Adiabaticity in a time-dependent trap: A universal limit for the loss by touching the continuum

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We consider a time-dependent trap externally manipulated in such a way that one of its bound states is brought into an instant contact with the continuum threshold and then down again. It is shown that, in the limit of slow evolution, the probability to remain in the bound state, P^{stay} tends to a universal limit and is determined only by the manner in which the adiabatic bound state approaches and leaves the threshold. The task of evaluating P^{stay} in the adiabatic limit can be reduced to studying the loss from a zero-range well and is performed numerically. Various types of trapping potentials are considered. Applications of this theory to cold atoms in traps and to propagation of transversal modes in tapered waveguides are proposed.

I. INTRODUCTION

Recent progress in laser-based techniques has led to the creation of various methods for trapping cold atoms. The laser-induced potentials, used for this purpose, range from extended optical lattices [1] to individual quasi-one-dimensional traps [2]. Such single traps, designed specifically in order to achieve single-site control, can be used, for example, for production of atomic Fock states [3]. These states, containing a known number of atoms, find numerous applications in fields as diverse as metrology, few-body quantum physics [4,5], quantum entanglement [6], and quantum computing [3]. Production of Fock states may be achieved by external manipulation of the trapping potential [3,7–10], and the question of whether the adiabatic limit is reached in its evolution plays a central role here. The presence of continuum states makes the problem somewhat more complicated [11–13] than the Landau-Zener case [14] where only two discrete levels are involved.

In a process similar to "laser culling" [3] or "laser squeezing" [7], the depth or the width of the trap is manipulated in such a way that its bound states move closer to the continuum threshold and some of the trapped particles are ejected into the continuum. For example, in culling, if the trap is made shallower and then deeper again, an adiabatic bound state may make a U turn before reaching the threshold. If the evolution is slow, the adiabatic theorem (see, for example, Ref. [15]) guarantees that the particle will remain trapped, and the probability of losing it to the continuum will be exponentially small [11]. Alternatively, the trap can be made so shallow that it temporarily ceases to support the original bound state and only recovers it once its depth begins to increase again. A slow evolution of this type will almost certainly lose the particle as almost nothing will be recaptured by deepening the well. Separating the two regimes is the borderline case where the adiabatic bound state only touches the continuum threshold and immediately resumes its downward journey. Relatively little is known about the probability to retain the particle within the trap if the threshold is approached slowly, and the adiabatic limit achieved in such evolution is the main subject of this paper.

Almost 50 years ago, Devdariani [16] considered the chance to retain the particle in a three-dimensional zero-range (ZR)

well whose magnitude, quadratic in time, vanished at $t = 0$. It was shown that the retention probability in this case is about 38% and is independent of the speed of the evolution. More recently, it was demonstrated that this 38% rule holds in the adiabatic limit for any bound state of an arbitrary one-dimensional trap, subject to a similar quadratic-in-time evolution [13]. This result may suggest the existence of a universal adiabatic limit for the loss of particles caused by touching the continuum, at least, in a one-dimensional cullinglike process. It would be reasonable to expect such a limit to depend only on the manner in which an adiabatic bound state approaches and touches the threshold and be common to all shapes of trapping potentials, masses of the particle, and to the ground and excited states alike [13]. In this paper we will demonstrate the existence of the limit and evaluate the retention probabilities P^{stay} for evolutions of different kinds. Our task is somewhat simpler than the one usually performed when the adiabatic limit is known beforehand, and the deviations from it are of interest. In what follows the subject is the limit itself, rather than the manner in which it may be reached.

The rest of the paper is organized as follows. In Sec. II we will formulate the problem in the case of culling. In Sec. III, in the adiabatic limit, we will reduce it to solving a time-dependent problem for a ZR potential. Sections IV and V show the central results of our paper where we demonstrate the existence of the universal limit for the bound states and numerically evaluate P^{stay}. Additionally we check the validity of our conclusions for several realistic potentials and extend our analysis to evolutions of arbitrary type. In Sec. VI the case of squeezing is discussed. Finally, Sec. VII contains the summary of our results.

II. LOSS BY TOUCHING THE CONTINUUM: CULLING

We start by considering the Schrödinger equation (SE) for a particle of a mass μ in a one-dimensional time-dependent potential well ($\hbar = 1$),

$$i\,\partial_t \Psi(x,t) = -\partial_x^2 \Psi/2\mu - W(x,t)\Psi, \qquad (1)$$
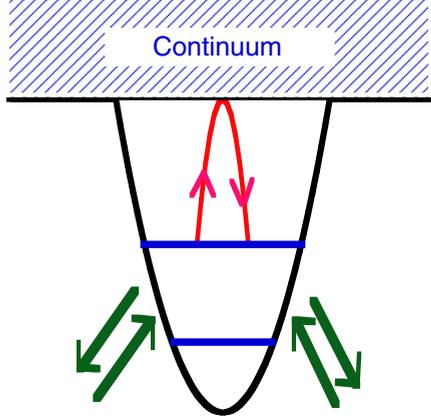
FIG. 1. Evolution of the potential $W(x,t)$ lifts its nth adiabatic bound state, makes it briefly touch the continuum, and then brings it down again. We wish to evaluate the probability for a particle to remain in state P_n^{stay} in case the evolution is slow.

where

$$W(x,t) = (\rho^{(n)} + v^\gamma |t|^\gamma)W(x). \quad (2)$$

The well has a finite range so that $W(x)$, normalized by the condition $\int_{-\infty}^{\infty} W(x)dx = 1$, vanishes for $|x| > a$. It may support several adiabatic states $\phi_n(x,t)$, $n = 0,1,2,\ldots$, with the energies $E_n(t)$, and the constant $\rho^{(n)}$ is chosen in such a way that $E_n(t=0) = 0$. Thus, the time evolution of the potential brings the nth adiabatic state up to the continuum threshold and then brings it down the same way it came up (see Fig. 1). The type of the evolution depends on the exponent γ, which can be any positive real number. The speed of the evolution is controlled by the parameter v.

In the spirit of the adiabatic theorem [15], we wish to know how many particles, if any, will be lost to the continuum if the state is approaching the threshold very slowly, i.e., in the limit $v \to 0$. With this in mind, we will prepare the particle in a deep-lying nth bound state at some large negative $t = -T$,

$$\Psi(x,-T) = \phi_n(x,-T), \qquad T \to \infty, \quad (3)$$

and then evaluate the retention probability $P_n^{\text{stay}}(v,\mu,\gamma,W)$ to still find it in the same state at $t = T$. The probability is given by the square of the modulus of the corresponding amplitude,

$$
\begin{aligned}
&P_n^{\text{stay}}(v,\mu,\gamma,W) \\
&= |\langle\phi_n(T)| \exp\left[-i\int_{-T}^{T} \hat{H}(t)dt\right]|\phi_n(-T)\rangle|^2, \quad (4)
\end{aligned}
$$

where $\hat{H}$ is the operator on the right-hand side of Eq. (1) and the exponential is understood to be the time-ordered product of noncommuting terms $\exp[-i\hat{H}(t)dt]$. To find the final P_{stay}, instead of evolving the initial state until $t = T$, we may evolve it to $t = 0$, thus obtaining $\Psi(x,0)$. We will also need to evolve the final state backwards in time to the same $t = 0$. By time reversal [17], this backward evolution can be replaced by a forward one, accompanied by the appropriate complex conjugation. Since $\phi_n(x,-T) = \phi_n(x,T)$, $\hat{H}(t) = \hat{H}(-t)$, and $\phi_n(x,T) = \phi_n^*(x,T)$, the result is $\Psi^*(x,0)$, and

we have

$$P_n^{\text{stay}}(v,\mu,\gamma,W) = \left|\int \Psi(x,0)^2 dx\right|^2, \quad (5)$$

which gives the probability P_n^{stay} in terms of the wave function at $t = 0$.

III. REDUCTION TO THE ZERO-RANGE MODEL IN THE ADIABATIC LIMIT

A scaling transformation [18] $t \to \alpha t$, $x \to \beta x$, where

$$
\begin{aligned}
\alpha(\mu',v'|\mu,v) &= (\mu'/\mu)^{1/(2\gamma+1)}(v'/v)^{2\gamma/(2\gamma+1)}, \\
\beta(\mu',v'|\mu,v) &= (\mu'/\mu)^{(\gamma+1)/(2\gamma+1)}(v'/v)^{\gamma/(2\gamma+1)}
\end{aligned}
\quad (6)
$$

converts Eq. (1) conditioned by (3) into

$$i\,\partial_t\Psi(x,t) = -\partial_x^2\Psi/2\mu' - \left(\frac{\alpha}{\beta}\rho^{(n)} + v'^\gamma|t|^\gamma\right)\tilde{W}(x')\Psi, \quad (7)$$

where $\tilde{W}(x') = \beta W(\beta x')$, with a new initial condition,

$$\Psi(x,-T) = \beta^{-1/2}\phi_n(\beta x,-T), \qquad T \to \infty. \quad (8)$$

Choosing $\mu'(v) = \mu(v'/v)^{1+\epsilon}$ with $\epsilon > 0$ ensures that in the adiabatic limit $v \to 0$, we have $\beta = v^{-[1+\epsilon(\gamma+1)/(2\gamma+1)]} \to \infty$, whereas the constant term multiplying $W(x)$ vanishes, $\alpha/\beta = (v/v')^{\epsilon\gamma/2\gamma+1} \to 0$.

As a result of the scaling, the well becomes narrower and deeper, whereas the initial distribution of the particle's position also narrows

$$
\begin{aligned}
\lim_{v\to 0}\tilde{W}(x') &= \delta(x'), \\
\lim_{v\to 0}\lim_{T\to\infty}|\Psi(x,-T)|^2 &\to \beta^{-1}\delta(x'),
\end{aligned}
\quad (9)
$$

where $\delta(x)$ is the Dirac δ.

Thus, after scaling, we have to solve the SE for a ZR potential,

$$i\,\partial_{t'}\Psi(x',t') = -\partial_{x'}^2\Psi/2\mu' - v'^\gamma|t'|^\gamma\delta(x')\Psi, \quad (10)$$

which no longer depends on the particular shape of $W(x)$. A scaling transformation cannot alter the value of a dimensionless quantity, so for the retention probability we should have

$$P_n^{\text{stay}}(v \to 0,\mu,\gamma,W) = P_\delta^{\text{stay}}(\gamma), \quad (11)$$

where $P_\delta^{\text{stay}}(\gamma)$ is the retention probability for a very heavy particle $\mu' \to \infty$, trapped in a ZR well evolving at a rate v'. Since v' was chosen arbitrarily, $P_\delta^{\text{stay}}(\gamma)$ should not depend on its choice if we expect Eq. (11) to be correct. We just have to check that the transformations (6) leave the form of the SE for a ZR well (10) unchanged, except for replacing $\mu' \to \mu''$ and $v' \to v''$. For $\rho < 0$, a ZR well $\rho(t)\delta(x)$ supports a single adiabatic bound state with an energy $E_0(t) = -\mu\rho(t)^2/2 = -\mu v^{2\gamma}|t|^{2\gamma}/2$ (see the inset in Fig. 2),

$$
\begin{aligned}
\varphi_0(x,t) &= \sqrt{-ik_0(t)}\exp[ik_0(t)|x|], \\
k_0(t,\mu,v) &= -i\mu\rho(t) = i\mu v^\gamma|t|^\gamma. \quad (12)
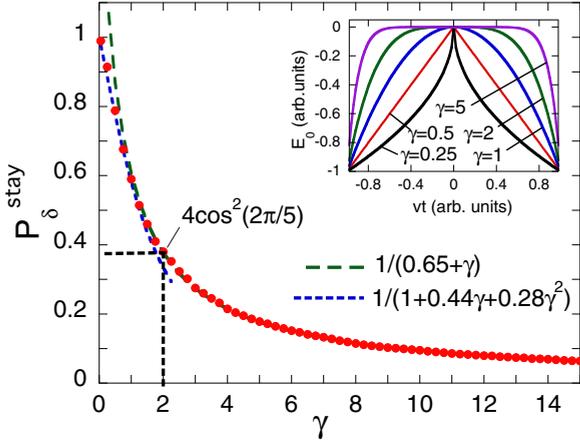\end{aligned}
$$

FIG. 2. Probability for a particle to remain in the bound state of a zero-range well $W(x,t) = -v^\gamma |t|^\gamma \delta(x)$, $P_\delta^{\text{stay}}(\gamma)$, obtained by numerical integration of Eq. (10). Also shown are the approximations (20) and (21) as well as the value $P_\delta^{\text{stay}}(\gamma = 2) \approx 0.38$, which can be obtained analytically [13,16]. The inset shows the adiabatic energy of the state for different kinds of evolution.

Thus, the initial condition (3) also remains unchanged under the scaling (6), acquiring only an inessential constant factor,

$$\varphi_0(x,T) \to \beta^{-1/2} \varphi_0(x,T). \tag{13}$$

Since scaling leaves the values of dimensionless quantities unaltered, P_δ^{stay} may not depend on the choice of the particle's mass μ or the speed of evolution v,

$$P_\delta^{\text{stay}}(v, \mu, \gamma) = P_\delta^{\text{stay}}(\gamma), \tag{14}$$

which is the desired result.

To provide an additional check, we note also that probability densities, such as the energy distribution of the emitted particles $w(E)$ must change when transformations (6) are applied. With t scaling as αt, the energy must scale as $E \to E/\alpha$, and we should have (for a more detailed proof see also Appendix A)

$$w(E|\mu'', v'') = \alpha(\mu'', v''|\mu', v')^{-1} w(E/\alpha(\mu'', v''|\mu', v')|\mu', v'), \tag{15}$$

which, since $P_\delta^{\text{stay}} = 1 - \int w(E) dE$, would confirm the validity of (14) (cf. Fig. 3).

The limit (11) can now be seen as universal in the following sense: In a culling process, the loss by touching the continuum in a slowly evolving well is determined only by the manner in which the state approaches the threshold, i.e., on the exponent γ. It is independent of the shape of the well and the particle's mass and equals the loss from a ZR well.

It remains to obtain the function $P_\delta^{\text{stay}}(\gamma)$, preferably in the most general and transparent way. Unfortunately, both of the existing analytical techniques, based on expanding the wave function in the Sturmian [12,13,19–22] or Siegert [11,23–25] states, fail to provide a simple answer except in the well-studied cases of $\gamma = 1$ [12] and $\gamma = 2$ [11,13,16]. (For details we refer the reader to the Appendices B and C.) The Siegert-state method does, however, establish the existence of
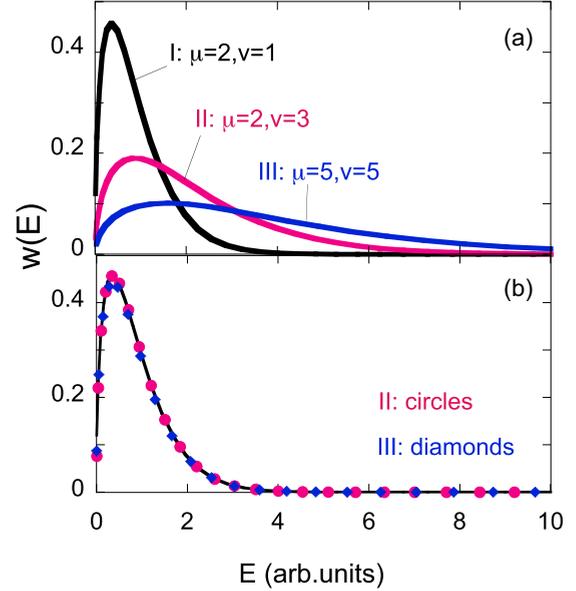

FIG. 3. (a) Energy distribution $w(E)$ (arb. units) of the particles ejected from a zero-range well $v^2 |t|^2 \delta(x)$ for various masses μ and speeds of evolution v, obtained by numerical integration of Eq. (10); (b) the result of scaling the curves II and III as prescribed by Eq. (15).

a universal adiabatic limit for a broad class of evolutions as will be discussed in the next section.

IV. UNIVERSALITY OF THE ADIABATIC LIMIT

Even though the Siegert-state approach does not offer an analytic solution to the problem, it allows us to prove the validity of Eq. (5) beyond the particular type of the culling evolution considered so far. Next we will show that, in the adiabatic limit, the loss by touching the continuum always depends only on the manner in which an adiabatic eigenstate approaches the continuum threshold. For the adiabatic energy $E_n(t)$ at $t \approx 0$, without loss of generality, we can write

$$E_n(vt) = k_n(vt)^2 / 2\mu \approx -Cv^{2\gamma} |t|^{2\gamma}, \qquad C > 0 = \text{const}. \tag{16}$$

It is necessary to demonstrate that it is the power γ alone, which determines the loss to the continuum as $v \to 0$, and the proof is as follows. As $v \to 0$, we may neglect all Siegert states [11], except ϕ_n, which is to touch the continuum, and look for the solution in the form

$$\Psi(x,t) \approx a_n(t, \gamma, v) \phi_n(x,t). \tag{17}$$

The coefficient $a(t, \gamma, v)$ satisfies Eq. (30a) of Ref. [11], which in the limit $v \to 0$, reduces to (cf. Eqs. (31), (34), and (36) of Ref. [11]),

$$\hat{\lambda}_t a_n(t, v) - i k_n(vt, \gamma) a_n(t, v) = 0, \tag{18}$$

with $k_n(vt) = i\sqrt{2\mu C} v^\gamma |t|^\gamma$. For $v \to 0$, the factor $\sqrt{\mu C}$ determines how rapidly $a(t)$ tends to its limit $a_n(t, \gamma, v \to 0)$ but not the limit itself. The limit must, therefore, be the same for a particle of any mass and for any C in Eq. (16). Finally, since $\phi_n(x,t)$ is normalized to unity, $\int \phi_n^2(x,t) = 1$, (cf. Eq. (22) of

Ref. [25]), insertion of (17) into Eq. (5) yields

$$P_n^{\text{stay}}(v \to 0, \mu, \gamma, W) \to |a_n(t \to 0, \gamma, v \to 0)|^4. \quad (19)$$

Since we have shown that the right-hand side of Eq. (19) depends only on the power γ, Eq. (11) must hold in general for a particle of any mass and for any state of a finite range well of any shape. Note that the argument can be extended to the case of an asymmetric evolution where the state approaches the continuum and then leaves it in a different manner (see Appendix D).

The above still does not offer a simple way for calculating $P^{\text{stay}}(\gamma)$ as $v \to 0$ analytically, but makes us free to choose the simple ZR model for the purpose. The corresponding SE (10) can be easily solved numerically, and we will do it in the next section.

V. THE UNIVERSAL ADIABATIC LIMIT: EVALUATION OF P_{stay}

Equation (1) is solved by the finite difference method [26] for a particle of $\mu = 1$ in a ZR well $W(x,t) = -v^\gamma |t|^\gamma \delta(x)$ placed between two infinite walls at $x = \pm L$. Since the solution is symmetric around the origin, it is sufficient to consider only the right half-space with the boundary conditions $\partial_x \log(\Psi(0,t)) = -v^\gamma |t|^\gamma$ and $\Psi(L) = 0$ at $x = 0$ and $x = L$, respectively. The initial condition (3) is imposed at T large enough to make $P_\delta^{\text{stay}}(\gamma)$ independent of the choice, and L is chosen sufficiently large to avoid unphysical reflections. The calculation is made easier by the freedom of choosing v without changing the value of P_δ^{stay}, which is then obtained
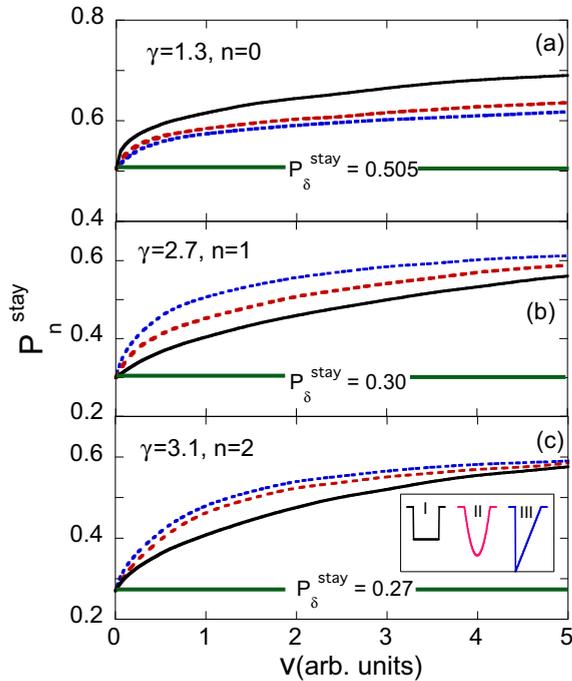

FIG. 4. Culling: the probability to remain in the nth excited state P_n^{stay} vs v (arb. units) for the three potential wells (22) shown in the inset: (a) $\mu = 1$, $\gamma = 1.3$, and $n = 0$; (b) $\mu = 1$, $\gamma = 2.7$, and $n = 1$; (c) $\mu = 1$, $\gamma = 3.1$, and $n = 2$. Also shown by horizontal lines are the corresponding results for the zero-range well.

with the help of Eq. (5). The results are shown in Fig. 2, which is the central result of this paper. We note that, for $\gamma \gtrsim 1$, $P_\delta^{\text{stay}}(\gamma)$ is reasonably well described by a rational function,

$$P_\delta^{\text{stay}}(\gamma) \approx [0.65 + \gamma]^{-1} \quad (20)$$

whereas for $\gamma \lesssim 1$,

$$P_\delta^{\text{stay}}(\gamma) \approx [1 + 0.44\gamma + 0.28\gamma^2]^{-1} \quad (21)$$

provides a suitable approximation.

Validity of Eq. (11) is also checked numerically for the particles trapped in the ground and excited states of three potentials [$\theta(x) = 1$ for $x \geqslant a$ and 0 otherwise],

$$W_{\text{I}}(x) = (2a)^{-1}\theta(x - a)\theta(x + a), \text{ square well}, \quad (22)$$

$$W_{\text{II}}(x) = (4a^3/3)^{-1}(a^2 - x^2)\theta(x - a)\theta(x + a), \text{ parabolic},$$

$$W_{\text{III}}(x) = (2a^2)^{-1}(a - x)\theta(x - a)\theta(x + a), \text{ asymmetric},$$

and the results are shown in Fig. 4 for different values of γ and n.

VI. LOSS BY TOUCHING THE CONTINUUM: SQUEEZING

A different type of evolution, called squeezing in Ref. [7], proceeds by making the trap narrower while leaving its depth unchanged. Before concluding, we will show that the chance to remain in a bound state brought to a brief contact with the continuum by a squeezed potential,

$$W(x,t) = W(x/L(t)) = W(x/(L^{(n)} + v^\gamma t^\gamma)),$$
$$E_n(t = 0) = E_n(L^{(n)}) = 0 \quad (23)$$

is the same as in the case of culling (2). As discussed in Sec. IV, it is sufficient to demonstrate that in both cases a bound state approaches the continuum in the same manner. Let $E_n(\rho, L)$ be the energy of the bound state in a potential $\rho W(x/L)$. In the case of culling, we have $\rho(t) = \rho^{(n)} + v^\gamma |t|^\gamma$ and $L = \text{const}$ so that

$$E_n(t) \approx -C(\rho - \rho^{(n)})^2 = -Cv^{2\gamma}|t|^{2\gamma}, \quad (24)$$

where $C = \partial_\rho^2 E(\rho^{(n)}, L)/2$. Consider the SE describing a bound state in a potential $\rho W(x/L)$, written in some dimensionless variables,

$$\left[-\partial_x^2/2 - \rho W(x/L) - E_n(\rho, L) \right]\phi_n(x, \rho, L) = 0. \quad (25)$$

By scaling the x variable $x \to xL'/L$, we can express E_n for a squeezed well of a fixed depth ρ in terms of that for a culled well of a fixed width L',

$$E_n(\rho, L) = \eta^{-2}E_n(\eta^2 \rho, L'), \quad \eta \equiv L/L'. \quad (26)$$

Let $\rho^{(n)}(L')$ and $L^{(n)}(\rho)$ be the values of the corresponding parameters for which the bound state disappears, i.e., $E_n(\rho^{(n)}, L') = 0$ and $E_n(\rho, L^{(n)}) = 0$. It follows that $L^{(n)}(\rho) = L'\sqrt{\rho^{(n)}(L')/\rho}$, and expanding $E_n(\rho, L)$ in Eq. (26) around $L = L^{(n)}(\rho)$ yields

$$E_n(\rho, L) \approx -C'[L - L^{(n)}(\rho)]^2 = -C'v^{2\gamma}|t|^{2\gamma}, \quad (27)$$

where $C' = -\partial_L^2 E_n(\rho, L^{(n)})/2 = -2\rho^2 \partial_\rho^2 E(\rho^{(n)}, L')/L'^2$. Equations (24) and (27) differ only by inessential constant factors, and the loss by touching the continuum must be the
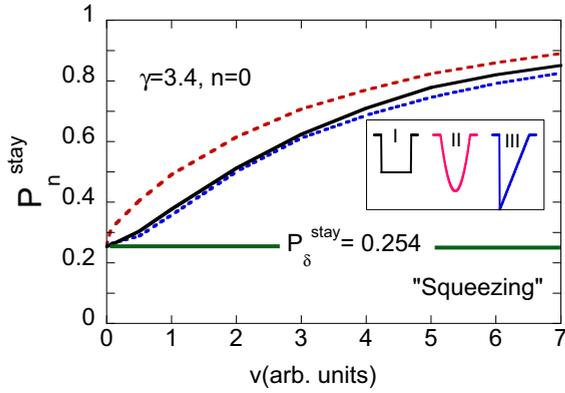
FIG. 5. Squeezing: the probability to remain in the ground state P_0^{stay} vs v (arb. units) for the three potentials (22) with $\mu = 1$, $\gamma = 3.4$, and $n = 0$. Also shown by a horizontal line is the corresponding result for the zero-range well.

same for the potentials in Eqs. (2) and (23). This result is easily verified numerically as Fig. 5 shows.

VII. SUMMARY AND DISCUSSION

In one dimension, there exists a universal adiabatic limit for the probability to remain in a bound state of a slowly evolving trap P^{stay} as the state briefly touches the continuum threshold. The limiting value of P^{stay} is determined only by the manner in which the state approaches the continuum and is independent of the particle's mass, the particular shape of the trapping potential, or the details of the trap's evolution. More precisely, if the adiabatic energy of the particle near the threshold changes as $E_n(t) \approx -(v|t|)^{2\gamma}$, then for $v \to 0$ the probability $P^{\text{stay}}(\gamma)$ tends to the adiabatic limit, which depends only on the exponent γ. In a way, this an expected result. If the evolution is slow, the particle is exchanged between the bound state, whereas it is close to the threshold $E = 0$ and a few low-lying continuum states. The presence of other bound states in the well and the overall structure of the continuum should, therefore, play no role for the outcome of this exchange.

Evaluation of the limiting values of $P^{\text{stay}}(\gamma)$ is a separate matter. Given that the result should hold for all potentials, we may take the simplest case of a zero-range well as a reference. With the help of either the Sturmian or the Siegert state's approach, the problem can be reduced to solving an ordinary differential equation. However, in the general case the equation is of fractional order and has an analytic solution known (at least to us) only for $\gamma = 1$ and $\gamma = 2$. Although it is possible that further insight can be gained by using the methods of fractional calculus [27], we chose to solve the problem numerically with the results presented in Fig. 2. For $\gamma \ll 1$, the adiabatic state passes almost no time near the threshold (cf. the inset in Fig. 2), and $P^{\text{stay}}(\gamma)$ tends to unity. As this time increases with an increase in the value of γ, we have $P^{\text{stay}}(\gamma) \sim 1/\gamma$ as prescribed by Eq. (20). The analysis is easily extended to asymmetric evolutions as is illustrated in Fig. 6.

With recent technological developments, it should be possible to verify our predictions in an experiment. One straightforward choice would be the use of cold atoms in
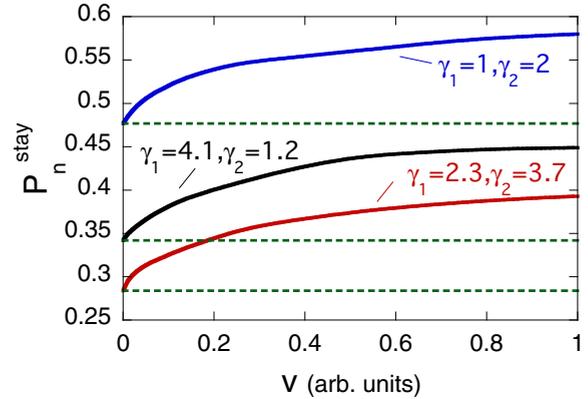

FIG. 6. The probability to remain in the ground state $n = 0$ for $\mu = 1$ and $W(x,t) = [(vt)^{\gamma_1}\theta(-t) + (vt)^{\gamma_2}\theta(t)]W(x)$, $W(x) = W_I(x)$ (solid curves). Also shown by dashed lines are the corresponding results for the zero-range wells $W(x) = \delta(x)$, independent of both v and μ.

a laser-induced quasi-one-dimensional trap [2], which is manipulated as in culling or squeezing in order to bring one of its states to the continuum threshold. Another possibility is offered by studying the propagation of transverse modes in tapered waveguides [28–30]. Since narrowing of the guide lifts the energies of the quantized transverse motion, a massive particle or a photon trapped in such a mode would have a similar chance of being lost to the continuum while passing the narrow region. A detailed analysis of waveguide propagation will be given in our forthcoming work.

APPENDIX A: THE ENERGY DISTRIBUTION FOR A ZERO-RANGE WELL

Consider a particle of mass μ prepared in the bound state of a ZR well, evolving at a rate v at $t = -T \to -\infty$. At $t \to \infty$, a deep impenetrable well divides the space at $x = 0$ so that the continuum energy eigenstates are given by

$$\phi_E(x,\mu) = (\mu/2\pi^2 E)^{1/4} \sin(\sqrt{2\mu E}|x|), \qquad \text{(A1)}$$

$\langle \phi_E | \phi_{E'} \rangle = \delta(E - E')$. The deep-lying bound state is decoupled from the continuum, and the ejected particles are described by the wave function $\psi_{\text{cont}}(x,t) = \int_0^\infty C(E)\phi(x,E)\exp(-iEt)dE$. For the (time- independent) energy distribution we have

$$w(E|\mu,v) \equiv \left| \int \phi_E(x,\mu)\psi_{\text{cont}}(x,t)dx \right|^2. \qquad \text{(A2)}$$

Consider next another particle of a mass μ' in a ZR well evolving at a different rate v'. We can also describe the new system by applying the transformations (6) to the old one. Thus, the ejected particles are described by $\psi'_{\text{cont}}(x,t) = $

$\beta^{1/2}\psi_{\text{cont}}(\beta x, \alpha t)$. Inserting $\psi'_{\text{cont}}(x,t)$ into Eq. (A2) together with $\phi(x,E,\mu')$ and noting that $\mu' = \beta^2\mu/\alpha$, we obtain

$$w(E|\mu'',v'') = \alpha^{-1}w(E/\alpha|\mu',v'), \qquad (A3)$$

where $\alpha(\mu',v''\mu,v)$ is given by Eq. (6).

APPENDIX B: ZERO-RANGE WELL: THE STURMIAN APPROACH

The main difficulty in solving Eq. (1) analytically is the presence of the continuum states also affected by the change in the potential well. One way to simplify the problem is to use the discrete basis of the Sturmian states satisfying the outgoing waves' boundary conditions [12,13,19–22]. For a real energy E, the method finds the values $\rho_n(E)$, real or complex valued, such that the potential $\rho_n W(x)$ supports a suitable Sturmian state. The case of a ZR potential $W(x) = \delta(x)$ is particularly simple: There is only one such value $\rho_0 = i\sqrt{2E}$, and a single Sturmian so that the solution of Eq. (1) with $v = \mu = 1$ can be found in the form

$$\Psi(x,t) = \int dE \, \exp(-iEt)B(E)S_0(x,E), \qquad (B1)$$

where $S_0(x,E)$ is the Sturmian function,

$$S_0(x,E) = \exp(i\sqrt{2E}|x|). \qquad (B2)$$

In Eq. (B1) the integration contour runs just above the real axis on the first sheet of the Riemann surface of $\sqrt{E}$ cut along the positive semiaxis, where as a function of x, $S_0(x,E)$ decays for $E < 0$ and oscillates for $E > 0$ [13]. Inserting (B1) into (10) yields the equation for the unknown function $B(E)$,

$$\int \mathcal{G}(E-E')B(E')dE' + i\sqrt{2E}B(E) = 0, \qquad (B3)$$

where the kernel $\mathcal{G}(E)$ is formally defined as the Fourier transform of $|t|^\gamma$,

$$\mathcal{G}(E) = (2\pi)^{-1}\int |t|^\gamma \exp(iEt)dt. \qquad (B4)$$

Equation (B3) is most useful whenever γ is an integer. For an odd γ, $\gamma = 1,3,5,\ldots$, we may replace $|t|$ with $-t$, follow the evolution until $t = 0$, obtain $\Psi(x,0)$, and then evaluate the integral (5). With this, the kernel (B4) becomes $\mathcal{G}(E) = -(-i)^\gamma \partial_E^\gamma \delta(E)$, and integration by parts of Eq. (B3) yields

$$i^\gamma \partial_E^\gamma B + i\sqrt{2E}B = 0. \qquad (B5)$$

In the simplest case of the evolution linear in time $\gamma = 1$, the resulting first-order equation is easily solved, giving $\Psi(x,t \leqslant 0)$ by the quadrature (B1) [12].

For an even γ, $\gamma = 2,4,6,\ldots$, we can replace $|t|$ with t and again obtain for $B(E)$ Eq. (B5). In the quadratic-in-time case $\gamma = 2$, Eq. (B5) is of second order, has an exact solution expressed in terms of the Hankel functions, and gives a retention probability (4) of about 38% [13,16]. For an even integer $\gamma > 2$ one faces a similar but more difficult task of finding the correct boundary condition for the ordinary differential equation (ODE) (B5), expressing the solution in terms of "incoming" and "outgoing" waves for $E \to -\infty$ [13]. In the general case of a noninteger γ, the kernel $\mathcal{G}(E)$ is obviously related to a fractional derivative of the δ function

(see, for example, Ref. [27]), which makes the resulting fractional order ODE satisfied by $B(E)$ even less tractable.

APPENDIX C: ZERO-RANGE WELL: THE SIEGERT-STATE APPROACH

A different and more general approach, based on the expansion of the time-dependent state in terms of Siegert rather than Sturmian states has recently been developed in Refs. [11,23–25] for a class of finite range potentials which vanish for $|x| \geqslant a$. Next we will explore its usefulness for treating the ZR problem at hand. In our case, for a given (real) potential $\rho W(x)$ the method looks for the Siegert states associated with the poles $k_n(\rho)$ of the transmission and reflection amplitudes T and R on the complex plane of the momentum $k = \sqrt{2\mu E}$. The technique is based on imposing an outgoing wave's boundary condition at $x = \pm a$ and expanding the wave function $\Psi_I(x,t)$ in the inner region $|x| < a$ in terms of the corresponding (Siegert) eigenstates. Once the wave function in the inner region is known, and the solution in the outer regions are $x > a$ and $x < -a$, $\Psi_O(x,t)$ is obtained by solving the free-particle SE with the boundary conditions $\Psi_O(\pm a,t) = \Psi_I(\pm a,t)$. Treatment of the wave vector k rather than the energy $E = k^2/2\mu$ as an eigenvalue requires linearization of the problem [23], doubling the dimension of the Hilbert space in the inner region, and introducing the fractional time derivative $\hat{\lambda}_t = \exp(3\pi i/4)\sqrt{2\partial_t}$.

The transmission amplitude for the ZR potential $\rho\delta(x)$ is well known [31] to be $T(k,t) = k/(k + i\mu\rho)$. Associated with the pole at $k = -i\mu\rho$ is the single Siegert state (12). As the width of well a tends to zero, the inner region contracts to a single point $x = 0$, and the method allows us to find $\Psi_I(0,t)$, which is sought in the form (cf. Eq. (29) of Ref. [11])

$$\Psi_I(0,t) = a_0(t)\varphi_0(0,t) = \sqrt{-ik_0(t)}a_0(t).$$

The unknown function $a_0(t)$ satisfies Eqs. (30) and (31) of Ref. [11] (without the factor of 2 in the denominator of Eq. (31) since our problem is on the whole x axis [25]). In the limit $a \to 0$ and with $m = n = 0$, they reduce to a single equation for $\Psi_I(0,t)$,

$$\hat{\lambda}_t \Psi_I(0,t) - ik_0\Psi_I(0,t) = 0. \qquad (C1)$$

Taking the Fourier transform $\Psi_I(0,t) = \int dE \, \exp(-iEt)\Psi_I(0,E)$ and recalling that $\hat{\lambda}_t \exp(-iEt) = i\sqrt{2E}\exp(-iEt)$ [11], for $\mu = v = 1$, we have

$$i\sqrt{2E}\Psi(0,E) + \int \mathcal{G}(E-E')\Psi(0,E')dE' = 0, \qquad (C2)$$

where $\mathcal{G}(E)$ is defined by Eq. (B4). Comparison with Eqs. (B1)–(B3) shows that we arrived at the equation for the value of the wave function at the origin $\Psi(x = 0,E)$ obtained earlier in the Sturmian approach and, therefore, face the same problem of solving it. The result is not unexpected. There is a close relation between the Sturmian eigenvalues ρ_n and the Siegert energies $E_n = k_n^2/2$, similar to the relation between the Regge and the complex energy poles of a scattering matrix (see, e.g., Ref. [32]). Whereas the Sturmian approach of Appendix B uses the analytical function $\rho_0(E) = i\sqrt{2E}$, the Siegert-state method employs its inverse $E_0(\rho) = -\rho^2/2$, and both techniques lead in the end to the same equation.

APPENDIX D: ASYMMETRIC EVOLUTIONS

Let the trap be manipulated in such a way that the energy of the adiabatic bound state $E_n(t)$ changes with time according to

$$E_n(t) = -v^{2\gamma}|t|^{2\gamma}, \quad \text{for } t < 0,$$
$$-v^{2\gamma'}|t|^{2\gamma'} \quad \text{for } t > 0. \tag{D1}$$

Arguing as in Sec. II it is easy to show that the retention probability $P^{\text{stay}}(\gamma,\gamma')$ is given by

$$P_n^{\text{stay}}(v,\mu,\gamma,\gamma',W) = \left| \int \Psi(x,0,\gamma)\Psi(x,0,\gamma')dx \right|^2, \tag{D2}$$

where $\Psi(x,t,\gamma)$ is the result of evolving the initial state in such a manner that for $t < 0$ its adiabatic energy changes according to $E_n(t) = -v^{2\gamma}|t|^{2\gamma}$. As in Sec. IV as $v \to 0$,

we have $\Psi(x,t,\gamma) \approx a_n(t,\gamma,v)\phi_n(x,t,\gamma)$ and $\Psi(x,t,\gamma') \approx a_n(t,\gamma',v)\phi_n(x,t,\gamma')$, where $\phi_n(x,t,\gamma)$ denotes the corresponding adiabatic bound state. The coefficients $a_n(t,\gamma,v)$ and $a_n(t,\gamma',v)$ satisfy Eq. (18) with $k_n(vt,\gamma)$ and $k_n(vt,\gamma')$, respectively, and in the limit $v \to 0$ may depend only on γ and γ'. Since $\phi_n(x,t,\gamma)$ and $\phi_n(x,t,\gamma')$ coincide at $t = 0$, we should have $\int \phi_n(x,0,\gamma)\phi_n(x,0,\gamma')dx = 1$ and

$$P_n^{\text{stay}}(v \to 0,\mu,\gamma,\gamma',W) \to |a_n(t \to 0,\gamma,v \to 0)|^2$$
$$\times |a_n(t \to 0,\gamma',v \to 0)|^2. \tag{D3}$$

Thus, also in the case of an asymmetric evolution the probability to remain in the bound state depends only on the powers γ and γ' and not on the particular shape of the potential or the particle's mass. Numerical examples are shown in Fig. 6.

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