

Matching universal behavior with potential models

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Two-, three-, and four-boson systems are studied close to the unitary limit using potential models constructed to reproduce the minimal information given by the two-body scattering length a and the two-body binding energy or virtual-state energy E_2 . The particular path used to reach the unitary limit is given by varying the potential strength. In this way the energy spectrum in the three- and four-boson systems is computed. The lowest-energy states show finite-range effects absorbed in the construction of level functions that can be used to study real systems. Higher-energy levels are free from finite-range effects, therefore the corresponding level functions tend to the zero-range universal function. Using this property, a zero-range equation for the four-boson system is proposed and the four-boson universal function is computed.

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I. INTRODUCTION

The Efimov effect was predicted by Efimov in a series of papers [1,2] and experimentally confirmed more than 35 years after its prediction [3]. At present there is intense experimental activity [4–7] aimed at understanding the behavior of few-body systems close to the unitary limit in which the two-body scattering length a diverges. Around this limit the systems show universal behavior; very different systems such as atomic or nuclear systems present similar features. The study of these characteristics is a very active field of research nowadays. From a theoretical point of view, the behavior of a few-body system in the limit of large scattering length can be formulated in the framework of the renormalization group using an effective field theory (EFT) [8,9] (for recent reviews see Refs. [10,11]). Using this language, one of the main striking properties of three identical bosons in the unitary limit is the discrete scaling invariance (DSI) shown by the spectrum: An infinite series of bound states (trimers) appears to be distributed geometrically with the accumulation point at zero energy. The ratio of binding energies for two consecutive states is $E_3^n/E_3^{n+1} = e^{2\pi/s_0}$, with the universal number $s_0 \approx 1.00624$. Explicitly, the total angular momentum $L = 0$ spectrum of three identical bosons in the zero-range limit can be described by the Efimov radial law

$$E_3^n/(\hbar^2/ma^2) = \tan^2 \xi, \quad (1a)$$

$$\kappa_* a = e^{(n-n_*)\pi/s_0} \frac{e^{-\Delta_3(\xi)/2s_0}}{\cos \xi}. \quad (1b)$$

The main ingredients in these equations are the universal function $\Delta_3(\xi)$ and the binding momentum κ_* , called the three-body parameter, defining the energy $\hbar^2\kappa_*^2/m$ of level n_* at the unitary limit, m being the boson mass. Discrete scaling

invariance manifests from the fact that the function $\Delta_3(\xi)$ is the same for all n levels. Furthermore, the spectrum described by these equations is not bounded from below; this characteristic is known as the Thomas collapse [12].

The determination of the universal function $\Delta_3(\xi)$ in the interval $-\pi \leq \xi \leq -\pi/4$ limited by the two- and three-cluster continua can be obtained by solving the Skorniakov-Termartirosian (STM) equation or equivalently by using EFT as reported in Ref. [8]. This equation describes the $L = 0$ state of a three-boson system in the zero-range limit. To avoid the Thomas collapse it is a common practice to introduce a cutoff in the solution of the STM equation and the universal function $\Delta_3(\xi)$ is computed looking at the second or even third excited state [13] where the cutoff effects are negligible. A parametrization of it can be found in Ref. [10]. In the same way, potential models can be used to solve the Schrödinger equation looking at the high part of the spectrum, where finite-range effects are negligible.

The extension of the zero-range theory to four bosons has been discussed in Refs. [14–17]. The main conclusion of these works is that the four-boson spectrum presents a two-level tree structure. For each three-body level E_3^n there are two four-body states (tetramers), one deep ($m = 0$) and one shallow ($m = 1$), with binding energies $E_4^{n,m}$. The universal ratios of these binding energies in the unitary limit have been calculated in Ref. [18] and they are $E_4^{n,0}/E_3^n = 4.611$ and $E_4^{n,1}/E_3^n = 1.0023$. In the present work we analyze the complete interval between the four-body continuum and the dimer-dimer thresholds in order to extend Eq. (1) to the four-boson system and, if possible, to determine the corresponding universal function.

The present study is done using potential models with variable strength. In this way a path to reach the unitary limit is defined. Close to this limit the low-energy dynamics is largely independent of the specific interaction of the system.

For example, in Ref. [19] the trimer of helium atoms has been described using a soft interaction model. Always limiting the discussion to the low-energy regime, it has been recently shown that a two-parameter potential captures the essential ingredients of the few-boson dynamics close to the unitary limit [20]. Accordingly, here we define two different potentials, a local Gaussian potential and a nonlocal Gaussian potential used to solve the Schrödinger equation along the path. For the lowest states finite-range effects are appreciable. For this case it is possible to define level functions that absorb those effects and can be used to estimate the spectrum of a real system close to the unitary limit. For higher states finite-range effects can be neglected and the solution tends to the zero-range limit; therefore the level functions tend to the universal zero-range function. In order to illustrate the procedure we first analyze the two-body system and then the analysis is extended to the three- and four-boson systems.

The paper is organized as follows. In Sec. II the particular path used to reach the unitary limit is studied in the two-body system. In Secs. III and IV the spectra of the three- and four-boson systems are computed using the two potential models, local and nonlocal. From the results a zero-range equation for the four-boson system is proposed. A summary and perspectives are given in Sec. V.

II. REACHING THE UNITARY LIMIT WITH POTENTIAL MODELS

In the two-body system the $L = 0$ spectrum within the zero-range theory can be defined through the relations

$$E_2 = \frac{\hbar^2}{ma^2}, \quad (2a)$$

$$k \cot \delta_0 = -\frac{1}{a}. \quad (2b)$$

The first relation establishes that there is a bound state ($a > 0$) or a virtual state ($a < 0$) with binding energy E_2 fixed by the scattering length a . At positive energies $E = \hbar^2 k^2/m$, the s -wave phase shift δ_0 is determined by the scattering length as well. Accordingly, the scattering length a emerges as a control parameter in terms of which the observables such as the cross section or mean square radius can be computed. The zero-range theory describes the extreme situation in which the two particles are always outside the interaction range. If the two-body quantum system, interacting through a short-range potential, has a shallow state, there is a big probability of finding the particles outside the interaction range. In fact, when E_2 is very small or, equivalently, $a_B = \hbar/\sqrt{mE_2} \gg r_0$, with r_0 the interaction range, the two-body wave function has a very long tail and the two particles have a large probability of being at relative distances greater than r_0 . When a shallow state is present, the scattering length verifies $a \gg r_0$ too and $a \approx a_B$, with Eq. (2a) approximately fulfilled. The extension of Eq. (2) in the case of finite-range interactions and at low energies is

$$E_2 = \frac{\hbar^2}{ma_B^2}, \quad (3a)$$

$$k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} k^2, \quad (3b)$$

with r_{eff} being the effective range. In the case of shallow states Eq. (3b) can be used to relate the effective range to a_B ,

$$\frac{1}{a_B} = \frac{1}{a} + \frac{r_{\text{eff}}}{2a_B^2}, \quad (4)$$

from which we obtain the relation

$$r_{\text{eff}} a = 2r_B a_B, \quad (5)$$

with $r_B = a - a_B$. The above discussion stressed the fact that in the low-energy limit or large scattering length limit the dynamics of the two-body system depends on two parameters: the scattering length and the effective range r_{eff} (or the length r_B).

The scaling limit is defined by $r_B \rightarrow 0$, whereas in the unitary limit $1/a$ and $1/a_B \rightarrow 0$. In the first case, for each value of a the two-body energy is determined by the zero-range condition $a = a_B$. When $r_B \neq 0$, the unitary limit can be reached by different paths determined by the functional relation $a_B = a_B(a)$. Having in mind that we intend to study the structure of few-boson systems, we construct a two-parameter potential able to reproduce the minimal information given by one specific set of a and a_B values. Then the potential strength can be varied in order to reach the unitary limit. We define local (given in coordinate space) and nonlocal (given in momentum space) Gaussian potentials

$$V_\lambda^L(r) = -\lambda V_0^L e^{-r^2/r_0^2}, \quad (6)$$

$$V_\lambda^{NL}(k, k') = -\lambda V_0^{NL} e^{-k^2/k_0^2} e^{-k'^2/k_0^2}, \quad (7)$$

respectively, with the strengths V_0^L, V_0^{NL} and the ranges r_0, k_0^{-1} determined to describe the particular values of a and a_B of a two-boson system. The parameter λ can be varied in order to reach the unitary limit. With this procedure the functional relation $a_B = a_B(a)$ is determined. It should be noted that with the potentials defined above the lengths, momenta, and energy scale with r_0, k_0 , and \hbar^2/mr_0^2 (or $\hbar^2 k_0^2/m$), respectively. Accordingly, the local Gaussian potential defines a particular path to the unitary limit that encompasses all local Gaussian potentials and the same for the nonlocal one. In particular, the values of the effective range and strength at unitary are given in Table I for the cases in which there are n bound states in the two-body system. Note that the nonlocal potential, being a rank-one separable potential, supports only one bound state at most. This potential could be considered as a simplification of a finite-range two-channel model for the Feshbach resonance, but it also simulates quite well the

TABLE I. Universal values of the effective range (in units of r_0 or k_0^{-1}) and potential strength (in units of \hbar^2/mr_0^2 or $\hbar^2/k_0 m$) for local and nonlocal Gaussian potentials at the unitary limit for different bound-state numbers n .

n	Local		Nonlocal	
	r_{eff}/r_0	$\lambda V_0^L m r_0^2 / \hbar^2$	$r_{\text{eff}} k_0$	$\lambda V_0^{NL} m k_0 / \hbar^2$
0	1.43522	2.6840	3.19154	0.126987
1	2.41303	17.7957		
2	2.89034	45.5735		
3	3.20006	85.9632		

features of the open-channel-dominated Feshbach resonance (see, for example, Refs. [21,22]).

In the following we consider the $n = 0$ case, however the other cases can be analyzed in a similar way. We are interested in the functional relation $a_B(a)$ as $1/a \rightarrow 0$. Defining r_u , the value of the effective range at the unitary limit, from Eq. (5) we can define

$$\frac{r_{\text{eff}}}{r_u} = \frac{2r_B}{r_u} - \frac{x}{2} \left(\frac{2r_B}{r_u} \right)^2, \quad (8)$$

with $x = r_u/a$. Moreover, the quantity $2r_B/r_u$ can be expanded around the unitary limit as

$$\frac{2r_B}{r_u} = 1 + \mathcal{A}x + \mathcal{B}x^2 + \dots \quad (9)$$

Inserting the above expansion in Eq. (8), the effective range expansion in terms of the inverse of the scattering length becomes

$$\frac{r_{\text{eff}}}{r_u} = 1 - \mathcal{C}x + \mathcal{D}x^2 + \dots \quad (10)$$

If the length r_B were constant along the path, the coefficient \mathcal{C} would be 0.5 and $\mathcal{D} = 0$ as well as higher terms. Differences from these values indicate a nonconstant behavior of r_B . For example, the study of the van der Waals (vdW) potential shows a quadratic relation between r_{eff} and the inverse of the scattering length. Explicitly it results in (see Ref. [23] and references therein)

$$\frac{r_{\text{eff}}}{r_u} = 1 - \frac{12\pi^2}{\Gamma(1/4)^4}x + \frac{72\pi^4}{\Gamma(1/4)^8}x^2, \quad (11)$$

with $\Gamma(1/4)^4/6\pi^2 \approx 2.9179$. Therefore, the coefficients in Eq. (10) are $\mathcal{C} \approx 0.685$ and $\mathcal{D} \approx 0.235$. In the case of the local Gaussian (LG) and nonlocal Gaussian (NLG) potentials the behavior is almost linear ($\mathcal{D} \approx 0$), with the coefficients $\mathcal{C} \approx 0.504$ and 0.393, respectively. The relation $a_B = a_B(a)$ can be analyzed starting from the definition

$$\frac{r_u}{a_B} = \frac{r_u}{a} \frac{1}{1 - r_B/a}. \quad (12)$$

Defining $x = r_u/a$ and $y = r_u/a_B$ and considering the expansion of $2r_B/r_u$ given above, this relation, at first order in $1/a$, results in

$$y = \frac{x}{1 - 0.5x}. \quad (13)$$

In Fig. 1 the quantity r_u/a_B is shown for different potential models around the unitary limit. The results of the different potential models collapse in the curve given by Eq. (13), showing that up to first order the relation $a - a_B \approx 0.5r_u$ is well verified. In the study we have included the LM2M2 helium-helium interaction from Aziz and Slaman [24]. We can conclude that Eq. (13) can be seen as a universal relation describing the path to the unitary limit fixed by the relation $a - a_B = \text{const}$. Potentials with variable strength follow this path with reasonable accuracy, however first-order corrections could be of the order of a few percent for the LG potential and up to 10% in the case of the NLG potential.

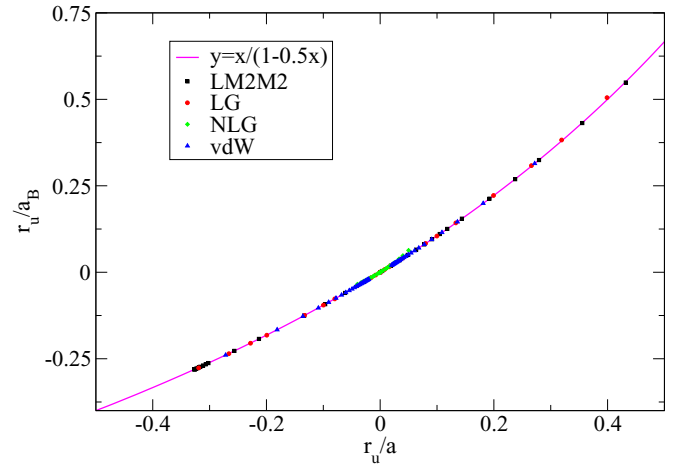


FIG. 1. Relation between the inverse of a_B and a (in units of $1/r_u$) for the different potential models. All the computed values collapse on the curve $y = x/(1 - 0.5x)$.

III. UNIVERSAL BEHAVIOR IN THE THREE-BODY SECTOR

The analysis of the two-boson system indicates that a two-parameter potential such as a local or a nonlocal Gaussian potential can be used to study the low-energy dynamics around the unitary limit. We extend the analysis to the three- and four-boson systems. The numerical results for the local Gaussian potential are obtained by solving the Schrödinger equation in the coordinate space framework using the hyperspherical harmonic expansion [25–27], while the predictions for the nonlocal Gaussian potential are obtained by solving Faddeev-Yakubovsky (FY) [28] or Alt-Grassberger-Sandhas (AGS) equations [29] using momentum-space methods from Refs. [18,30].

The $L = 0$ spectrum of three identical bosons in the zero-range limit can be described by the Efimov radial law given in Eq. (1). In the case of a system with a finite-range interaction, this equation can be seen as describing the asymptotic spectrum of the three-boson system close to the unitary limit. In fact, the solution of the Schrödinger equation can be used to determine the universal function $\Delta_3(\xi)$ looking at the excited states of the spectrum as, for these states, finite-range effects are negligible. In this context the description of few-body systems with potential models close to the unitary limit can be seen as a particular regularization scheme. Accordingly, it is possible to modify the Efimov radial law as (see Refs. [20,31])

$$E_3^n/E_2 = \tan^2 \xi, \quad (14a)$$

$$\kappa_3^n a_B = \frac{e^{-\tilde{\Delta}_3^n(\xi)/2s_0}}{\cos \xi}, \quad (14b)$$

where $E_2 = \hbar^2/ma_B^2$ is the dimer binding energy for positive values of a , whereas for negative values it is the two-body virtual-state energy. Modifications at the three-body level are introduced by the parameters κ_3^n , which absorb the scaling factor $e^{n\pi/s_0}$ defining the energy of level n at the unitary limit $E_u^n = (\hbar^2/m)(\kappa_3^n)^2$. Furthermore, the finite-range character of the interaction slightly modifies the ratio $\kappa_3^n/\kappa_3^{n+1}$ from

its universal value of ≈ 22.7 . The main modification in the above equations is the introduction of the level function $\tilde{\Delta}_3^n(\xi)$. For the ground state ($n = 0$) it could be very different from the zero-range function $\Delta_3(\xi)$. As we will see below, the differences are greatly reduced by considering the first excited state ($n = 1$) and, starting from $n > 1$, both functions almost coincide. The level function can be calculated using the corresponding solutions of the Schrödinger equation as

$$\tilde{\Delta}_3^n(\xi) = s_0 \ln \left(\frac{E_3^n + E_2}{E_u^n} \right). \quad (15)$$

It should be noticed that for $n = 0, 1$ this function depends on the particular potential used to calculate the spectrum or, in the case of the STM equation, the cutoff. It depends also on the particular path selected to reach the unitary limit, for example, potentials with variable strength as discussed in the previous section. However, as shown in Ref. [20], following this particular path, different potentials do not produce too much spread in $\tilde{\Delta}_3^n(\xi)$ and $\tilde{\Delta}_3^1(\xi)$. Furthermore, the LG potential defines a unique Gaussian function $\tilde{\Delta}_3^n(\xi)$ for each level n independent of the range r_0 of the potential. In particular, for the first two levels $n = 0, 1$, the binding momenta at the unitary limit are $\kappa_3^0 = 0.4874/r_0$ and $\kappa_3^1 = 0.0212/r_0$ and the ratio $\kappa_3^0/\kappa_3^1 \approx 23.0$. These values have been obtained with the potential acting only in s waves; they are slightly different when the LG potential is taken to act in all waves (see, for example, Refs. [20,32]). Also, the NLG potential acting in s waves defines a unique nonlocal Gaussian function with the universal ratios $\kappa_3^0/k_0 = 0.2127$, $\kappa_3^1/k_0 = 0.009085$ and $\kappa_3^0/\kappa_3^1 \approx 23.4$. From the above discussion the following picture emerges: A two-parameter potential such as the local or nonlocal Gaussian potentials can be used to construct level functions for each level n of the three-boson system. For $n = 0, 1$ these functions are different from the zero-range universal function and are also different among themselves. For $n > 1$ they converge to the zero-range universal function showing a universal behavior. In order to analyze this fact quantitatively, in Fig. 2 the level functions $\tilde{\Delta}_3^0(\xi)$ and $\tilde{\Delta}_3^1(\xi)$ are shown for the case of the LG, NLG, and LM2M2 potentials. The trend discussed above is visible in the figure, for the ground

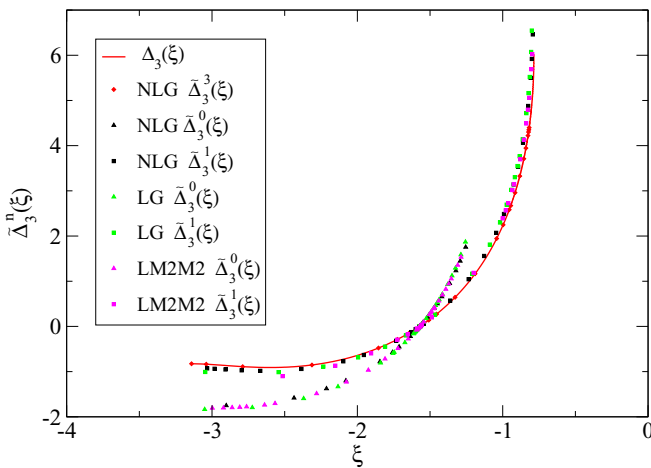


FIG. 2. Level functions $\tilde{\Delta}_3^n(\xi)$ for different levels and potentials. The zero-range universal function $\Delta_3(\xi)$ is shown as the red solid curve.

state the level functions $\tilde{\Delta}_3^0(\xi)$ spread in a narrow band and are rotated with respect to the universal function $\Delta_3(\xi)$, given by the red solid curve. In the case of the first excited state the level functions $\tilde{\Delta}_3^1(\xi)$ spread also in a very narrow band very close to $\Delta_3(\xi)$. The level function $\tilde{\Delta}_3^3(\xi)$ calculated using the NLG third excited state (red diamonds) completely overlaps with the zero-range universal function.

The Efimov radial law is a one-parameter equation. The knowledge of the universal function $\Delta_3(\xi)$ allows for a complete determination of the spectrum after assigning a value to κ_* (or to one of the energies E_3^n). Equation (14) applies mostly to $n = 0, 1$ and works slightly differently. First, it is necessary to calculate the level functions for the ground and first excited state. This can be done using, for example, the LG or NLG potentials. At this point the equation is a one-parameter equation and the $n = 0$ and $n = 1$ spectra can be completely determined from the knowledge of one energy. One can argue that if we use Eq. (14) to describe a particular system, the respective potential could be used to compute the level function $\tilde{\Delta}_3^0(\xi)$ or $\tilde{\Delta}_3^1(\xi)$ by varying the strength in order to reach the unitary limit. However, as it is shown in Fig. 2, close to the unitary limit the three-boson system has universal behavior and therefore a two-parameter potential captures the essential ingredients of the dynamics absorbing finite-range effects. As an example we can use $\tilde{\Delta}_0(\xi)$ computed using the LG or NLG potentials to estimate the three-body parameter κ_3^0 of a system composed of three ^4He atoms. As given in Ref. [20], the result using the LG potential is $\kappa_3^0 \approx 0.0438a_0^{-1}$ ($a_0 = 0.529177 \dots \text{Å}$ is the Bohr radius). Using the NLG potential we obtain $\kappa_3^0 \approx 0.0442a_0^{-1}$, whereas with the LM2M2 potential the result is $0.0440a_0^{-1}$. This shows that the level functions produce a description with the accuracy better than 1%. The results for the three-body parameter corresponding to the first excited state are $\kappa_3^1 \approx 0.0018a_0^{-1}$ and $\kappa_3^1 \approx 0.0019a_0^{-1}$ for the LG and NLG potentials, respectively, to be compared to the LM2M2 value of $\kappa_3^1 = 0.0019a_0^{-1}$. As expected, finite-range effects are reduced in this level.

IV. UNIVERSAL BEHAVIOR IN THE FOUR-BODY SECTOR

The previous analysis can be extended to the four-body case. In this case a two-level structure, with energies $E_4^{n,0}$ and $E_4^{n,1}$, is attached to each E_3^n level [14–17]. As in the three-body case, the four-body system can be studied using potential models; also in this case there is a modification of the universal ratios due to finite-range effects (see, for example, Refs. [30,33]). Following the previous discussion, the equations describing the four-boson spectrum can be written as

$$E_4^{n,m}/E_2 = \tan^2 \xi, \quad (16a)$$

$$\kappa_4^{n,m} a_B = \frac{e^{-\tilde{\Delta}_4^{n,m}(\xi)/2s_0}}{\cos \xi}, \quad (16b)$$

with $m = 0, 1$ and n identifying the corresponding three-body branch. The four-body parameters $\kappa_4^{n,m}$ are related to the energy of the level at the unitary limit $E_u^{n,m} = \hbar^2(\kappa_4^{n,m})^2/m$. It should be stressed that only the branch $n = 0$, with energies

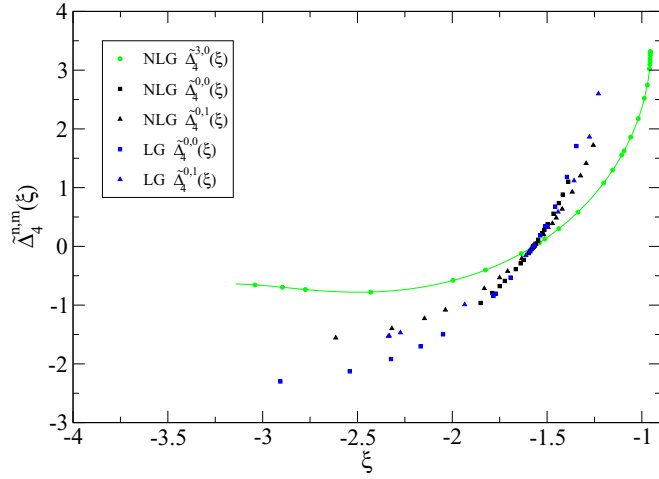


FIG. 3. Level functions $\tilde{\Delta}_4^{n,m}(\xi)$ for different levels and potentials. The solid green line is an interpolation of the $\tilde{\Delta}_4^{3,0}(\xi)$ values and should represent the universal function $\Delta_4^0(\xi)$.

$E_4^{0,0}$ and $E_4^{0,1}$, corresponds to true bound states. The other states, corresponding to branches with $n > 0$, are above the trimer ground-state threshold and therefore are unstable bound states (UBSs) [30,34]. In the above equation we have introduced the level function $\tilde{\Delta}_4^{n,m}(\xi)$ that governs the four-body spectrum in levels n, m . They can be computed using potential models using the definition

$$\tilde{\Delta}_4^{n,m}(\xi) = s_0 \ln \left(\frac{E_4^{n,m} + E_2}{E_u^{n,m}} \right). \quad (17)$$

For $n = 0$ it could be very different from the universal function $\Delta_4^m(\xi)$ that governs the four-body spectrum in the zero-range limit. However, as n increases finite-range effects become negligible and $\tilde{\Delta}_4^{n,m}(\xi)$ should tend to that function. The results are shown in Fig. 3, where the function $\tilde{\Delta}_4^{n,m}(\xi)$ has been calculated using the LG and NLG potentials for the ground-state $n = 0, m = 0$ level (squares) and first excited-state $n = 0, m = 1$ level (triangles). For the sake of comparison the function corresponding to the $n = 3, m = 0$ level (circles) and calculated using the NLG potential is also shown. From the figure we can see that the results of both potentials for the $n = 0, m = 0$ and $n = 0, m = 1$ levels are close to each other. However, there is a difference between the functions with different m values, more pronounced for $\xi < -\pi/2$.

Equation (16) works very similarly to the three-body case discussed before. The knowledge of $\tilde{\Delta}_4^{n,m}(\xi)$ makes this equation a one-parameter equation. We would like to stress that $\tilde{\Delta}_4^{n,m}(\xi)$ can be computed using a LG or a NLG potential and then is used to determine the spectrum of a real system around the unitary limit. As an example we discuss the spectrum of four He atoms. A very complete discussion of this system has been given in Refs. [35,36] using realistic potentials. The binding energies of the tetramers using the LM2M2 interaction are $E_4^{0,0} = 559.22$ mK and $E_4^{0,1} = 127.42$ mK. The dimer energy is $E_2 = 1.3094$ mK and therefore the angles are $\xi = -1.52244$ and -1.46977 , respectively. Using the NLG $\tilde{\Delta}_4^{n,m}(\xi)$ function, we estimate $E_u^{0,0} \approx 0.443$ K and $E_u^{0,1} \approx 0.0865$ K, in a very good agreement with the quoted

values for the LM2M2 potential of $E_u^{0,0} \approx 0.4449$ K and $E_u^{0,1} \approx 0.0870$ K given in Ref. [36]. We can conclude that the estimates obtained using the level functions are given with an accuracy well below 1%.

In Fig. 3 the level function $\tilde{\Delta}_4^{3,0}(\xi)$, calculated using the NLG potential, is shown (circles). For this level finite-range effects can be neglected and therefore we can consider this function a good representation of the zero-range four-body universal function in the level $m = 0$. For a generic level m we define this function $\Delta_4^m(\xi)$. It does not depend on the three-body branch n , as DSI, with the geometrical factor $e^{\pi/s_0} \approx 22.7$, has already been verified among these branches [16,18,37]. In the following we study its dependence on the levels m and its relation to the three-boson universal function $\Delta_3(\xi)$. To this aim the level functions $\tilde{\Delta}_4^{3,0}(\xi)$ and $\tilde{\Delta}_4^{3,1}(\xi)$, calculated using the NLG potential, are shown in Fig. 4 together with the zero-range universal function $\Delta_3(\xi)$ and the level function $\tilde{\Delta}_3^3(\xi)$. We consider $\tilde{\Delta}_3^3(\xi)$ a representation of the zero-range universal function $\Delta_3(\xi)$ and we consider $\tilde{\Delta}_4^{3,m}(\xi)$ a representation of $\Delta_4^m(\xi)$. For the $m = 1$ case the range of ξ values in which this level results in an inelastic virtual state (IVS) is explicitly shown. From the figure we can see that $\Delta_3(\xi)$ and $\Delta_4^m(\xi)$ are very close to each other around the unitary limit. As the functions approach the different thresholds differences appear. In the case of the threshold at $\xi = -\pi$ we have $\Delta_4^0(-\pi) = \tilde{\Delta}_4^0(-\pi) = -0.645$, appreciably different from $\Delta_3(\xi) = -0.8266$. Defining $a_{4,-}^{n,m}$, the two-body scattering length at which the four-boson system disappears into the four-body continuum, the first relation establishes that

$$\kappa_4^{n,m} a_{4,-}^{n,m} = 1.378 \quad (18)$$

is a universal number. Defining $a_{3,-}^n$ to be the two-body scattering length at which the trimer disappears into the three-body continuum, the second relation results in

$$\kappa_3^n a_{3,-}^n = 1.508, \quad (19)$$

confirming the highly accurate value given in Ref. [38]. Within the zero-range theory, the set of values at which the different

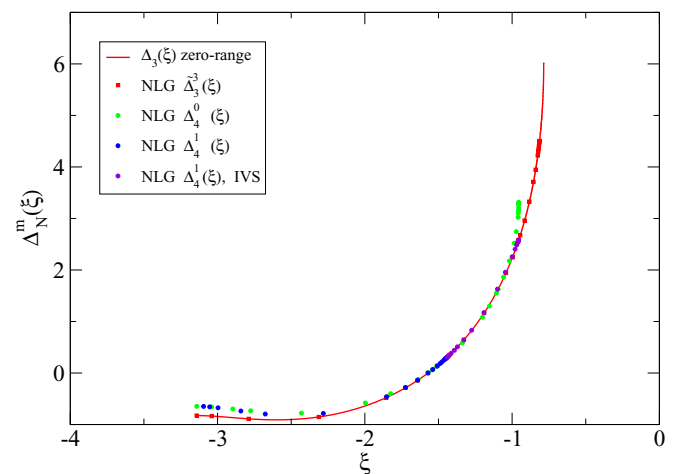


FIG. 4. Universal functions $\Delta_4^0(\xi)$ (green solid points) and $\Delta_4^1(\xi)$ (blue solid points and violet solid points in the IVS region). For the sake of comparison, the three-boson universal function $\Delta_3(\xi)$ and level function $\Delta_3^3(\xi)$ are also shown.

branches disappear into the three- and four-body continua can be determined from the above relations using the universal ratios [37] $a_{3,-}^{n+1}/a_{3,-}^n = 22.694$, $a_{4,-}^{n,0}/a_{3,-}^n = 0.4254$, and $a_{4,-}^{n,1}/a_{3,-}^n = 0.9125$.

The dimer-dimer threshold is defined by $\tan \xi_c = -\sqrt{2}$ corresponding to $\xi_c = -0.9553$. In Fig. 4 we can observe that around this value $\Delta_4^0(\xi)$ differs from $\Delta_4^1(\xi)$ and $\Delta_3(\xi)$, with these last two functions close to each other. It should be noticed that in this region $\Delta_4^1(\xi)$ is obtained from the energy of the IVS except for the vicinity of ξ_c where the shallow tetramer again becomes an UBS before decaying through the dimer-dimer threshold. The values of the universal functions at the critical value ξ_c can be calculated using the universal ratios of Ref. [39]: $a_{dd}^{n,0}/a_{dd}^n = 0.3235$, $a_{dd}^{n,1}/a_{dd}^n = 0.99947$, $a_{dd}^n/a_d^n = 6.789$, and the relation $\kappa_3^n a_n^d = 0.0707645$, with $a_{dd}^{n,m}$, a_{dd}^n , and a_n^d the scattering lengths at which the level n, m of the tetramer intersects the dimer-dimer threshold and at which the level n of the trimer intersects the dimer-dimer and dimer thresholds, respectively. Using these ratios in Eq. (16) results in $\Delta_4^0(\xi_c) = 3.316$ and $\Delta_4^1(\xi_c) = 2.580$, respectively, in complete agreement with the computed values shown in Fig. 4. We can conclude that the small differences between $\Delta_4^0(\xi)$ and $\Delta_4^1(\xi)$ around the critical value are due to threshold effects.

From the above discussion we propose the following zero-range equation for the $L = 0$ spectrum of four equal bosons:

$$E_4^{n,m}/(\hbar^2/ma^2) = \tan^2 \xi, \quad (20a)$$

$$\kappa_*^m a = e^{(n-n_*)\pi/s_0} \frac{e^{-\Delta_4^m(\xi)/2s_0}}{\cos \xi}, \quad (20b)$$

with κ_*^m the binding momentum of the level n_*, m at the unitary limit verifying the universal ratio $\kappa_*^0/\kappa_*^1 = 2.1449$ and $\Delta_4^m(\xi)$ the universal four-boson universal function given in Fig. 4. This equation extends the Efimov radial law for three bosons to the four-boson system.

V. CONCLUSION

In the present work we have analyzed the low-energy spectrum of two-, three-, and four-boson systems close to the unitary limit. To this aim we have solved the Schrödinger equation (or FY and AGS equations) using potential models with variable strength constructed to reproduce the minimal information given by the two-body scattering length a and the two-body binding energy or virtual-state energy E_2 . It has been shown that a two-parameter interaction as a Gaussian potential can capture the main ingredients of the dynamics in this region. Moreover, these types of

potentials define level functions independent of the range used to compute it. This property can be used to construct level functions of general validity that can be used to predict some characteristics of real systems along the particular path used to reach the unitary limit. The level functions Δ_3^n and $\Delta_4^{n,m}$ have two properties. When they are used in the lowest branches $n = 0, 1$, they absorb finite-range effects. This portion of the spectrum does not show a perfect DSI since finite-range effects are visible. So the interest here is to use the level functions to describe the dynamics of real systems close to the unitary limit. For example, potentials with variable strength describe with reasonable accuracy the variation of the interatomic potential using broad Feshbach resonances in ultracold atomic traps.

The second property is given by the description of the asymptotic part of the spectrum. For levels with $n > 1$ the spectrum calculated with the potentials starts to show DSI and coincides with the spectrum in the zero-range limit. Accordingly, the level functions for $n > 1$ do not depend any more on the level number n and on the particular potential used to compute it as well as the path selected to reach the unitary limit. They are good representations of the universal functions Δ_3 and Δ_4^m . This property has been used here to propose Eq. (20) as an extension of the Efimov radial law for four bosons.

In the present work we have studied a particular path to reach the unitary limit based on a single-channel potential with variable strength. Other possibilities could be, for example, the study of coupled-channel interactions as in molecular systems. In this way different level functions can be constructed, allowing a systematic study of finite-range effects. Other improvements of the present work could be the study of the spectrum as the number of bosons increases. Preliminary results along this line have been obtained [33]. Finally, we would like to mention the recent study of Efimov physics in the three-body system having spin-isospin degrees of freedom [40]. The extension to the four-body system is left to future investigation.

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