# Intrinsic upper bound on two-qubit polarization entanglement predetermined by pump polarization correlations in parametric down-conversion

Girish Kulkarni, V. Subrahmanyam, and Anand K. Jha*
*Department of Physics, Indian Institute of Technology, Kanpur 208016, India*

(Received 1 December 2015; published 24 June 2016)

We study how one-particle correlations transfer to manifest as two-particle correlations in the context of parametric down-conversion (PDC), a process in which a pump photon is annihilated to produce two entangled photons. We work in the polarization degree of freedom and show that for any two-qubit generation process that is both trace-preserving and entropy-nondecreasing, the concurrence $C(\rho)$ of the generated two-qubit state $\rho$ follows an intrinsic upper bound with $C(\rho) \leqslant (1+P)/2$, where $P$ is the degree of polarization of the pump photon. We also find that for the class of two-qubit states that is restricted to have only two nonzero diagonal elements such that the effective dimensionality of the two-qubit state is the same as the dimensionality of the pump polarization state, the upper bound on concurrence is the degree of polarization itself, that is, $C(\rho) \leqslant P$. Our work shows that the maximum manifestation of two-particle correlations as entanglement is dictated by one-particle correlations. The formalism developed in this work can be extended to include multiparticle systems and can thus have important implications towards deducing the upper bounds on multiparticle entanglement, for which no universally accepted measure exists.

DOI: 10.1103/PhysRevA.93.063842

## I. INTRODUCTION

The wave-particle duality, that is, the simultaneous existence of both particle and wave properties, is the most distinguishing feature of a quantum system. A quantum system is characterized in terms of physical quantities such as energy, momentum, etc., as well as in terms of correlations, the degree of which can be measured in terms of the contrast with which a system produces interference patterns [1–3]. In the context of quantum systems consisting of more than one particle, the wave-particle duality can manifest as entanglement [4]. Quantum entanglement refers to intrinsic multiparticle correlations in a system and is quite often referred to as the quintessential feature of quantum systems [5]. There are many processes in which a quantum system gets annihilated to produce a new quantum system consisting of either equal or more number of particles. An example is the nonlinear optical process of parametric down-conversion (PDC), in which an input pump photon gets annihilated to produce two entangled photons called the signal and idler photons [6]. Another example is the four-wave mixing process, in which two input pump photons get annihilated to produce two new photons [7]. In such processes, it is known that certain physical quantities remain conserved [6,8]. For example, in parametric down-conversion, the energy of the pump photon remains equal to the sum of the energies of the down-converted signal and idler photons [6]. However, it is not very well understood as to how in such processes the intrinsic correlations in the annihilated quantum system get transferred to the generated new quantum system.

One of the main difficulties in addressing questions related to correlation transfer is the lack of a mathematical framework for quantifying correlations in multidimensional systems in terms of a single scalar quantity, although more recently there have been a lot of research efforts with the aim of quantifying coherence [9–13]. For a one-particle quantum system with a two-dimensional Hilbert space, the correlation in the system can be completely specified. For example, polarization is a degree of freedom that provides a two-dimensional basis and the correlations in an arbitrary state of a one-photon system can be uniquely quantified in terms of the degree of polarization [1,14]. Two-photon systems have a four-dimensional Hilbert space in the polarization degree of freedom and are described by two-qubit states [15]. In the last several years much effort has gone into quantifying the entanglement of the two-qubit states [16–24], and among the available entanglement quantifiers, Wootters's concurrence [21,22] is the most widely used one. However, when the Hilbert-space dimensionality of individual quantum particles is more than two, there is no prescription for quantifying the correlations in the entire system; one can at best quantify correlations in a two-dimensional subspace [25]. So, with the current mathematical framework, as far as quantifying intrinsic correlations in a quantum system in terms of a single quantity is concerned, it can only be done when the Hilbert space is two-dimensional. The polarization degree of freedom provides such a two-dimensional space and the intrinsic correlations can therefore be completely quantified in this degree of freedom.

In the context of signal-idler photons produced by PDC, the two-qubit polarization-entangled states have been extensively studied [15,26,27] and are now seen to hold a lot of promise for practical quantum-information protocols [28,29]. However, to the best of our knowledge, the correlations in the down-converted polarization-entangled states have not been studied from the perspective of how these correlations are dictated by the polarization correlations in the pump field. In degrees of freedom other than polarization, some aspects of how one-photon pump correlations transfer to two-photon signal-idler correlations have previously been investigated [30–34]. In particular, Ref. [31] studied correlation transfer in PDC in the spatial degree of freedom. Although spatial degree of freedom provides an infinite dimensional basis, correlations in Ref. [31] were quantified in restricted two-dimensional subspaces only. More specifically, the spatial correlations in

*akjha9@gmail.com

 

the pump field were quantified in terms of a spatial two-point correlation function; for quantifying spatial correlations of the signal and idler fields, spatial two-qubit states with only two nonzero diagonal elements were considered. It was then shown that the maximum achievable concurrence of spatial two-qubit states is bounded by the degree of spatial correlations of the pump field. In this article, we study correlation transfer from one-photon to two-photon systems, not in any restricted subspace, but in the complete space of the polarization degree of freedom. We quantify intrinsic one-photon correlations in terms of the degree of polarization and the two-photon correlations in terms of concurrence.

The paper is organized as follows. In Sec. II, we present our derivation of the upper bound on two-qubit polarization entanglement. We first present the most general bound on two-qubit polarization entangled states and then discuss the bound for two-qubit states that have only two nonzero diagonal elements. In Sec. III, we discuss an example experimental setup in which a wide variety of two-qubit states can be produced. By numerically varying different tunable parameters of the setup, we simulate a large number of two-qubit states, calculate the corresponding concurrences, and illustrate how the bounds derived in Sec. II are obeyed. Section IV presents our conclusions.

## II. UPPER BOUND ON TWO-QUBIT POLARIZATION ENTANGLEMENT

### A. The degree of polarization of the pump field

We begin by noting that the state of a normalized quasi-monochromatic pump field may be described by a $2 \times 2$ density matrix [1] given by

$$J = \begin{bmatrix} \langle E_H E_H^* \rangle & \langle E_H E_V^* \rangle \\ \langle E_H^* E_V \rangle & \langle E_V E_V^* \rangle \end{bmatrix}, \tag{1}$$

which is referred to as the "polarization matrix." The complex random variables $E_H$ and $E_V$ denote the horizontal and vertical components of the electric field, respectively, and $\langle \cdots \rangle$ denotes an ensemble average. By virtue of a general property of $2 \times 2$ density matrices, $J$ has a decomposition of the form,

$$J = P\,|\psi_{\text{pol}}\rangle\langle\psi_{\text{pol}}| + (1-P)\,\bar{\mathbb{1}}, \tag{2}$$

where $|\psi_{\text{pol}}\rangle$ is a pure state representing a completely polarized field, and $\bar{\mathbb{1}}$ denotes the normalized $2 \times 2$ identity matrix representing a completely unpolarized field [1]. This means that any arbitrary field can be treated as a unique weighted mixture of a completely polarized part and a completely unpolarized part. The fraction $P$ corresponding to the completely polarized part is called the degree of polarization and is a basis-invariant measure of polarization correlations in the field. If we denote the eigenvalues of $J$ as $\epsilon_1$ and $\epsilon_2$, then it can be shown that $P = |\epsilon_1 - \epsilon_2|$ [1]. Furthermore, the eigenvalues are connected to $P$ as $\epsilon_1 = (1+P)/2$ and $\epsilon_2 = (1-P)/2$.

### B. The general upper bound

We now investigate the PDC-based generation of polarization entangled two-qubit signal-idler states $\rho$ from a quasimonochromatic pump field $J$ (see Fig. 1). The nonlinear optical process of PDC is a very low-efficiency process [7]. Most of the pump photons do not get down-converted and just pass through the nonlinear medium. Only a very few pump photons do get down-converted, and in our description, only these photons constitute the ensemble containing the pump photons. We further assume that the probabilities of the higher-order down-conversion processes are negligibly small so that we do not have in our description the down-converted state containing more than two photons. With these assumptions, we represent the state of the down-converted signal and idler photons by a $4 \times 4$, two-qubit density matrix in the polarization basis $\{|H\rangle_s|H\rangle_i, |H\rangle_s|V\rangle_i, |V\rangle_s|H\rangle_i, |V\rangle_s|V\rangle_i\}$. In what follows, we will be applying some results from the theory of majorization [35] in order to study the propagation of correlations from the $2 \times 2$ pump density matrix $J$ to the $4 \times 4$ two-qubit density matrix $\rho$. This requires us to equalize the dimensionalities of the pump and the two-qubit states. We therefore represent the pump field by a $4 \times 4$ matrix $\sigma$, where

$$\sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes J. \tag{3}$$

We denote the eigenvalues of $\sigma$ in nonascending order as $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \equiv ((1+P)/2, (1-P)/2, 0, 0)$ and the eigenvalues of $\rho$ in nonascending order as $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Let us represent the two-qubit generation process $\sigma \to \rho$ by a completely positive map $\mathcal{E}$ (see Fig. 1) such that $\rho = \mathcal{E}(\sigma) = \sum_i M_i \sigma M_i^\dagger$, where $M_i$'s are the Sudarshan-Kraus operators for the process [36–39]. We restrict our analysis only to maps that satisfy the following two conditions for all $\sigma$: (i) No part of the system can be discarded, that is, there must be no postselection. This means that the map must be trace-preserving, which leads to the condition that $\sum_i M_i^\dagger M_i = \mathbb{1}$; (ii) coherence may be lost to, but not gained from degrees of freedom external to the system. In other words, the von Neumann entropy cannot decrease. This condition holds if and only if the map is unital, that is, $\sum_i M_i M_i^\dagger = \mathbb{1}$. The above two conditions together imply that the process $\sigma \to \rho$ is doubly stochastic [40]. The characteristic implication of double stochasticity is that the two-qubit state is majorized by the pump state, that is, $\rho \prec \sigma$. This means that the eigenvalues of $\rho$ and $\sigma$ satisfy the following relations:

$$\lambda_1 \leqslant \epsilon_1, \tag{4a}$$

$$\lambda_1 + \lambda_2 \leqslant \epsilon_1 + \epsilon_2, \tag{4b}$$

$$\lambda_1 + \lambda_2 + \lambda_3 \leqslant \epsilon_1 + \epsilon_2 + \epsilon_3, \tag{4c}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4. \tag{4d}$$

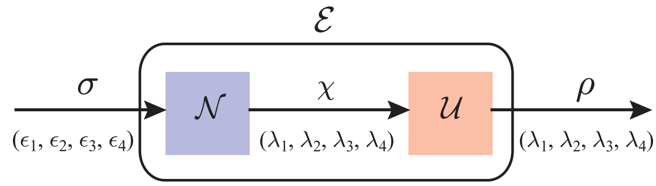


FIG. 1. Modeling the generation of two-qubit states $\rho$ from $\sigma$ through a doubly stochastic process.

We must note that condition (i) may seem not satisfied in some of the experimental schemes for producing polarization entangled two-qubit states. For example, in the scheme for producing a polarization Bell state using type-II phase matching [15], only one of the polarization components of the pump photon is allowed to engage in the down-conversion process; the other polarization component, even if present, simply gets discarded away. Nevertheless, our formalism is valid even for such two-qubit generation schemes. In such schemes, the state $\sigma$ represents that part of the pump field which undergoes the down-conversion process so that condition (i) is satisfied.

Now, for a general realization of the process $\sigma \rightarrow \rho$, the generated density matrix $\rho$ can be thought of as arising from a process $\mathcal{N}$, that can have a nonunitary part, followed by a unitary-only process $\mathcal{U}$, as depicted in Fig. 1. This means that we have $\sigma \rightarrow \chi \equiv \mathcal{N}(\sigma) \rightarrow \rho \equiv \mathcal{U}(\chi)$. The process $\mathcal{N}$ generates the two-qubit state $\chi$ with eigenvalues $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ which are different from the eigenvalues $\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}$ of $\sigma$, except when $\mathcal{N}$ consists of unitary-only transformations, in which case the eigenvalues of $\chi$ remain the same as that of $\sigma$. The unitary part $\mathcal{U}$ transforms the two-qubit state $\chi$ to the final two-qubit state $\rho$. This action does not change the eigenvalues but can change the concurrence of the two-qubit state. The majorization relations of Eq. (4) dictate how the two sets of eigenvalues are related and thus quantify the effects due to $\mathcal{N}$. We quantify the effects due to $\mathcal{U}$ by using the result from Refs. [20,41,42] for the maximum concurrence achievable by a two-qubit state under unitary transformations. According to this result, for a two-qubit state $\rho$ with eigenvalues in nonascending order denoted as $\lambda_1,\lambda_2,\lambda_3,\lambda_4$, the concurrence $C(\rho)$ obeys the inequality:

$$C(\rho) \leqslant \max\{0,\lambda_1 - \lambda_3 - 2\sqrt{\lambda_2\lambda_4}\}; \tag{5}$$

the bound is saturable in the sense that there always exists a unitary transformation $\mathcal{U}(\chi) = \rho$ for which the equality holds true [42]. Now, from Eq. (5), we clearly have $C(\rho) \leqslant \lambda_1$. And, from the majorization relation of Eq. (4a), we find that $\lambda_1 \leqslant \epsilon_1 = (1+P)/2$. Therefore, for a general doubly stochastic process $\mathcal{E}$, we arrive at the inequality:

$$C(\rho) \leqslant \frac{1+P}{2}. \tag{6}$$

We stress that this bound is tight, in the sense that there always exists a pair of $\mathcal{N}$ and $\mathcal{U}$ for which the equality in the above equation holds true. In fact, the saturation of Eq. (6) is achieved when $\mathcal{N}$ consists of unitary-only process and when $\mathcal{U}$ is such that it yields the maximum concurrence for $\rho$ as allowed by Eq. (5). This can be verified, first, by noting that when $\mathcal{N}$ is unitary the process $\chi = \mathcal{N}(\sigma)$ preserves the eigenvalues to yield $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = ((1+P)/2,(1-P)/2,0,0)$, and second, by substituting these eigenvalues in Eq. (5) which then yields $(1+P)/2$ as the maximum achievable concurrence. Equation (6) is the central result of this article which clearly states that the intrinsic polarization correlations of the pump field in PDC predetermine the maximum entanglement that can be achieved by the generated two-qubit signal-idler states. We note that while Eq. (6) has been derived keeping in mind the physical context of parametric down-conversion, the derivation does not make any specific reference to the PDC process or to any explicit details of the two-qubit generation scheme. As a result, Eq. (6) is also applicable to processes other than PDC that would produce a two-qubit state from a single source qubit state via a doubly stochastic process.

### C. The restricted bound for "2D states"

We now recall that our present work is directly motivated by previous studies in the spatial degree of freedom for two-qubit states with only two nonzero diagonal entries in the computational basis [31]. Therefore, we next consider this special class of two-qubit states in the polarization degree of freedom. We refer to such states as "2D states" in this article and represent the corresponding density matrix as $\rho^{(2D)}$. Since such states can only have two nonzero eigenvalues, the majorization relations of Eq. (4) reduce to $\lambda_1 \leqslant \epsilon_1$ and $\lambda_1 + \lambda_2 = \epsilon_1 + \epsilon_2 = 1$. Owing to its $2 \times 2$ structure, the state $\rho^{(2D)}$ has a decomposition of the form [1],

$$\rho^{(2D)} = \tilde{P}|\psi^{(2D)}\rangle\langle\psi^{(2D)}| + (1-\tilde{P})\bar{\mathbb{1}}^{(2D)}, \tag{7}$$

where $|\psi^{(2D)}\rangle$ is a pure state and $\bar{\mathbb{1}}^{(2D)}$ is a normalized $2 \times 2$ identity matrix. As in Eq. (2), the pure state weightage $\tilde{P}$ can be shown to be related to the eigenvalues as $\tilde{P} = \lambda_1 - \lambda_2$. It is known that the concurrence is a convex function on the space of density matrices [22], that is, $C(\sum_i p_i\rho_i) \leqslant \sum_i p_i C(\rho_i)$, where $0 \leqslant p_i \leqslant 1$ and $\sum_i p_i = 1$. Applying this property to Eq. (7) along with the fact that $C(\bar{\mathbb{1}}^{(2D)}) = 0$, we obtain that the concurrence $C(\rho^{(2D)})$ of a 2D state satisfies $C(\rho^{(2D)}) \leqslant \tilde{P}$. Now since $\tilde{P} = \lambda_1 - \lambda_2 = 2\lambda_1 - 1$, and $\lambda_1 \leqslant \epsilon_1$, we get $\tilde{P} \leqslant 2\epsilon_1 - 1 = \epsilon_1 - \epsilon_2 = P$, or $\tilde{P} \leqslant P$. We therefore arrive at the inequality,

$$C(\rho^{(2D)}) \leqslant P. \tag{8}$$

Thus, for 2D states the upper bound on concurrence is the degree of polarization itself. This particular result is in exact analogy with the result shown previously for 2D states in the spatial degree of freedom that the maximum achievable concurrence is bounded by the degree of spatial correlations of the pump field itself [31].

Our entire analysis leading up to Eqs. (6) and (8) describes the transfer of one-particle correlations, as quantified by $P$, to two-particle correlations and their eventual manifestation as entanglement, as quantified by concurrence. For 2D states, which have a restricted Hilbert space available to them, the maximum concurrence that can get manifested is $P$. Thus, restricting the Hilbert space appears to restrict the degree to which pump correlations can manifest as the entanglement of the generated two-qubit state. However, when there are no restrictions on the available Hilbert space, the maximum concurrence that can get manifested is $(1+P)/2$. This leads to the somewhat nonintuitive consequence that even an unpolarized pump field ($P = 0$) can produce two-qubit states with nonzero concurrence [up to $C(\rho) = 0.5$]. This is attributed to the fact that the one-particle correlations of the pump field are allowed to manifest in the full unrestricted Hilbert space of the two-particle state of the signal-idler photons. We note that our general result as derived in Eq. (6) remains applicable even in situations where the entanglement in a generated two-qubit state is transferred to another two-qubit state [43–46] or where a two-qubit state is made to go through a turbulent atmosphere [47]. As long as the trace-preserving and entropy nondecreasing conditions are satisfied the upper bound on entanglement in such transfers still remain dictated by Eq. (6).

### III. AN ILLUSTRATIVE EXPERIMENTAL SCHEME

We now illustrate the bounds derived in this article in an example experimental scheme. The scheme shown in Fig. 2(a) can produce a wide range of two-qubit states in a doubly stochastic manner. A pump field with the degree of polarization $P$ is split into two arms by a nonpolarizing beam splitter (BS) with splitting ratio $t:1-t$. We represent the horizontal and vertical polarization components of the field hitting the PDC crystals in arm (1) as $E_{H1}$ and $E_{V1}$, respectively. The phase retarder (PR1) introduces a phase difference $\alpha_1$ between $E_{H1}$ and $E_{V1}$. The rotation plate (RP1) rotates the polarization vector by angle $\theta_1$. The corresponding quantities in arm (2) have similar representations. The stochastic variable $\gamma$ introduces a decoherence between the pump fields in the two arms. Its action is described as $\langle e^{i\gamma}\rangle = \mu\, e^{i\gamma_0}$, where $\langle\cdots\rangle$ represents the ensemble average, $\mu$ is the degree of coherence, and $\gamma_0$ is the mean value of $\gamma$ [1]. The entangled photons in each arm are produced using type-I PDC in a two-crystal geometry [26]. The purpose of the half-wave plate (HP) is to convert the two-photon state vectors $|H\rangle_s|H\rangle_i$ and $|V\rangle_s|V\rangle_i$, into $|V\rangle_s|H\rangle_i$ and $|H\rangle_s|V\rangle_i$, respectively. Therefore, a typical realization $|\psi_\gamma\rangle$ of the two-qubit state in the ensemble detected at $D_s$ and $D_i$ can be represented as $|\psi_\gamma\rangle = E_{V1}|H\rangle_s|H\rangle_i + E_{H1}|V\rangle_s|V\rangle_i + e^{i\gamma}(E_{V2}|H\rangle_s|V\rangle_i + E_{H2}|V\rangle_s|H\rangle_i)$. The two-qubit density matrix is then $\rho = \langle|\psi_\gamma\rangle\langle\psi_\gamma|\rangle =$

$$\begin{bmatrix} \langle E_{V1}E_{V1}^*\rangle & \langle E_{V1}E_{V2}^*e^{-i\gamma}\rangle & \langle E_{V1}E_{H2}^*e^{-i\gamma}\rangle & \langle E_{V1}E_{H1}^*\rangle \\ \langle E_{V2}E_{V1}^*e^{i\gamma}\rangle & \langle E_{V2}E_{V2}^*\rangle & \langle E_{V2}E_{H2}^*\rangle & \langle E_{V2}E_{H1}^*e^{i\gamma}\rangle \\ \langle E_{H2}E_{V1}^*e^{i\gamma}\rangle & \langle E_{H2}E_{V2}^*\rangle & \langle E_{H2}E_{H2}^*\rangle & \langle E_{H2}E_{H1}^*e^{i\gamma}\rangle \\ \langle E_{H1}E_{V1}^*\rangle & \langle E_{H1}E_{V2}^*e^{-i\gamma}\rangle & \langle E_{H1}E_{H2}^*e^{-i\gamma}\rangle & \langle E_{H1}E_{H1}^*\rangle \end{bmatrix}.$$

For calculating the matrix elements of $\rho$, we represent the polarization vector of the pump field before the BS as $(E_H,E_V)^T$ and thus write $E_{H1}$ and $E_{V1}$ as

$$\begin{bmatrix} E_{H1} \\ E_{V1} \end{bmatrix} = \eta_1 \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha_1} \end{bmatrix} \begin{bmatrix} E_H \\ E_V \end{bmatrix}, \quad (9)$$

where $\eta_1 = \sqrt{t}$, and the two matrices represent the transformations by PR1 and RP1. $E_{H2}$ and $E_{V2}$ are calculated in a similar manner, with the corresponding quantity $\eta_2 = \sqrt{1-t}\ e^{i\gamma}$. Without the loss of generality, we assume $\langle E_H^*E_H\rangle = \langle E_V^*E_V\rangle = 1/2$ and $\langle E_H^*E_V\rangle = P/2$, and calculate the matrix elements to be

$$\langle E_{V1(2)}E_{V1(2)}^*\rangle = |\eta_{1(2)}|^2(1 - P\cos\alpha_{1(2)}\sin 2\theta_{1(2)})/2,$$

$$\langle E_{H1(2)}E_{H1(2)}^*\rangle = |\eta_{1(2)}|^2(1 + P\cos\alpha_{1(2)}\sin 2\theta_{1(2)})/2,$$

$$\langle E_{V1(2)}E_{H1(2)}^*\rangle = |\eta_{1(2)}|^2\, P(\cos\alpha_{1(2)}\cos 2\theta_{1(2)} + i\sin\alpha_{1(2)})/2,$$

$$\begin{aligned}\langle E_{V1}E_{V2}^*e^{-i\gamma}\rangle = {}& \mu|\eta_1\eta_2|(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 e^{i(\alpha_1-\alpha_2)} \\ & -P\cos\theta_1\sin\theta_2 e^{i\alpha_1} \\ & -P\sin\theta_1\cos\theta_2 e^{-i\alpha_2})e^{-i\gamma_0}/2,\end{aligned}$$

$$\begin{aligned}\langle E_{V1}E_{H2}^*e^{-i\gamma}\rangle = {}& \mu|\eta_1\eta_2|(-\sin\theta_1\cos\theta_2 \\ & +\cos\theta_1\sin\theta_2 e^{i(\alpha_1-\alpha_2)} + P\cos\theta_1\cos\theta_2 e^{i\alpha_1} \\ & -P\sin\theta_1\sin\theta_2 e^{-i\alpha_2})e^{-i\gamma_0}/2,\end{aligned}$$

$$\begin{aligned}\langle E_{V2}E_{H1}^*e^{i\gamma}\rangle = {}& \mu|\eta_1\eta_2|(-\cos\theta_1\sin\theta_2 \\ & +\sin\theta_1\cos\theta_2 e^{-i(\alpha_1-\alpha_2)} - P\sin\theta_1\sin\theta_2 e^{-i\alpha_1} \\ & +P\cos\theta_1\cos\theta_2 e^{i\alpha_2})e^{i\gamma_0}/2,\end{aligned}$$

$$\begin{aligned}\langle E_{H2}E_{H1}^*e^{i\gamma}\rangle = {}& \mu|\eta_1\eta_2|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 e^{-i(\alpha_1-\alpha_2)} \\ & +P\sin\theta_1\cos\theta_2 e^{-i\alpha_1} \\ & +P\cos\theta_1\sin\theta_2 e^{i\alpha_2})e^{i\gamma_0}/2.\end{aligned}$$

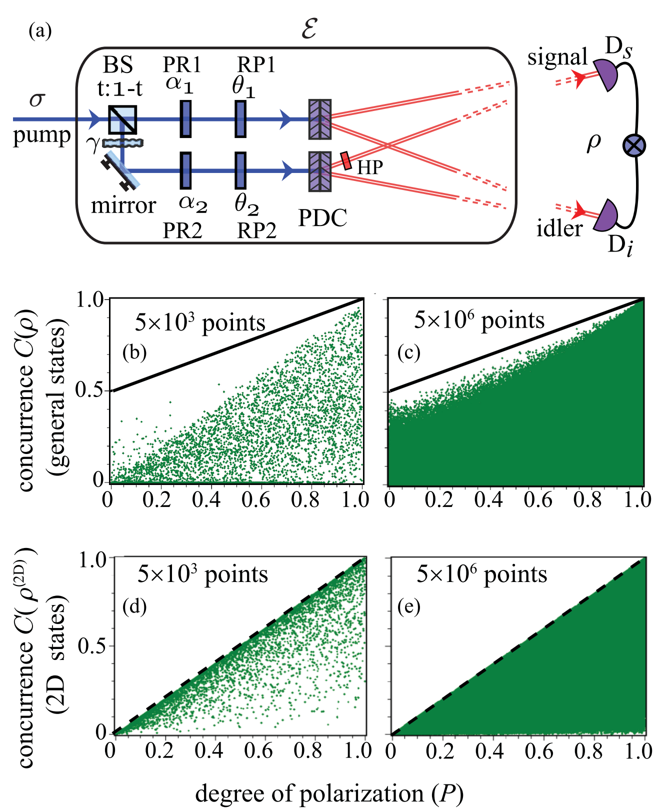


FIG. 2. (a) An example of an experimental scheme for producing a wide range of two-qubit states. BS, beam splitter; PR, phase retarder; RP, rotation plate; HP, half-wave plate. $D_s$ and $D_i$ are photon detectors in a coincidence-counting setup. (b) and (c) are the scatter plots of concurrences of states numerically generated by randomly varying all the tunable parameters. (d) and (e) are the scatter plots of concurrence of 2D states, numerically generated by keeping $t = 1$ and varying all the remaining tunable parameters.

Here, $t, \alpha_1, \alpha_2, \theta_1, \theta_2, \mu$, and $\gamma_0$ are the tunable parameters. We numerically vary these parameters with a uniform random sampling and simulate a large number of two-qubit states. Figures 2(b) and 2(c) are the scatter plots of concurrences of $5 \times 10^3$ and $5 \times 10^6$ two-qubit states, respectively, numerically generated by varying all the tunable parameters. Figures 2(d) and 2(e) are the scatter plots of concurrence of $5 \times 10^3$ and $5 \times 10^6$ 2D states, respectively, numerically generated by keeping $t = 1$ and varying all the remaining tunable parameters. The solid black lines are the general upper bound $C(\rho) = (1 + P)/2$ and the dashed black lines are the upper bound $C(\rho) = P$ for 2D states. The unfilled gaps in the scatter plots can be filled in either by sampling more data points or by adopting a different sampling strategy. To this end, we note that one possible setting for which the general upper bound $C(\rho) = (1 + P)/2$ can be achieved for any value of $P$ is $t = 0.5$, $\theta_1 = -\pi/4$, $\theta_2 = 0$, $\alpha_1 = \pi/2$, $\alpha_2 = \pi$, $\mu = 1$, and $\gamma_0 = 0$. Thus, even an unpolarized pump field ($P = 0$) can be made to produce two-qubit signal-idler states with nonzero entanglement (up to $C(\rho) = 0.5$) by a suitable choice of the tunable parameters. This is due to the fact that the setup is capable of producing a wide variety of two-qubit states, which in general, reside in the full unrestricted four-dimensional Hilbert space.

## IV. CONCLUSIONS AND DISCUSSIONS

In conclusion, we have investigated how one-particle correlations transfer to manifest as two-particle correlations in the physical context of PDC. We have shown that if the generation process is trace preserving and entropy nondecreasing, the concurrence $C(\rho)$ of the generated two-qubit state $\rho$ follows an intrinsic upper bound with $C(\rho) \leqslant (1 + P)/2$, where $P$ is the degree of polarization of the pump photon. For the special class of two-qubit states $\rho^{(2D)}$ that is restricted to have only two nonzero diagonal elements, the upper bound on concurrence is the degree of polarization itself, that is, $C(\rho^{(2D)}) \leqslant P$. The surplus of $(1 + P)/2 - P = (1 - P)/2$ in the maximum achievable concurrence for arbitrary two-qubit states can be attributed to the availability of the entire $4 \times 4$ computational space, as opposed to 2D states which only have a $2 \times 2$ computational block available to them.

We note that the polarization correlations of a pump field do not impose serious limitations on the degree of entanglement of the signal and idler photons, insofar as its practical achievability in realistic experiments is concerned, since most available laboratory sources can be made to have nearly perfect degree of polarization. The main motivation behind this study is from the fundamental perspective of understanding how one-particle correlations transfer to manifest as two-particle correlations. The results derived in this paper can have two important implications. The first one is towards exploring whether or not correlations, too, follow a quantifiable conservation principle just as physical observables such as energy and momentum do. The second one could be towards deducing the upper bound on the correlations in a generated high-dimensional quantum system, purely from the knowledge of the correlations in the source. In light of the recent experiment on generation of three-photon entangled states from a single source photon [48], this formalism may prove useful in determining upper bounds on the entanglement of such multipartite systems, for which no well-accepted measure exists. In this context, we believe that this approach based on intrinsic source correlations could also complement the existing information-theoretic approaches [16–24] towards quantifying entanglement.

## ACKNOWLEDGMENTS

G.K. acknowledges helpful discussions with Martin. A.K.J. acknowledges financial support through an initiation grant from Indian Institute of Technology (IIT) Kanpur, India.

[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge university Press, New York, 1995).
[2] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[3] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[4] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[5] E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935).
[6] D. C. Burnham and D. L. Weinberg, Phys. Rev. Lett. **25**, 84 (1970).
[7] R. W. Boyd, *Nonlinear Optics*, 2nd ed. (Academic Press, New York, 2003).
[8] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, Nature (London) **412**, 313 (2001).
[9] W. Vogel and J. Sperling, Phys. Rev. A **89**, 052302 (2014).
[10] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[11] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[12] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[13] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Phys. Rev. A **92**, 022112 (2015).
[14] E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge University Press, New York, 2007).
[15] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, Phys. Rev. Lett. **75**, 4337 (1995).
[16] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[17] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[18] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[19] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
[20] W. K. Wootters, Quantum Inf. Comput. **1**, 27 (2001).
[21] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[22] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[23] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[24] J. Svozilík, A. Vallés, J. Peřina, and J. P. Torres, Phys. Rev. Lett. **115**, 220501 (2015).

[25] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, 1999).

[26] P. G. Kwiat, E. Waks, A. G. White, I. Appelbaum, and P. H. Eberhard, Phys. Rev. A **60**, R773 (1999).

[27] C. E. Kuklewicz, M. Fiorentino, G. Messin, F. N. C. Wong, and J. H. Shapiro, Phys. Rev. A **69**, 013807 (2004).

[28] T. Jennewein, C. Simon, G. Weihs, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **84**, 4729 (2000).

[29] R. Ursin, F. Tiefenbacher, T. Schmitt-Manderbach, H. Weier, T. Scheidl, M. Lindenthal, B. Blauensteiner, T. Jennewein, J. Perdigues, P. Trojek *et al.*, Nat. Phys. **3**, 481 (2007).

[30] A. K. Jha, M. N. O'Sullivan, K. W. C. Chan, and R. W. Boyd, Phys. Rev. A **77**, 021801(R) (2008).

[31] A. K. Jha and R. W. Boyd, Phys. Rev. A **81**, 013828 (2010).

[32] A. K. Jha, J. Leach, B. Jack, S. Franke-Arnold, S. M. Barnett, R. W. Boyd, and M. J. Padgett, Phys. Rev. Lett. **104**, 010501 (2010).

[33] C. H. Monken, P. H. Souto Ribeiro, and S. Pádua, Phys. Rev. A **57**, 3123 (1998).

[34] M. A. Olvera and S. Franke-Arnold, arXiv:1507.08623.

[35] R. Bhatia, *Matrix Analysis* (Springer Science & Business Media, New York, 2013).

[36] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, New York, 2010).

[37] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, Phys. Rev. **121**, 920 (1961).

[38] T. F. Jordan and E. Sudarshan, J. Math. Phys. (Cambridge, MA) **2**, 772 (1961).

[39] K. Kraus, Ann. Phys. (NY) **64**, 311 (1971).

[40] M. A. Nielsen, Lecture Notes, Department of Physics, University of Queensland, Australia (2002).

[41] S. Ishizaka and T. Hiroshima, Phys. Rev. A **62**, 022310 (2000).

[42] F. Verstraete, K. Audenaert, and B. De Moor, Phys. Rev. A **64**, 012316 (2001).

[43] S. Ramelow, A. Fedrizzi, A. Poppe, N. K. Langford, and A. Zeilinger, Phys. Rev. A **85**, 013845 (2012).

[44] M. Titov, B. Trauzettel, B. Michaelis, and C. W. J. Beenakker, New J. Phys. **7**, 186 (2005).

[45] V. Cerletti, O. Gywat, and D. Loss, Phys. Rev. B **72**, 115316 (2005).

[46] E. Altewischer, M. Van Exter, and J. Woerdman, Nature (London) **418**, 304 (2002).

[47] A. A. Semenov and W. Vogel, Phys. Rev. A **81**, 023835 (2010).

[48] D. R. Hamel, L. K. Shalm, H. Hübel, A. J. Miller, F. Marsili, V. B. Verma, R. P. Mirin, S. W. Nam, K. J. Resch, and T. Jennewein, Nat. Photon. **8**, 801 (2014).