Theory of TE-polarized waves in a lossless cubic-quintic nonlinear planar waveguide

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TE-polarized electromagnetic waves, guided by a three-layer slab structure consisting of a central film with quartic permittivity placed between two half spaces with Kerr permittivity, are studied. Traveling-wave solutions of Maxwell's equations are expressed in terms of Weierstrass's elliptic function $\wp$. A general dispersion relation is derived and evaluated by using a phase diagram analysis. Emphasis is placed on the conditions of existence and solvability of the dispersion relation. Numerical results are presented.

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I. INTRODUCTION

Several articles [1–9] have been published on the propagation of monochromatic TE waves in a slab waveguide structure consisting of a central film f placed between two half spaces (substrate s and cladding c) as shown in Fig. 1. Corresponding review articles [10–12] of this field summarize the results and present specific references. A major part of the papers is devoted to layers with nonlinear permittivities of quadratic (Kerr-like) order. Furthermore, only few [6,13–15] consider structures with all layers exhibiting nonlinear permittivities leading to solutions in terms of elliptic functions (instead of hyperbolic functions if, e.g., the film permittivity is linear).

Within the theory of spatial solutions higher-order nonlinearities have been investigated [16–18], leading (in particular) to a nonlinear Schrödinger equation with a cubic-quintic nonlinearity [19]. Recently [20], analytical solutions of the cubic-quintic-septimal Schrödinger equation together with conditions for stable propagation of one-dimensional bright spatial solitons were reported.

In the following, we consider a slab structure with a quartic nonlinear permittivity of the film and a Kerr-nonlinear permittivity of substrate and cladding. The analysis used is different from the analysis in the aforementioned literature. It is based on a combination of two approaches: Weierstrass's famous general solution of the ordinary differential equation $(\frac{dJ}{dx})^2 = R_4(J), R_4(J)$ being a fourth-degree polynomial and the use of phase diagrams $R(J)$ to represent solutions of certain nonlinear partial differential equations (see [21,22]).

The paper is organized as follows. Section II states the problem. The solution is presented in Sec. III. In Sec. IV the solution is specified by physical (real, non-negative, and bounded field intensities) conditions using a phase diagram analysis. Evaluation of the dispersion relation is presented in Sec. V. Results are elucidated by numerical examples in Sec. VI. A summary with comments concludes the paper in Sec. VII.

II. STATEMENT OF THE PROBLEM

A planar waveguide structure, shown in Fig. 1, with lossless, isotropic, homogeneous, and nonmagnetic material and a (local) permittivity according to

$$\epsilon = \begin{cases} \bar{\epsilon}_s = \epsilon_s + a_s|\mathcal{E}_y|^2, & x < 0 \\ \bar{\epsilon}_f = \epsilon_f + a_f|\mathcal{E}_y|^2 + b_f|\mathcal{E}_y|^4, & 0 \leqslant x \leqslant h \\ \bar{\epsilon}_c = \epsilon_c + a_c|\mathcal{E}_y|^2, & x > h \end{cases} \tag{1}$$

is considered. A stationary tentative solution for TE-polarized waves $\mathcal{E}_y(x,z,t)$,

$$\mathcal{E}_y(x,z,t) = E_y(x,\gamma^2)e^{i(\gamma z - \omega t)}, \tag{2}$$

must satisfy Maxwell's equations. The amplitude $E_y(x,\gamma^2)$ and the propagation constant γ are assumed to be real. Here ω denotes the circular frequency of the wave [for a discussion of modeling the problem by (1) and (2) see [23], Sec. 1]. Inserting (2) into Maxwell's equations, one obtains

$$E_y'' = \begin{cases} (\gamma^2 - \omega^2\mu_0\bar{\epsilon}_s)E_y, & x < 0 \\ (\gamma^2 - \omega^2\mu_0\bar{\epsilon}_f)E_y, & 0 \leqslant x \leqslant h \\ (\gamma^2 - \omega^2\mu_0\bar{\epsilon}_c)E_y, & x > h. \end{cases} \tag{3}$$

Defining $k_0^2 = \omega^2\mu_0\epsilon_0$, rescaling x and γ by k_0 and ϵ by ϵ_0, and replacing x and γ in $E_y(x,\gamma^2)$ by rescaled arguments, Helmholtz equations (3) can be written as

$$E'' = \begin{cases} (\gamma^2 - \bar{\epsilon}_s)E, & x < 0 \\ (\gamma^2 - \bar{\epsilon}_f)E, & 0 \leqslant x \leqslant h \\ (\gamma^2 - \bar{\epsilon}_c)E, & x > h, \end{cases} \tag{4}$$

where E denotes E_y (with rescaled arguments) and the $\bar{\epsilon}_\nu$ $(\nu = s,f,c)$ are dimensionless.

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Using (1), multiplying (4) by E', and integrating (with respect to E), one obtains

$$[J'(x)]^2 = \begin{cases} -2a_s J^3(x) + 4(\gamma^2 - \epsilon_s)J^2(x) + 4C_s J(x) := R_s(J), & x < 0 \\ -\frac{4b_f}{3}J^4(x) - 2a_f J^3(x) + 4(\gamma^2 - \epsilon_f)J^2(x) + 4C_f J(x) := R_f(J), & 0 \leqslant x \leqslant h \\ -2a_c J^3(x) + 4(\gamma^2 - \epsilon_c)J^2(x) + 4C_c J(x) := R_c(J), & x > h, \end{cases} \quad (5)$$

where the intensity $J(x) = E^2(x)$ was introduced. Here the C_ν ($\nu = s,f,c$) denote constants of integration, to be determined below.

The problem is to find physical (real, non-negative, and bounded) solutions $J(x)$ to Eqs. (5) that satisfy the boundary conditions at the interfaces $x = 0$ and $x = h$ together with the conditions at infinity

$$J(x) \to 0, \quad \frac{dJ(x)}{dx} \to 0, \quad |x| \to \infty.$$

Obviously these conditions imply that $C_s = C_c = 0$ in Eqs. (5).

III. SOLUTION

Disregarding the solution $J(x) = \text{const}$ [24] in Eqs. (5), the general solution for the substrate, film, and cladding can be presented by using a formula due to Weierstrass [21]:

$$J_{\nu\pm}(x) = J_{0,\nu} + \frac{\frac{1}{2}R'_\nu(J_{0,\nu})[\wp(x;g_{2,\nu},g_{3,\nu}) - \frac{1}{24}R''_\nu(J_{0,\nu})] \pm \wp'(x;g_{2,\nu},g_{3,\nu})\sqrt{R_\nu(J_{0,\nu})} + \frac{1}{24}R_\nu(J_{0,\nu})R'''_\nu(J_{0,\nu})}{2[\wp(x;g_{2,\nu},g_{3,\nu}) - \frac{1}{24}R''_\nu(J_{0,\nu})]^2 - \frac{1}{48}R_\nu(J_{0,\nu})R''''_\nu(J_{0,\nu})}, \quad (6)$$

where $\nu = s,f,c$ and

$$\begin{aligned} g_{2,\nu} &= \tfrac{4}{3}(\gamma^2 - \epsilon_\nu)^2, \quad \nu = s,c \\ g_{3,\nu} &= -\tfrac{8}{27}(\gamma^2 - \epsilon_\nu)^3, \quad \nu = s,c \\ g_{2,f} &= 2a_f C_f + \tfrac{4}{3}(\gamma^2 - \epsilon_f)^2, \\ g_{3,f} &= \tfrac{2}{3}a_f C_f(\epsilon_f - \gamma^2) + \tfrac{4}{3}b_f C_f^2 - \tfrac{8}{27}(\gamma^2 - \epsilon_f)^3 \end{aligned} \quad (7)$$

are the invariants of Weierstrass's function $\wp(x;g_{2,\nu},g_{3,\nu})$. The prime denotes differentiation with respect to x for $\wp(x;g_{2,\nu},g_{3,\nu})$ and differentiations with respect to J for $R_\nu(J)$. The $J_{0,\nu}$ are real constants to be chosen appropriately so that physical solutions are possible (see below).

Evaluating Eq. (6) for $\nu = s,c$ and taking the limit $x \to \pm\infty$, one obtains

$$J_{s,c\pm} \to 0, \quad J'_{s,c\pm} \to 0,$$

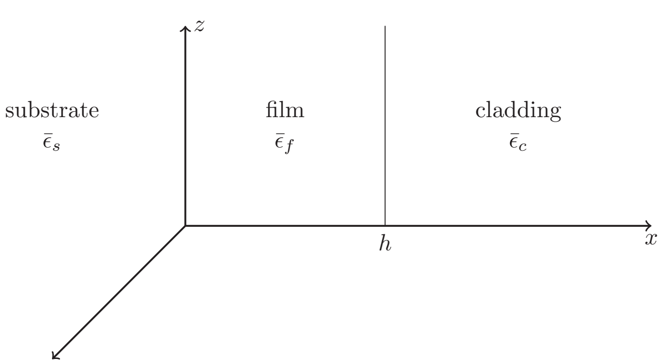

FIG. 1. Geometry of the problem.

so the conditions at infinity are satisfied. If $x = 0$, Eq. (6) yields

$$J_{s\pm}(0) = J_{f\pm}(0) = J_{0,s} = J_{0,f} := J_0. \quad (8)$$

If $x = h$, Eq. (6) implies

$$J_{c\pm}(h) = J_{f\pm}(h) = J_{0,c} := J^{(h)}. \quad (9)$$

As is well known from the cubic case ($b_f = 0$) [8], the solution of the problem requires investigation of the dispersion relation (DR) that relates the parameters of the problem (material parameters ϵ_ν,a_ν,b_f and free parameters γ^2,J_0,h). As in the cubic case, the boundary conditions must be evaluated to obtain the DR.

The electric field $\mathcal{E}_y$ according to Eq. (2) and its derivative with respect to x and z must be continuous at the interfaces $x = 0$ and $x = h$. Continuity with respect to z is obvious. Continuity of $E(x,\gamma^2)$ and $E'(x,\gamma^2)$ implies continuity of $J(x,\gamma^2)$ and $J'(x,\gamma^2)$, respectively. Hence, in Eqs. (5),

$$R_s(J_0) = R_f(J_0), \quad R_f(J^{(h)}) = R_c(J^{(h)}) \quad (10)$$

lead to

$$C_f = \frac{J_0^3}{3}b_f + \frac{J_0^2}{2}(a_f - a_s) + J_0(\epsilon_f - \epsilon_s) \quad (11)$$

and to

$$\frac{(J^{(h)})^3}{3}b_f + \frac{(J^{(h)})^2}{2}(a_f - a_c) + J^{(h)}(\epsilon_f - \epsilon_c) = C_f, \quad (12)$$

respectively. Equation (11) determines the integration constant C_f (if $J_0,b_f,\epsilon_f,a_f,\epsilon_s,a_s$ are prescribed) and Eq. (12) is a relation between the constants $J^{(h)}$ and J_0. According to (6), J_0, $J^{(h)}$, and $J_{\nu\pm}(x)$ must be physical (real, non-negative, and bounded). This condition conveniently can be represented by the graphs of $R_\nu(J)$, usually referred to as phase diagrams (PDs).

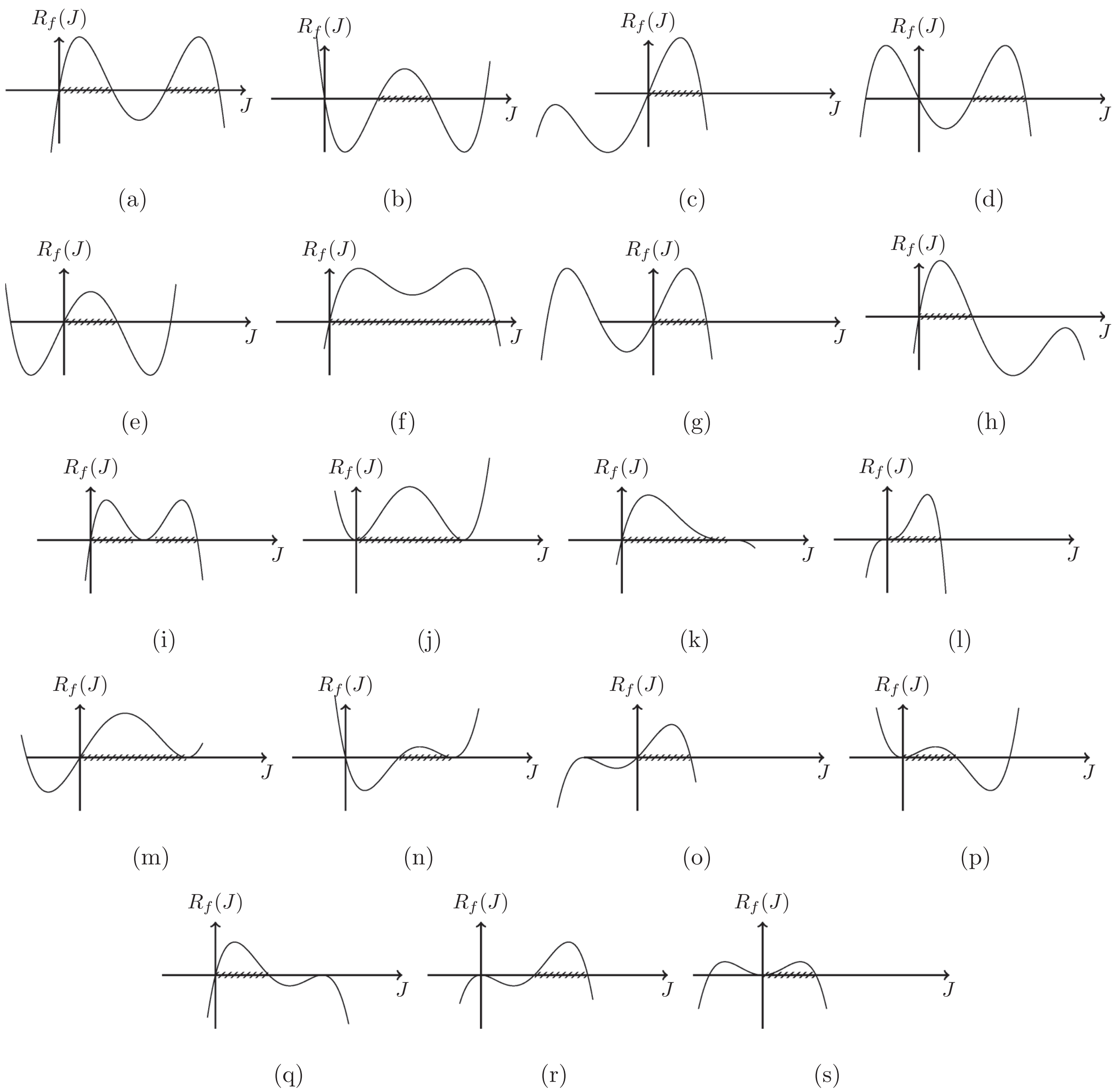

FIG. 2. Phase diagrams for physical $J_f(x)$.

IV. PHASE DIAGRAM CONDITIONS FOR PHYSICAL SOLUTIONS

As is well known, a phase diagram analysis is useful for investigation of solutions of the nonlinear Schrödinger equation [22,25]. Since $R_\nu(J) \geqslant 0$ must hold, with J varying monotonically until $J' = 0$, it is obvious that the roots of $R_\nu(J) = 0$ are essential for the behavior of $J(x)$. A little thought shows that the roots of $R_\nu(J) = 0$ must fall into one of the 19 categories of phase diagrams for $R_f(J)$ depicted in Fig. 2. The PDs for $R_s(J)$ and $R_c(J)$ are shown in Fig. 3.

Physical solutions occur only if $J_{\nu\pm}(x)$ lies in the intervals $[J_1, J_2]$ hatched in Figs. 2 and 3. This condition is referred to as the phase diagram condition (PDC); J_1, J_2 are referred to as PDC roots in the following: Eqs. (5) must be solved subject to the PDC.

Weierstrass's elliptic function $\wp(x; g_2, g_3)$ and its derivative $\wp'(x; g_2, g_3)$ are holomorphic with respect to $x \neq 0$ (mod 2ω, 2ω being the real period of $\wp$) and with respect to g_2 and g_3 (see [26], p. 635, formulas 18.5.1–18.5.4). The dependence of g_2 and g_3 with respect to the parameters $\gamma^2, \epsilon_\nu, a_\nu, b_f, J_0, \nu = s, f, c$ [according to (7) and (11)] is also holomorphic. Since only physical solutions of (5) are considered, then $J_{\nu\pm}$ from

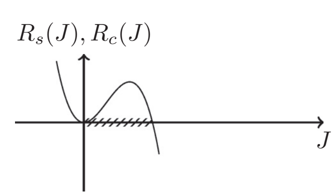

FIG. 3. Phase diagram for physical $J_s(x), J_c(x)$.

Eq. (6) is a continuous function with respect to x [including 0 (mod 2ω)] and with respect to all parameters $\gamma^2,\epsilon_\nu,a_\nu,b_f,J_0$ ($\nu = s,f,c$) as a superposition of the continuous functions [see Eqs. (5), (7), and (11)].

V. SOLUTIONS OF THE DISPERSION RELATION

As is well known from the cubic case (see [7–11]) the DR is a relation between all parameters of the problem. Neither γ nor h is contained in Eq. (12), but if $J^{(h)}$ is replaced by $J_{f\pm}$ according to Eq. (6),

$$J^{(h)}(J_0,b_f,\epsilon_\nu,a_\nu) = J_{f\pm}(\gamma^2,J_0,b_f,\epsilon_\nu,a_\nu,h), \quad \nu = s,f,c \tag{13}$$

then one obtains the equation

$$\frac{[J_{f\pm}(\gamma^2,J_0,b_f,\epsilon_\nu,a_\nu,h)]^3}{3}b_f + \frac{[J_{f\pm}(\gamma^2,J_0,b_f,\epsilon_\nu,a_\nu,h)]^2}{2}(a_f - a_c) + J_{f\pm}(\gamma^2,J_0,b_f,\epsilon_\nu,a_\nu,h)(\epsilon_f - \epsilon_c) = C_f \tag{14}$$

that satisfies the requirements for a dispersion relation if (13) can be fulfilled subject to the PDC. The first condition for Eq. (13) to hold is the existence of a positive root $J^{(h)}$ of Eq. (12). Assuming lossless media, the material parameters in Eq. (12) are real, so Eq. (12) has at least one real root $J^{(h)}$; if there is at least one change of sign in the sequence of coefficients, at least one positive root $J^{(h)}$ exists. The second condition for the validity of Eq. (13) is that both $J^{(h)}$ [as a root of Eq. (12)] and $J_{f\pm}$ [given by Eq. (6)] satisfy the PDC. Due to the continuity of $J_{f\pm}(\gamma^2,J_0,b_f,\epsilon_\nu,a_\nu,h)$, Eq. (14) is continuous with respect to all parameters of the problem.

We now examine (13) (omitting $\pm$ for simplicity) for the PDs in Figs. 2 and 3. There are three categories of PDs. First we assume that the discriminant Δ of $R_f(J)$ [$\Delta = 256(g_{2,f}^3 - 27g_{3,f}^2)$] does not vanish [Figs. 2(a), 2(b), 2(d), and 2(e)], so there are at least two simple PDC roots J_1,J_2 of $R_f = 0$ [see, for example, Fig. 2(d)]. Near J_1 we have

$$\left(\frac{dJ_f}{dx}\right)^2 = [J_f(x) - J_1]R_f'(J_1) + O([J_f(x) - J_1]^2) \tag{15}$$

as $J_f(x) \to J_1$. Hence

$$J_f(x) = J_1 + \left(\frac{x - x_1}{2}\right)^2 R_f'(J_1) + O((x - x_1)^3) \tag{16}$$

as $x \to x_1$, with $J_f(x_1) = J_1$. The same equation holds near x_2: $J_f(x_2) = J_2$. Since $R_f'(J_1) > 0$ and $R_f'(J_2) < 0$, $J_f(x)$ has a (local) minimum at x_1 and a (local) maximum at x_2, so $\frac{dJ_f}{dx}$ changes the sign at x_1 and x_2. Hence $J_f(x) \in [J_1,J_2]$ for all x. If at any point x_0 (without loss of generality we may assume that $x_0 = 0$) $J_f(0) = J_0 \in [J_1,J_2]$ and the sign of $\frac{dJ_f}{dx}$ is prescribed, the solution $J_f(x)$ is uniquely determined. Returning to (13), since J_f is continuous, it takes all values in $[J_1,J_2]$. If (due to PDs in Fig. 3)

$$a_s,a_c > 0, \quad \gamma^2 > \max\{\epsilon_s,\epsilon_c\}, \quad J_0 \in \left(0,\frac{2(\gamma^2 - \epsilon_s)}{a_s}\right] \tag{17}$$

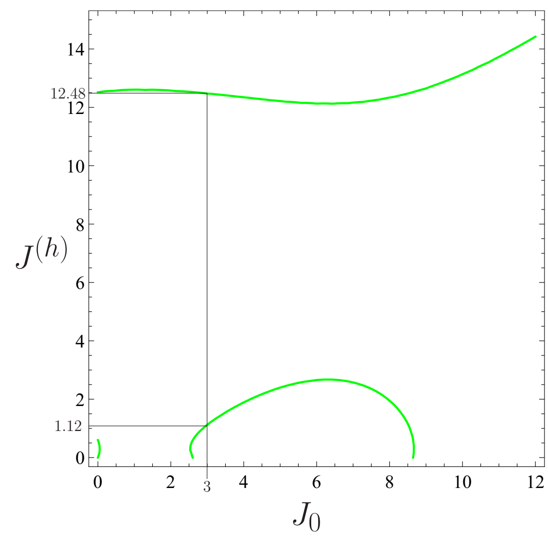

FIG. 4. Relation of $J^{(h)}$ on J_0 (the parameters are given in the text).

and if

$$J^{(h)} \in [J_1,J_2] \cap \left[0,\frac{2(\gamma^2 - \epsilon_c)}{a_c}\right] \neq \emptyset \tag{18}$$

holds for any of the PDs of Fig. 2, then (13) can be satisfied and thus a solution of the DR exists.

Summing up, first, for the case $\Delta \neq 0$, sufficient conditions for the existence and solvability (ESC) of solution tuples of the DR (14) are the existence of at least one positive root of Eq. (12), $J^{(h)} \in [J_1,J_2] \cap [0,\frac{2(\gamma^2-\epsilon_c)}{a_c}] \neq \emptyset$, and the conditions (17).

The DR can be evaluated with J_{f+} or J_{f-} leading to different solutions of the DR. According to the foregoing analysis, both $J_{f+}(x)$ and $J_{f-}(x)$ oscillate between J_1 and J_2 with the same period

$$\omega_f = 2\int_{J_1}^{J_2} \frac{dJ}{\sqrt{R_f(J)}},$$

having the same shape and different only by the sign of $\frac{dJ_f}{dx}|_{x=0}$.

If, second, $\Delta = 0$, $g_{2,f} > 0$, and $g_{3,f} < 0$, at least one double PDC root of $R_f(J) = 0$ exists. Considering, e.g., Fig. 2(j) (which describes a kink solitary wave in bulk media [25]) the behavior of $J_f(x)$ near any double root $\tilde{J}$ is given by

$$\left(\frac{dJ_f}{dx}\right)^2 = [J_f(x) - \tilde{J}]^2 R_f''(0) + O([J_f(x) - \tilde{J}]^3). \tag{19}$$

Hence $R_f''(0) > 0$ is necessary. For bounded $J_f(x)$ one obtains from (19)

$$J_f(x) \approx \tilde{J} + \text{const}e^{\pm\sqrt{R_f''(0)}x}$$

as $x \to \mp\infty$. Thus, $J_f(x)$ asymptotically reaches a minimum $J_1 = 0$ [Fig. 2(j)] for $x \to -\infty$ and a maximum J_2 for $x \to +\infty$.

If, third, $\Delta = 0$, $g_{2,f} = 0$, and $g_{3,f} = 0$, the polynomial $R_f(J)$ has a triple root J_1 [see Figs. 2(k) and 2(l)]. In this case

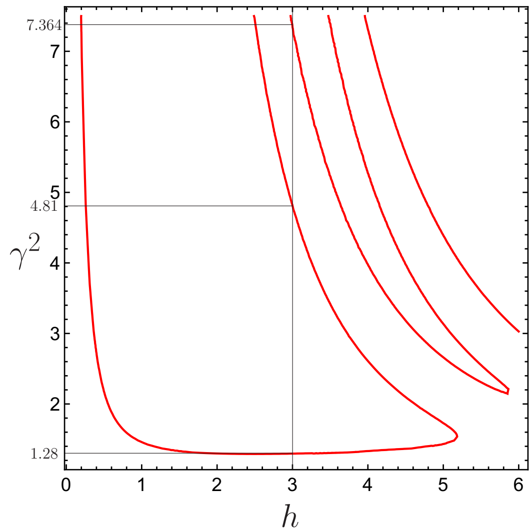

FIG. 5. Branches of the DR (14) and solutions $h = 3$ and $\gamma_1^2 = 1.289$, $\gamma_2^2 = 4.817$, and $\gamma_3^2 = 7.364$ for the same parameters as in Fig. 4.

one obtains

$$\left(\frac{dJ_f}{dx}\right)^2 = (J_f(x) - J_1)^3 R_f'''(J_1) + O((J_f(x) - J_1)^4). \quad (20)$$

Due to the PD [Figs. 2(k) and 2(l)] $b_f > 0$ holds; due to the PDC $a_f < 0$ must hold. Since $R_f'''(J_1) = -4(8b_f J_1 + 3a_f)$, Eq. (19) implies $J_f(x) > 0$ if $J_1 = 0$ and $J_f(x) < J_1$ ($J_1 = -\frac{3a_f}{2b_f}$). Thus $J_f(x)$ takes values according to the PDC from $(0, J_1]$ in Fig. 2(k) and from $(0, J_2]$ in Fig. 2(l).

Summing up, in all cases of PDs according to Fig. 2 the continuous function $J_f(x)$ takes values between two successive PDC roots J_1, J_2 of $R_f(J) = 0$, so the ESCs are valid for each of the three categories. Thus, if the ESCs are satisfied Eq. (13) holds, so tuples $\{\gamma^2, h, J_0, b_f, \epsilon_\nu, a_\nu\}$ exist that are consistent with the DR (14).

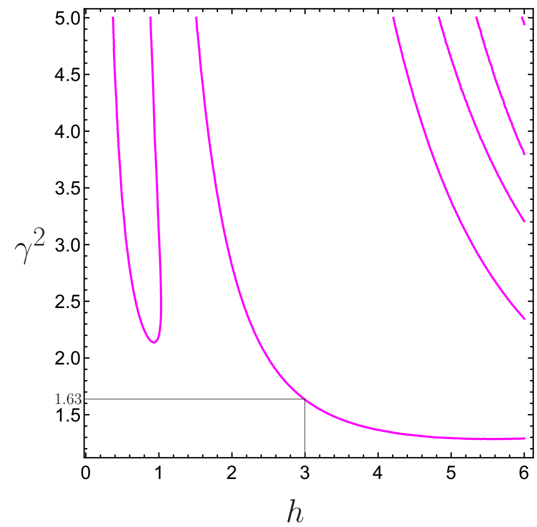

FIG. 7. Branches of the DR 14 with $J(0) = 3$ (for the same parameters as in Fig. 5), evaluated with $J_{f-}(h)$ according to Eq. (6) (for the same parameters as in Fig. 4).

It should be emphasized that, with the ansatz (2), the solution (6) and the DR (14) are valid in general for a permittivity according to (1). Clearly, if, e.g., $b_f = 0$ and $a_f \neq 0$, the set of all allowed PDs (there are five) is different from those in Fig. 2 and thus are the PDC roots J_1, J_2. Nevertheless, the PDs can be classified by the discriminant Δ as before (the case $\Delta = 0$, $g_2 = 0$, and $g_3 = 0$ is impossible) and the proof of the ESC is the same with the same result. If, e.g., $b_f \neq 0$ and $a_f = 0$, the same conclusion holds.

The particular case $b_f = 0$, $\epsilon_s = 1$, $\epsilon_f = 9$, $\epsilon_c = 4$, $a_s = a_c = 0$, and $J_0 = 1$ and the relation of the subcases $a_f \neq 0$ and $a_f = 0$ was considered in [9]. A result was the claim of a new propagating regime ($a_f \neq 0$) with no connections to the linear case ($a_f = 0$). Due to the continuity of (14) with respect to a_f, without going into details, we note that the linear DR is the limit of the nonlinear DR as $a_f \to 0$, so the claim is unfounded.

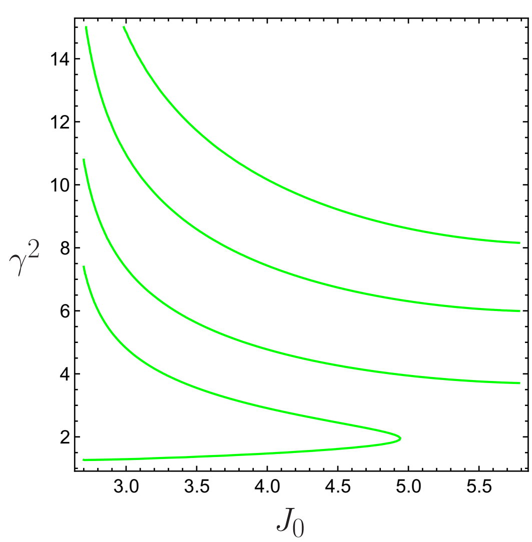

FIG. 6. Branches of the DR (14) with $h = 3$ for the same parameters as in Fig. 4.

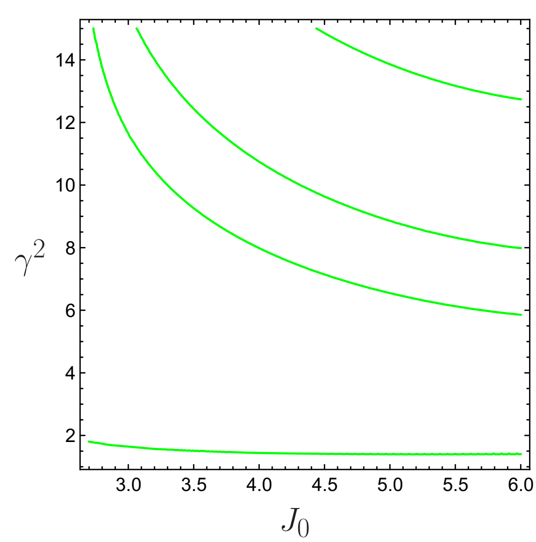

FIG. 8. Branches of the DR (14), evaluated with $J_{f-}(h)$, $h = 3$, for the same parameters as in Fig. 4.

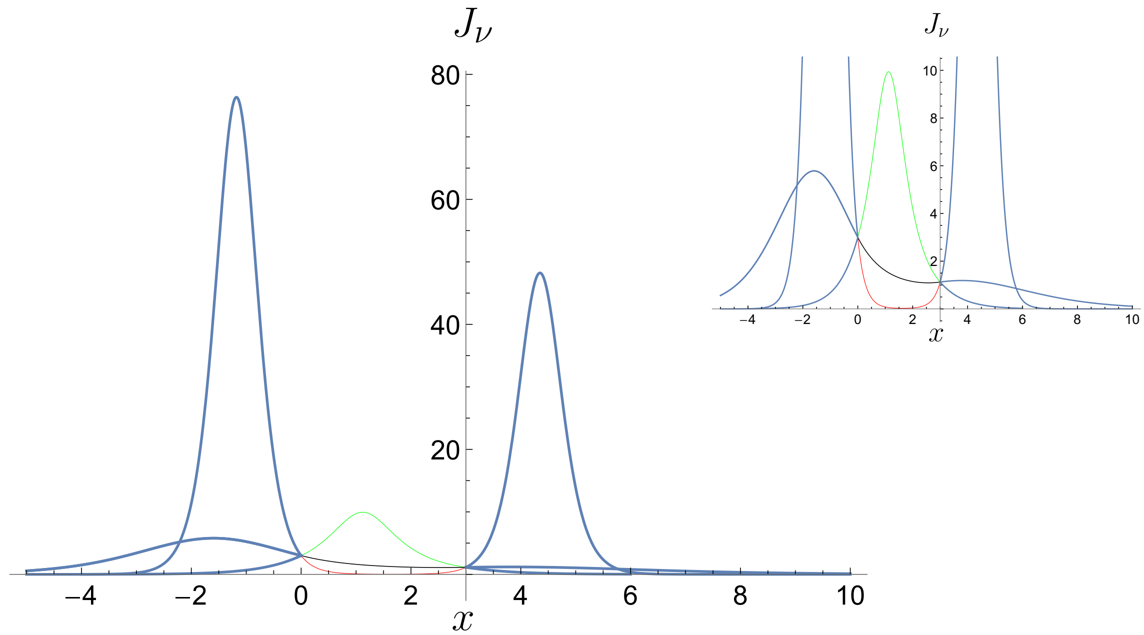

FIG. 9. Intensity patterns $J_{\nu\pm}(x,\gamma^2)$. Inside the film $J_{f+}(x,\gamma_1^2)$ (black curve), $J_{f+}(x,\gamma_2^2)$ (red curve), and $J_{f-}(x,\gamma_3^2)$ (green curve). Values for $\gamma_1^2,\gamma_2^2,\gamma_3^2$ are in the text.

A final remark seems appropriate with respect to the choice of the sought parameters of (14). Usually, the material parameters ϵ_ν, a_ν, and b_f and J_0 or h are prescribed and the relation $\gamma^2(h)$ or $\gamma^2(J_0)$ is sought. Subject to the ESCs, Eq. (14) relates all quantities or parameters. Even if, e.g., only ϵ_f and ϵ_c are chosen, free solutions of (14) exist. The same holds if, e.g., J_0 and h or if J_0 and ϵ_f are free, with other conditions remaining the same. Numerical examples for these cases are presented in the following section.

VI. NUMERICAL EVALUATIONS

The results of Sec. V can be elucidated numerically. Let [see (16)] $b_f = 0.04$, $a_f = -0.2$, $\epsilon_f = 1.3$, $a_s = 0.1$, $\epsilon_s = 1$, $a_c = 0.15$, and $\epsilon_c = 1.2$. The intensities (scaled in units of the intensity $J(0) = J_0$ at the boundary $x = 0$) J_0 and $J^{(h)}$ in (17) and (18) are related (see Fig. 4) by Eq. (12). The PDC roots J_1, J_2 of $R_f(J) = 0$ depend on $\gamma^2, J_0, \epsilon_\nu, a_\nu$ ($\nu = s, f$). Let the ESC be satisfied according to one of the PDs in Figs. 2 and 3. To find a solution triple $\{\gamma^2, J_0, h\}$ consistent with the parameters above, there are two possibilities. If J_0 is fixed (e.g., $J_0 = 3$), two $J^{(h)}$ are possible (see Fig. 4). Disregarding intersections with higher branches, the DR (γ^2, h) yields three values $\gamma_1^2, \gamma_2^2, \gamma_3^2$ ($= 7.364$) at $h = 3$ (see Fig. 5). For $\gamma^2 = \gamma_1^2$, (17) is satisfied only if $J^{(h)} = 1.125$. For $\gamma^2 = \gamma_2^2$, (17) and (18) are satisfied for $J^{(h)} = 1.127$ and for $J^{(h)} = 12.480$). Consistent with this result, Eq. (6) yields $J_{f+}(3) = 1.125$ for γ_1^2 and γ_2^2 and $J_{f+}(3) = 12.480$ for γ_3^2.

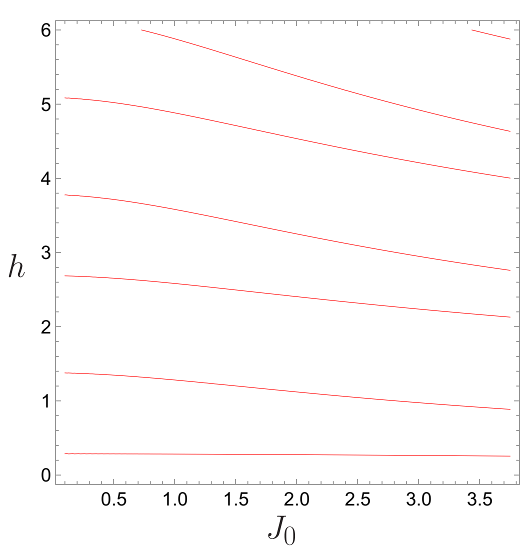

FIG. 10. Branches of the DR (14), evaluated with $J_{f-}(h)$, for the parameters $\gamma = 2.7$, $b_f = 10^{-2}$, $a_f = 10^{-2}$, $\epsilon_f = 9$, $a_c = a_s = 0$, $\epsilon_s = 1$, and $\epsilon_c = 6$.

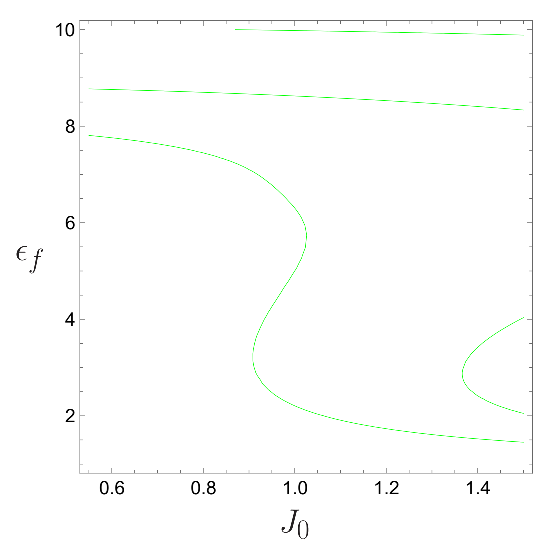

FIG. 11. Branches of the DR (14), evaluated with $J_{f-}(h)$, for the same parameters as in Fig. 10, but $h = 4$ and ϵ_f free.

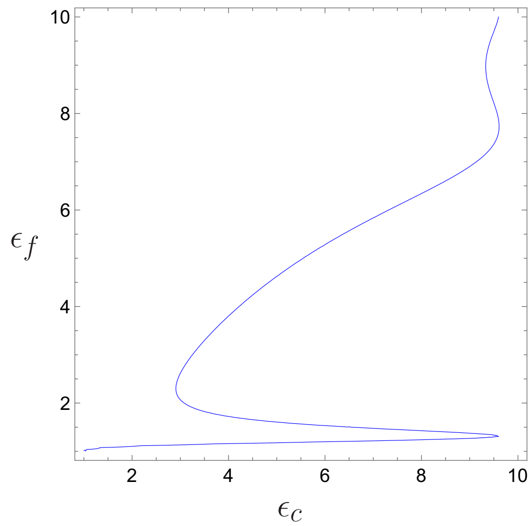

FIG. 12. Branches of the DR (14), evaluated with $J_{f-}(h)$, for the same parameters as in Fig. 10, but $\gamma = 3.1$ and ϵ_c, ϵ_f free.

If instead of J_0, h is fixed (e.g., $h = 3$), the graph of the DR (γ^2, J_0) is shown in Fig. 6. Choosing $J_0 = 3$, one obtains $\gamma_1^2, \gamma_2^2, \gamma_3^2$ as before (as expected). So there is no essential difference by using the DR (γ^2, h) or the DR (γ^2, J_0).

The foregoing results were obtained by using J_{f+} in the DR (14). If J_{f-} is used, evaluation of the corresponding DR (γ^2, h) leads to a graph shown in Fig. 7. Following the lines as before (for J_{f+}) and assuming the same parameters as before, one obtains the DR (γ^2, J_0) graphically presented in Fig. 8 and, remarkably, $J_{f-}(h) = J_{f+}(h) = 1.125$ for $h = 3$ and $\gamma_4^2 = 1.63$ (see the intensity patterns in Fig. 9). The present example corresponds to the phase diagram in Fig. 2(d). Figures 10–12 refer to the final remark in Sec. V, illustrating the large scope of the DR (14).

VII. SUMMARY

The basis of the foregoing analysis is Weierstrass's seminal treatment [21] of the elliptic differential $\frac{dx}{\sqrt{R_4(x)}}$ (with R_4 as a fourth-degree polynomial) and a series of articles by Gagnon *et al.* [22] on exact solutions of the nonlinear Schrödinger equation, where phase diagram considerations were used. To be more specific, the combination of Weierstrass's formula (6) with a phase diagram analysis of $R_\nu(J)$ yields an exact solution of the problem addressed in this paper.

The following are the main results:

(i) The PDC represents all physical solutions $J_\nu(x)$, corresponding to PDs in Figs. 2 and 3. If, finally, a solution has been found, the corresponding PDs in Figs. 2 and 3 must occur.

(ii) The $J_{\nu\pm}(x)$ are given in general by Eq. (6).

(iii) The dispersion relation (14) is valid in general (including $a_f = 0$ and $b_f = 0$), subject to the PDC.

(iv) A (sufficient) existence and solvability condition for the dispersion relation (14) is presented using the PDC.

(v) Due to the compact representation of $J_{\nu\pm}(x)$ and of the dispersion relation according to (6) and (14), numerical evaluation is straightforward and simple.

Additionally, it should be emphasized that, in contrast to [7–11], where fields and DRs (for the Kerr nonlinearity) are presented in terms of Jacobi elliptic functions, the foregoing results are given [according to Eq. (6)] in terms of $J_{\nu\pm}$ and thus by Weierstrass's elliptic function $\wp(x; g_2, g_3)$. Despite the equivalence of the Jacobi and Weierstrass elliptic functions, this is *not* a matter of preference. In [7], for instance, a case distinction is necessary (with respect to λ and K) to describe field amplitudes and DRs, because varying λ and K in the modulus $m(\lambda, K)$ leads to different Jacobi functions. The use of Eq. (6) avoids these complications, thus simplifying evaluation considerably. The fields $J_{\nu\pm}(x)$ $(\nu = s, f, c)$ are represented by Eq. (6) in general, so no case distinction is necessary for the DR. Equation (14) is valid in general for a fourth-degree nonlinearity; it comprises the Kerr and constant permittivity due to the continuity of (14) with respect to a_f [27] and b_f (see remarks at the end of Sec. V). Without presenting numerical examples, we note that the material parameters can be chosen so that waveguides with metamaterial can be modeled by the present method.

In conclusion, two points should be mentioned. The present method does not work if the media are lossy (see [8], p. 1047, and [7], p. 537). For this case, an integral equation procedure has been proposed [23]. Definitely, it is not necessary to explain the occurrence of singular field intensities [7] by the absence of loss. The PDC exclude singular intensities from the beginning.

The occurrence of different solutions of the DR (see Fig. 9) leads to the question of stability. It may be that only certain branches of the DR in Figs. 5–8 correspond to stable modes. Thus, a stability analysis would be worthwhile for applications [20].

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