is given from Eq. (60) as

$$\begin{aligned}\mathcal{L}^{\boldsymbol{\alpha}}_{\text{pre}} &= \rho_{\boldsymbol{\alpha}}(\Pi_{\boldsymbol{\alpha}}\partial_t\Pi^*_{\boldsymbol{\alpha}} - \Pi^*_{\boldsymbol{\alpha}}\partial_t\Pi_{\boldsymbol{\alpha}}) + b_{\boldsymbol{\alpha}}|\nabla\Pi_{\boldsymbol{\alpha}}|^2 \\ &= 2i\rho_{\boldsymbol{\alpha}}\left(\pi^I_{\boldsymbol{\alpha}}\partial_t\pi^R_{\boldsymbol{\alpha}} - \pi^R_{\boldsymbol{\alpha}}\partial_t\pi^I_{\boldsymbol{\alpha}}\right) + b_{\boldsymbol{\alpha}}\left(\nabla\pi^R_{\boldsymbol{\alpha}}\right)^2 + b_{\boldsymbol{\alpha}}\left(\nabla\pi^I_{\boldsymbol{\alpha}}\right)^2.\end{aligned} \tag{79}$$

From the effective Lagrangian, we find that two fields $\pi^R_{\boldsymbol{\alpha}}$ and $\pi^I_{\boldsymbol{\alpha}}$ associated with the generators $E^R_{\boldsymbol{\alpha}}$ and $E^I_{\boldsymbol{\alpha}}$ form a canonical conjugate pair, producing a precession mode of the generalized magnetization $\boldsymbol{S}_{\boldsymbol{\alpha}}$,

$$\boldsymbol{S}_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \begin{pmatrix} \sin\Delta'\cos(\boldsymbol{k}\cdot\boldsymbol{x}) \\ \sin\Delta'\sin(\boldsymbol{k}\cdot\boldsymbol{x}) \\ \cos\Delta' \end{pmatrix}, \tag{80}$$

where $\Delta'(\ll 1)$ represents a precession angle. This mode is reminiscent of a magnon in the FM, where one magnon representing the precession of the magnetization appears.

E. Distinctions among three types of NG modes in decay processes

Although both the phason and the oscillaton have linear dispersions, they play distinct roles in decay processes. To see this, let us consider μ-SB with ODLRO and $\boldsymbol{\mu}_0$. Similarly to the proof of Theorem 2, under the unitary transformation generated by $H_{\boldsymbol{s}} \in \mathfrak{h}(\boldsymbol{s} \in \mathbb{R}^{\bar{r}})$, π_I, $\Pi^*_{\boldsymbol{\alpha}}$, and $\Pi_{\boldsymbol{\alpha}}$ are transformed into π_I, $\Pi^*_{\boldsymbol{\alpha}}e^{-i(\boldsymbol{s},\boldsymbol{\alpha})}$, and $\Pi_{\boldsymbol{\alpha}}e^{i(\boldsymbol{s},\boldsymbol{\alpha})}$, respectively. Up to the third order in π_I, $\pi^R_{\boldsymbol{\alpha}}$, and $\pi^I_{\boldsymbol{\alpha}}$, the interaction part of the Lagrangian between NG modes, $\mathcal{L}_{\text{int}}$, which is invariant under the transformations generated by $H_{\boldsymbol{s}} \in \mathfrak{h}$ for any $\boldsymbol{s} \in \mathbb{R}^{\bar{r}}$, should be a linear combination of the following four terms:

$$\begin{aligned}&(\pi_I)^3,\ \pi_I\Pi^*_{\boldsymbol{\alpha}}\Pi_{\boldsymbol{\alpha}} \quad \text{for } \boldsymbol{\alpha} \in R^{\text{OD}}_0, \\ &\Pi^*_{\boldsymbol{\beta}}\Pi^*_{\boldsymbol{\gamma}}\Pi_{\boldsymbol{\alpha}},\ \Pi_{\boldsymbol{\beta}}\Pi_{\boldsymbol{\gamma}}\Pi^*_{\boldsymbol{\alpha}} \quad \text{for } \boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}.\end{aligned} \tag{81}$$

Hence, we obtain

$$\begin{aligned}\mathcal{L}_{\text{int}} &= c(\pi_I)^3 + \pi_I\sum_{\boldsymbol{\alpha}\in R^{\text{OD}}_0} c_{\boldsymbol{\alpha}}\Pi^*_{\boldsymbol{\alpha}}\Pi_{\boldsymbol{\alpha}} \\ &\quad + \sum_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in R^{\text{OD}}_0,\boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\gamma}} \text{Re}(c_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\Pi^*_{\boldsymbol{\beta}}\Pi^*_{\boldsymbol{\gamma}}\Pi_{\boldsymbol{\alpha}}),\end{aligned} \tag{82}$$

where c and $c_{\boldsymbol{\alpha}}$ are real constants, $c_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ are complex numbers, and $\text{Re}(x)$ denotes the real part of x. Thus, one phason decays into two phasons π_I or two oscillatons with the same root vector, whereas one oscillaton $\Pi_{\boldsymbol{\alpha}}$ decays into one phason π_I and one oscillaton $\Pi_{\boldsymbol{\alpha}}$ or two oscillatons, $\Pi_{\boldsymbol{\beta}}$ and $\Pi_{\boldsymbol{\gamma}}$ with $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$.

Similarly, for μ-SB with ODLRO and $\boldsymbol{\mu}_H$, the interaction Lagrangian between NG modes, $\mathcal{L}_{\rm int}$, can be written up to the third order in π_I, $\pi_{\boldsymbol{\alpha}}^R$, and $\pi_{\boldsymbol{\alpha}}^I$ as

$$\mathcal{L}_{\rm int} = c(\pi_I)^3 + \pi_I \sum_{\boldsymbol{\alpha}\in R_H} c_{\boldsymbol{\alpha}} \Pi_{\boldsymbol{\alpha}}^* \Pi_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in R_H, \boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\gamma}} \mathrm{Re}(c_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \Pi_{\boldsymbol{\beta}}^* \Pi_{\boldsymbol{\gamma}}^* \Pi_{\boldsymbol{\alpha}}), \quad (83)$$

where c and $c_{\boldsymbol{\alpha}}$ are real constants, and $c_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ are complex numbers. Thus, one precesson $\Pi_{\boldsymbol{\alpha}}$ decays into two phasons π_I and a precesson $\Pi_{\boldsymbol{\alpha}}$ or two precessons, $\Pi_{\boldsymbol{\beta}}$ and $\Pi_{\boldsymbol{\gamma}}$ with $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$.

V. HOMOTOPY GROUPS OF TOPOLOGICAL EXCITATIONS FOR μ-SB

Let us now calculate the homotopy groups for μ-SB states to find their topological excitations. An element of the first homotopy group $\pi_1(G/H)$ characterizes the topological charge of a vortex and that of the second homotopy group $\pi_2(G/H)$ characterizes the topological charge of a point defect and that of a skyrmion [40]. Usually, the homotopy groups are calculated separately for individual cases. We here show that not only the homotopy groups but also the textures of topological excitations can be calculated systematically for μ-SB.

We first briefly review the theory of an integral lattice and a coroot lattice [41,42] as it is needed for the calculation of the second homotopy group. We define integral lattices L_G and L_H for a compact Lie group G and its subgroup H and a lattice $L_R(S)$ for a subset S of R_+ as

$$L_G := \{\boldsymbol{t} \in \mathbb{R}^r \,|\, \exp(2\pi i H_{\boldsymbol{t}}) = e,\ H_{\boldsymbol{t}} \in \mathfrak{g}\}, \quad (84)$$

$$L_H := \{\boldsymbol{t} \in \mathbb{R}^r \,|\, \exp(2\pi i H_{\boldsymbol{t}}) = e,\ H_{\boldsymbol{t}} \in \mathfrak{h}\}, \quad (85)$$

$$L_R(S) := \mathrm{span}_{\mathbb{Z}} \left\{ \frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})} \middle| \boldsymbol{\alpha} \in S \right\}, \quad (86)$$

where e is the identity element of G, $H_{\boldsymbol{t}}$ for $\boldsymbol{t} \in \mathbb{R}^r$ is defined by

$$H_{\boldsymbol{t}} = \sum_{b=1}^{r} t_b H_b, \quad (87)$$

$\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively, and $\mathrm{span}_{\mathbb{Z}} X$ denotes a vector space spanned by elements of X with integer coefficients:

$$\mathrm{span}_{\mathbb{Z}} X := \left\{ \sum_{k=1}^{m} n_k x_k \middle| x_k \in X,\ n_k \in \mathbb{Z},\ m \in \mathbb{N} \right\}. \quad (88)$$

The vector $2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})$ and $L_R(R_+)$ are called the coroot vector (the inverse root vector) of a root vector $\boldsymbol{\alpha}$ and the coroot lattice (the inverse root lattice) of G, respectively.

Under the addition of r-dimensional vectors, L_G, L_H, and $L_R(S)$ $(S \subset R_+)$ form Abelian groups. Let us examine this point by discussing an example of L_G. Let $\boldsymbol{t}$ and $\boldsymbol{s}$ be elements of L_G. The sum $\boldsymbol{t} + \boldsymbol{s}$ satisfies

$$\exp(2\pi i H_{\boldsymbol{t}+\boldsymbol{s}}) = \exp(2\pi i H_{\boldsymbol{t}}) \exp(2\pi i H_{\boldsymbol{s}}) = ee = e, \quad (89)$$

and hence we have $\boldsymbol{t} + \boldsymbol{s} \in L_G$. The identity element of L_G is the zero vector $\boldsymbol{0}$ and the inverse element of $\boldsymbol{t}$ $(\in L_G)$ is $-\boldsymbol{t}$. Since $L_R(S)$ is an Abelian group generated by a subset S of R_+, $L_R(S)$ is an Abelian subgroup of the coroot lattice $L_R(R_+)$. Therefore, the coset space $L_R(R_+)/L_R(S)$ becomes an Abelian group. Writing the element of the coset space $L_R(R_+)/L_R(S)$ as $[\boldsymbol{t}]$ $(\boldsymbol{t} \in L_R(R_+))$, the sum in the coset space $L_R(R_+)/L_R(S)$ is defined as follows:

$$[\boldsymbol{t}] + [\boldsymbol{s}] := [\boldsymbol{t} + \boldsymbol{s}]. \quad (90)$$

It is known that any coroot vector $2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})$ is an element of L_G [41,42]. The coroot vector $2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in R_+\backslash R_H$ is an element of L_H because $H_{\boldsymbol{\alpha}} \in \mathfrak{h}$ from Theorem 1, where $R_+\backslash R$ is the set of elements that belong to R_+ but not to R. Therefore, $L_R(R_+\backslash R_H)$ is a common subgroup of $L_R(R_+)$ and L_H, and hence the coset space $[L_H \cap L_R(R_+)]/L_R(R_+\backslash R_H)$ forms an Abelian group.

Then the following theorem holds.

Theorem 3. (1) Provided that H is a connected subgroup of G, the first homotopy group for μ-SB is given as follows:

$$\pi_1(G/H) = \begin{cases} 0 & \text{for DLRO}, \\ \mathbb{Z}_l & \text{for ODLRO and } \boldsymbol{\mu}_H, \\ \mathbb{Z} & \text{for ODLRO and } \boldsymbol{\mu}_0, \end{cases} \quad (91)$$

where $\mathbb{Z}_l = \{g^i | i = 0,1,2,\ldots,l-1\}$ is the cyclic group of order l (g is the generator of $\mathbb{Z}_l$), l is a positive integer which is uniquely determined from $\boldsymbol{\mu}_H$, and $\mathbb{Z}_{l=1}$ is a trivial group consisting of the identity element alone. Let $\theta \in [0,2\pi]$ and $O(\theta)$ be the azimuth angle around the vortex and the value of the order parameter at the angle θ. The vortex with topological charge g represents a vortex around which the phase of the order parameter rotates by 2π:

$$O(\theta) = e^{i\theta} O_0, \quad (92)$$

where O_0 is the value of the order parameter at $\theta = 0$.

(2) The second homotopy group for μ-SB is isomorphic to the coset space constructed from the coroot lattice as follows:

$$\pi_2(G/H) = \begin{cases} \frac{L_R(R_+)}{L_R(R_+\backslash R_H)} & \text{for DLRO and } \boldsymbol{\mu}_H, \\ \frac{L_R(R_+)}{L_R(R_+\backslash R_0^{\rm D})} & \text{for DLRO and } \boldsymbol{\mu}_0, \\ \frac{L_H \cap L_R(R_+)}{L_R(R_+\backslash R_H)} & \text{for ODLRO and } \boldsymbol{\mu}_H, \\ \frac{L_R(R_+)}{L_R(R_+\backslash R_0^{\rm OD})} & \text{for ODLRO and } \boldsymbol{\mu}_0. \end{cases} \quad (93)$$

Let $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ be the three-dimensional polar coordinates surrounding the point defect, $O(\theta,\phi)$ be the value of the order parameter at (θ,ϕ). The texture of the point defect with topological charge $[2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})]$ is given by

$$O(\theta,\phi) = \exp\left[i\phi \frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})} \right] \circ \exp\left[i\theta \frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})} \right] \circ O_0, \quad (94)$$

where O_0 and $g \circ O_0$ $(g \in G)$ is the value of the order parameter at $\theta = \phi = 0$ and that of the order parameter obtained by the symmetry transformation g from O_0, respectively ($\circ$ denotes the group operaton).

The proof of this theorem is given in Appendix.

Physically, nontrivial elements $\mathbb{Z}_l$ and $\mathbb{Z}$ in Eq. (91) represent quantum vortices. We note that a quantum vortex does not necessarily have an integer charge in a system with internal degrees of freedom [43]. Let us examine this point as yet another example of μ-SB. Consider the three-dimensional representation of $\bar{\mathfrak{g}} = \mathfrak{su}(N=2)$ and a phase with ODLRO and $\boldsymbol{\mu}_H$. This example will appear as a special case of $N = 2$ of a U(N)-symmetric system in Sec. VI. In this representation, the order parameter $\langle \mathbf{\Delta}_s \rangle$ is a 2×2 symmetric complex matrix which transforms under the symmetry transformation of U(2) as follows [20]:

$$\langle \mathbf{\Delta}_s \rangle \mapsto U \langle \mathbf{\Delta}_s \rangle^t U \quad \text{for } U \in \mathrm{U}(2). \tag{95}$$

In μ-SB with ODLRO and $\boldsymbol{\mu}_H$, the expectation value of $\langle \mathbf{\Delta}_s \rangle$ is given by

$$\langle \mathbf{\Delta}_s \rangle = \langle \mathbf{\Delta}_0 \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{96}$$

Since this representation is the three-dimensional representation of U(2), which is equivalent to the three-dimensional representation of U(1) $\times$ SO(3), the order parameter $\langle \mathbf{\Delta}_s \rangle$ is related to the three-dimensional complex vector $\langle \boldsymbol{\phi} \rangle = (\langle \phi_1 \rangle, \langle \phi_0 \rangle, \langle \phi_{-1} \rangle)$, which is the order parameter of a spin-1 BEC [21,22], as

$$\langle \mathbf{\Delta}_s \rangle = \begin{pmatrix} \langle \phi_1 \rangle & \langle \phi_0 \rangle \\ \langle \phi_0 \rangle & \langle \phi_{-1} \rangle \end{pmatrix}. \tag{97}$$

The first homotopy group $\pi_1(G/H)$ of this phase is given by

$$\pi_1(G/H) = \mathbb{Z}_2 = \{e, g\}, \tag{98}$$

where e and g are the identity element and the generator of $\mathbb{Z}_2$ with $g^2 = e$. From Eq. (92), a vortex with topological charge g is given by

$$\langle \mathbf{\Delta}_s^g \rangle(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 0 \end{pmatrix}, \tag{99}$$

where $\theta \in [0, 2\pi]$ represents the azimuth angle around the vortex. We can show $g^2 = e$ as follows. Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli matrices. From Eq. (92), the texture of the vortex with topological charge g^2 is given by

$$\langle \mathbf{\Delta}_s^{g^2} \rangle(\theta) = \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & 0 \end{pmatrix} = U(\theta) \langle \mathbf{\Delta}_0 \rangle^t U(\theta), \tag{100}$$

$$U(\theta) = e^{i\theta\sigma_z}. \tag{101}$$

By the continuous deformation defined by

$$\langle \mathbf{\Delta}_{s,t} \rangle(\theta) = U(\theta,t) \langle \mathbf{\Delta}_0 \rangle^t U(\theta,t), \tag{102}$$

$$U(\theta,t) = e^{-i\pi t \frac{\sigma_y}{2}} e^{i\theta \frac{\sigma_z}{2}} e^{i\pi t \frac{\sigma_y}{2}} e^{i\theta \frac{\sigma_z}{2}} \quad \text{for } 0 \leqslant t \leqslant 1, \tag{103}$$

$\langle \mathbf{\Delta}_{s,t} \rangle$ is transformed from $\langle \mathbf{\Delta}_{s,t=0} \rangle = \langle \mathbf{\Delta}_s^{g^2} \rangle$ to a uniform order $\langle \mathbf{\Delta}_{s,t=1} \rangle = \langle \mathbf{\Delta}_0 \rangle$, which implies $g^2 = e$.

From Eqs. (86) and (93), $\pi_2(G/H)$ is generated by a set of representative elements $\{[2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})] | \boldsymbol{\alpha} \in R_+\}$ of the coset space. This shows that any point defect in μ-SB can be represented as a composite of the point defects with topological charge $[2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})]$ ($\boldsymbol{\alpha} \in R_+$). In Eq. (93), the coset spaces of $L_R(R_+)$ by their subgroups $L_R(S)$ ($S = R_+ \backslash R_H, R_+ \backslash R_0^{\mathrm{D}}, R_+ \backslash R_0^{\mathrm{OD}}$) are considered instead of the numerators $L_R(R_+)$ or $L_H \cap L_R(R_+)$. This is because the element of the denominators $L_R(S)$ of the coset does not give nontrivial topological excitations. Let us clarify this point for the case of $\boldsymbol{\mu}_H$. Since $E_{\boldsymbol{\alpha}}^{R,I}$ and $H_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in R_+ \backslash R_H$ are unbroken generators for $\boldsymbol{\mu}_H$, the successive actions of $\exp[i\phi \frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}]$ and $\exp[i\theta \frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}]$ leave the order parameter invariant:

$$\exp\left[i\phi \frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\right] \circ \exp\left[i\theta \frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\right] \circ O_0 = O_0 \quad \text{for } \forall\, (\theta,\phi) \in [0,\pi] \times [0,2\pi]. \tag{104}$$

Therefore, Eq. (94) does not give a nontrivial point defect but a uniform order for $\boldsymbol{\alpha} \in R_+ \backslash R_H$ for $\boldsymbol{\mu}_H$.

The point defect in Eq. (94) with the topological charge $2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})$ is similar to the point defect in the FM and the AFM in that the former is obtained by replacing $\mathfrak{su}(2)$ spin $\boldsymbol{S}$ by the generalized magnetization $\boldsymbol{S}_{\boldsymbol{\alpha}}$. In fact, let $\boldsymbol{M}$ be the magnetization of the FM or that of the sublattice in the AFM. A point defect can be described as a hedgehog configuration of $\boldsymbol{M}$:

$$\boldsymbol{M}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{105}$$

where θ and ϕ are three-dimensional polar coordinates. This hedgehog configuration is obtained by the successive rotation of the spin around the y axis by angle θ followed by the rotation around the z axis by angle ϕ,

$$\boldsymbol{M}(\theta,\phi) = \exp(i\phi S_z) \circ \exp(i\theta S_y) \circ \boldsymbol{M}_0, \tag{106}$$

where $\boldsymbol{M}_0 = (0,0,1)$ is the magnetization of the FM or that of the sublattice in the AFM parallel to the z axis. Comparing with Eq. (94), we can conclude that the point defect in Eq. (94) is the generalization of the point defect in the FM and the AFM obtained by replacing $\boldsymbol{S}$ by $\boldsymbol{S}_{\boldsymbol{\alpha}}$.

VI. APPLICATION OF μ-SB TO U(N)-SYMMETRIC SYSTEMS

In this section, we apply μ-SB to U(N)-symmetric systems [13,14,44,45]. Here, we consider up to the third lowest-dimensional representation of the $\mathfrak{su}(N)$-Lie algebra since symmetry-broken phases are not necessarily characterized by μ-SB in higher-dimensional representations. Up to and including the third lowest-dimensional representation, the irreducible representations of the $\mathfrak{su}(N)$-Lie algebra are given by the following three representations, the N-dimensional representation, the $N(N-1)/2$-dimensional representation, and the $N(N+1)/2$-dimensional representation. To make this paper self-contained, we briefly review the symmetry transformation, the highest weight, and the set of weight vectors of these three representations. See Refs. [20,46] for details on the $\mathfrak{su}(N)$-Lie algebra and its representations.

Let $\{\boldsymbol{\nu}_j\}_{j=1}^N$ be a set of real $(N-1)$-dimensional vectors that satisfy

$$\sum_{j=1}^{N} \boldsymbol{\nu}_j = 0, \tag{107}$$

$$(\boldsymbol{\nu}_i,\boldsymbol{\nu}_j) = \delta_{ij} - \frac{1}{N}. \tag{108}$$

Defining $\boldsymbol{\alpha}_{i,j}$ by $\boldsymbol{\alpha}_{i,j} := \boldsymbol{\nu}_i - \boldsymbol{\nu}_j$, the set of positive root vectors R_+ of the $\mathfrak{su}(N)$ Lie algebra is given by

$$R_+ = \{\boldsymbol{\alpha}_{i,j} | \boldsymbol{\alpha}_{i,j} := \boldsymbol{\nu}_i - \boldsymbol{\nu}_j, 1 \leqslant i < j \leqslant N\}. \tag{109}$$

The weight vector $\boldsymbol{\nu}_i$ is normalized in Eq. (108) so that the magnitude of the root vectors is $\sqrt{2}$. For example, $\{\boldsymbol{\nu}_j\}_{j=1}^N$ for $N=2$ and 3 are given by

$$N=2: \boldsymbol{\nu}_1 = \frac{1}{\sqrt{2}}, \quad \boldsymbol{\nu}_2 = -\frac{1}{\sqrt{2}}, \tag{110}$$

$$N=3: \boldsymbol{\nu}_1 = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}}\right), \quad \boldsymbol{\nu}_2 = \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}}\right),$$
$$\boldsymbol{\nu}_3 = \left(0,\frac{1}{\sqrt{2}}\right). \tag{111}$$

For $N=2$ and $N=3$, the schematic illustrations are presented in Figs. 1(a) and 1(b), respectively.

The lowest-dimensional representation is the N-dimensional representation. The element of this representation is an N-dimensional complex vector $\boldsymbol{v}$. The symmetry transformation of this representation acts on $\boldsymbol{v}$ as an action of the matrix from the left:

$$\boldsymbol{v} \mapsto U\boldsymbol{v} \quad \text{for } U \in \mathrm{U}(N). \tag{112}$$

For the ODLRO, $\boldsymbol{v}$ is the order parameter of a BEC in degenerate N-component bosons. The highest weight $\boldsymbol{\mu}_H$ and the set of weight vectors $W[\boldsymbol{\mu}_H]$ are given by

$$\boldsymbol{\mu}_H = \boldsymbol{\nu}_1, \tag{113}$$

$$W[\boldsymbol{\nu}_1] = \{\boldsymbol{\nu}_j | j = 1,2,\ldots,N\}. \tag{114}$$

The weight vector state $|\boldsymbol{\nu}_j)$ is a unit vector whose components vanish except for the jth component:

$$|\boldsymbol{\nu}_j) = {}^t(0,\ldots,0,\overset{j}{1},0,\ldots,0). \tag{115}$$

The second lowest-dimensional representation is the $N(N-1)/2$-dimensional representation. The element of this representation is an $N\times N$ complex skew-symmetric matrix $\boldsymbol{\Delta}_a$. In fact, the dimension of the set of $N\times N$ complex skew-symmetric matrices is $N(N-1)/2$. The symmetry transformation of this representation acts on $\boldsymbol{\Delta}_a$ as an action of the matrix and that of its transpose from the left and the right, respectively:

$$\boldsymbol{\Delta}_a \mapsto U\boldsymbol{\Delta}_a{}^tU \quad \text{for } U \in \mathrm{U}(N). \tag{116}$$

For the ODLRO, this order parameter corresponds to that of an s-wave superfluid phase in degenerate N-component fermions in a nonrelativistic system [24–27]. In fact, let $\{\psi_i\}_{i=1}^N$ be the fields of the degenerate N-component fermions. Since the N-components are degenerate and the total number of fermion is conserved in a nonrelativistic system, this system is invariant under the U(N)-symmetry transformation:

$$\psi_i \mapsto U_{ij}\psi_j \quad \text{for } U \in \mathrm{U}(N). \tag{117}$$

Here, repeated indices are assumed to be summed over $i = 1,\ldots,N$. The order parameter of the phase is given by the following $N\times N$ matrix:

$$\widetilde{\boldsymbol{\Delta}}_a = \{\langle\psi_i\psi_j\rangle\}_{i,j=1}^N. \tag{118}$$

The antisymmetric nature of $\widetilde{\boldsymbol{\Delta}}_a$ arises from the anticommutation relation of the fermions. Under the U(N)-symmetric transformation, ψ_i and $\widetilde{\boldsymbol{\Delta}}_a$ transform as

$$\psi_i \mapsto U_{ij}\psi_j, \tag{119}$$

$$(\widetilde{\boldsymbol{\Delta}}_a)_{ij} \mapsto U_{ik}\langle\psi_k\psi_l\rangle({}^tU)_{lj} = (U\widetilde{\boldsymbol{\Delta}}_a{}^tU)_{ij}$$
$$\text{for } U \in \mathrm{U}(N), \tag{120}$$

which coincides with the transformation (116). The highest weight $\boldsymbol{\mu}_H$ and the set of weight vectors $W[\boldsymbol{\mu}_H]$ are given by

$$\boldsymbol{\mu}_H = \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2, \tag{121}$$

$$W[\boldsymbol{\nu}_1+\boldsymbol{\nu}_2] = \{\boldsymbol{\nu}_i+\boldsymbol{\nu}_j | 1 \leqslant i < j \leqslant N\}. \tag{122}$$

The weight vector state $|\boldsymbol{\nu}_i+\boldsymbol{\nu}_j)$ is the skew-symmetric matrix whose elements vanish except for the (i,j)th and (j,i)th elements:

$$|\boldsymbol{\nu}_i+\boldsymbol{\nu}_j) = \boldsymbol{\Delta}_a^{(i,j)}, \tag{123}$$

$$\left[\boldsymbol{\Delta}_a^{(i,j)}\right]_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{124}$$

The third lowest-dimensional representation is the $N(N+1)/2$-dimensional representation. The element of this representation is an $N\times N$ complex symmetric matrix $\boldsymbol{\Delta}_s$. In fact, the dimension of the set of $N\times N$ complex symmetric matrices is $N(N+1)/2$. Similarly to Eq. (116), the symmetry transformation of this representation acts on $\boldsymbol{\Delta}_s$ as

$$\boldsymbol{\Delta}_s \mapsto U\boldsymbol{\Delta}_s{}^tU \quad \text{for } U \in \mathrm{U}(N). \tag{125}$$

Although the transformation in Eq. (125) coincides with Eq. (116), $\boldsymbol{\Delta}_a$ has the odd parity, ${}^t\boldsymbol{\Delta}_a = -\boldsymbol{\Delta}_a$, and $\boldsymbol{\Delta}_s$ the even parity, ${}^t\boldsymbol{\Delta}_s = \boldsymbol{\Delta}_s$. An $N\times N$ skew-symmetric (symmetric) matrix $\boldsymbol{\Delta}$ transforms into a skew-symmetric (symmetric) matrix under the transformation $\boldsymbol{\Delta} \mapsto U\boldsymbol{\Delta}{}^tU$ $(U \in \mathrm{U}(N))$. For the ODLRO, the order parameter $\boldsymbol{\Delta}_s$ is related to that of a p-wave superfluid phase in degenerate N-component fermions [47]. In fact, let $\{\psi_i\}_{i=1}^N$ be the fields of the degenerate N-component fermions. The order parameter of the p-wave superfluid phase is given as follows [47]:

$$\langle\psi_{i,\boldsymbol{k}}\psi_{j,-\boldsymbol{k}}\rangle = \sum_{\alpha=x,y,z} k_\alpha \Delta_{\alpha,ij}. \tag{126}$$

where three $N\times N$ matrices $\boldsymbol{\Delta}_\alpha := \{\Delta_{\alpha,ij}\}_{i,j=1}^N$ $(\alpha = x,y,z)$ are symmetric matrices [47]. The symmetric nature of $\boldsymbol{\Delta}_\alpha$ arises from the anticommutation relation of the fermions and the odd parity of the orbital part of the p-wave pairing. Under the symmetry transformation that mixes the degenerate N-components, $\psi_i \mapsto U_{ij}\psi_j$, the $N\times N$ symmetric matrix $\boldsymbol{\Delta}_s$

transforms according to Eq. (125). The highest weight $\boldsymbol{\mu}_H$ and the set of weight vectors $W[\boldsymbol{\mu}_H]$ are given by

$$\boldsymbol{\mu}_H = 2\boldsymbol{\nu}_1, \tag{127}$$

$$W[2\boldsymbol{\nu}_1] = \{\boldsymbol{\nu}_i + \boldsymbol{\nu}_j | 1 \leqslant i \leqslant j \leqslant N\}. \tag{128}$$

The weight vector state $|\boldsymbol{\nu}_i + \boldsymbol{\nu}_j)$ is the symmetric matrix whose elements vanish except for the (i,j)th and (j,i)th elements:

$$|\boldsymbol{\nu}_i + \boldsymbol{\nu}_j) = \boldsymbol{\Delta}_s^{(i,j)}, \tag{129}$$

$$\left[\boldsymbol{\Delta}_s^{(i,j)}\right]_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{130}$$

A. Classification of μ-SB phases

We first classify the μ-SB phases for $\bar{\mathfrak{g}} = \mathfrak{su}(N)$ that appear up to and including the third lowest-dimensional representation.

1. Lowest-dimensional representation

For μ-SB with ODLRO and $\boldsymbol{\mu}_H$, the expectation value of the order parameter coincides with the highest weight of this representation in Eq. (114):

$$\langle \boldsymbol{\phi} \rangle = |\boldsymbol{\nu}_1) = (1,0,\dots,0). \tag{131}$$

The remaining symmetry H of the state is given by

$$\begin{aligned} H &:= \{U \in \mathrm{U}(N) | U|\boldsymbol{\nu}_1) = |\boldsymbol{\nu}_1)\} \\ &= \{U \in \mathrm{U}(N) | U_{1j} = U_{j1} = \delta_{j1}\ (j = 1,2,\dots,N)\} \\ &\simeq \mathrm{U}(N-1), \end{aligned} \tag{132}$$

where $\simeq$ represents the group isomorphism. Thus, H is a connected group. In this representation, the Casimir invariant $C_2^{\bar{\mathfrak{g}}}(\langle \boldsymbol{\phi} \rangle)$ coincides with $|\langle \boldsymbol{\phi} \rangle|^4$ and hence $C_2^{\bar{\mathfrak{g}}}(\langle \boldsymbol{\phi} \rangle) \neq 0$ so long as the order parameter has a nonzero expectation value. Therefore the pair of ODLRO and $\boldsymbol{\mu}_0$ is absent in this representation.

We next consider the case of DLRO. There are N-fold degenerate states in each site which are referred to as color or flavor. For μ-SB with DLRO and $\boldsymbol{\mu}_H$, the mean-field ground state is given by

$$|\mathrm{GS}\rangle = \bigotimes_{i \in L} |\boldsymbol{\nu}_1)_i. \tag{133}$$

This is the mean-field of an SU(N)-ferromagnet [44]. The unitary operator $\widehat{U}$ associated with the symmetry transformation $U \in \mathrm{U}(N)$ leaves $|\mathrm{GS}\rangle$ unchanged up to a global phase factor if and only if U leaves the weight vector $|\boldsymbol{\nu}_1)$ on each sites unchanged up to a global phase:

$$U|\boldsymbol{\nu}_1) = e^{i\phi}|\boldsymbol{\nu}_1) \quad \text{for} \ \exists\, \phi \in \mathbb{R}. \tag{134}$$

This condition is equivalent to the condition that the first row and the first column of U vanish except for a diagonal element:

$$U_{1j} = U_{j1} = 0 \quad \text{for } j = 2,3,\dots,N. \tag{135}$$

The remaining symmetry H of the state is given by

$$\begin{aligned} H &:= \{U \in \mathrm{U}(N) | U_{1j} = U_{j1} = 0\ (j = 2,3,\dots,N)\} \\ &\simeq \mathrm{U}(1) \times \mathrm{U}(N-1), \end{aligned} \tag{136}$$

and hence H is a connected group. For the case of $\boldsymbol{\mu}_0$, from Eq. (32) the expectation value of the Casimir invariant within the unit cell u is given by

$$C_2^{\bar{\mathfrak{g}}} = \left\| \sum_{i \in u} \boldsymbol{\mu}_i \right\|^2. \tag{137}$$

From Eq. (107), the right-hand side of Eq. (137) vanishes when the unit cell consists of N-different weight vectors, $\{\boldsymbol{\mu}_i\}_{i \in u} = \{\boldsymbol{\nu}_j\}_{j=1}^N$. Thus, the ground state is given by

$$|\mathrm{GS}\rangle = \bigotimes_{u \in \mathcal{U}} \bigotimes_{i \in u} |\boldsymbol{\mu}_i)_i, \tag{138}$$

$$\{\boldsymbol{\mu}_i\}_{i \in u} = \{\boldsymbol{\nu}_j\}_{j=1}^N. \tag{139}$$

This is the mean-field ground state of the so-called N-color density wave (N-CDW) phase [48–51]. This state is a generalization of the SU(2)-antiferromagnet to a general SU(N)-spin system. In the former, two states, a spin-up state and a spin-down state, constitute a unit cell, while in the latter N states do. The unitary operator $\widehat{U}$ associated with the symmetry transformation $U \in \mathrm{U}(N)$ leaves $|\mathrm{GS}\rangle$ unchanged up to a global phase factor if and only if U leaves all of the N-different weight vectors $|\boldsymbol{\nu}_i)$ within each unit cell unchanged up to a global phase. In other words, the condition

$$U|\boldsymbol{\nu}_i) = e^{i\phi}|\boldsymbol{\nu}_i) \quad \text{for} \ \exists\, \phi \in \mathbb{R}, \tag{140}$$

must be satisfied for all of N-weight vectors $\{\boldsymbol{\nu}_j\}_{j=1}^N$. This condition is satisfied if and only if U is a diagonal matrix. Thus, the remaining symmetry H of the state is given by

$$\begin{aligned} H &:= \{U \in \mathrm{U}(N) | U = \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, \dots, e^{i\phi_N}),\ \phi_i \in \mathbb{R}\} \\ &\simeq \mathrm{U}(1)^N, \end{aligned} \tag{141}$$

and hence H is a connected group.

2. Second-lowest-dimensional representation

For μ-SB with ODLRO and $\boldsymbol{\mu}_H$, the expectation value of the order parameter is given from Eqs. (8) and (53) by

$$\boldsymbol{\Delta}_a = |\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2) = \boldsymbol{\Delta}_a^{(1,2)}. \tag{142}$$

The remaining symmetry H of the state is determined by a straightforward calculation as

$$\begin{aligned} H &:= \left\{U \in \mathrm{U}(N) | U \boldsymbol{\Delta}_a^{(1,2)t} U = \boldsymbol{\Delta}_a^{(1,2)}\right\} \\ &= \{U \in \mathrm{U}(N) | U_{11}U_{22} - U_{12}U_{21} = 1, \\ &\qquad U_{ij} = U_{ji} = 0\ (i = 1,2,\ j = 3,4,\dots,N)\} \\ &\simeq \mathrm{SU}(2) \times \mathrm{U}(N-2), \end{aligned} \tag{143}$$

and hence H is a connected group. On the other hand, μ-SB with ODLRO and $\boldsymbol{\mu}_0$ does not necessarily exist for arbitrary N. The same point is discussed in the derivation of the mean fields of μ-SB in Sec. II. The set $W[\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2]$ does not necessarily include the zero-weight vector [20]. In fact, $\boldsymbol{\nu}_i + \boldsymbol{\nu}_j = \mathbf{0}$ ($1 \leqslant i < j \leqslant N$) only when the following conditions are met: $N = 2$, $\boldsymbol{\nu}_i = \boldsymbol{\nu}_1$, and $\boldsymbol{\nu}_j = \boldsymbol{\nu}_2$. For N greater than 2, $W[\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2]$ does not include the zero-weight vector.

We next consider the case of DLRO. There are $N(N-1)/2$-degenerate states labeled by a weight vector $\boldsymbol{\mu} \in W[\boldsymbol{\nu}_1 +$

$\boldsymbol{\nu}_2]$ in each site. For μ-SB with DLRO and $\boldsymbol{\mu}_H$, the mean-field ground state is given from Eqs. (10) and (53) by

$$|\mathrm{GS}\rangle = \bigotimes_{i\in L} |\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2)_i. \quad (144)$$

Similarly to the DLRO and $\boldsymbol{\mu}_H$ in the lowest-dimensional representation, the unitary operator $\widehat{U}$ associated with the symmetry transformation $U \in \mathrm{U}(N)$ leaves $|\mathrm{GS}\rangle$ unchanged up to a global phase factor if and only if U leaves the weight vector $|\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2)$ on each site unchanged up to a global phase:

$$U|\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2) = e^{i\phi}|\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2) \quad \text{for } \exists\, \phi \in \mathbb{R}$$
$$\Leftrightarrow U_{ij} = U_{ji} = 0 \quad (i = 1,2,\ j = 3,4,\ldots,N). \quad (145)$$

Thus, the remaining symmetry H of the state is given by

$$\begin{aligned} H &:= \{U \in \mathrm{U}(N)| \\ &\qquad U_{ij} = U_{ji} = 0\, (i = 1,2,\ j = 3,4,\ldots,N)\} \\ &\simeq \mathrm{U}(2) \times \mathrm{U}(N-2), \end{aligned} \quad (146)$$

and hence H is a connected group. For the case of $\boldsymbol{\mu}_0$, the right-hand side of Eq. (137) vanishes when the set of weight vectors $\{\boldsymbol{\mu}_i\}_{i\in u}$ on the unit cell u is

$$\begin{cases} \{\boldsymbol{\mu}_i\}_{i\in u} = \{\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j}\}_{j=1}^{N/2} & \text{for even } N, \\ \{\boldsymbol{\mu}_i\}_{i\in u} = \{\boldsymbol{\nu}_j + \boldsymbol{\nu}_{j+1}\}_{j=1}^{N-1} \cup \{\boldsymbol{\nu}_1 + \boldsymbol{\nu}_N\} & \text{for odd } N. \end{cases} \quad (147)$$

For even N, $N/2$-sites within each unit cell are sufficient because we have from Eq. (107)

$$\sum_{j=1}^{N/2} (\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j}) = \sum_{j=1}^{N} \boldsymbol{\nu}_j = 0. \quad (148)$$

On the other hand, $(N-1)/2$-sites within the unit cell are not sufficient because the sum of the weight vectors within each set $\{\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j}\}_{j=1}^{(N-1)/2}$ is

$$\sum_{j=1}^{(N-1)/2} (\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j}) = \sum_{j=1}^{N-1} \boldsymbol{\nu}_j = -\boldsymbol{\nu}_N \neq 0. \quad (149)$$

We have to consider the unit cell with N sites. In fact, the set $\{\boldsymbol{\nu}_j + \boldsymbol{\nu}_{j+1}\}_{j=1}^{N-1} \cup \{\boldsymbol{\nu}_1 + \boldsymbol{\nu}_N\}$ satisfies

$$\sum_{j=1}^{N-1} (\boldsymbol{\nu}_j + \boldsymbol{\nu}_{j+1}) + (\boldsymbol{\nu}_1 + \boldsymbol{\nu}_N) = 2\sum_{j=1}^{N} \boldsymbol{\nu}_j = 0 \quad (150)$$

Thus, the ground state is given by

$$|\mathrm{GS}\rangle = \bigotimes_{u\in\mathcal{U}} \bigotimes_{i\in u} |\boldsymbol{\mu}_i)_i, \quad (151)$$

$$\begin{cases} \{\boldsymbol{\mu}_i\}_{i\in u} = \{\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j}\}_{j=1}^{N/2} & \text{for even } N, \\ \{\boldsymbol{\mu}_i\}_{i\in u} = \{\boldsymbol{\nu}_j + \boldsymbol{\nu}_{j+1}\}_{j=1}^{N-1} \cup \{\boldsymbol{\nu}_1 + \boldsymbol{\nu}_N\} & \text{for odd } N. \end{cases} \quad (152)$$

The remaining symmetry H of the state can be calculated in a manner similar to the derivation of Eq. (141). A unitary matrix U is included in H if and only if $U|\boldsymbol{\mu}_i) = e^{i\phi}|\boldsymbol{\mu}_i)$ $(\exists\, \phi \in \mathbb{R})$ for all of the weight vectors within each unit cell. For even N, $U|\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j}) = e^{i\phi}|\boldsymbol{\nu}_{2j-1} + \boldsymbol{\nu}_{2j})$ $(\exists\, \phi \in \mathbb{R})$ is satisfied when both the $(2j-1)$ and the $(2j)$th columns and the $(2j-1)$ and the $(2j)$th rows vanish except for the $(2j-1,2j-1)$, $(2j-1,2j)$, $(2j,2j-1)$, and $(2j,2j)$ elements. Therefore, U is block-diagonalized into a direct product of 2×2 matrices and we obtain

$$\begin{aligned} H &= \left\{ U \in \mathrm{U}(N) \,\middle|\, U = \bigoplus_{j=1}^{N/2} U_i \ (U_i \in \mathrm{U}(2)) \right\}, \quad (153) \\ &\simeq [\mathrm{U}(2)]^{N/2}. \quad (154) \end{aligned}$$

For odd N and the set of weight vectors $\{\boldsymbol{\nu}_j + \boldsymbol{\nu}_{j+1}\}_{j=1}^{N-1} \cup \{\boldsymbol{\nu}_1 + \boldsymbol{\nu}_N\}$, a unitary matrix U is included in H if and only if U is a diagonal matrix. Therefore, we obtain

$$\begin{aligned} H &:= \{U \in \mathrm{U}(N)|U = \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_N}), \phi_i \in \mathbb{R}\} \\ &\simeq \mathrm{U}(1)^N, \end{aligned} \quad (155)$$

For both even and odd N, H is a connected group.

3. Third lowest-dimensional representation

For μ-SB with ODLRO and $\boldsymbol{\mu}_H$, the expectation value of the order parameter is given by

$$\boldsymbol{\Delta}_s = |2\boldsymbol{\nu}_1) = \boldsymbol{\Delta}_s^{(1,1)}. \quad (156)$$

The remaining symmetry H of the state is given by

$$\begin{aligned} H &:= \left\{U \in \mathrm{U}(N)|U\boldsymbol{\Delta}_s^{(1,1)t}U = \boldsymbol{\Delta}_s^{(1,1)}\right\} \\ &= \{U \in \mathrm{U}(N)|U_{1j} = U_{j1} = 0\, (j = 1,2,\ldots,N)\} \\ &\simeq \mathrm{U}(N-1), \end{aligned} \quad (157)$$

and hence H is a connected group. Similar to the case of the second lowest-dimensional representation, μ-SB with ODLRO and $\boldsymbol{\mu}_0$ does not necessarily exist for arbitrary N. The set $W[2\boldsymbol{\nu}_1]$ includes the weight vector $\boldsymbol{\nu}_i + \boldsymbol{\nu}_j = \mathbf{0}$ $(1 \leqslant i \leqslant j \leqslant N)$ only when the following conditions are met: $N = 2$, $\boldsymbol{\nu}_i = \boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_j = \boldsymbol{\nu}_2$ [20]. For N greater than 2, $W[2\boldsymbol{\nu}_1]$ does not include the zero-weight vector. For the ODLRO and $\boldsymbol{\mu}_0$ with $N = 2$, H is not a connected group. In fact, the order parameter of this phase and H are given by

$$\boldsymbol{\Delta}_s = \sigma_x, \quad (158)$$

$$\begin{aligned} H &= \{e^{it}I_2|t \in \mathbb{R}\} \rtimes \left\{I_2, e^{i\frac{\pi}{2}}\sigma_y\right\} \\ &= \mathrm{U}(1) \rtimes \mathbb{Z}_2, \end{aligned} \quad (159)$$

where $N \rtimes N'$ is a semidirect product whose product is given by

$$\begin{aligned} &(n,h) \rtimes (n',h') = (nhn'h^{-1}, hh') \\ &\text{for } \forall\, n,n' \in N, \quad \forall\, h,h' \in N'. \end{aligned} \quad (160)$$

We next consider the case of DLRO. There are $N(N+1)/2$ states labeled by a weight vector $\boldsymbol{\mu} \in W[2\boldsymbol{\nu}_1]$ in each site. For μ-SB with DLRO and $\boldsymbol{\mu}_H$, the mean-field ground state is given by

$$|\mathrm{GS}\rangle = \bigotimes_{i\in L} |2\boldsymbol{\nu}_1)_i. \quad (161)$$

The remaining symmetry H of the state is calculated in a manner similar to Eq. (136) as

$$\begin{aligned} H &:= \left\{U \in \mathrm{U}(N) | U \boldsymbol{\Delta}_s^{(1,1)t} U = e^{i\phi} \boldsymbol{\Delta}_s^{(1,1)} \quad \text{for } \exists\, \phi \in \mathbb{R}\right\} \\ &= \{U \in \mathrm{U}(N) | U_{1j} = U_{j1} = 0\ (j = 2,3,\dots,N)\} \\ &\simeq \mathrm{U}(1) \times \mathrm{U}(N-1), \end{aligned} \tag{162}$$

and hence H is a connected group. For the case of $\boldsymbol{\mu}_0$, the ground state and the remaining symmetry H of the state is obtained in a manner similar to the case of the DLRO and $\boldsymbol{\mu}_0$ for the lowest-dimensional representation. the right-hand side of Eq. (137) vanishes when the set of weight vectors $\{\boldsymbol{\mu}_i\}_{i\in u}$ within the unit cell u are

$$\{\boldsymbol{\mu}_i\}_{i\in u} = \{2\boldsymbol{\nu}_j\}_{j=1}^N. \tag{163}$$

Thus, the ground state is given by

$$|\mathrm{GS}\rangle = \bigotimes_{u\in\mathcal{U}} \bigotimes_{i\in u} |\boldsymbol{\mu}_i\rangle_i, \tag{164}$$

$$\{\boldsymbol{\mu}_i\}_{i\in u} = \{2\boldsymbol{\nu}_j\}_{j=1}^N. \tag{165}$$

The remaining symmetry H of the state can be calculated in a manner similar to the derivation of Eq. (141). A unitary matrix U is included in H if and only if U is a diagonal matrix. Thus, the remaining symmetry H of the state is given by

$$\begin{aligned} H &:= \{U \in \mathrm{U}(N) | U = \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, \dots, e^{i\phi_N}),\ \phi_i \in \mathbb{R}\} \\ &\simeq \mathrm{U}(1)^N, \end{aligned} \tag{166}$$

and hence H is a connected group.

B. Numbers of NG modes

We next calculate the number of NG modes in μ-SB phases classified in the previous subsection. From the quadratic part of effective Lagrangians in Eq. (59), the numbers of type-1 and type-2 NG modes, n_1 and n_2, are given by

$$(n_1,n_2) = \begin{cases} (0,|R_H|) & \text{for DLRO and } \boldsymbol{\mu}_H, \\ \left(2\left|R_0^{\mathrm{D}}\right|,0\right) & \text{for DLRO and } \boldsymbol{\mu}_0, \\ (1,|R_H|) & \text{for ODLRO and } \boldsymbol{\mu}_H, \\ \left(2\left|R_0^{\mathrm{OD}}\right| + 1,0\right) & \text{for ODLRO and } \boldsymbol{\mu}_0, \end{cases} \tag{167}$$

where $|X|$ denotes the number of elements in the set X.

1. Lowest-dimensional representation

Combining the set $W[\boldsymbol{\mu}_H]$ of the weight vectors in Eq. (114) and the set $\{\boldsymbol{\mu}_i\}_{i\in u}$ of the weight vectors in the unit cell u in Eq. (139), the sets R_H and R_0^{D} are given from Eqs. (47) and (48) by

$$R_H = \{\boldsymbol{\alpha}_{1,j} | 2 \leqslant j \leqslant N\}, \tag{168}$$

$$R_0^{\mathrm{D}} = \{\boldsymbol{\alpha}_{i,j} | 1 \leqslant i < j \leqslant N\} = R_+. \tag{169}$$

Substituting these equations into Eq. (167), we obtain

$$(n_1,n_2) = \begin{cases} (0,N-1) & \text{for DLRO and } \boldsymbol{\mu}_H, \\ (N(N-1),0) & \text{for DLRO and } \boldsymbol{\mu}_0, \\ (1,N-1) & \text{for ODLRO and } \boldsymbol{\mu}_H. \end{cases} \tag{170}$$

2. Second-lowest-dimensional representation

Combining the set $W[\boldsymbol{\mu}_H]$ of the weight vectors in Eq. (122) and the set $\{\boldsymbol{\mu}_i\}_{i\in u}$ of the weight vectors in the unit cell u in Eq. (152), the sets R_H and R_0^{D} are given from Eqs. (47) and (48) by

$$R_H = \{\boldsymbol{\alpha}_{i,j} | i = 1,2,\ 3 \leqslant j \leqslant N\}, \tag{171}$$

$$R_0^{\mathrm{D}} = R_+ \backslash \{\boldsymbol{\alpha}_{2j-1,2j} | j = 1,2,\dots,N/2\} \quad \text{for even } N, \tag{172}$$

$$R_0^{\mathrm{D}} = \{\boldsymbol{\alpha}_{i,j} | 1 \leqslant i < j \leqslant N\} = R_+ \quad \text{for odd } N. \tag{173}$$

For $N = 2$, there exists the μ-SB phase with ODLRO and $\boldsymbol{\mu}_0$. From Eq. (49), the set R_0^{OD} in this phase is empty,

$$R_0^{\mathrm{OD}} = \emptyset, \tag{174}$$

where $\emptyset$ denotes the empty set. Substituting the above three equations into Eq. (167), we obtain

$$(n_1,n_2) = \begin{cases} (0,2(N-2)) & \text{for DLRO and } \boldsymbol{\mu}_H, \\ (N(N-2),0) & \text{for DLRO, } \boldsymbol{\mu}_0, \text{and even } N, \\ (N(N-1),0) & \text{for DLRO, } \boldsymbol{\mu}_0, \text{and odd } N, \\ (1,2(N-2)) & \text{for ODLRO and } \boldsymbol{\mu}_H, \\ (1,0) & \text{for ODLRO, } \boldsymbol{\mu}_0, \text{and } N = 2. \end{cases} \tag{175}$$

3. Third-lowest-dimensional representation

Combining the set $W[\boldsymbol{\mu}_H]$ of the weight vectors in Eq. (128) and the set $\{\boldsymbol{\mu}_i\}_{i\in u}$ of the weight vectors in the unit cell u in Eq. (165), the sets R_H and R_0^{D} are given from Eqs. (47) and (48) by

$$R_H = \{\boldsymbol{\alpha}_{1,j} | 2 \leqslant j \leqslant N\}, \tag{176}$$

$$R_0^{\mathrm{D}} = \{\boldsymbol{\alpha}_{i,j} | 1 \leqslant i < j \leqslant N\} = R_+. \tag{177}$$

For $N = 2$, there exists a μ-SB phase with ODLRO and $\boldsymbol{\mu}_0$. The set R_0^{OD} in this phase is given from the definition of R_0^{OD} in Eq. (49) as

$$R_0^{\mathrm{OD}} = \{\boldsymbol{\alpha}_{1,2}\}. \tag{178}$$

Substituting the above equations into Eq. (167), we obtain

$$(n_1,n_2) = \begin{cases} (0,N-1) & \text{for DLRO and } \boldsymbol{\mu}_H, \\ (N(N-1),0) & \text{for DLRO and } \boldsymbol{\mu}_0, \\ (1,N-1) & \text{for ODLRO and } \boldsymbol{\mu}_H, \\ (3,0) & \text{for ODLRO, } \boldsymbol{\mu}_0, \text{and } N = 2. \end{cases} \tag{179}$$

C. Homotopy groups of topological excitations

Finally, we calculate the first and second homotopy groups for μ-SB phases. Since H is a connected group for all the cases except for the ODLRO and $\boldsymbol{\mu}_0$ with $N = 2$ in the third lowest-dimensional representation, we can apply Theorem 3 except for this case.

1. Lowest-dimensional representation

For the DLRO, from Eq. (91), we obtain

$$\pi_1(G/H) = 0. \tag{180}$$

For the DLRO and $\boldsymbol{\mu}_H$, we obtain from R_+ in Eq. (109)

$$\begin{aligned} L_R(R_+) &= \mathrm{span}_{\mathbb{Z}}\left\{ \frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})} \middle| \boldsymbol{\alpha} \in R_+ \right\} \\ &= \mathrm{span}_{\mathbb{Z}}\{\boldsymbol{\alpha}_{i,j} | 1 \leqslant i < j \leqslant N\}. \end{aligned} \tag{181}$$

$$L_R(R_+ \backslash R_H) = \mathrm{span}_{\mathbb{Z}}\{\boldsymbol{\alpha}_{1,j} | 2 \leqslant j \leqslant N\}, \tag{182}$$

In deriving the first equality of Eq. (181), we use Eq. (109) and $(\boldsymbol{\alpha}_{i,j},\boldsymbol{\alpha}_{i,j}) = 2$ for any i,j. Therefore we obtain from Theorem 3

$$\begin{aligned} \pi_2(G/H) &= \frac{L_R(R_+)}{L_R(R_+ \backslash R_H)} \\ &= \mathrm{span}_{\mathbb{Z}}\{\boldsymbol{\alpha}_{1,2}\} \simeq \mathbb{Z}. \end{aligned} \tag{183}$$

For the DLRO and $\boldsymbol{\mu}_0$, since $R_0^{\mathrm{D}} = R_+$, we obtain from Theorem 3

$$\begin{aligned} \pi_2(G/H) &= L_R(R_+) \\ &= \left\{ \sum_{j,k=1}^{N} m_{jk}\boldsymbol{\alpha}_{j,k} \middle| m_{jk} \in \mathbb{Z} \right\} \\ &= \left\{ \sum_{j=1}^{N} m_j \boldsymbol{v}_j \middle| m_j \in \mathbb{Z}, \sum_{j=1}^{N} m_j = 0 \right\} \\ &\simeq \mathbb{Z}^{N-1}. \end{aligned} \tag{184}$$

For the ODLRO and $\boldsymbol{\mu}_H$, it is easier to calculate $\pi_1(G/H)$ and $\pi_2(G/H)$ directly rather than using Theorem 3 because the order parameter manifold is isomorphic to a higher-dimensional sphere:

$$G/H = \mathrm{U}(N)/\mathrm{U}(N-1) = S^{2N-1}. \tag{185}$$

Since $N \geqslant 2$, we obtain the following two equations from the standard results of homotopy groups [40]:

$$\pi_1(G/H) = \pi_1(S^{2N-1}) = 0, \tag{186}$$

$$\pi_2(G/H) = \pi_2(S^{2N-1}) = 0. \tag{187}$$

2. Second-lowest-dimensional representation

For the DLRO, from Eq. (91), we obtain

$$\pi_1(G/H) = 0. \tag{188}$$

For the DLRO and $\boldsymbol{\mu}_H$, from Theorem 3 and Eq. (173), we obtain

$$\begin{aligned} \pi_2(G/H) &= \frac{L_R(R_+)}{L_R(R_+ \backslash R_H)} \\ &= \mathrm{span}_{\mathbb{Z}}\{\boldsymbol{\alpha}_{2,3}\} \simeq \mathbb{Z}. \end{aligned} \tag{189}$$

For the DLRO, $\boldsymbol{\mu}_0$ and even N, from Theorem 3 and Eq. (152), we obtain

$$L_R\big(R_+ \backslash R_0^{\mathrm{OD}}\big) = \left\{ \sum_{j=1}^{N/2} m_j \boldsymbol{\alpha}_{2j-1,2j} \middle| m_j \in \mathbb{Z} \right\}, \tag{190}$$

$$\begin{aligned} \pi_2(G/H) &= \frac{L_R(R_+)}{L_R\big(R_+ \backslash R_0^{\mathrm{OD}}\big)} \\ &= \left\{ \sum_{j=1}^{N/2-1} m_j \boldsymbol{\alpha}_{2j,2j+1} \middle| m_j \in \mathbb{Z} \right\} \\ &\simeq \mathbb{Z}^{\frac{N}{2}-1}. \end{aligned} \tag{191}$$

For the DLRO, $\boldsymbol{\mu}_0$ and odd N, we obtain the same result $\pi_2(G/H) = \mathbb{Z}^{N-1}$ as in the case of the lowest-dimensional representation since R_0^{D} is the same in both cases of the lowest-dimensional and second lowest-dimensional representations.

For the ODLRO and $\boldsymbol{\mu}_H$, we can prove $\pi_1(G/H) = 0$ as follows. To show this, it is sufficient to show that the following vortex-like texture analogous to Eq. (92) can be deformed into a uniform one:

$$\langle \boldsymbol{\Delta}_a \rangle = e^{i\theta} \boldsymbol{\Delta}_a^{(1,2)}. \tag{192}$$

Here $\theta \in [0,2\pi]$ is the azimuth angle around the vortex-like object. Let λ_z and λ_y be two matrices defined by

$$\lambda_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{193}$$

$$\lambda_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{194}$$

and define $N \times N$ matrices $\widetilde{\lambda}_x$ and $\widetilde{\lambda}_y$ by a direct product of λ_z and λ_y with the identity matrix I_{N-3} of size $(N-3)$, respectively:

$$\widetilde{\lambda}_z = \lambda_z \oplus I_{N-3}, \tag{195}$$

$$\widetilde{\lambda}_y = \lambda_y \oplus I_{N-3}. \tag{196}$$

By using $\widetilde{\lambda}_z$, Eq. (192) can be written as

$$\langle \boldsymbol{\Delta}_a \rangle = U(\theta) \boldsymbol{\Delta}_a^{(1,2)t} U(\theta), \tag{197}$$

$$U(\theta) = e^{i\theta\widetilde{\lambda}_z}. \tag{198}$$

Consider the continuous deformation defined by

$$\langle \boldsymbol{\Delta}_{a,t} \rangle(\theta) = U(\theta,t) \big\langle \boldsymbol{\Delta}_a^{(1,2)} \big\rangle^t U(\theta,t), \tag{199}$$

$$U(\theta,t) = e^{-i\pi t\frac{\widetilde{\lambda}_y}{2}} e^{i\theta\frac{\widetilde{\lambda}_z}{2}} e^{i\pi t\frac{\widetilde{\lambda}_y}{2}} e^{i\theta\frac{\widetilde{\lambda}_z}{2}} \quad \text{for } 0 \leqslant t \leqslant 1. \tag{200}$$

The unitary matrix $U(\theta,t)$ satisfies

$$U(\theta,t=0) = \exp(i\theta\widetilde{\lambda}_z) = U(\theta), \tag{201}$$

$$\begin{aligned} U(\theta,t=1) &= e^{-i\pi\frac{\widetilde{\lambda}_y}{2}} e^{i\theta\frac{\widetilde{\lambda}_z}{2}} e^{i\pi\frac{\widetilde{\lambda}_y}{2}} e^{i\theta\frac{\widetilde{\lambda}_z}{2}} \\ &= e^{-i\theta\frac{\widetilde{\lambda}_z}{2}} e^{i\theta\frac{\widetilde{\lambda}_z}{2}} = I_N, \end{aligned} \tag{202}$$

where I_N denotes the identity matrix of size N. In the third line, we use the relation

$$e^{-i\pi\frac{\widetilde{\lambda}_y}{2}}\widetilde{\lambda}_z e^{i\pi\frac{\widetilde{\lambda}_y}{2}} = -\widetilde{\lambda}_z. \tag{203}$$

Therefore, $\langle\mathbf{\Delta}_{a,t}\rangle$ is transformed from Eq. (192) to a uniform order $\langle\mathbf{\Delta}_{a,t=1}\rangle = \mathbf{\Delta}_a^{(1,2)}$, which implies the triviality of the vortex-like texture in Eq. (192). From Eq. (173), we obtain

$$L_H = \left\{ t\boldsymbol{\alpha}_{1,2} + \sum_{j,k=3}^{N} m_{jk}\boldsymbol{\alpha}_{j,k} \,\middle|\, t, m_{jk} \in \mathbb{R}, 3 \leqslant i < j \leqslant N \right\}, \tag{204}$$

$$\begin{aligned} L_H \cap L_R(R_+) &= \mathrm{span}_{\mathbb{Z}}[\{\boldsymbol{\alpha}_{i,j} | 3 \leqslant i < j \leqslant N\} \cup \{\boldsymbol{\alpha}_{1,2}\}] \\ &= L_R(R_+ \backslash R_H). \end{aligned} \tag{205}$$

Therefore, we obtain $\pi_2(G/H) = 0$ from Eq. (93). For the ODLRO and $\boldsymbol{\mu}_0$ with $N = 2$, by using Eq. (91) and substituting $R_0^{\mathrm{OD}} = \emptyset$ into Eq. (93), we obtain

$$\pi_1(G/H) = \mathbb{Z}, \tag{206}$$

$$\pi_2(G/H) = 0. \tag{207}$$

3. *Third-lowest-dimensional representation*

For the DLRO, from Eq. (91) we obtain

$$\pi_1(G/H) = 0. \tag{208}$$

For the DLRO and $\boldsymbol{\mu}_H$, we obtain the same result $\pi_2(G/H) = \mathbb{Z}$ as in the case of the lowest-dimensional representation since R_H coincides in both cases of the lowest-dimensional and third lowest-dimensional representations. For the DLRO and $\boldsymbol{\mu}_0$, we obtain the same result $\pi_2(G/H) = \mathbb{Z}^{N-1}$ as in the case of the lowest-dimensional representation since R_0^{D} coincides in both cases of the lowest-dimensional and third lowest-dimensional representations.

For the ODLRO and $\boldsymbol{\mu}_H$, we can prove $\pi_1(G/H) = \mathbb{Z}_2$ in a manner similar to the discussion in Sec. V. Let g be the generator of $\mathbb{Z}_2$. The vortex with topological charge g^2 is described as

$$\mathbf{\Delta}_s(\theta) = e^{2i\theta}\mathbf{\Delta}_s^{(1,1)}, \tag{209}$$

where $\theta \in [0,2\pi]$ is the azimuth angle around the vortex. We can prove $g^2 = e$ in a manner similar to the case of $N = 2$ in Sec. V by replacing Pauli matrices $\sigma_i (i = x,y,z)$ into

$$\widetilde{\sigma}_i := \sigma_i \oplus I_{N-2} (i = x,y,z). \tag{210}$$

For the ODLRO and $\boldsymbol{\mu}_H$, we obtain the same result $\pi_2(G/H) = 0$ as in the case of the lowest-dimensional representation since L_H and R_H coincide in both cases of the lowest-dimensional and third lowest-dimensional representations. For the ODLRO and $\boldsymbol{\mu}_0$ with $N = 2$, Eq. (91) is not applicable because H is not a connected group. From the correspondence with the spin-1 BEC in Eq. (97), this phase coincides with the polar phase in the spin-1 BEC [21,22]. The first and second homotopy groups of this phase are given as follows [52,53]:

$$\pi_1(G/H) = \mathbb{Z}, \tag{211}$$

$$\pi_2(G/H) = \mathbb{Z}. \tag{212}$$

We list the results obtained in this section in Table III together with the examples of the classified phases. Examples with $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{so}(3)$ are included because $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$. From the fifth column of Table III, we can see that a large class of symmetry-broken phases are described in terms of μ-SB.

VII. DISCUSSION ON THE CASE OF HIGHER-DIMENSIONAL REPRESENTATION

So far we have confined our discussions to low-dimensional representations. We next turn to the case of a higher-dimensional representation. In the case of ODLRO in a higher-dimensional representation, there appears more than one Casimir invariant in the energy functional. Due to the competition between different Casimir invariants, the phases that arise from the minimization of the energy functional are, in general, described by neither μ-SB nor inert states, where an inert state is a state in which the order parameter is independent of the coupling constants [55,56]. In this section, we focus on the case of a higher-dimensional representation in which two Casimir invariants appear in the energy functional. In this case, many of the ground states are described by inert states despite the competition between Casimir invariants. Let us examine this point by discussing examples of spin-2 BECs [15–17] and spin-1 color superconductors [18,19].

A. Spin-2 BEC

First, we consider the example of spin-2 BECs. As we will see below, all of the ground states are inert states. The symmetry group of the system is U(1) × SO(3). The order parameter of a spin-2 BEC is a five-dimensional complex vector

$$\langle\boldsymbol{\phi}\rangle = {}^t(\langle\phi_2\rangle,\langle\phi_1\rangle,\langle\phi_0\rangle,\langle\phi_{-1}\rangle,\langle\phi_{-2}\rangle), \tag{213}$$

and the Cartan generator of SO(3) is the S_z operator defined by

$$S_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{214}$$

The three eigenstates of the S_z operator

$$\langle\boldsymbol{\phi}\rangle_H = {}^t(1,0,0,0,0), \tag{215}$$

$$\langle\boldsymbol{\phi}\rangle_Z = {}^t(0,0,1,0,0), \tag{216}$$

$$\langle\boldsymbol{\phi}\rangle_L = {}^t(0,0,0,0,1), \tag{217}$$

are the highest-weight, the zero-weight, and the lowest-weight states, respectively. For $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{so}(3)$, the lowest-weight state is obtained by applying π-rotation around the x-axis to

TABLE III. Classification of μ-symmetry breaking in systems without Lorentz invariance. The Lie algebras of the systems are assumed to take the form of $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{su}(N)$ with $N \geqslant 2$. Here $\{\boldsymbol{\nu}_i\}_{i=1}^N$ is the set of weight vectors in an N-dimensional representation of $\mathfrak{su}(N)$, which satisfy Eqs. (107) and (108). The first column shows the irreducible representation of the order parameter $\langle\boldsymbol{\phi}\rangle$ for ODLRO and that of the field of particles on each site of the lattice L for DLRO. The second column shows whether the system is characterized by $\boldsymbol{\mu}_H$ or $\boldsymbol{\mu}_0$ and by ODLRO or DLRO, respectively. The expectation value of the order parameter for ODLRO can be written in the form of Eq. (8) or Eq. (9), while the ground state for DLRO can be written in the form of Eq. (10) or Eq. (11). The row with $\boldsymbol{\mu}_0$ and ODLRO appears only for $N = 2$. For the representation of the $\boldsymbol{\mu}_H = \boldsymbol{\nu}_1$ case, the pair of $\boldsymbol{\mu}_0$ and ODLRO is absent since this representation is the lowest-dimensional one. The third column (n_1,n_2) shows the numbers of type-1 and type-2 NG modes. The fourth column (π_1,π_2) lists the first and second homotopy groups of G/H. FM, AFM, CDW, SF, and VBS stand for ferromagnet, antiferromagnet, color density wave, superfluid, and valence bond solid, respectively. In the rows with $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2$, $\boldsymbol{\mu}_0$, and DLRO, the upper (lower) row corresponds to the case of even (odd) N.

Representation $\boldsymbol{\mu}_H$	Classification	(n_1,n_2)	(π_1,π_2)	Example
$\boldsymbol{\nu}_1$	ODLRO, $\boldsymbol{\mu}_H$	$(1,N-1)$	$(0,0)$	SU(N)-FM BEC
	DLRO, $\boldsymbol{\mu}_H$	$(0,N-1)$	$(0,\mathbb{Z})$	spin-1/2 FM, SU(N)-FM [44]
	DLRO, $\boldsymbol{\mu}_0$	$(N(N-1),0)$	$(0,\mathbb{Z}^{N-1})$	spin-1/2 AFM, N-CDW [48–51]
$\boldsymbol{\nu}_1+\boldsymbol{\nu}_2$	ODLRO, $\boldsymbol{\mu}_H$	$(1,2(N-2))$	$(0,0)$	s-wave SF in three-component fermion [24–26]
	ODLRO, $\boldsymbol{\mu}_0$	$(1,0)$	$(\mathbb{Z},0)$	s-wave SF in two-component fermion [3]
	DLRO, $\boldsymbol{\mu}_H$	$(0,2(N-2))$	$(0,\mathbb{Z})$	
	DLRO, $\boldsymbol{\mu}_0$	$(N(N-2),0)$	$(0,\mathbb{Z}^{\frac{N}{2}-1})$	VBS in SU(4)-spin model [31,54]
		$(N(N-1),0)$	$(0,\mathbb{Z}^{N-1})$	
$2\boldsymbol{\nu}_1$	ODLRO, $\boldsymbol{\mu}_H$	$(1,N-1)$	$(\mathbb{Z}_2,0)$	FM phase in spin-1 BEC [21,22]
	ODLRO, $\boldsymbol{\mu}_0$	$(3,0)$	$(\mathbb{Z},\mathbb{Z})$	polar phase in spin-1 BEC [21,22]
	DLRO, $\boldsymbol{\mu}_H$	$(0,N-1)$	$(0,\mathbb{Z})$	spin-1 FM
	DLRO, $\boldsymbol{\mu}_0$	$(N(N-1),0)$	$(0,\mathbb{Z}^{N-1})$	spin-1 AFM

the highest-weight state. The mean-field energy functional can be constructed from the norm $|\langle\boldsymbol{\phi}\rangle|$ of the order parameter and the Casimir invariants of SO(5) and SO(3) as

$$V(\langle\boldsymbol{\phi}\rangle) = -c|\langle\boldsymbol{\phi}\rangle|^2 + c_0'|\langle\boldsymbol{\phi}\rangle|^4 + c_1' C_2^{\mathfrak{so}(5)}(\langle\boldsymbol{\phi}\rangle) + c_2' C_2^{\mathfrak{so}(3)}(\langle\boldsymbol{\phi}\rangle), \tag{218}$$

where $C_2^{\mathfrak{so}(5)}(\langle\boldsymbol{\phi}\rangle)$ and $C_2^{\mathfrak{so}(3)}(\langle\boldsymbol{\phi}\rangle)$ are the Casimir invariants of SO(5) and SO(3), respectively [23]. Here $C_2^{\mathfrak{so}(5)}(\langle\boldsymbol{\phi}\rangle)$ is related to the spin-singlet pair amplitude

$$A_{00}(\boldsymbol{\phi}) := \frac{1}{\sqrt{5}}[\phi_2\phi_{-2} - \phi_1\phi_{-1} + (\phi_0)^2] \tag{219}$$

as

$$C_2^{\mathfrak{so}(5)}(\langle\boldsymbol{\phi}\rangle) = |\langle\boldsymbol{\phi}\rangle|^4 - 5|A_{00}(\langle\boldsymbol{\phi}\rangle)|^2. \tag{220}$$

In the energy functional, there are two competing Casimir invariants, namely $C_2^{\mathfrak{so}(5)}(\langle\boldsymbol{\phi}\rangle)$ and $C_2^{\mathfrak{so}(3)}(\langle\boldsymbol{\phi}\rangle)$. By minimizing the energy functional, the following four phases are obtained [15–17]:

$$\begin{cases} \text{ferromagnetic phase, } \langle\boldsymbol{\phi}\rangle = \langle\boldsymbol{\phi}\rangle_H = {}^t(1,0,0,0,0) \\ \quad \text{for } c_1' < 0 \text{ and } c_2' < 0; \\ \text{cyclic phase, } \langle\boldsymbol{\phi}\rangle = {}^t\left(\frac{1}{2},0,\frac{i}{\sqrt{2}},0,\frac{1}{2}\right) \\ \quad \text{for } c_1' < 0 \text{ and } c_2' > 0; \\ \text{uniaxial nematic phase, } \langle\boldsymbol{\phi}\rangle = \langle\boldsymbol{\phi}\rangle_Z = {}^t(0,0,1,0,0) \\ \quad \text{for } c_1' > 0; \\ \text{biaxial nematic phase, } \langle\boldsymbol{\phi}\rangle = {}^t\left(\frac{1}{\sqrt{2}},0,0,0,\frac{1}{\sqrt{2}}\right) \\ \quad \text{for } c_1' > 0; \end{cases} \tag{221}$$

where the order parameters are normalized such that $|\langle\boldsymbol{\phi}\rangle| = 1$. We note that the uniaxial nematic and biaxial nematic phases are energetically degenerate at the mean-field level.

While the ferromagnetic phase and the uniaxial nematic phase are described by μ-SB with $\boldsymbol{\mu}_H$ and $\boldsymbol{\mu}_0$, respectively, the cyclic phase and the biaxial nematic phase are not. However, both the cyclic phase and the biaxial nematic phase are inert states. Moreover, they are both described by linear combinations of the highest-weight, zero-weight, and lowest-weight states with simple ratios between the coefficients, 1 and $\sqrt{2}i$.

B. Spin-1 color superconductor

Next, we consider the example of spin-1 color superconductors. As we will see below, the four ground states are obtained from the minimization of the energy functional and three of them are inert while one of them is not. Color superconducting phases are the superconducting phases in which Cooper pairs formed by quarks are condensed [57]. Since each quark field $q_{f,s}^c$ has three different types of internal degrees of freedom, flavor f, spin s, and color c, the resulting Cooper pair has these three internal degrees of freedom. In the spin-1 color superconducting phases, quarks form a Cooper pair in a single flavor, a spin SO(3)-triplet, and a color SU(3)-antitriplet channel [18,19]. The single flavor and the spin SO(3) triplet imply that the Cooper pair does not have an internal degree of freedom in the flavor but has the spin 1, respectively. The color SU(3) antitriplet implies that the Cooper pair has three color charges, antired, antiblue, and antigreen. Let $\Delta_{c,l}$ be the field of the Cooper pair with color c $(= 1,2,3)$ and the spin direction parallel to the l $(= x,y,z)$ axis, respectively, where colors 1, 2, and 3 denote the color charge antired, antiblue, and antigreen, respectively. The term "anti" implies that $\Delta_{c,l}$ transforms in the conjugate representation of the three-dimensional representation of SU(3):

$$\Delta_{c,l} \mapsto (U_{cc'})^* \Delta_{c,l}$$
$$\text{when } q^c_{f,s} \mapsto U_{cc'} q^{c'}_{f,s} \;\; (U \in \mathrm{SU}(3)). \tag{222}$$

Under the spin rotation, $\mathbf{\Delta}_{c,l}$ transforms in the vector representation of SO(3):

$$\mathbf{\Delta}_{c,l} \mapsto R_{ll'} \mathbf{\Delta}_{c,l'} \quad \text{for } R \in \mathrm{SO}(3). \tag{223}$$

Also, there is the U(1)-symmetry associated with the baryon-number conservation which acts on the order parameter $\mathbf{\Delta}$ as

$$\mathbf{\Delta} \mapsto e^{i\phi} \mathbf{\Delta} \quad \text{for } \phi \in \mathbb{R}. \tag{224}$$

Based on the above discussions, the order parameter of the spin-1 color superconducting phase is given by a 3×3 complex matrix

$$\mathbf{\Delta} = \{\Delta_{c,l} | c = 1,2,3, l = x,y,z\}, \tag{225}$$

and the symmetry group G is $G = \mathrm{U}(3) \times \mathrm{SO}(3)$, which consists of three symmetries, the color SU(3) symmetry, the spin SO(3) symmetry, and the U(1) symmetry associated with the baryon number conservation, respectively. Combining Eqs. (222), (223), and (224), the order parameter $\mathbf{\Delta}$ transforms under G as

$$\mathbf{\Delta} \mapsto U^* \mathbf{\Delta}^t R \quad \text{for } U \in \mathrm{U}(3), R \in \mathrm{SO}(3). \tag{226}$$

We note that the system has a combined symmetry of U(3) and SO(3), resulting in the two Casimir invariants in the energy functional. The Cartan generators of the Lie algebra of G consist of three generators: two generators, λ_3 and λ_8, of $\mathfrak{su}(3)$, and one generator, S_z, of $\mathfrak{so}(3)$. They are defined as

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{227}$$

$$S_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{228}$$

We note that from Eq. (226) the actions of λ_3, λ_8, and S_z on the order parameter commute because the SU(3) group and its generators act on the order parameter $\mathbf{\Delta}$ from the left, while the SO(3) group and its generators act on it from the right. The mean-field energy functional of spin-1 color superconductors can be constructed from the Hilbert-Schmit norm $\sqrt{\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger)}$ of the matrix $\mathbf{\Delta}$ and the Casimir invariants of SU(3) and SO(3) as

$$V(\mathbf{\Delta}) = -\bar{c}\,\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger) + \bar{c}_0[\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger)]^2 + \bar{c}_1 C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}) + \bar{c}_2 C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}). \tag{229}$$

Here $C_2^{\mathfrak{su}(3)}(\mathbf{\Delta})$ and $C_2^{\mathfrak{so}(3)}(\mathbf{\Delta})$ are the Casimir invariants of SU(3) and SO(3) defined by

$$C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}) = \sum_{a=1}^{8} [\mathrm{Tr}(\mathbf{\Delta}\lambda_a\mathbf{\Delta}^\dagger)]^2, \tag{230}$$

$$C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}) = \sum_{a=1}^{3} [\mathrm{Tr}(\mathbf{\Delta}S_a\mathbf{\Delta}^\dagger)]^2, \tag{231}$$

where $\{\lambda_a\}_{a=1}^8$ and $\{S_a\}_{a=1}^3$ are the set of the Gell-Mann matrices of $\mathfrak{su}(3)$ [20] and the set of the generators of the vector representation of $\mathfrak{so}(3)$ defined by $(S_a)_{bc} = i\epsilon^{abc}$, respectively. In the energy functional, there are two competing Casimir invariants, namely $C_2^{\mathfrak{su}(3)}(\mathbf{\Delta})$ and $C_2^{\mathfrak{so}(3)}(\mathbf{\Delta})$. These Casimir invariants are related to the quartic invariants $\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger\mathbf{\Delta}\mathbf{\Delta}^\dagger)$ and $\mathrm{Tr}[\mathbf{\Delta}^t\mathbf{\Delta}(\mathbf{\Delta}^t\mathbf{\Delta})^\dagger]$ used in Ref. [19] as

$$C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}) = 2\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger\mathbf{\Delta}\mathbf{\Delta}^\dagger) - \tfrac{2}{3}[\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger)]^2, \tag{232}$$

$$C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}) = \mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger\mathbf{\Delta}\mathbf{\Delta}^\dagger) - \mathrm{Tr}[\mathbf{\Delta}^t\mathbf{\Delta}(\mathbf{\Delta}^t\mathbf{\Delta})^\dagger]. \tag{233}$$

From the analysis of the quartic invariants in Ref. [19], they satisfy the inequalities

$$0 \leqslant C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}) \leqslant \tfrac{4}{3}, \tag{234}$$

$$0 \leqslant C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}) \leqslant 1, \tag{235}$$

for a 3×3 matrix normalized as $\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger) = 1$. By minimizing the energy functional, four phases are obtained [19],

$$\text{A phase, } \mathbf{\Delta}_A = \frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\text{for } 2\bar{c}_1 + \bar{c}_2 < 0 \text{ and } \bar{c}_2 < 0; \tag{236}$$

$$\text{polar phase, } \mathbf{\Delta}_P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\text{for } \bar{c}_1 < 0 \text{ and } \bar{c}_2 > 0; \tag{237}$$

$$\text{color}-\text{spin}-\text{locked phase, } \mathbf{\Delta}_C = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\text{for } \bar{c}_1 > 0 \text{ and } \bar{c}_1 + \bar{c}_2 > 0; \tag{238}$$

$$\epsilon \text{ phase, } \mathbf{\Delta}_\epsilon = \begin{pmatrix} \epsilon_1 & i\epsilon_1 & 0 \\ -i\epsilon_1 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}$$
$$\text{for } 2\bar{c}_1 + \bar{c}_2 > 0 \text{ and } \bar{c}_1 + \bar{c}_2 < 0; \tag{239}$$

where ϵ_1 and ϵ_2 are the constant values defined as

$$\epsilon_1 = \frac{1}{2}\sqrt{\frac{2\bar{c}_1}{4\bar{c}_1 + \bar{c}_2}}, \quad \epsilon_2 = \sqrt{\frac{2\bar{c}_1 + \bar{c}_2}{4\bar{c}_1 + \bar{c}_2}}. \tag{240}$$

The order parameters are normalized such that $\mathrm{Tr}(\mathbf{\Delta}\mathbf{\Delta}^\dagger) = 1$.

Let us analyze these phases from the viewpoint of μ-SB, the inert state, and the Casimir invariants. Since the symmetry group of this system is no longer a simple Lie group, we generalize the concept of μ-SB for ODLRO to the case in which the symmetry group of the system is a semisimple Lie group. Since a semisimple Lie algebra can be decomposed into the direct sum of the one-dimensional commutative Lie algebras and simple Lie algebras, we can assign the Casimir invariants to each simple Lie algebra. When the order parameter is a simultaneous eigenstate of all of the Cartan generators and each Casimir invariant is minimized or maximized for the state, we refer to such a symmetry breaking as μ-symmetry breaking. Among the four ground states, the A phase, the polar phase, and the color-spin-locked phase are inert states, while the ϵ phase is not. The order parameter of the A phase is a simultaneous eigenstate of λ_3, λ_8, and S_z:

$$\lambda_3 \mathbf{\Delta}_A = \mathbf{\Delta}_A, \quad \lambda_8 \mathbf{\Delta}_A = \frac{\mathbf{\Delta}_A}{\sqrt{3}}, \quad \mathbf{\Delta}_A S_z = \mathbf{\Delta}_A. \tag{241}$$

We note from Eq. (226) that the generator S_z of $\mathfrak{so}(3)$ acts on the order parameter from the right. In the A phase, both $C_2^{\mathfrak{su}(3)}(\mathbf{\Delta})$ and $C_2^{\mathfrak{so}(3)}(\mathbf{\Delta})$ are maximized:

$$C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}_A) = \tfrac{4}{3}, \quad C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}_A) = 1. \tag{242}$$

Therefore, the A phase is described by μ-SB. The order parameter of the polar phase is a simultaneous eigenstate of λ_3, λ_8, and S_z,

$$\lambda_3 \mathbf{\Delta}_P = \frac{2}{\sqrt{3}}\mathbf{\Delta}_P, \quad \lambda_8 \mathbf{\Delta}_P = 0, \quad \mathbf{\Delta}_P S_z = 0, \tag{243}$$

and $C_2^{\mathfrak{su}(3)}(\mathbf{\Delta})$ is maximized, whereas $C_2^{\mathfrak{so}(3)}(\mathbf{\Delta})$ is minimized:

$$C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}_P) = \tfrac{4}{3}, \quad C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}_P) = 0. \tag{244}$$

Therefore, the polar phase is described by μ-SB. The color-spin-locked phase is an inert state but is not μ-SB. However, we can see from Eq. (238) that the ratios between the components are all simple numbers similarly to the case of the spin-2 BECs; they are all one. In the color-spin-locked phase, both $C_2^{\mathfrak{su}(3)}(\mathbf{\Delta})$ and $C_2^{\mathfrak{so}(3)}(\mathbf{\Delta})$ are minimized:

$$C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}_C) = 0, \quad C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}_C) = 0. \tag{245}$$

The ϵ phase is not an inert state. This phase is an intermediate phase between the A phase and the polar phase. In the limit $\bar{c}_1/\bar{c}_2 \to -1/2$ ($\bar{c}_1/\bar{c}_2 \to 0$), it coincides with the A phase (the polar phase). In the ϵ phase, the Casimir invariants takes intermediate values between their minimum and maximum:

$$C_2^{\mathfrak{su}(3)}(\mathbf{\Delta}_\epsilon) = \frac{4}{3} - \frac{8\bar{c}_1(2\bar{c}_1 + \bar{c}_2)}{(4\bar{c}_1 + \bar{c}_2)^2}, \tag{246}$$

$$C_2^{\mathfrak{so}(3)}(\mathbf{\Delta}_\epsilon) = \left(\frac{2\bar{c}_1}{4\bar{c}_1 + \bar{c}_2}\right)^2. \tag{247}$$

In spin-1 color superconductors, a noninert state emerges as a consequence of the competition between different Casimir invariants.

VIII. CONCLUSION

In conclusion, we have proposed a Lie-algebraic approach to systematically finding mean fields of quantum many-body systems on the basis of the dynamical symmetry. The mean fields of μ-symmetry breaking is derived through the minimization of the energy functional constructed from the Casimir invariants. We have introduced a concept of μ-symmetry breaking as a phase that is characterized by a weight vector in the representation of the Lie algebra. For μ-SB, the quadratic part of an effective Lagrangian of NG modes is block-diagonalized as in Eq. (59) in terms of the Cartan canonical form. In μ-SB there appear three types of NG modes as listed in Table II. Also, homotopy groups of topological excitations are calculated systematically for μ-SB as summarized in Eqs. (91) and (93). The textures of NG modes and topological excitations are described in terms of the generalized magnetization $\boldsymbol{S}_{\boldsymbol{\alpha}}$. By applying μ-SB to a U(N)-symmetric system, we have demonstrated that μ-SB involves a large class of symmetry-broken phases as listed in Table III.

In Sec. VII, we have seen from the examples of spin-2 BECs and spin-1 color superconductors that many of the ground states obtained by the minimization of the energy functional are inert, despite the fact that there is a competition between different Casimir invariants. Moreover, these states are described by linear combinations of weight vectors with simple ratios between the coefficients. The physics behind this fact has yet to be fully understood and merits further study.

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APPENDIX: PROOF OF THEOREM V ON THE HOMOTOPY GROUPS IN μ-SB

In this Appendix, we prove Theorem 3 for μ-SB on the basis of the theory of an integral lattice and a coroot lattice [41,42].

Let r and R_+ be the rank and the set of positive roots of a compact Lie group G. We define an integral lattice L_G for a compact Lie group G and a lattice $L_R(S)$ for a subset S of R_+ as [41,42]

$$L_G := \{\boldsymbol{t} \in \mathbb{R}^r \,|\, \exp(2\pi i H_t) = e\}, \tag{A1}$$

$$L_R(S) := \mathrm{span}_{\mathbb{Z}} \left\{ \frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})} \middle| \boldsymbol{\alpha} \in S \right\}, \tag{A2}$$

where $\mathrm{span}_{\mathbb{Z}} X$ denotes a vector space spanned by elements of X with integer coefficients:

$$\mathrm{span}_{\mathbb{Z}} X := \left\{ \sum_{k=1}^{n'} n_k x_k \middle| x_k \in X,\, n_k \in \mathbb{Z},\, n' \in \mathbb{N} \right\}. \tag{A3}$$

Under a addition of r-dimensional vectors, L_G and $L_R(S)$ of R_+ become Abelian groups for any subset S. It is known that

$L_R(S)$ is an Abelian subgroup of L_G for any subset S of R_+ [41,42].

The proof of Theorem 3 proceeds in four steps.

First, we prove the following lemma on the general formulas of homotopy groups.

Lemma 1. Let $i_n^* : \pi_n(H) \to \pi_n(G)$ be the induced homomorphism of the inclusion map $i : H \to G$. The homotopy groups of the homogeneous space $\pi_i(G/H)$ $(i = 1,2)$ are calculated as follows.

(1) For a Lie group G and its connected subgroup H,

$$\pi_1(G/H) = \text{Coker}\{i_1^* : \pi_1(H) \to \pi_1(G)\}, \tag{A4}$$

where $\text{Coker}\{f : X \to Y\}$ for a homomorphism $f : X \to Y$ is defined as

$$\text{Coker}\{f : X \to Y\} := Y/\text{Im}\{f : X \to Y\}. \tag{A5}$$

Let $[a]$ $(a \in \pi_1(G))$ and $\theta \in [0,2\pi]$ be a representative element of the coset space Coker i_1^* and the azimuth angle around the vortex. The texture $O(\theta)$ of the vortex associated with $[a]$ is given by

$$O(\theta) = a(\theta) \circ O_0, \tag{A6}$$

where O_0 is the value of the order parameter at $\theta = 0$ and $g \circ O_0$ $(g \in G)$ is the action of g on O_0.

(2) For a compact Lie group G and its subgroup H,

$$\pi_2(G/H) = \text{Ker}\{i_1^* : \pi_1(H) \to \pi_1(G)\}. \tag{A7}$$

Let σ and σ_θ $(0 \leqslant \theta \leqslant \pi)$ be an element of Ker i_1^* and a continuous deformation from $\sigma_{\theta=0} = \sigma$ to $\sigma_{\theta=\pi} = e$ (trivial loop). The texture $O(\theta,\phi)$ of the point defect associated with σ is given by

$$O(\theta,\phi) = \sigma_\theta(\phi) \circ O_0, \tag{A8}$$

where $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ are the three-dimensional polar coordinates and O_0 is the value of the order parameter at $\theta = \phi = 0$.

Proof of Lemma 1. We first prove the formula

$$\frac{\pi_n(G/H)}{\text{Coker } i_n^*} = \text{Ker } i_{n-1}^* \tag{A9}$$

by using the homotopy exact sequence [58]

$$\xrightarrow{i_2^*} \pi_2(G) \xrightarrow{p_2^*} \pi_2(G/H) \xrightarrow{\partial_2^*} \pi_1(H) \xrightarrow{i_1^*} \pi_1(G) \xrightarrow{p_1^*}$$
$$\xrightarrow{p_1^*} \pi_1(G/H) \xrightarrow{\partial_1^*} \pi_0(H) \xrightarrow{i_0^*} \pi_0(G), \tag{A10}$$

where $p_n^* : \pi_n(G) \to \pi_n(G/H)$ and $\partial_n^* : \pi_n(G/H) \to \pi_{n-1}(G/H)$ are the induced homomorphism of the projection map $p : G \to G/H$ and the boundary map $\partial : G/H \to H$.

By using the homomorphism theorem and the exact sequence, we obtain

$$\frac{\pi_n(G/H)}{\text{Ker } \partial_n^*} = \text{Im } \partial_n^*, \tag{A11}$$

$$\text{Im } \partial_n^* = \text{Ker } i_{n-1}^*, \tag{A12}$$

$$\text{Ker } \partial_n^* = \text{Im } p_n^* = \frac{\pi_n(G)}{\text{Ker } p_n^*} = \text{Coker } i_n^*. \tag{A13}$$

Combining Eqs. (A11)–(A13), we obtain Eq. (A9). For a connected subgroup H, we have $\pi_0(H) = 0$ and hence Ker $i_0^* = 0$. Thus, we obtain Eq. (A4). Let O and $[a]$ $(a \in \pi_1(G))$ be an element of $\pi_1(G/H)$ and the element of Coker i_1^* that corresponds to O in Eq. (A4). Let O_0 be a value of the order parameter. The projection $p : G \to G/H$ coincides with an action on O_0:

$$p(g) = g \circ O_0. \tag{A14}$$

In fact, any element h in H is mapped by p to O_0, $p(h) = O_0$ because an element of H acts on O_0 trivially. Since $p_1^* : \pi_1(G) \to \pi_1(G/H)$ is the induced homomorphism of the projection map $p : G \to G/H$, the element O of $\pi_1(G/H)$ corresponding to the element of $[a]$ $(a \in \pi_1(G))$ is obtained by the projection p as follows [58]:

$$O(\theta) = p[a(\theta)] = a(\theta) \circ O_0. \tag{A15}$$

Thus, we obtain Eq. (A6). Since $\pi_2(G) = 0$ for a compact Lie group G [41], we obtain Coker $i_2^* = 0$ and hence Eq. (A7). Let σ and σ_θ $(0 \leqslant \theta \leqslant \pi)$ be the element of Ker i_1^* and the path of a continuous deformation from $\sigma_{\theta=0} = \sigma$ to the trivial loop $\sigma_{\theta=\pi} = e$. We can obtain the element O of $\pi_2(G/H)$ that corresponds to σ in Eq. (A7) as

$$O(\theta,\phi) = \sigma_\theta(\phi) \circ O_0, \tag{A16}$$

where $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ are the three-dimensional polar coordinates and O_0 is the value of the order parameter at $\theta = \phi = 0$. Thus, we obtain Eq. (A8), which completes the proof of Lemma 1.

Second, we prove Eq. (93) on the second homotopy group. Let L_H be an integral lattice of H. We define the coroot lattice $L_{R,G}$ for G by

$$L_{R,G} := \text{span}_{\mathbb{Z}}\left\{\frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\middle|\boldsymbol{\alpha} \in R_+\right\}. \tag{A17}$$

It is known that the coroot lattice $L_{R,G}$ is a subgroup of the integral lattice L_G and that the coset space $L_G/L_{R,G}$ is equivalent to the first homotopy group $\pi_1(G)$ [41,42]:

$$\pi_1(G) = L_G/L_{R,G}. \tag{A18}$$

Also, we define $L_{R,H}$ as the coroot lattice of the subgroup H. For μ-SB, L_H is a subgroup of L_G and $L_{R,H}$ is a subgroup of $L_{R,G}$. Writing an element of $\pi_1(H) = L_H/L_{R,H}$ as $\boldsymbol{t} + L_{R,H} (\boldsymbol{t} \in L_H)$, the inclusion map $i_1^* : \pi_1(H) \to \pi_1(G)$ satisfies

$$i_1^*(\boldsymbol{t} + L_{R,H}) = \boldsymbol{t} + L_{R,G}. \tag{A19}$$

From Eq. (A7), we have

$$\begin{aligned}\pi_2(G/H) &= \text{Ker}\{i_1^* : \pi_1(H) \to \pi_1(G)\}\\ &= \{\boldsymbol{t} + L_{R,H} | \boldsymbol{t} \in L_H, i_1^*(\boldsymbol{t} + L_{R,H}) \in L_{R,G}\}\\ &= \{\boldsymbol{t} + L_{R,H} | \boldsymbol{t} \in L_H, \boldsymbol{t} \in L_{R,G}\}\\ &= (L_H \cap L_{R,G})/L_{R,H}.\end{aligned} \tag{A20}$$

For μ-SB with $\boldsymbol{\mu}_H$, $L_{R,G}$ and $L_{R,H}$ can be rewritten from Theorem 1 as

$$L_{R,G} = L_R(R_+), \tag{A21}$$

$$L_{R,H} = L_R(R_+ \backslash R_H). \tag{A22}$$

Therefore, we obtain the third row of Eq. (93). Since $L_R(R_+)$ is a subgroup of L_G and all of the Cartan generators are not broken ones for DLRO, we have $L_H \cap L_{R,G} = L_{R,G} = L_R(R_+)$. Therefore, we obtain the first row of Eq. (93). For μ-SB with $\boldsymbol{\mu}_0$, $L_{R,G}$ and $L_{R,H}$ can be rewritten from Theorem 1 as

$$L_{R,G} = L_R(R_+), \tag{A23}$$

$$L_{R,H} = \begin{cases} L_R\big(R_+ \backslash R_0^{\mathrm{D}}\big) & \text{for DLRO and } \boldsymbol{\mu}_0, \\ L_R\big(R_+ \backslash R_0^{\mathrm{OD}}\big) & \text{for ODLRO and } \boldsymbol{\mu}_0, \end{cases} \tag{A24}$$

and L_H is a subgroup of $L_{R,G}$. Thus, we obtain the second and fourth rows of Eq. (93). The texture of the point defect in Eq. (94) is obtained by using Eq. (A8). Let σ be an element of Ker i_1^* with coroot vector $2\boldsymbol{\alpha}/(\boldsymbol{\alpha},\boldsymbol{\alpha})$. Here σ represents the loop on H defined by

$$\sigma(\phi) = \exp\left[i\phi\frac{2H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\right] \quad \text{for } \phi \in [0,2\pi]. \tag{A25}$$

Since $\sigma \in$ Ker i_1^*, there exists a continuous deformation from σ to the trivial loop. In fact,

$$\sigma_\theta(\phi) = e^{-i\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}}$$
$$\text{for } (\theta,\phi) \in [0,\pi] \times [0,2\pi] \tag{A26}$$

describes a continuous deformation from $\sigma_{\theta=0} = \sigma$ to the trivial loop $\sigma_{\theta=\pi} = e$:

$$\sigma_{\theta=0}(\phi) = e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} = \sigma(\phi), \tag{A27}$$

$$\begin{aligned}\sigma_{\theta=\pi}(\phi) &= e^{-i\pi\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\pi\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} \\ &= e^{-i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} = e.\end{aligned} \tag{A28}$$

Here we use

$$e^{-i\pi\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} H_{\boldsymbol{\alpha}} e^{i\pi\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} = -H_{\boldsymbol{\alpha}}. \tag{A29}$$

From Eq. (A8), the element $O' \in \pi_2(G/H)$ corresponding to σ can be obtained by acting $\sigma_\theta(\phi)$ on O_0,

$$\begin{aligned}O'(\theta,\phi) &= e^{-i\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} \circ O_0 \\ &= e^{-i\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\phi\frac{H_{\boldsymbol{\alpha}}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} e^{i\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} \circ O_0,\end{aligned} \tag{A30}$$

where we use the fact that $H_{\boldsymbol{\alpha}}$ is an unbroken generator. Through the continuous deformation

$$O'_u(\theta,\phi) = e^{iu\theta\frac{E_{\boldsymbol{\alpha}}^I}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}} O'(\theta,\phi) \quad \text{for } 0 \leqslant u \leqslant 1, \tag{A31}$$

O'_u is deformed from $O'_{u=0} = O'$ to $O_{u=1} = O$. Thus, Eq. (94) is obtained.

Third, we show Coker $i_1^* = 0$ for DLRO. From Eqs. (A4) and (A19), we have

$$\begin{aligned}\mathrm{Im}\, i_1^* &= \{i_1^*(\boldsymbol{t} + L_{R,H}) | \boldsymbol{t} \in L_H\} \\ &= \{\boldsymbol{t} + L_{R,G} | \boldsymbol{t} \in L_H\} \\ &= \mathrm{span}_{\mathbb{Z}}\{L_H \cup L_{R,G}\}/L_{R,G},\end{aligned} \tag{A32}$$

and hence we obtain from Eq. (A18)

$$\mathrm{Coker}\, i_1^* = \frac{L_G}{\mathrm{span}_{\mathbb{Z}}\{L_H \cup L_{R,G}\}}. \tag{A33}$$

For a μ-SB with DLRO, no Cartan generators are broken ones. Therefore, we obtain

$$L_G = L_H = L_{R,G}, \tag{A34}$$

$$\mathrm{Coker}\, i_1^* = \frac{L_G}{\mathrm{span}_{\mathbb{Z}}\{L_G \cup L_G\}} = 0. \tag{A35}$$

Finally, we calculate Coker i_1^* for a μ-SB with ODLRO. For a μ-SB with ODLRO, the Cartan subalgebra $\mathfrak{g}_C$ of G is included in $\mathfrak{h}$ except for one direction $\mathrm{span}_{\mathbb{Z}}\{I\} \simeq \mathbb{Z}$ from Theorem 1. Thus, Coker i_1^* can be written as

$$\mathrm{Coker}\, i_1^* = \frac{\mathrm{span}_{\mathbb{Z}}\{I\}}{\mathrm{span}_{\mathbb{Z}}\{I\} \cap \mathrm{span}_{\mathbb{Z}}\{L_H \cup L_{R,G}\}}. \tag{A36}$$

Let a_n ($n \in \mathbb{Z}$) be an element of $\mathrm{span}_{\mathbb{Z}}\{I\}$ in the numerator of Eq. (A36). Since the action of $a_n(\theta)$ on the order parameter represents the rotation of the phase $2n\pi$ around the vortex, Eq. (A6) reduces to Eq. (92).

For the ODLRO with $\boldsymbol{\mu}_0$, L_H is a subgroup of $L_{R,G}$ from Theorem 1. We obtain from Theorem 3

$$\begin{aligned}\mathrm{span}_{\mathbb{Z}}\{I\} \cap \mathrm{span}_{\mathbb{Z}}(L_H \cap L_{R,G}) &= \mathrm{span}_{\mathbb{Z}}\{I\} \cap L_{R,G} \\ &= \emptyset,\end{aligned} \tag{A37}$$

$$\mathrm{Coker}\, i_1^* = \mathrm{span}_{\mathbb{Z}}\{I\}. \tag{A38}$$

Therefore, we obtain Coker $i_1^* = \mathbb{Z}$ from Eq. (A36). For the ODLRO with $\boldsymbol{\mu}_H$,Coker i_1^* is a subgroup of $\mathbb{Z}$. To show that Coker i_1^* is a finite group, it is sufficient to show

$$nI \in \mathrm{span}_{\mathbb{Z}}(L_H \cup L_{R,G}) \quad \text{for } \exists\, n \in \mathbb{Z}. \tag{A39}$$

Let $\bar{r}$ be the rank of the Lie algebra $\bar{\mathfrak{g}}$ and let $C = \{C_{ij}\}_{i,j=1}^{\bar{r}}$, $\{\boldsymbol{\alpha}^{(j)}\}_{j=1}^{\bar{r}}$, and $\{m_i\}_{i=1}^{\bar{r}}$ be the Cartan matrix of $\bar{\mathfrak{g}}$, a set of simple roots of $\bar{\mathfrak{g}}$, and the Dynkin index of $\boldsymbol{\mu}_H$, respectively. The generator I can be written as

$$\begin{aligned}I = {}& \frac{1}{|\boldsymbol{\mu}_H|^2}\big(|\boldsymbol{\mu}_H|^2 I - H_{\boldsymbol{\mu}_H}\big) \\ &+ \sum_{j=1}^{\bar{r}}\left[\frac{1}{|\boldsymbol{\mu}_H|^2}\sum_{i=1}^{\bar{r}} m_i C_{ij}\frac{(\boldsymbol{\alpha}^{(j)},\boldsymbol{\alpha}^{(j)})}{2}\right]\frac{2H_{\boldsymbol{\alpha}^{(j)}}}{(\boldsymbol{\alpha}^{(j)},\boldsymbol{\alpha}^{(j)})}.\end{aligned} \tag{A40}$$

Since $|\boldsymbol{\mu}_H|^2 I - H_{\boldsymbol{\mu}_H}$ and $\frac{2H_{\boldsymbol{\alpha}^{(j)}}}{(\boldsymbol{\alpha}^{(j)},\boldsymbol{\alpha}^{(j)})}$ are the elements of $L_H \cup L_{R,G}$ and the coefficients are all rational numbers, the greatest common divisor of the denominator of the coefficients satisfies Eq. (A39). The positive integer l in Eq. (91) is determined to be the minimum number that satisfies Eq. (A39), which completes the proof of Theorem 3.

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