Quantum reference frames associated with noncompact groups: The case of translations and boosts and the role of mass

Alexander R. H. Smith,[1,2,*] Marco Piani,[3,4] and Robert B. Mann[1,5]

[1]*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1*

[2]*Department of Physics and Astronomy, Macquarie University, New South Wales 2109, Australia*

[3]*SUPA and Department of Physics, University of Strathclyde, Glasgow G4 0NG, United Kingdom*

[4]*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1*

[5]*Perimeter Institute for Theoretical Physics, 31 Caroline St. N., Waterloo, Ontario, Canada, N2L 2Y5*

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Quantum communication without a shared reference frame or the construction of a relational quantum theory requires the notion of a quantum reference frame. We analyze aspects of quantum reference frames associated with noncompact groups, specifically, the group of spatial translations and Galilean boosts. We begin by demonstrating how the usually employed group average, used to dispense of the notion of an external reference frame, leads to unphysical states when applied to reference frames associated with noncompact groups. However, we show that this average does lead naturally to a reduced state on the relative degrees of freedom of a system, which was previously considered by Angelo *et al.* [J. Phys. A: Math. Theor. **44**, 145304 (2011)]. We then study in detail the informational properties of this reduced state for systems of two and three particles in Gaussian states.

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I. INTRODUCTION

The central lesson of relativity is that all observable quantities are relations: length, time, and energy, which were once thought to be absolute, only have meaning with respect to an observer. The same is true of a quantum state. For example, when we write the quantum state $|\uparrow\rangle$, say up in z, what we mean is somebody in a laboratory with an appropriately aligned measuring apparatus will measure a specific outcome. This is the description of a quantum state with respect to a classical object, in this example, the macroscopic laboratory.

This state of affairs is not fully satisfactory, since a quantum system is being described with respect to a classical system, that is, by mixing elements of conceptually different frameworks. If we believe that our world is completely described by quantum mechanics, we should seek a theory in which quantum systems are described with respect to quantum systems. Much work has been done on this subject, known as quantum reference frames [1], and it has found applications in quantum interferometry [2], quantum communication [3], and cryptography [4], as well as offering an explanation of previously postulated superselection rules [5,6].

Additionally, treating reference frames quantum mechanically is a crucial step towards the goal of constructing a relational quantum theory [7,8]. By relational it is meant a theory that does not make use of an external reference frame to specify its elements. The main motivation for this is general relativity, which does not use an external reference frame in its construction. It is believed that a theory of quantum gravity will inherent this property, and thus, a theory of quantum gravity will necessarily include a theory of quantum reference frames [9,10].

The natural language of reference frames is that of group theory, owing to the fact that the transformations that describe the act of changing reference frames form a group. Most discussion of quantum reference frames revolves around reference frames defined with respect to compact groups. For example, the relevant group used to describe a phase reference in quantum optics is U(1) or the group used to describe the transformation between orientations of a laboratory is SO(3).

However, if we would like to apply the established formalism to more general groups, such as the Poincaré group and more generally to systems in curved spacetimes, we will need to understand quantum reference frames that are associated with noncompact groups. The purpose of this paper is to embark on such an inquiry.

We begin in Sec. II by introducing the G-twirl, which is a group average over all possible orientations of a system with respect to an external reference frame, and demonstrate its failure when naively applied to situations involving the noncompact groups of translations in position and velocity. However, we find that the G-twirl over these groups naturally introduces a reduced state obtained by tracing out the center-of-mass degrees of freedom of a composite system. In Sec. III we examine informational properties of this reduced state for systems of two and three particles in fully separable Gaussian states with respect to an external frame. Specifically, we study the entanglement that appears when moving from a description of the system with respect to an external frame to a fully relational description, which can alternatively be interpreted in terms of noise. This study is motivated by the need to determine how best to prepare states in the external partition in order to encode information in relational degrees of freedom, which will be useful for various communications tasks [11]. We conclude in Sec. IV with a discussion and summary of the results presented.

II. RELATIONAL DESCRIPTIONS

In constructing a relational quantum theory, one essential task will be the description of a quantum system with respect to another quantum system. We thus seek a way in which

*a14smith@uwaterloo.ca

to remove any information contained in a quantum state that makes reference to an external reference frame. This is accomplished by the G-twirl, which we introduce in Sec. II A and apply to the group of translations and boosts[1] in Sec. II B.

A. Relational description for compact groups

When the state of a system is described with respect to an external reference frame, such that the transformations that generate a change of this reference frame form a compact group, the relational description is well studied [1].

Suppose we have a quantum system in the state $\rho \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the space of bounded linear operators on the Hilbert space $\mathcal{H}$, described with respect to an external reference frame. Changes of the orientation of the system with respect to the external frame are generated by $U(g)$ acting on ρ, where $U(g)$ is the unitary representation of the group element $g \in G$, and G is the compact group of all possible changes of the external reference frame. The relational description of ρ, that is, the quantum state that does not contain any information about the external frame, is given by an average over all possible orientations of ρ with respect to the external frame, with each possible orientation given an equal weight

$$\mathcal{G}[\rho] := \int d\mu(g)\, U(g)\rho U^{\dagger}(g), \tag{1}$$

where $d\mu(g)$ is the Haar measure of the group G; this averaging is referred to as the G-twirl. By averaging over all elements of the group, the G-twirl removes any relation to the external reference frame that was implicitly made use of in the description of ρ. What remains is information about only the relational degrees of freedom within the system. For example, if $\rho \in \mathcal{B}(\mathcal{H})$ describes a composite system of two particles such that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, what remains in $\mathcal{G}[\rho]$ is information about the relational degrees of freedom between the two particles. Notice that the G-twirl is performed via the product representation $U(g) = U_1(g) \otimes U_2(g)$, where U_1 and U_2 are representations of the group G for system 1 and system 2, respectively.

This relational description is used extensively in the study of quantum reference frames involving compact groups [1–3,12,13]. However, when the G-twirl operation is generalized to the case where the group G is noncompact, and thus does not admit a normalized Haar measure, it results in un-normalized states.

For example, let us consider the G-twirl of the state $\rho \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H} \cong L_2(\mathbb{R})$, over the noncompact group of spatial translations T generated by the momentum operator $\hat{P}$. Expressing ρ in the momentum basis we find

$$\begin{aligned}\mathcal{G}_{\mathrm{T}}[\rho] &= \int dg\, e^{-ig\hat{P}} \left(\int dp dp'\, \rho(p,p') \left|p\right\rangle \left\langle p'\right| \right) e^{ig\hat{P}} \\ &= 2\pi \int dp\, \rho(p,p) \left|p\right\rangle \left\langle p\right|, \end{aligned}\tag{2}$$

where dg is the Haar measure associated with T and in going from the first to the second line we have used the definition of the Dirac δ function $2\pi\delta(p-p') = \int dg\, e^{ig(p-p')}$. Although the averaging operation is mathematically well defined, the resulting state $\mathcal{G}[\rho]$ is not normalized, as the trace of $\mathcal{G}_{\mathrm{T}}[\rho]$ is infinite. This is a result of the Haar measure associated with T not being normalized, i.e., the integral $\int dg$ is infinite. This issue does not arise when twirling over a compact group for which there exists a normalized Haar measure. Thus the relational description constructed by averaging a system over all possible orientations of a reference frame fails when the group describing changes of the reference frame is noncompact.

One may try to remedy this problem by introducing a measure $p(g)$ on the group such that $\int dg\, p(g) = 1$ and interpreting $p(g)$ as representing *a priori* knowledge of how the average should be performed [14]. However, in general there is no objective way to choose $p(g)$—if we want a normalized measure it cannot be invariant.

B. Relational description for noncompact groups

We now construct a relational description of quantum states suitable for systems described with respect to reference frames associated with the noncompact groups of boosts and translations. We begin by twirling the state of a system of particles $\rho \in \mathcal{B}(\mathcal{H})$ over all possible boosts and translations of the external reference frame ρ is specified with respect to. The result of this twirling is an un-normalized state proportional to $\mathbb{I}_{CM} \otimes \rho_R$, where $\mathbb{I}_{CM}$ is the identity on the center-of-mass degrees of freedom and $\rho_R = \mathrm{tr}_{CM}\, \rho$ is a normalized density matrix describing the relative degrees of freedom of the system. In doing so, we connect two approaches to quantum reference frames that have been studied in the past, specifically, the approach introduced by Bartlett *et al.* [1], which makes use of the twirl to remove any information the state may have about an external reference frame, and the approach of Angelo *et al.* [15], in which they trace over center-of-mass degrees of freedom to obtain a relational state.

Consider a composite system of N particles each with mass m_n. We may partition the Hilbert space $\mathcal{H}$ of the entire system as $\mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n$ where $\mathcal{H}_n \cong L_2(\mathbb{R}^3)$, which spans the degrees of freedom defined with respect to an external frame associated with the nth particle; we will refer to this as the external partition of the Hilbert space. We may alternatively partition the Hilbert space as $\mathcal{H} = \mathcal{H}_{CM} \otimes \mathcal{H}_R$, where $\mathcal{H}_{CM} \cong L_2(\mathbb{R}^3)$ is associated with the degrees of freedom of the center of mass defined with respect to an external frame, and $\mathcal{H}_R \cong L_2(\mathbb{R}^{3N-3})$ is associated with the relative degrees of freedom of the system defined with respect to a chosen reference particle; we will refer to this partition as the center-of-mass and relational partition of the Hilbert space.

As was done in Sec. II A for reference frames associated with compact groups, to obtain a relational state we will average the state of our system over all possible orientations—intended in a generic sense, meant here to be about translations and boosts—with respect to the external frame. Here we consider the system to be described with respect to an inertial external frame. Thus a change of the external frame corresponds to acting on the system with an element of the Galilean group, and the average over all possible orientations

[1]By boost it is meant Galilean boost as opposed to a Lorentz boost.

of the system with respect to the external frame will be an average over the Galilean group.

The Galilean group Gal is a semidirect product of the translation group T_4, the group of boosts B_3, and the rotation group SO(3):

$$\mathrm{Gal} \cong \mathrm{T}_4 \rtimes (\mathrm{B}_3 \rtimes \mathrm{SO}(3)). \tag{3}$$

We will restrict our analysis to an average over spatial translations T_3, where $T_4 \cong T_1 \rtimes T_3$, and boosts B_3, as averages over SO(3), the orientation of a system with respect to an external frame, have been well studied in literature [1], and we are primarily interested in issues associated with noncompact groups. Further, we do not average over time translations T_1, as this would require us to introduce a Hamiltonian to generate time translations, and for now we are interested only in a relative description of the state at one instant of time and not its dynamics. Suppose the state of a system is given with respect to an external reference frame with a specific position and velocity. The operator that results from these restricted averages is related to the state as seen from an observer who is ignorant of both the position and velocity of the external reference frame.

The position and momentum operators associated with the center of mass, $\hat{\mathbf{X}}_{CM}$ and $\hat{\mathbf{P}}_{CM}$, and relational degrees of freedom, $\hat{\mathbf{X}}_{i|1}$ and $\hat{\mathbf{P}}_{i|1}$, may be expressed in terms of the operators $\hat{\mathbf{X}}_n$ and $\hat{\mathbf{P}}_n$ associated with the position and momentum operators of each of the N particles with respect to the external frame as

$$\hat{\mathbf{X}}_{CM} = \frac{1}{M}\sum_{n=1}^{N} m_n \hat{\mathbf{X}}_n, \tag{4a}$$

$$\hat{\mathbf{P}}_{CM} = \sum_{n=1}^{N} \hat{\mathbf{P}}_n, \tag{4b}$$

$$\hat{\mathbf{X}}_{i|1} = \hat{\mathbf{X}}_i - \hat{\mathbf{X}}_1 \quad \text{for} \quad i \in \{2,\dots,N\}, \tag{4c}$$

$$\hat{\mathbf{P}}_{i|1} = \hat{\mathbf{P}}_i - \frac{m_i}{M}\hat{\mathbf{P}}_{CM} \quad \text{for} \quad i \in \{2,\dots,N\}, \tag{4d}$$

where $M := \sum_{n=1}^{N} m_n$ is the total mass, and without loss of generality we have chosen to define the relative degrees of freedom with respect to particle 1. The above operators satisfy the canonical commutation relations $[\hat{\mathbf{X}}_{CM},\hat{\mathbf{P}}_{CM}] = [\hat{\mathbf{X}}_{i|1},\hat{\mathbf{P}}_{i|1}] = i$, with all other combinations vanishing.

With the exception of the two-particle case, $\hat{\mathbf{P}}_{i|1}$ is not equal to the usually defined relative momentum

$$\hat{\mathbf{P}}_{r_i} := \mu_{1i}\left(\frac{\hat{\mathbf{P}}_i}{m_i} - \frac{\hat{\mathbf{P}}_1}{m_1}\right) \neq \hat{\mathbf{P}}_{i|1} \quad \text{for} \quad i \in \{2,\dots,N\}, \tag{5}$$

where $\mu_{1i} := m_1 m_i/(m_1+m_i)$ is the reduced mass of particle 1 and the ith particle, as one might expect. Alternatively, one may begin with the set relative momentum operators $\{\hat{\mathbf{P}}_{r_i} \mid i = 2,\dots,N\}$ and construct canonically conjugate relative position operators. However, we restrict ourselves to considering the operators given in Eq. (4) and refer the reader to [15] for a more detailed discussion of the nonuniqueness of canonically conjugate operators on $\mathcal{H}_R$.

The action of a translation $\mathbf{g} \in \mathbb{R}^3 \cong \mathrm{T}_3$ and boost $\mathbf{h} \in \mathbb{R}^3 \cong \mathrm{B}_3$ of the external frame in the external partition $\mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n$ is given by

$$\mathrm{U}_\mathrm{T}(\mathbf{g}) = \bigotimes_{n=1}^{N} e^{-i\mathbf{g}\cdot\hat{\mathbf{P}}_n}, \tag{6a}$$

$$\mathrm{U}_\mathrm{B}(\mathbf{h}) = \bigotimes_{n=1}^{N} e^{i m_n \mathbf{h}\cdot\hat{\mathbf{X}}_n}, \tag{6b}$$

and in the center-of-mass and relational partition $\mathcal{H}_{CM} \otimes \mathcal{H}_R$ is given by

$$\mathrm{U}_\mathrm{T}(\mathbf{g}) = e^{-i\mathbf{g}\cdot\hat{\mathbf{P}}_{CM}} \otimes \mathbb{I}_R, \tag{7a}$$

$$\mathrm{U}_\mathrm{B}(\mathbf{h}) = e^{iM\mathbf{h}\cdot\hat{\mathbf{X}}_{CM}} \otimes \mathbb{I}_R. \tag{7b}$$

To carry out the average over T_3 and B_3, let us express ρ in the $\mathcal{H}_{CM} \otimes \mathcal{H}_R$ partition in the momentum basis

$$\rho = \int d\,\mathbf{p}_{CM}\, d\,\mathbf{p}'_{CM}\, d\,\mathbf{p}_R\, d\,\mathbf{p}'_R\, \rho(\mathbf{p}_{CM},\mathbf{p}'_{CM},\mathbf{p}_R,\mathbf{p}'_R) \times |\mathbf{p}_{CM}\rangle\langle\mathbf{p}'_{CM}| \otimes |\mathbf{p}_R\rangle\langle\mathbf{p}'_R|, \tag{8}$$

where $\mathbf{p}_{CM}$ and $\mathbf{p}'_{CM}$ denote the momentum vector of the center of mass and $\mathbf{p}_R$ and $\mathbf{p}'_R$ denote the $N-1$ relative momentum vectors. Making use of Eq. (7a), we may average over all possible spatial translations of the external frame,

$$\begin{aligned}\mathcal{G}_\mathrm{T}[\rho] &= \int d\,\mathbf{p}_{CM}\, d\,\mathbf{p}'_{CM}\, d\,\mathbf{p}_R\, d\,\mathbf{p}'_R\, \rho(\mathbf{p}_{CM},\mathbf{p}'_{CM},\mathbf{p}_R,\mathbf{p}'_R) \int d\,\mathbf{g}\, U_\mathrm{T}(\mathbf{g})\, |\mathbf{p}_{CM}\rangle\langle\mathbf{p}'_{CM}|\, U_\mathrm{T}(\mathbf{g})^\dagger \otimes |\mathbf{p}_R\rangle\langle\mathbf{p}'_R| \\ &= 2\pi \int d\,\mathbf{p}_{CM}\, d\,\mathbf{p}_R\, d\,\mathbf{p}'_R\, \rho(\mathbf{p}_{CM},\mathbf{p}_{CM},\mathbf{p}_R,\mathbf{p}'_R)\, |\mathbf{p}_{CM}\rangle\langle\mathbf{p}_{CM}| \otimes |\mathbf{p}_R\rangle\langle\mathbf{p}'_R|.\end{aligned} \tag{9}$$

From Eq. (9) we see the effect of twirling over the group of translations T_3 is to project ρ into a charge sector of definite center-of-mass momentum.

Similarly, we can average ρ over all possible boosts of the external frame with the result

$$\begin{aligned}\mathcal{G}_\mathrm{B}[\rho] &= \int d\,\mathbf{x}_{CM}\, d\,\mathbf{x}'_{CM}\, d\,\mathbf{x}_R\, d\,\mathbf{x}'_R\, \rho(\mathbf{x}_{CM},\mathbf{x}'_{CM},\mathbf{x}_R,\mathbf{x}'_R) \int d\,\mathbf{h}\, U_\mathrm{B}(\mathbf{h})\, |\mathbf{x}_{CM}\rangle\langle\mathbf{x}'_{CM}|\, U_\mathrm{B}(\mathbf{h})^\dagger \otimes |\mathbf{x}_R\rangle\langle\mathbf{x}'_R| \\ &= \frac{2\pi}{M} \int d\,\mathbf{x}_{CM}\, d\,\mathbf{x}_R\, d\,\mathbf{x}'_R\, \rho(\mathbf{x}_{CM},\mathbf{x}_{CM},\mathbf{x}_R,\mathbf{x}'_R)\, |\mathbf{x}_{CM}\rangle\langle\mathbf{x}_{CM}| \otimes |\mathbf{x}_R\rangle\langle\mathbf{x}'_R|,\end{aligned} \tag{10}$$

where $\mathbf{x}_{CM}$ and $\mathbf{x}'_{CM}$ denote the position vector of the center of mass, $\mathbf{x}_R$ and $\mathbf{x}'_R$ denote the $N-1$ relative position vectors, and $\rho(\mathbf{x}_{CM},\mathbf{x}'_{CM},\mathbf{x}_R,\mathbf{x}'_R) = \langle \mathbf{x}_{CM}| \langle \mathbf{x}_R| \rho |\mathbf{x}_{CM}\rangle |\mathbf{x}_R\rangle$. From Eq. (10) we see the effect of twirling over the group of boosts B_3 is to project ρ into a charge sector of definite center-of-mass position.

Now, averaging Eq. (9) over all boosts, using Eq. (7b), yields

$$\begin{aligned}\mathcal{G}_{\mathrm{B}} \circ \mathcal{G}_{\mathrm{T}}[\rho] &= 2\pi \int d\,\mathbf{h} \int d\,\mathbf{p}_{CM}\, d\,\mathbf{p}_R\, d\,\mathbf{p}'_R\, \rho(\mathbf{p}_{CM},\mathbf{p}_{CM},\mathbf{p}_R,\mathbf{p}'_R) U_{\mathrm{B}}(\mathbf{h}) |\mathbf{p}_{CM}\rangle \langle \mathbf{p}_{CM}| U_{\mathrm{B}}(\mathbf{h})^\dagger \otimes |\mathbf{p}_R\rangle \langle \mathbf{p}'_R| \\ &= 2\pi \int d\,\mathbf{h} \int d\,\mathbf{p}_{CM}\, d\,\mathbf{p}_R\, d\,\mathbf{p}'_R\, \rho(\mathbf{p}_{CM} - M\mathbf{h},\mathbf{p}_{CM} - M\mathbf{h},\mathbf{p}_R,\mathbf{p}'_R) |\mathbf{p}_{CM}\rangle \langle \mathbf{p}_{CM}| \otimes |\mathbf{p}_R\rangle \langle \mathbf{p}'_R| \\ &= \frac{2\pi}{M} \int d\,\mathbf{h} \int d\,\mathbf{p}_{CM}\, d\,\mathbf{p}_R\, d\,\mathbf{p}'_R\, \rho(\mathbf{h},\mathbf{h},\mathbf{p}_R,\mathbf{p}'_R) |\mathbf{p}_{CM}\rangle \langle \mathbf{p}_{CM}| \otimes |\mathbf{p}_R\rangle \langle \mathbf{p}'_R| \\ &= \frac{2\pi}{M} \int d\,\mathbf{p}_{CM} |\mathbf{p}_{CM}\rangle \langle \mathbf{p}_{CM}| \otimes \int d\,\mathbf{p}_R\, d\,\mathbf{p}'_R \left(\int d\,\mathbf{h}\, \rho(\mathbf{h},\mathbf{h},\mathbf{p}_R,\mathbf{p}'_R) \right) |\mathbf{p}_R\rangle \langle \mathbf{p}'_R| \\ &= \frac{2\pi}{M} \mathbb{I}_{CM} \otimes \rho_R, \end{aligned} \tag{11}$$

where in the last line

$$\begin{aligned}\rho_R &:= \int d\,\mathbf{p}_R\, d\,\mathbf{p}'_R \left(\int d\,\mathbf{h}\, \rho(\mathbf{h},\mathbf{h},\mathbf{p}_R,\mathbf{p}'_R) \right) |\mathbf{p}_R\rangle \langle \mathbf{p}'_R| \\ &= \mathrm{tr}_{CM}\, \rho, \end{aligned} \tag{12}$$

and we have made use of the resolution of the identity $\mathbb{I}_{CM} = \int d\,\mathbf{p}_{CM}\ |\mathbf{p}_{CM}\rangle \langle \mathbf{p}_{CM}|$. The action of $\mathcal{G}_{\mathrm{B}} \circ \mathcal{G}_{\mathrm{T}}$ may be expressed as

$$\mathcal{G}_{\mathrm{B}} \circ \mathcal{G}_{\mathrm{T}}[\rho] = \frac{2\pi}{M} (\mathcal{D}_{CM} \otimes \mathcal{I}_R)[\rho], \tag{13}$$

where $\mathcal{D}_{CM}$ denotes the operation that takes every operator on $\mathcal{H}_{CM}$ to the identity operator on that space and $\mathcal{I}_R$ denotes the identity map on $\mathcal{H}_R$. Note, as the generators of T_3 and B_3 commute to a multiple of the identity, $[\hat{\mathbf{X}}_{CM},\hat{\mathbf{P}}_{CM}] = i\mathbb{I}_{CM}$, by application of the Baker-Campbell-Hausdorff equality, it can be shown that $\mathcal{G}_{\mathrm{B}} \circ \mathcal{G}_{\mathrm{T}} = \mathcal{G}_{\mathrm{T}} \circ \mathcal{G}_{\mathrm{B}}$.

From the appearance of $\mathcal{D}_{CM}$, the analog of the complete depolarizing channel on $\mathcal{H}_{CM} \cong L_2(\mathbb{R}^3)$, in Eq. (13), we see that $\mathcal{G}_{\mathrm{B}} \circ \mathcal{G}_{\mathrm{T}}[\rho]$ contains no information about the center of mass, and thus no information about the external frame. However, all the information about the relational degrees of freedom of the system is encoded in ρ_R, which is normalized.

By twirling over all possible boosts and translations of the system, we see from Eq. (11) that the reduced state ρ_R naturally appears. We have thus connected the use of ρ_R that is made by Angelo *et al.* [15,16] when analyzing absolute and relative degrees of freedom, with the usual quantum reference formalism [1].

In general, when transforming from the external partition $\mathcal{H} = \bigotimes_{n=1}^N \mathcal{H}_n$, to the center-of-mass and relational partition $\mathcal{H} = \mathcal{H}_{CM} \otimes \mathcal{H}_R$, entanglement will appear between the center-of-mass and relational degrees of freedom, as well as within the relational Hilbert space $\mathcal{H}_R$. Thus the state ρ_R will be mixed, reflecting the fact that information about the external degrees of freedom has been lost. This is analogous to information about the external frame being lost in Eq. (1) when averaging over all elements of a compact group.

III. GAUSSIAN QUANTUM MECHANICS AND THE RELATIONAL DESCRIPTION

We now examine in detail the informational properties of the reduced state ρ_R of the relational degrees of freedom given in Eq. (12) by examining systems of two and three particles in one dimension distinguished by their masses. As mentioned earlier, in general, entanglement will appear when moving from the external partition, $\mathcal{H} = \bigotimes_{n=1}^N \mathcal{H}_n$, to the center-of-mass and relational partition, $\mathcal{H} = \mathcal{H}_{CM} \otimes \mathcal{H}_R$. This entanglement is crucial in determining how to describe physics relative to a particle within the system [15]. For example, if there is entanglement between the center-of-mass and the relational degrees of freedom, an observer identified with the reference particle, particle 1 as chosen in Eq. (4), will describe the rest of the system as being in a mixed state.

As a concrete example of the entanglement that can emerge when changing from the external partition to the center-of-mass and relational partition of the Hilbert space, we consider systems of two and three particles in Gaussian states in the external partition. The advantage of considering Gaussian states in the external partition is that the transformation which takes the state from being specified in the external partition to being specified in the center-of-mass and relational partition is a Gaussian unitary, that is, a state which is Gaussian in the external partition will also be Gaussian in the center-of-mass and relational partition. Further, if we are interested in the reduced state ρ_R defined in Eq. (12), and the state of the particles in either partition is a Gaussian state, then the trace over the center-of-mass degrees of freedom also results in a Gaussian state. Thus, by considering Gaussian states in the external partition we are able to make use of the extensive tools developed in the field of Gaussian quantum information. We begin here by briefly reviewing relevant aspects of Gaussian quantum information; for more detail the reader may consult one of the many good references on the topic [17–19].

A. The Wigner function and Gaussian states

Any density operator has an equivalent representation as a quasiprobability distribution over phase space. To see this, we introduce the Weyl operator

$$D(\boldsymbol{\xi}) := \exp(i\hat{\mathbf{x}}^T\boldsymbol{\Omega}\boldsymbol{\xi}), \tag{14}$$

where $\hat{\mathbf{x}} := (\hat{q}_1,\hat{p}_1,\ldots,\hat{q}_n,\hat{p}_n)$ is a vector of phase-space operators, $\boldsymbol{\xi} \in \mathbb{R}^{2n}$, and Ω is the symplectic form defined as

$$\Omega = \bigoplus_{i=1}^{n} \omega, \quad \text{with} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15}$$

A density operator $\rho \in \mathcal{B}(\mathcal{H})$ has an equivalent representation as a Wigner characteristic function $\chi(\boldsymbol{\xi}) := \mathrm{tr}[\rho D(\boldsymbol{\xi})]$, or by its Fourier transform, known as the Wigner function,

$$W(\mathbf{x}) := \int_{\mathbb{R}^{2n}} \frac{d^{2n}\xi}{(2\pi)^{2n}} \exp(-i\mathbf{x}^T\boldsymbol{\Omega}\boldsymbol{\xi})\chi(\boldsymbol{\xi}), \tag{16}$$

where $\mathbf{x} := (q_1,p_1,\ldots,q_n,p_n)$ is a vector of phase-space variables.

An n-particle Gaussian state is a state whose Wigner function is Gaussian, that is,

$$W(\mathbf{x};\bar{\mathbf{x}},\mathbf{V}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{x}-\bar{\mathbf{x}})^T\mathbf{V}^{-1}(\mathbf{x}-\bar{\mathbf{x}})\right]}{(2\pi)^n\sqrt{\det \mathbf{V}}}, \tag{17}$$

where $\bar{\mathbf{x}} := (\bar{q}_1,\bar{p}_1,\ldots,\bar{q}_n,\bar{p}_n)$ is given by a vector of averages

$$\bar{x}_i := \langle \hat{x}_i \rangle = \mathrm{tr}[\hat{x}_i\rho], \tag{18}$$

and $\mathbf{V}$ is the real $2n \times 2n$ covariance matrix with components

$$V_{ij} := \tfrac{1}{2}\,\mathrm{tr}[\{\hat{x}_i - \bar{x}_i,\hat{x}_j - \bar{x}_j\}\rho], \tag{19}$$

where we have made use of the anticommutator $\{A,B\} := AB + BA$.

B. Two particles

We begin our analysis by considering two particles with masses m_1 and m_2 to be in a tensor product of Gaussian states $\rho_E = \rho_1 \otimes \rho_2$, where $\rho_1 \in \mathcal{B}(\mathcal{H}_1)$ and $\rho_2 \in \mathcal{B}(\mathcal{H}_2)$ in the external partition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Due to the tensor product structure of ρ_E, the Wigner function of the composite system is a product of the Wigner functions associated with particles 1 and 2:

$$W(\mathbf{x};\bar{\mathbf{x}}_E,\mathbf{V}_E) = W(\mathbf{x};\bar{\mathbf{x}}_1,\mathbf{V}_1)W(\mathbf{x};\bar{\mathbf{x}}_2,\mathbf{V}_2). \tag{20}$$

The reason for considering factorized states in the external partition, apart from their common usage in the literature [3,13], is that if we are to use the composite system for communication, the tensor product structure is easily prepared as it does not require an entangling operation. Further, if one party wishes to communicate a string of classical bits (or qubits), they can try to encode one bit (or qubit) per physical qubit, and this string can be decoded sequentially. The sender does not need to know at the outset the entire message they wish to communicate, and the receiver does not need to store the entire message before decoding it [3].

As we will only be interested in the entanglement generated in moving from the external partition to the center-of-mass and relational partition, we may, without loss of generality, set $\bar{\mathbf{x}}_1 = \bar{\mathbf{x}}_2 = 0$, as these averages can be arbitrarily adjusted via local unitary operations in either partition and thus do not affect the entanglement properties under consideration.

Making use of Eq. (17), we find the covariance matrix associated with ρ_E is given by $\mathbf{V}_E = \mathbf{V}_1 \oplus \mathbf{V}_2$; the direct sum structure resulting from the fact that we chose ρ_E to be a tensor product state in the external partition. Using Williamson's theorem [20], one can show that the most general form of the covariance matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ is given by

$$\mathbf{V}_i = \frac{1}{\mu_i}\mathbf{R}(\theta_i)\mathbf{S}(2r_i)\mathbf{R}(\theta_i)^T = \frac{1}{\mu_i}\begin{pmatrix} \cosh 2r_i - \cos 2\theta_i \sinh 2r_i & \sin 2\theta_i \sinh 2r_i \\ \sin 2\theta_i \sinh 2r_i & \cosh 2r_i + \cos 2\theta_i \sinh 2r_i \end{pmatrix}, \tag{21}$$

where the free parameter $\mu_i = 1/\sqrt{\det \mathbf{V}_i} \in (0,1]$ is the purity $\mathrm{tr}(\rho_i^2)$ of the state ρ_i, $\mathbf{R}(\theta_i)$ is a rotation matrix specifying a phase rotation by an angle $\theta_i \in [0,\pi/4]$, and $\mathbf{S}(2r_i)$ is a diagonal symplectic matrix specifying a squeezing of the Wigner function parameterized by $r_i \in \mathbb{R}$.

1. *Transforming to the center-of-mass and relational partition*

For two particles in one dimension the transformation from the external degrees of freedom $\mathbf{x}_E := (x_1,p_1,x_2,p_2)$, where x_i and p_i denote the position and momentum of the ith particle with respect to an external frame, to the center-of-mass and relational degrees of freedom $\mathbf{x}_{CMR} := (x_{cm},p_{cm},x_{2|1},p_{2|1})$, where x_{cm} and p_{cm} are the position and momentum of the center of mass with respect to an external frame and $x_{2|1}$ and $p_{2|1}$ are the position and momentum of particle 2 with respect to particle 1, is given by Eq. (4) with $N = 2$ and vectors of operators replaced by a single operator. Under this transformation the external covariance matrix $\mathbf{V}_E$ transforms to $\mathbf{V}_{CMR} = \mathbf{M}_2\mathbf{V}_E\mathbf{M}_2^T$, where $\mathbf{M}_2$ is given by

$$\mathbf{M}_2 := \begin{pmatrix} \frac{m_1}{m_1+m_2} & 0 & \frac{m_2}{m_1+m_2} & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -\frac{m_2}{m_1+m_2} & 0 & 1-\frac{m_2}{m_1+m_2} \end{pmatrix}. \tag{22}$$

As both the external and center-of-mass and relational position and momentum operators obey the canonical commutation relations, it follows that $\mathbf{M}_2$ is a symplectic transformation, i.e., it preserves the symplectic form $\mathbf{M}_2\boldsymbol{\Omega}\mathbf{M}_2^T = \boldsymbol{\Omega}$. Since $\mathbf{M}_2$ is symplectic, the associated transformation preserves the Gaussianity of the state, that is, if a state is Gaussian in the external partition, it will also be Gaussian in the center-of-mass and relational partition.

The relational state ρ_R, given in Eq. (12), is a Gaussian state whose covariance matrix $\mathbf{V}_{2|1}$ is obtained by deleting the first and second rows and columns of $\mathbf{V}_{CMR}$. Taking the most general form of $\mathbf{V}_1$ and $\mathbf{V}_2$ yields

$$\mathbf{V}_{2|1}=\frac{1}{\mu_1\mu_2}\begin{pmatrix}\mu_2 f_1^- + \mu_1 f_2^- & -\mu_2\tilde{m}_2 g_1 + \mu_1\tilde{m}_1 g_2\\ -\mu_2\tilde{m}_2 g_1 + \mu_1\tilde{m}_1 g_2 & \mu_2\tilde{m}_2^2 f_1^+ + \mu_1\tilde{m}_1^2 f_2^+\end{pmatrix}, \tag{23}$$

where

$$f_i^{\pm} := \cosh 2r_i \pm \cos 2\theta_i \sinh 2r_i,$$
$$g_i := \sin 2\theta_i \sinh 2r_i,$$

and $\tilde{m}_i := m_i/(m_1+m_2)$.

2. Entanglement between the center-of-mass and relational degrees of freedom

As a measure of entanglement we will employ the logarithmic negativity [21]

$$E_{\mathcal{N}}(\rho) := \log \|\rho^{\Gamma_A}\|_1, \tag{24}$$

where Γ_A is the partial transpose and $\|\cdot\|_1$ denotes the trace norm, with $\log(\cdot)$ denoting the natural logarithm. The logarithmic negativity is a measure of the failure of the partial transpose of a quantum state to be a valid quantum state and is a faithful measure of entanglement for the $1\times N$ mode Gaussian states [22].

For Gaussian states the logarithmic negativity is given by

$$E_{\mathcal{N}} := -\sum_k \log\tilde{\nu}_k \quad \forall\, \tilde{\nu}_k < 1, \tag{25}$$

where $\{\tilde{\nu}_k\}$ is the symplectic spectrum of the partially transposed covariance matrix $\tilde{\mathbf{V}}$, i.e., the eigenspectrum of $|i\boldsymbol{\Omega}\tilde{\mathbf{V}}|$. The partial transpose of a covariance matrix is

$$\tilde{\mathbf{V}} = \boldsymbol{\theta}_{1|2}\mathbf{V}\boldsymbol{\theta}_{1|2}, \tag{26}$$

where $\boldsymbol{\theta}_{1|2} = \mathrm{diag}(1,1,1,-1)$.

We will use the logarithmic negativity to quantify the entanglement between the center-of-mass and relational degrees of freedom in $\mathbf{V}_{CMR} = \mathbf{M}_2\mathbf{V}_E\mathbf{M}_2^T$, for $\mathbf{V}_E = \mathbf{V}_1\oplus\mathbf{V}_2$, which corresponds to the two particles being in a factorized state $\rho_1\otimes\rho_2$ in the external partition. $\mathbf{V}_1$ and $\mathbf{V}_2$ will necessarily be of the form given in Eq. (21).

Plots of the logarithmic negativity of the state associated with $\mathbf{V}_{CMR}$ for different choices of $\mathbf{V}_1$ and $\mathbf{V}_2$ are given in Figs. 1 (identical state parameters), 2 (differing purity), and 3 (differing squeezing). Several trends emerge from a perusal of these figures.

We first note that equal-mass systems suppress entanglement between center-of-mass and relational degrees of freedom. When particles in the external partition are prepared such that they have identical covariance matrices we find vanishing entanglement in the equal-mass case, regardless of the amount of squeezing and rotation. This occurs for both pure and mixed situations, respectively, illustrated in Figs. 1 and 2. As one of the masses gets larger, center-of-mass and relational entanglement increases for any fixed value of the squeezing parameter r.

The next trend we observe is that phase rotation, corresponding to squeezing along a rotated axis in phase space, appears to play a more important role than squeezing. For a phase rotation $\theta=\theta_1=\theta_2=0$, we find the entanglement

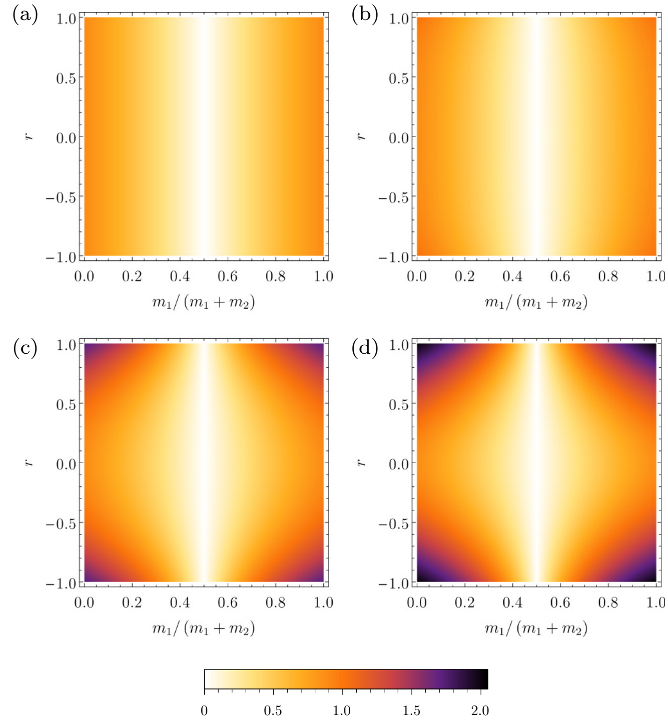

FIG. 1. The logarithmic negativity, as a measure of the entanglement between the center-of-mass and relation degrees of freedom, of the state associated with $\mathbf{V}_{CMR}$, when $\mathbf{V}_1=\mathbf{V}_2$ and both ρ_1 and ρ_2 are pure, i.e., $\det\mathbf{V}_1=\det\mathbf{V}_2=1$, is plotted as a function of the squeezing parameter $r=r_1=r_2$ and the ratio of masses $m_1/(m_1+m_2)$ for different phase rotations $\theta=\theta_1=\theta_2$: (a) $\theta=0$, (b) $\theta=\pi/32$, (c) $\theta=\pi/8$, and (d) $\theta=\pi/4$.

between the center-of-mass and relational degrees of freedom is insensitive to the amount of squeezing. As θ increases we see that squeezing plays an increasingly important role, particularly as the ratio of the masses increasingly departs from unity. Not surprisingly, entanglement is greater for the pure case, shown in Fig. 1, than for the mixed case, shown in Fig. 2.

Asymmetric squeezing, $r_2=\alpha r_1$ where $\alpha\in\mathbb{R}_+$, illustrated in Fig. 3, modifies this situation. When there is no squeezing, $r_1=r_2=0$, entanglement between the center-of-mass and relational degrees of freedom vanishes when the masses of the two particles are equal. However, as r_1 departs from zero, the ratio of masses, $m_1/(m_1+m_2)$, at which entanglement between the center-of-mass and relational degrees of freedom vanishes, increases [Fig. 3(a)], a trend that is less pronounced as α approaches unity [Fig. 3(b)]. Again, we see that phase rotation plays a significant role; Figs. 3(c) and 3(d) demonstrate that if the squeezing of the particles is different and along a rotated axis, entanglement between the center-of-mass and relational degrees of freedom can vanish entirely.

Decreasing the purity of the states of the particles in the external partition, shown in Fig. 2, indicates the same trends as for the pure case (Fig. 1). The main effects of decreased purity are to decrease the overall entanglement between the center-of-mass and relational degrees of freedom and to widen the range of ratio of masses for which this entanglement vanishes.

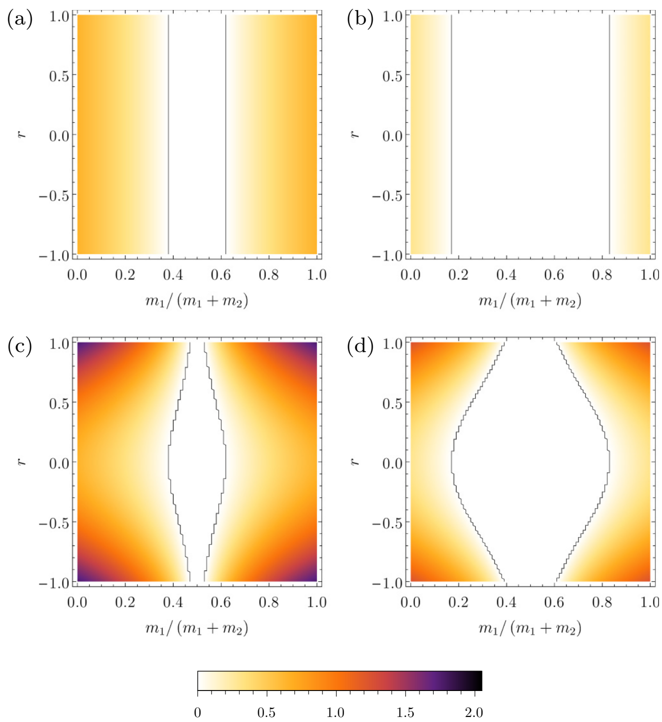

FIG. 2. The logarithmic negativity, as a measure of the entanglement between the center-of-mass and relation degrees of freedom, of the state associated with $\mathbf{V}_{CMR}$, when $r = r_1 = r_2$, $\theta = \theta_1 = \theta_2$, and particle 2 is a pure state, $\mu_2 = 1$, and particle 1 is not, for different purities of particle 1, μ_1, and phase rotations θ. In (a,b) $\theta = 0$ and (c,d) $\theta = \pi/4$. In (a,c) $\mu_1 = 0.6$ and (b,d) $\mu = 0.2$. Plots for $\theta = 0$ and $\mu_1 = 1$ and $\theta = \pi/4$ and $\mu_1 = 1$ are shown in Figs. 1(a) and 1(d), respectively.

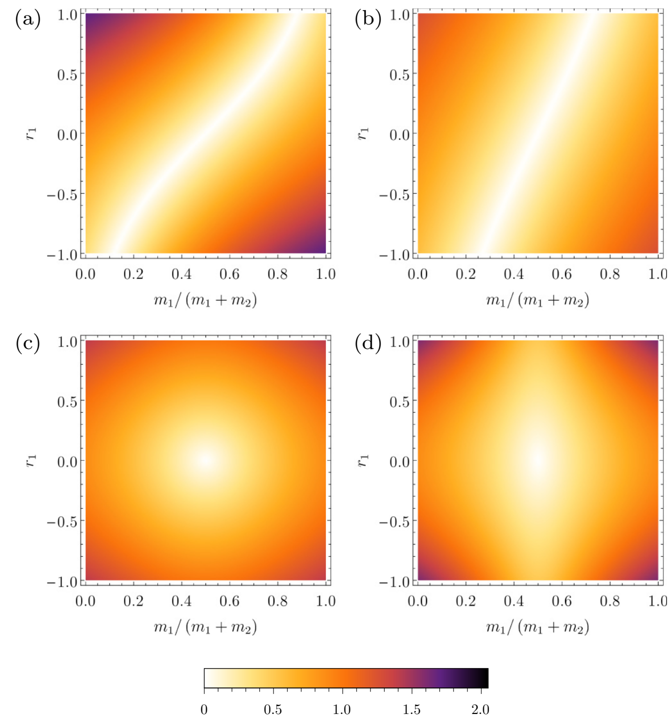

FIG. 3. The logarithmic negativity, as a measure of the entanglement between the center-of-mass and relation degrees of freedom, of the state associated with $\mathbf{V}_{CMR}$ when $\det \mathbf{V}_1 = \det \mathbf{V}_2 = 1$ for $r_2 = \alpha r_1$, and for different phase rotations $\theta = \theta_1 = \theta_2$ and values α. In (a,b) $\theta = 0$ and (c,d) $\theta = \pi/4$. In (a,c) $\alpha = 0$ and (b,d) $\alpha = 0.5$. Plots for $\theta = 0$ and $\alpha = 1$ and $\theta = \pi/4$ and $\alpha = 1$ are shown in Figs. 1(a) and 1(d), respectively.

In Figs. 1–3 we have plotted the logarithmic negativity as a measure of the entanglement between the center-of-mass and relational degrees of freedom for a wide variety of separable states in the external partition. The more entangled these degrees of freedom are, the more mixed the reduced state ρ_R of the relational degrees of freedom will be. The practical consequence of this is that if one wishes to encode quantum information in the relational degrees of freedom of two Gaussian states, perhaps to communicate this information to another party who does not have access to their external reference frame, then the purity and amount and direction of squeezing should be chosen in accordance with Figs. 1–3 as to minimize the entanglement between the center-of-mass and relational degrees of freedom.

C. Three particles

We consider now a similar analysis for a system of three particles with masses m_1, m_2, and m_3. When transforming a fully factorized state in the external partition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ to the center-of-mass and relational partition $\mathcal{H} = \mathcal{H}_{CM} \otimes \mathcal{H}_R$, there will again be entanglement generated between the center-of-mass and relational degrees of freedom. In addition, there will be entanglement generated among the relational degrees of freedom, a new feature not possible for the two-particle system considered above.

The center-of-mass position and momentum operators, along with the relative position and momentum operators, are again defined via Eq. (4). The transformed covariance matrix is given by $\mathbf{V}_{CMR} = \mathbf{M}_3 \mathbf{V}_E \mathbf{M}_3^T$, where

$$\mathbf{M}_3 := \begin{pmatrix} \frac{m_1}{M} & 0 & \frac{m_2}{M} & 0 & \frac{m_3}{M} & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{m_2}{M} & 0 & 1-\frac{m_2}{M} & 0 & -\frac{m_2}{M} \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{m_3}{M} & 0 & -\frac{m_3}{M} & 0 & 1-\frac{m_3}{M} \end{pmatrix}. \quad (27)$$

The relational state $\mathbf{V}_{23|1}$ of particles 2 and 3 as described by particle 1 is obtained by deleting the first and second rows and columns of $\mathbf{V}_{CMR}$. We observe that in the limit when m_3 vanishes and the columns and rows of $\mathbf{M}_3$ associated with particle 3 are deleted, that is, the last two rows and columns, $\mathbf{M}_2$ as defined in Eq. (22) is recovered.

We assume the state of the three-particle system in the external partition is a fully factorized Gaussian state with the covariance matrix $\mathbf{V}_E = \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \mathbf{V}_3$. For simplicity we restrict ourselves to the case when $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}_3$ and $\det \mathbf{V}_E = 1$, in other words, a pure state, with each of the three particles identically squeezed in the same direction.

In Fig. 4 the logarithmic negativity as a measure of entanglement between the center-of-mass and relational degrees of freedom in $\mathbf{V}_{CMR}$ is plotted for different choices of $\mathbf{V}_E$. In

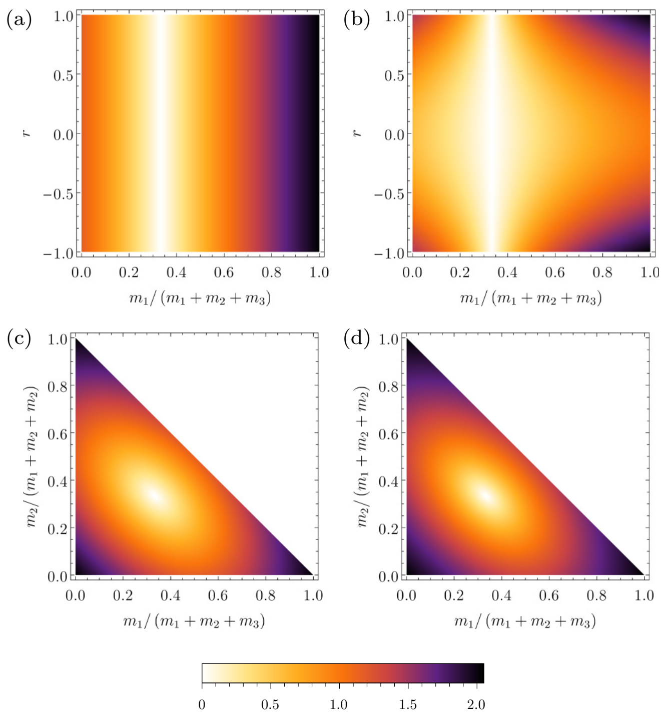

FIG. 4. The logarithmic negativity is plotted, as a measure of the entanglement between the center-of-mass and relation degrees of freedom, of the state associated with $\mathbf{V}_{CMR}$ describing three particles for different equal phase rotations $\theta_1 = \theta_2 = \theta_3 = \theta$ with $\det \mathbf{V}_1 = \det \mathbf{V}_2 = \det \mathbf{V}_3 = 1$. In (a) and (b) the logarithmic negativity is plotted for the case when $m_2 = m_3$, as a function of the ratio $m_1/(m_1 + m_2 + m_3)$ and equal squeezing parameter $r_1 = r_2 = r_3 = r$, with $\theta = 0$ and $\theta = \pi/4$, respectively. In (c) and (d) the logarithmic negativity is plotted as a function of the two mass ratios $m_1/(m_1 + m_2 + m_3)$ and $m_2/(m_1 + m_2 + m_3)$ for $\theta = 0$ and $\theta = \pi/4$, respectively, with the equal squeezing parameter fixed at $r = 0.7$.

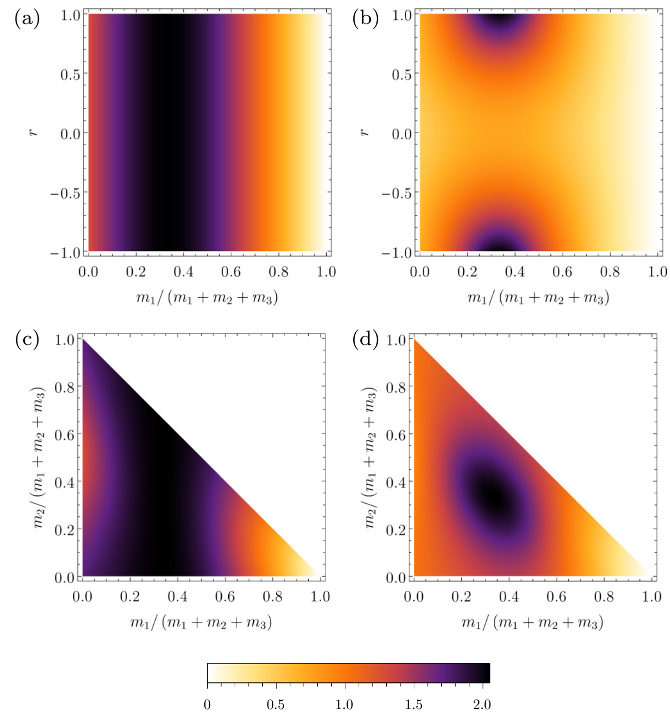

FIG. 5. The logarithmic negativity of the relative state of particles 2 and 3 described by $\mathbf{V}_{23|1}$ is plotted, characterizing the entanglement among the relational degrees of freedom, for different equal phase rotations $\theta_1 = \theta_2 = \theta_3 = \theta$ with $\det \mathbf{V}_1 = \det \mathbf{V}_2 = \det \mathbf{V}_3 = 1$. In (a) and (b) the logarithmic negativity is plotted for the case $m_2 = m_3$ as a function of the ratio $m_1/(m_1 + m_2 + m_3)$ and equal squeezing parameters $r_1 = r_2 = r_3 = r$ for $\theta = 0$ and $\theta = \pi/4$, respectively. In (c) and (d) logarithmic negativity is plotted as a function of the mass ratios $m_1/(m_1 + m_2 + m_3)$ and $m_2/(m_1 + m_2 + m_3)$ for equal squeezing parameter $r = 0.7$ and $\theta = 0$ and $\theta = \pi/4$, respectively.

Fig. 5 the logarithmic negativity between the relational degrees of freedom in $\mathbf{V}_{23|1}$ is plotted for different choices of $\mathbf{V}_E$.

We see similar trends for the center-of-mass and relational entanglement as for the two-particle case but qualitatively different behavior of the internal-relational entanglement, i.e., the entanglement generated among the relational degrees of freedom—in the case at hand, the entanglement between particles 2 and 3 as described by particle 1.

The internal-relational entanglement, illustrated in Fig. 5, shows strikingly different behavior. Such entanglement is maximized in the equal-mass case, shown in Figs. 5(b) and 5(d), provided there is some phase rotation. In the absence of phase rotation, this effect vanishes. For all values of the (equal) phase rotation parameter, we observe that as the mass of the reference particle m_1 becomes infinite, the entanglement between particles 2 and 3 vanishes. This is as expected, since this limit corresponds to particle 1 behaving as a classical reference frame with a large mass. Indeed, we notice that in the limit $m_1 \to \infty$, the 4×4 lower-right submatrix of $\mathbf{M}_3$ becomes the identity matrix, and the only effect of the change of coordinates is that of redefining the origin in space for the coordinates of the second and third particle.

IV. DISCUSSION AND OUTLOOK

We have highlighted issues involving quantum reference frames associated with noncompact groups. We began in Sec. II A by introducing the usually employed G-twirl as a relational description between quantum systems and demonstrated how it leads to un-normalized states when applied to noncompact groups. In Sec. II B we demonstrated that a lack of reference frame associated with the translation group and the group of Galilean boosts leads to a superselection rule on the respective momentum and position of the center of mass of a multiparticle system. Further, we saw how the G-twirl over these groups leads to the appearance of the reduced state on the relational degrees of freedom previously considered by Angelo *et al.* [15]. We then examined the consequences of this relational description in Sec. III by studying the entanglement that emerges between the center-of-mass degrees of freedom and the relational degrees of freedom, as well as the entanglement among the relational degrees of freedom, for a system of particles when moving from a description of the quantum system entirely with respect to an external frame, to a description in which only the center of mass is specified with respect to an external frame and all other degrees of freedom are relational.

Two main observations emerged from studying the reduced state ρ_R on the relational degrees of freedom, introduced in Eq. (12), for systems of two and three particles. First, for fully separable Gaussian states in the external partition with identical second moments, entanglement between the center-of-mass degrees of freedom and relational degrees of freedom is minimized when the masses of the particles are the same. Second, again for fully separable Gaussian states in the external partition with identical second moments, in the limit when the mass of the reference particle, that is, the particle for which the relational degrees of freedom are defined with respect to, becomes infinite, the entanglement among the relational degrees of freedom vanishes. This second observation suggests a meaningful way to interpret the external reference frame, with which we usually describe a quantum state with respect to, as the limit of a physical system, say a particle, in which its mass is taken to infinity [23]. The consequences of this second observation will be explored in future work.

It may be possible to gain further physical intuition into the behavior of the informational properties of ρ_R by comparing ρ_R with the behavior of nonclassical states of light passing through a beam splitter, as this scenario has been well studied in the field of quantum optics and the formalism of Gaussian quantum information was developed with this situation in mind.

The primary motivation for examining quantum reference frames associated with noncompact groups is to apply the quantum reference frame formalism to relativistic systems, in which the natural group associated with changes of a reference frame is the Poincaré group. We note that the approach taken in Sec. II B was to introduce the relative and center-of-mass partition of the Hilbert space and then show that the relative degrees of freedom form a decoherence-free subsystem, whereas the center-of-mass degree of freedom forms a decoherence-full subsystem [see Eq. (11)]. This approach may not be possible for the Poincaré group as the usually defined center of mass is not covariant [24]. In this case, the decoherence-free and decoherence-full subspaces will need to be identified from the structure of the Poincaré group.

Two other possible applications of the formalism introduced come to mind. The first is in constructing a relativity principle for quantum mechanics by studying changes of quantum reference frames, which was first suggested in Ref. [13]. The second is to construct a relational quantum theory, similar to what was done in Ref. [9], for the Galilean group using the relational description in Eq. (12) and examine how the usual "nonrelational" theory emerges.

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APPENDIX: PURITY OF THE RELATIONAL STATE

The covariance matrices considered in Secs. III B 1 and III B 2 were of the form $\mathbf{V}_E = \mathbf{V}_1 \oplus \mathbf{V}_2$, where both $\mathbf{V}_1$ and $\mathbf{V}_2$ were given by Eq. (21). The purity of $\mathbf{V}_{CMR} = \mathbf{M}_2\mathbf{V}_E\mathbf{M}_2^T$ is given by

$$\mu_{CMR} = \frac{1}{\sqrt{\det \mathbf{V}_{CMR}}} = \mu_1\mu_2, \tag{A1}$$

where μ_1 and μ_2 are the purities associated with $\mathbf{V}_1$ and $\mathbf{V}_2$, respectively.

The purity of the relational state $\mathbf{V}_{2|1}$ in Eq. (23), that is, the state obtained from $\mathbf{V}_{CMR}$ by taking the partial trace over the center-of-mass degrees of freedom, is

$$\begin{aligned}\mu_{2|1} &= \frac{1}{\sqrt{\det \mathbf{V}_{2|1}}}\\ &= \mu_1\mu_2\big[\mu_2^2\tilde{m}_2^2 f_1^- f_1^+ + \mu_1\mu_2\big(\tilde{m}_1^2 f_1^- f_2^+ + \tilde{m}_2^2 f_1^+ f_2^-\big)\\ &\quad\times \mu_1^2\tilde{m}_1^2 f_2^- f_2^+ - \mu_2^2\tilde{m}_2^2 g_1^2 + 2\mu_1\mu_2\tilde{m}_1\tilde{m}_2 g_1 g_2\\ &\quad - \mu_1^2\tilde{m}_1^2 g_2^2\big]^{-1/2},\end{aligned} \tag{A2}$$

where we have introduced the notation $\tilde{m}_i = m_i/(m_1+m_2)$.

If $\mathbf{V}_{CMR}$ is pure, which corresponds to both $\mathbf{V}_1$ and $\mathbf{V}_2$ being pure, then $\mu_{CMR} = 1$ and $\mu_{2|1}$ is a genuine measure of entanglement between the center-of-mass and relational degrees of freedom. In this case, $\mu_{2|1}^{-2}$ simplifies to

$$\begin{aligned}\mu_{2|1}^{-2} &= (\tilde{m}_2 - \tilde{m}_1)[\sinh(2r_1)\cosh(2r_2)\cos(2\theta_1)\\ &\quad - \sinh(2r_2)\cosh(2r_1)\cos(2\theta_2)]\\ &\quad + (2\tilde{m}_1\tilde{m}_2 + 1)\cosh(2r_1)\cosh(2r_2)\\ &\quad - \sinh(2r_1)\sinh(2r_2)[2\tilde{m}_1\tilde{m}_2\cos[2(\theta_1+\theta_2)]\\ &\quad + \cos(2\theta_1)\cos(2\theta_2)] + \tilde{m}_1^2 + \tilde{m}_2^2.\end{aligned} \tag{A3}$$

If the mass of the two particles are equal $m_1 = m_2$, $\mu_{2|1}^{-2}$ further simplifies to

$$\begin{aligned}\mu_{2|1}^{-2} &= \tfrac{1}{4}[-2\sinh(2r_1)\sinh(2r_2)\cos[2(\theta_1-\theta_2)]\\ &\quad + \cosh[2(r_1-r_2)] + \cosh[2(r_1+r_2)] + 2].\end{aligned} \tag{A4}$$

For the case when $m_1 \neq m_2, r_1 = r_2 = r$, and $\theta_1 = \theta_2 = \theta$, corresponding to Fig. 1, $\mu_{2|1}^{-2}$ becomes

$$\begin{aligned}\mu_{2|1}^{-2} &= 2\frac{m_1^2+m_2^2}{(m_1+m_2)^2} + \sin^2(2\theta)\\ &\quad\times\left(\frac{m_1^2+m_2^2}{(m_1+m_2)^2}\sinh^2(2r) - 2\frac{m_1 m_2}{(m_1+m_2)^2}\right).\end{aligned} \tag{A5}$$

From Eq. (A5), we observe that when the masses of the two particles are identical $m_1 = m_2$, the reduced state $\mathbf{V}_{2|1}$ is pure, i.e, $\mu_{2|1} = 1$, which corresponds to vanishing entanglement between the center-of-mass and relational degrees of freedom in $\mathbf{V}_{CMR}$. This agrees with the plots of the logarithmic negativity in Fig. 1.

When the mass of either particle becomes infinite we find

$$\mu_{2|1}^{-2} = 2 + \sinh^2(2r)\cos^2(2\theta). \tag{A6}$$

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