General monogamy of Tsallis q-entropy entanglement in multiqubit systems

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In this paper, we study the monogamy inequality of Tsallis q-entropy entanglement. We first provide an analytic formula of Tsallis q-entropy entanglement in two-qubit systems for $\frac{5-\sqrt{13}}{2} \leqslant q \leqslant \frac{5+\sqrt{13}}{2}$. The analytic formula of Tsallis q-entropy entanglement in $2 \otimes d$ system is also obtained and we show that Tsallis q-entropy entanglement satisfies a set of hierarchical monogamy equalities. Furthermore, we prove the squared Tsallis q-entropy entanglement follows a general inequality in the qubit systems. Based on the monogamy relations, a set of multipartite entanglement indicators is constructed, which can detect all genuine multiqubit entangled states even in the case of N-tangle vanishes. Moreover, we study some examples in multipartite higher-dimensional system for the monogamy inequalities.

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I. INTRODUCTION

Multipartite entanglement is an important physical resource in quantum mechanics, which can be used in quantum computation, quantum communication, and quantum cryptography. One of the most surprising phenomena for multipartite entanglement is that the monogamy property, which quantifies the resources of quantum entanglement, cannot be shared freely between different constituents in a multipartite system. Monogamy property may be as fundamental as the no-cloning theorem [1–4]. A simple example of monogamy property can be interpreted as the amount of entanglement between A and B, plus the amount of entanglement between A and C, cannot be greater than the amount of entanglement between A and the pair BC. Monogamy property has been considered in many areas of physics: One can estimate the quantity of information captured by an eavesdropper about the secret key to be extracted in quantum cryptography [3,5], the frustration effects observed in condensed matter physics [6,7], and even in black-hole physics [8,9].

Monogamy property of various entanglement measures have been discovered. Coffman *et al.* first considered three qubits A, B, and C which may be entangled with each other [2], who showed that the squared concurrence $\mathcal{C}^2$ follows this monogamy inequality. Osborne *et al.* proved the squared concurrence follows a general monogamy inequality for the N-qubit system [3]. Different kinds of monogamy inequalities for concurrence have been noted in Refs. [10–14]. Some similar monogamy inequalities were also discussed for entanglement of formation [12,15,16], negativity [17–21], relative entropy entanglement [22,23], continuous variable systems [24–26], Renyi α-entropy entanglement [27,28], and Tsallis q-entropy entanglement [29,30]. The monogamy property of other physical resources has also been discussed, such as discord [31,32] and steering [33,34].

Tsallis q entropy is an important entropic measure, which can be used in many areas of quantum information theory [35–40]. In this paper, we study the monogamy inequality of Tsallis q-entropy entanglement (TEE). We first provide an analytic formula of TEE in two-qubit systems for $\frac{5-\sqrt{13}}{2} \leqslant q \leqslant \frac{5+\sqrt{13}}{2}$. The analytic formula of TEE in the $2 \otimes d$ system is also obtained and we show that TEE satisfies a set of hierarchical monogamy equalities. Furthermore, we prove the squared TEE follows a general inequality in the qubit systems. As a corollary, we provide that the αth power of TEE satisfies the monogamy inequality for $\alpha \geqslant 2$. Based on the monogamy relations, a set of multipartite entanglement indicators is constructed, which can detect all genuine multiqubit entangled states even in the case of N-tangle vanishes. Moreover, we study some examples in the multipartite higher-dimensional system for the monogamy inequalities.

This paper is organized as follows. In Sec. II, we recall the definition of TEE and entanglement of formation. In Sec. III, we discuss the monogamy properties of TEE. In Sec. IV, we construct a set of multipartite entanglement indicators, and analysis of some examples. In Sec. V, we study some examples in the multipartite higher-dimensional system for the monogamy inequalities. We summarize our results in Sec. VI.

II. QUANTIFYING ENTANGLEMENT BY TSALLIS q ENTROPY

Quantifying entanglement is an important problem in quantum information. Given a bipartite state ρ_{AB} in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The Tsallis-$q$ entropy is defined as [41]

$$T_q(\rho) = \frac{1}{q-1}(1 - \mathrm{Tr}\rho^q) \tag{1}$$

for any $q > 0$ and $q \neq 1$. When q tends to 1, the Tsallis q entropy $T_q(\rho)$ converges to its von Neumann entropy [42]: $\lim_{q\to1} T_q(\rho) = -Tr(\rho \ln \rho)$. For any pure state $|\psi_{AB}\rangle$, the TEE is defined as

$$\mathcal{T}_q(|\psi_{AB}\rangle) = T_q(\rho_A) \tag{2}$$

for any $q > 0$. For a mixed state ρ_{AB}, the TEE can be defined as

$$\mathcal{T}_q(\rho_{AB}) = \min \sum_i p_i \mathcal{T}_q\left(\left|\psi^i_{AB}\right\rangle\right), \tag{3}$$

for any $q > 0$, where the minimum is taken over all possible pure state decompositions $\{p_i, \psi^i_{AB}\}$ of ρ_{AB}. TEE can be viewed as a general entanglement of formation when q tends

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to 1. The entanglement of formation is defined as [43,44]

$$E_f(\rho_{AB}) = \min \sum_i p_i E_f(|\psi^i_{AB}\rangle), \tag{4}$$

where $E_f(|\psi^i_{AB}\rangle) = -\mathrm{Tr}\rho^i_A \ln \rho^i_A = -\mathrm{Tr}\rho^i_B \ln \rho^i_B$ is the von Neumann entropy, the minimum is taken over all possible pure state decompositions $\{p_i,\psi^i_{AB}\}$ of ρ_{AB}. In Ref. [45], Wootters derived an analytical formula for a two-qubit mixed state ρ_{AB},

$$E_f(\rho_{AB}) = H\left(\frac{1+\sqrt{1-\mathcal{C}^2_{AB}}}{2}\right), \tag{5}$$

where $H(x) = -x\ln x - (1-x)\ln(1-x)$ is the binary entropy and $\mathcal{C}_{AB} = \max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}$ is the concurrence of ρ_{AB}, with λ_i being the eigenvalues, in decreasing order, of matrix $\sqrt{\rho_{AB}(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)}$ [45].

In particular, Kim found $\mathcal{T}_q(\rho_{AB})$ has an analytical formula for a two-qubit mixed state, which can be expressed as a function of the squared concurrence $\mathcal{C}^2_{AB}$ for $1 \leqslant q \leqslant 4$ [29],

$$\mathcal{T}_q(\rho_{AB}) = f_q(\mathcal{C}^2_{AB}), \tag{6}$$

where the function $f_q(x)$ has the form,

$$f_q(x) = \frac{1}{q-1}\left[1-\left(\frac{1+\sqrt{1-x}}{2}\right)^q-\left(\frac{1-\sqrt{1-x}}{2}\right)^q\right]. \tag{7}$$

In this paper, we further prove that the analytical formula also holds for $q\in[\frac{5-\sqrt{13}}{2},\frac{5+\sqrt{13}}{2}]$, where $\frac{5-\sqrt{13}}{2}\approx 0.697$ and $\frac{5+\sqrt{13}}{2}\approx 4.302$. We refer the interested readers to Appendix A for the detailed calculation.

III. MONOGAMY OF TEE IN MULTIQUBIT SYSTEMS

Before presenting our main results, we have the following properties for TEE $f_q(\mathcal{C}^2)$.

Property 1. The squared Tsallis q-entropy entanglement $f_q^2(\mathcal{C}^2)$ is an increase monotonic and convex function of the squared concurrence $\mathcal{C}^2$ for any two-qubit mixed states, where $q\in[\frac{5-\sqrt{13}}{2},\frac{5+\sqrt{13}}{2}]$.

Property 2. The Tsallis q-entropy entanglement $f_q(\mathcal{C}^2)$ is an increase monotonic and concave function of the squared concurrence $\mathcal{C}^2$, where $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$.

We refer the interested readers to Appendixes B and C for the detailed proof for properties above. The region of q we considered for the properties is $q\in[\frac{5-\sqrt{13}}{2},\frac{5+\sqrt{13}}{2}]$.

It's well known that for any pure state in a $2\otimes d$ system, TEE has an analytical expression for $q>0$ [29]. We have the following result for any mixed state in a $2\otimes d$ system:

Theorem 1. For a mixed state ρ_{AC} in a $2\otimes d$ system, TEE has an analytical expression,

$$\mathcal{T}_q(\rho_{AC}) = f_q[\mathcal{C}^2(\rho_{AC})], \tag{8}$$

for $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$.

Proof. First, we should prove $\mathcal{T}_q(\rho_{AC}) \leqslant f_q[\mathcal{C}^2(\rho_{AC})]$. For $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$, consider a mixed state ρ_{AC} in a $2\otimes d$ system. We use an optimal convex decomposition $\{p_i,|\phi^i_{AC}\rangle\}$ for the TEE $\mathcal{T}_q(\rho_{AC})$:

$$\begin{aligned}\mathcal{T}_q(\rho_{AC}) &= \sum_i p_i \mathcal{T}_q(|\phi^i_{AC}\rangle)\\ &= \sum_i p_i f_q[\mathcal{C}^2(|\phi^i_{AC}\rangle)]\\ &\leqslant \sum_j s_j f_q[\mathcal{C}^2(|\psi^j_{AC}\rangle)]\\ &\leqslant f_q\left[\sum_j s_j \mathcal{C}^2(|\psi^j_{AC}\rangle)\right]\\ &= f_q[\mathcal{C}^2(\rho_{AC})],\end{aligned} \tag{9}$$

where we have used an optimal convex decomposition $\{s_j,|\psi^j_{AC}\rangle\}$ for concurrence $\mathcal{C}^2(\rho_{AC}) = \min\sum_j s_j\mathcal{C}^2(|\psi^j_{AC}\rangle)$ in the first inequality. The second inequality holds due to the function $f_q(\mathcal{C}^2)$ is a concave function of the squared concurrence $\mathcal{C}^2$ for $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$.

Second, we will prove $\mathcal{T}_q(\rho_{AC}) \geqslant f_q[\mathcal{C}^2(\rho_{AC})]$. We can obtain

$$\begin{aligned}\mathcal{T}_q(\rho_{AC}) &= \sum_i p_i \mathcal{T}_q(|\phi^i_{AC}\rangle)\\ &= \sum_i p_i f_q[\mathcal{C}(|\phi^i_{AC}\rangle)]\\ &\geqslant f_q\left\{\left[\sum_j s_j\mathcal{C}(|\psi^j_{AC}\rangle)\right]^2\right\}\\ &\geqslant f_q\left\{\left[\sum_k r_k\mathcal{C}(|\psi^j_{AC}\rangle)\right]^2\right\}\\ &= f_q[\mathcal{C}^2(\rho_{AC})],\end{aligned} \tag{10}$$

where the first inequality holds due to the convexity of $f_q(\mathcal{C}^2)$ as the function of concurrence $\mathcal{C}$ for $q>0$ (see Appendix A), and we have used the optimal convex decomposition $\{r_k,|\psi^k_{AC}\rangle\}$ for concurrence $\mathcal{C}(\rho_{AC}) = \min\sum_k r_k\mathcal{C}(|\psi^k_{AC}\rangle)$ in the second inequality, thus proving Theorem 1. ■

A straightforward corollary of Theorem 1 is as follows.

Corollary 1. For any mixed state in a $2\otimes d$ system, TEE obeys the following relation:

$$\mathcal{T}_q(\rho_{AC}) \geqslant f_q[\mathcal{C}^2(\rho_{AC})], \tag{11}$$

where $q>0$.

The Eq. (11) provides a lower bound for TEE in the $2\otimes d$ system.

Now we will study the monogamy property of TEE. We have the following theorem first.

Theorem 2. For a mixed state $\rho_{A|BC}$ in a $2\otimes 2\otimes 2^{N-2}$ system, the following monogamy inequality holds:

$$\mathcal{T}_q^2(\rho_{A|BC}) \geqslant \mathcal{T}_q^2(\rho_{AB}) + \mathcal{T}_q^2(\rho_{AC}), \tag{12}$$

where $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$.

Proof. Consider a mixed state $\rho_{A|B\mathbf{C}}$ in a $2\otimes 2\otimes 2^{N-2}$ system for $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$; from Eq. (8) we have

$$\begin{aligned}\mathcal{T}_q^2(\rho_{A|B\mathbf{C}}) &= f_q^2[\mathcal{C}^2(\rho_{A|B\mathbf{C}})]\\ &\geqslant f_q^2[\mathcal{C}^2(\rho_{AB})+\mathcal{C}^2(\rho_{A\mathbf{C}})]\\ &\geqslant f_q^2[\mathcal{C}^2(\rho_{AB})]+f_q^2[\mathcal{C}^2(\rho_{A\mathbf{C}})]\\ &= \mathcal{T}_q^2(\rho_{AB})+\mathcal{T}_q^2(\rho_{A\mathbf{C}}),\end{aligned}$$

where the first inequality holds because $f_q^2(x)$ is an increase monotonic function of the squared concurrence $\mathcal{C}^2$ and $\mathcal{C}^2(\rho_{A|B\mathbf{C}})\geqslant\mathcal{C}^2(\rho_{AB})+\mathcal{C}^2(\rho_{A\mathbf{C}})$ for concurrence [3]. The second inequality holds because of convexity of $f_q^2(\mathcal{C}^2)$ as a function of $\mathcal{C}^2$.

From Theorem 2, a set of hierarchical monogamy inequalities of $\mathcal{T}_q^2(\rho_{A_1|A_2\ldots A_N})$ holds for any N-qubit mixed state $\rho_{A_1A_2\ldots A_N}$ in k-partite cases with $k=\{3,4,\ldots,N\}$:

$$\mathcal{T}_q^2(\rho_{A_1|A_2\ldots A_N})\geqslant\sum_{i=2}^{k-1}\mathcal{T}_q^2(\rho_{A_1A_i})+\mathcal{T}_q^2(\rho_{A_1|A_k\ldots A_N}),\quad(13)$$

where $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$. These sets of hierarchical relations can be used to detect the multipartite entanglement in these k partites. When $k=N$, we have the following monogamy inequality for $q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]$:

$$\mathcal{T}_q^2(\rho_{A_1|A_2\ldots A_N})\geqslant\mathcal{T}_q^2(\rho_{A_1A_2})+\cdots+\mathcal{T}_q^2(\rho_{A_1A_N}).\quad(14)$$

One can wonder whether the monogamy inequality Eq. (14) still holds for $q\in[2,3]$. Here, we give an affirmative answer. In Ref. [29], the author proved the following inequality for $q\in[2,3]$,

$$\mathcal{T}_q(\rho_{A_1|A_2\ldots A_N})\geqslant\mathcal{T}_q(\rho_{A_1A_2})+\cdots+\mathcal{T}_q(\rho_{A_1A_N}),\quad(15)$$

which is easy to check that the inequality Eq. (14) also holds for $q\in[2,3]$ from Eq. (15). Thus we have following result.

Theorem 3. For a mixed state $\rho_{A_1A_2\ldots A_N}$ in an N-qubit system, the following monogamy inequality holds:

$$\mathcal{T}_q^2(\rho_{A_1|A_2\ldots A_N})\geqslant\mathcal{T}_q^2(\rho_{A_1A_2})+\cdots+\mathcal{T}_q^2(\rho_{A_1A_N}),\quad(16)$$

for $q\in[\frac{5-\sqrt{13}}{2},\frac{5+\sqrt{13}}{2}]$.

Bai *et al.* show that the squared entanglement of formation follows the general monogamy inequality in multiqubit systems [15,16]. Here, we prove the monogamous property of multiqubit entanglement can also be characterized in terms of squared TEE, where the monogamy inequality in terms of the squared entanglement of formation can be viewed as a special case for $q=1$.

As a result of Theorem 3, we also have the following corollary.

Corollary 2. For a mixed state $\rho_{A_1A_2\ldots A_N}$ in an N-qubit system, the αth power of TEE satisfies the monogamy inequality,

$$\mathcal{T}_q^\alpha(\rho_{A_1|A_2\ldots A_N})\geqslant\mathcal{T}_q^\alpha(\rho_{A_1A_2})+\cdots+\mathcal{T}_q^\alpha(\rho_{A_1A_N}),\quad(17)$$

for $\alpha\geqslant 2$ and $q\in[\frac{5-\sqrt{13}}{2},\frac{5+\sqrt{13}}{2}]$.

The proof can be found in Appendix D. We can view the coefficient α as a kind of assigned weight to regulate the monogamy property [19,46,47].

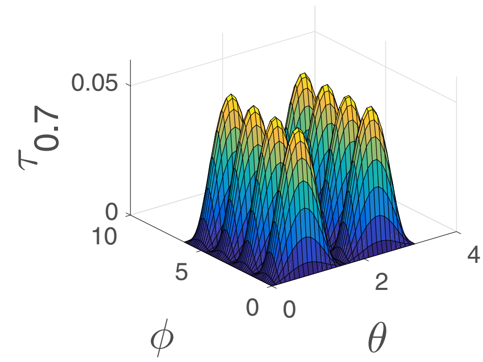

FIG. 1. The indicator $\tau_{0.7}(|W\rangle_G)$.

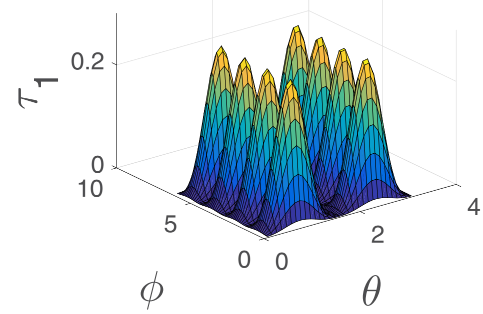

FIG. 2. The indicator $\tau_1(|W\rangle_G)$.

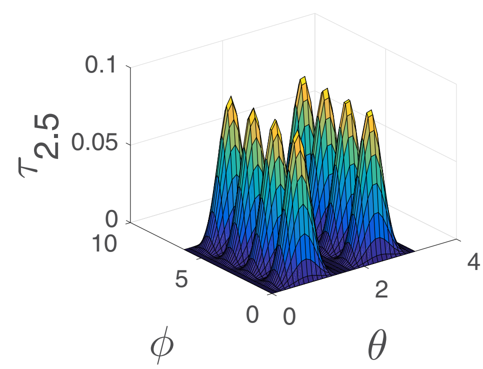

FIG. 3. The indicator $\tau_{2.5}(|W\rangle_G)$.

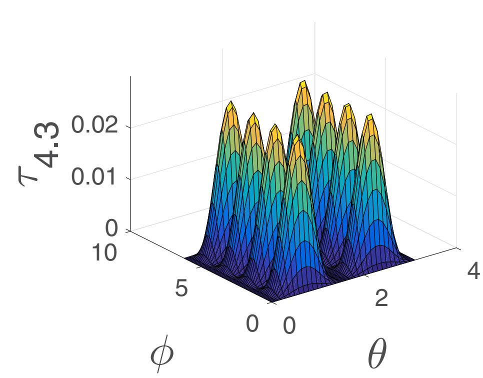

FIG. 4. The indicator $\tau_{4.3}(|W\rangle_G)$.

IV. A NEW KIND OF MULTIPARTITE ENTANGLEMENT INDICATOR

Based on Eq. (16), we can construct a class of multipartite entanglement indicator for $q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]$,

$$\tau_q\left(\rho_{A_1|A_2\ldots A_N}\right) = \min \sum_i p_i \tau_q\left(\left|\psi^i_{A_1|A_2\ldots A_N}\right\rangle\right), \tag{18}$$

where the minimum is taken over all possible pure state decompositions $\{p_i, \psi^i_{A_1|A_2\ldots A_N}\}$ of $\rho_{A_1A_2\ldots A_N}$ and $\tau_q(|\psi^i_{A_1|A_2\ldots A_N}\rangle = \mathcal{T}_q^2(\psi^i_{A_1|A_2\ldots A_N}) - \sum_{j=2}^N \mathcal{T}_q^2(\rho^i_{A_1A_j})$. Use the concavity of Tsallis q entropy for $q > 0$ [48], and follow the method of deriving the squared entanglement of formation in Ref. [15], we have following result.

Theorem 4. For any three-qubit mixed state ρ_{ABC}, the multipartite entanglement indicator $\tau_q(\rho_{A|BC})$ is zero if and only if ρ_{ABC} is biseparable, i.e., $\rho_{ABC} = \sum_i p_i \rho^i_{AB} \otimes \rho^i_C + \sum_j p_j \rho^j_{AC} \otimes \rho^j_B + \sum_k p_k \rho^k_A \otimes \rho^k_{BC}$.

We will show some examples as below.

Example 1. Coffman *et al*. considered a three-qubit general W state $|W\rangle_G = \sin\theta\cos\phi|001\rangle + \sin\theta\sin\phi|010\rangle + \cos\phi|100\rangle$ where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$, they found the three-tangle vanishes for every parameter θ and ϕ [2]. In this case, we consider the multipartite entanglement indicator shown in Eq. (18). For this state, the value of $\tau_q(|W\rangle_G)$ can be given by its analytical formula Eq. (6). In Figs. 1–4, we plot the indicator $\tau_q(|W\rangle_G)$ for $q = 0.7, 1, 2.5, 4.3$. The indicator $\tau_q(|W\rangle_G)$ shows that the $\tau_q(|W\rangle_G)$ is nonnegative for $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$, which vanishes when $|W\rangle_G$ is separable, thus the situation of $\theta = \frac{\pi}{2}, \pi$ and $\phi = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. For example, when $\theta = \frac{\pi}{2}$, the related state becomes $|W\rangle_G = \cos\phi|001\rangle + \sin\phi|010\rangle$ which is separable.

Example 2. We consider the N-qubit W state $|W\rangle_N = \frac{1}{\sqrt{N}}(|10\cdots0\rangle + |01\cdots0\rangle + |0\cdots01\rangle)$, the three-tangle cannot detect the entanglement of this state. By using the multipartite entanglement indicator shown in Eq. (18), we have $\tau_q(|W\rangle_N) = f_q^2(\frac{4(N-1)}{N^2}) - (N-1)f_q^2(\frac{4}{N^2})$. In Fig. 5, we plot the indicator $\tau_q(|W\rangle_N)$ for $N = 3, 6, 9, 11$, respectively. It shows that the indicator $\tau_q(|W\rangle)$ is always positive for $q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]$.

V. MONOGAMOUS EXAMPLES IN MULTIPARTITE HIGHER-DIMENSIONAL SYSTEM

In this section, let's consider several higher-dimensional examples to illustrate the monogamy inequality of TEE in Eq. (16). We define the "residual tangle" of TEE as

$$\tau_q\left(\left|\psi_{A_1A_2\ldots A_N}\right\rangle\right) = \mathcal{T}_q^2\left(\rho_{A_1|A_2\ldots A_N}\right) - \sum_{i=2}^N \mathcal{T}_q^2\left(\rho_{A_1A_i}\right). \tag{19}$$

Example 3 (Bai *et al*. [16]). Consider a tripartite pure state in a $4 \otimes 2 \otimes 2$ system,

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \beta|110\rangle + \alpha|201\rangle + \beta|311\rangle), \tag{20}$$

where $\alpha = \cos\theta$ and $\beta = \sin\theta$. Bai *et al*. point out the three-tangle is nonpositive for this state [16]. But the monogamy relation of squared TEE still works for this state when $q \in$

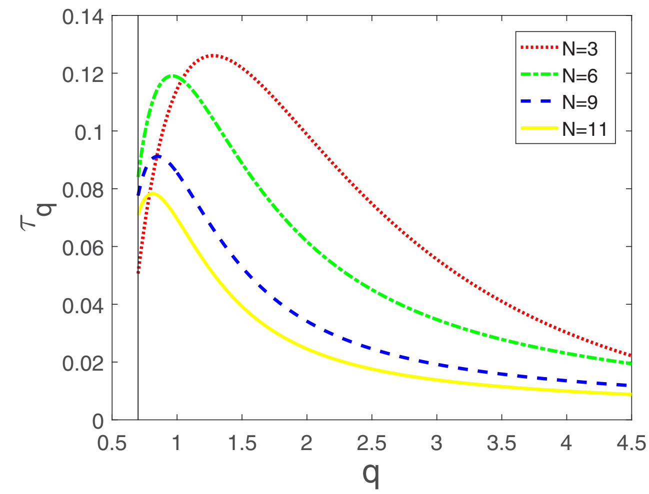

FIG. 5. The indicator $\tau_q(|W\rangle_N)$ is always positive for $q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]$.

$[1, \frac{5+\sqrt{13}}{2}]$:

$$\begin{aligned}\tau_q(|\psi_{A|BC}\rangle) &= \mathcal{T}_q^2(|\psi_{A|BC}\rangle) - \mathcal{T}_q^2(\rho_{AB}) + \mathcal{T}_q^2(\rho_{AC}) \\ &= \frac{(1-a)(1-b)}{(q-1)^2}[(1+a)(1+b) - 2] \\ &\geqslant 0,\end{aligned} \tag{21}$$

where $a = (\frac{1}{2})^{q-1}$ and $b = \alpha^{2q} + \beta^{2q}$. When $q = 1$, the TEE converges to entanglement of formation, which has been discussed in Ref. [16].

Example 4 (Ou [49]). Let $|\psi_{ABC}\rangle$ be a totally antisymmetric pure state on a three-qutrit system,

$$\begin{aligned}|\psi_{ABC}\rangle = \frac{1}{\sqrt{6}}(&|123\rangle - |132\rangle + |231\rangle \\ &- |213\rangle + |312\rangle - |321\rangle).\end{aligned} \tag{22}$$

Ou points out the CKW inequality in Ref. [2] does not work for this state [49]. However, for the squared TEE of this state,

$$\begin{aligned}&\tau_q(|\psi_{A|BC}\rangle) \\ &= \mathcal{T}_q^2(|\psi_{A|BC}\rangle) - \mathcal{T}_q^2(\rho_{AB}) + \mathcal{T}_q^2(\rho_{AC}) \\ &= \frac{1}{(q-1)^2}\left[\left(1 - \left(\frac{1}{3}\right)^{q-1}\right)^2 - 2\left(1 - \left(\frac{1}{2}\right)^{q-1}\right)^2\right],\end{aligned}$$

and the TEE can still work for this state when $q \in [\frac{5-\sqrt{13}}{2}, q_1]$, where $q_1 \approx 1.619$.

Example 5 (Kim *et al*. [17]). For a pure state $|\psi_{ABC}\rangle$ in a $3 \otimes 2 \otimes 2$ system,

$$|\psi_{ABC}\rangle = \tfrac{1}{6}(\sqrt{2}|121\rangle + \sqrt{2}|212\rangle + |311\rangle + |322\rangle). \tag{23}$$

Kim *et al*. shows that the CKW inequality does not work for this state [17].

The reduced state of subsystem A is

$$\rho_A = \frac{1}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{24}$$

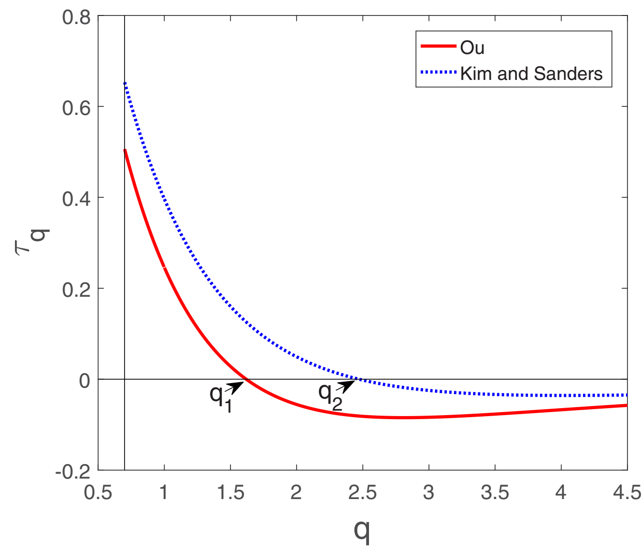

FIG. 6. The "residual tangle" $\tau_q(|\psi_{A|BC}\rangle)$ still works for $\frac{5-\sqrt{13}}{2} \leqslant q \leqslant q_1 \approx 1.619$ of Example 4 (solid red line) and for $\frac{5-\sqrt{13}}{2} \leqslant q \leqslant q_2 \approx 2.471$ of Example 5 (dashed blue line).

and the TEE of ρ_A is $\mathcal{T}_q(|\psi_{A|BC}\rangle) = \frac{1}{q-1}[1-(\frac{1}{3})^{q-1}]$. The bipartite reduced state of subsystem AB can be written as

$$\rho_{AB} = \tfrac{1}{2}(|x\rangle_{AB}\langle x| + |y\rangle_{AB}\langle y|), \tag{25}$$

where

$$|x\rangle_{AB} = \frac{\sqrt{2}}{\sqrt{3}}|12\rangle + \frac{1}{\sqrt{3}}|31\rangle, \tag{26}$$

$$|y\rangle_{AB} = \frac{\sqrt{2}}{\sqrt{3}}|21\rangle + \frac{1}{\sqrt{3}}|32\rangle. \tag{27}$$

It can be shown that for arbitrary pure states $|\phi_{AB}\rangle = c_x|x\rangle_{AB} + c_y|y\rangle_{AB}$ with $|c_x|^2 + |c_y|^2 = 1$; their reduced state $\rho_A = Tr_B(|\phi\rangle_{AB}\langle\phi|)$ has the same spectrum $\{0,1/3,2/3\}$. Then, the TEE of $|\phi_{AB}\rangle$ is $\mathcal{T}_q(|\phi_{AB}\rangle) = \frac{1}{q-1}[1-(1+2^q)(\frac{1}{3})^{q-1}]$. Thus, the TEE of ρ_{AB} is $\mathcal{T}_q(\rho_{AB}) = \frac{1}{q-1}[1-(1+2^q)(\frac{1}{3})^{q-1}]$. In the same way, the TEE of ρ_{AC} is $\mathcal{T}_q(\rho_{AC}) = \frac{1}{q-1}[1-(1+2^q)(\frac{1}{3})^{q-1}]$. We find the monogamy inequality of TEE still holds for $q \in [\frac{5-\sqrt{13}}{2}, q_2]$, where $q_2 \approx 2.471$.

As shown in Fig. 6, we have plotted "residual tangle" $\tau_q(|\psi_{A|BC}\rangle)$ as the function of q for the states of Examples 4 and 5, respectively. In the multipartite higher-dimensional system, the monogamy inequality Eq. (16) still works for the suitable parameter q.

VI. CONCLUSION

In this paper, we study the monogamy inequality of TEE. We provide an analytic formula of TEE in two-qubit systems for $\frac{5-\sqrt{13}}{2} \leqslant q \leqslant \frac{5+\sqrt{13}}{2}$. The analytic formula of TEE in $2 \otimes d$ system is also obtained and we show that TEE satisfies a set of hierarchical monogamy equalities. Furthermore, we prove the squared TEE follows a general inequality in the qubit systems. As a corollary, we provide the αth power of TEE satisfies the monogamy inequality for $\alpha \geqslant 2$. Based on the monogamy relations, a set of multipartite entanglement indicators is constructed, which can detect all genuine multiqubit entangled states even in the case of N-tangle vanishes. Moreover, we study some examples in the multipartite higher-dimensional system for the monogamy inequalities. Computing a variety of entanglement measures is NP hard [50], which implies (in a rigorous sense) that the analytical formulas of TEE for general mixed states are impossible unless $P = NP$. Thus, to find a useful method to compute general entanglement measures is still a problem. We may find other methods to derive new monogamy inequalities.

For entanglement of formation, its αth power satisfies the monogamy inequality in Eq. (17) for $\alpha \geqslant \sqrt{2}$ [12]. However, the monogamy inequality of the αth power of TEE does not work for $\alpha \geqslant \sqrt{2}$. To see this, we can consider the three-qubit W state $|W_{A|BC}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. Let $q = 0.7$ and $\alpha = \sqrt{2}$, we find that $\mathcal{T}_q^\alpha(|W_{A|BC}\rangle) - \mathcal{T}_q^\alpha(\rho_{AB}) - \mathcal{T}_q^\alpha(\rho_{Ac}) \approx -0.087 < 0$. Finally, we believe our results can be used in the quantum physics.

Note added in proof. Recently, we noted a similar work in Ref. [51].

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APPENDIX A: THE CRITICAL VALUE OF q FOR TWO-QUBIT STATE

In this section, we will discuss the analytic formula of TEE in two-qubit systems. Let us consider the monotonicity and convexity of $f_q(\mathcal{C}^2)$ as a function of $\mathcal{C}$, where $0 \leqslant \mathcal{C} \leqslant 1$. First, from Ref. [29], we obtain that $f_q(\mathcal{C}^2)$ is a monotonic increasing function of $\mathcal{C}$ for any $q > 0$ and $0 \leqslant \mathcal{C} \leqslant 1$. Second, we will consider the convexity of $f_q(\mathcal{C}^2)$ as a function of $\mathcal{C}$. Kim has proven the convexity of $f_q(\mathcal{C}^2)$ as a function of $\mathcal{C}$ for $1 \leqslant q \leqslant 4$ and the nonconvexity of $f_q(\mathcal{C}^2)$ as a function of $\mathcal{C}$ for $q \geqslant 5$ [29]. Thus, we only consider the situation of $0 < q < 1$ and $4 < q < 5$, respectively. The function $f_q(\mathcal{C}^2)$ is defined as

$$f_q(\mathcal{C}^2) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-\mathcal{C}^2}}{2}\right)^q - \left(\frac{1-\sqrt{1-\mathcal{C}^2}}{2}\right)^q\right]. \tag{A1}$$

The second derivative of $f_q(\mathcal{C}^2)$ is

$$\begin{aligned}\frac{\partial^2 f_q(\mathcal{C}^2)}{\partial \mathcal{C}^2} &= \alpha\Bigg[\frac{(1+\sqrt{1-\mathcal{C}^2})^{q-1}}{(1-\mathcal{C}^2)^{3/2}} - \frac{\mathcal{C}^2(q-1)(1+\sqrt{1-\mathcal{C}^2})^{q-2}}{(1-\mathcal{C}^2)} \\ &\quad - \frac{(1-\sqrt{1-\mathcal{C}^2})^{q-1}}{(1-\mathcal{C}^2)^{3/2}} - \frac{\mathcal{C}^2(q-1)(1-\sqrt{1-\mathcal{C}^2})^{q-2}}{(1-\mathcal{C}^2)}\Bigg],\end{aligned}$$

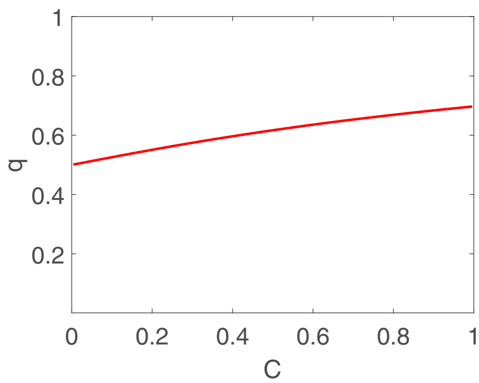

FIG. 7. The condition $\frac{\partial^2}{\partial \mathcal{C}^2} f_q(\mathcal{C}^2) = 0$ for $q \in [0,1]$.

where $\alpha = \frac{q}{2^q(q-1)}$. For the region $0 < q < 1$, the convexity of $f_q(\mathcal{C}^2)$ holds if $\frac{\partial^2}{\partial \mathcal{C}^2} f_q(\mathcal{C}^2) \geqslant 0$ for any concurrence $\mathcal{C}$. To find the region of q, we analyze the condition $\frac{\partial^2}{\partial \mathcal{C}^2} f_q(\mathcal{C}^2) = 0$. Numerical calculation shows that the value of q increases monotonically along with the increase of concurrence $\mathcal{C}$. As shown in Fig. 7, there may exist a critical point q_{c_1} corresponding to the limit $\mathcal{C} \to 1$ and the requirement that

$$\lim_{\mathcal{C}\to 1} \frac{\partial^2 f_q(\mathcal{C}^2)}{\partial \mathcal{C}^2} = 0. \tag{A2}$$

After some straightforward calculation, we derive the following equality:

$$-2(q-1)(q^2 - 5q + 3) = 0. \tag{A3}$$

The critical point of the region $0 < q < 1$ is $q_{c_1} = \frac{5-\sqrt{13}}{2} \approx 0.697$. The second derivative is nonnegative in this region, $q_{c_1} \leqslant q < 1$. For the region $4 < q < 5$, we obtain the critical point q_{c_2} with a similar method. As shown in Fig. 8, the value of q decreases monotonically along with the increase of concurrence $\mathcal{C}$; the critical point q_{c_2} can be obtained by the limit $\lim_{\mathcal{C}\to 1} \frac{\partial^2}{\partial \mathcal{C}^2} f_q(\mathcal{C}^2) = 0$. Thus the critical point of the region $4 < q < 5$ is $q_{c_2} = \frac{5+\sqrt{13}}{2} \approx 4.302$. The second derivative is nonnegative in this region, $4 < q \leqslant q_{c_1}$. Therefore, the second derivative is nonnegative for $q_{c_1} \leqslant q \leqslant q_{c_2}$ in the region of $0 < q < 5$. The analytic formula of TEE in two-qubit systems is in this region.

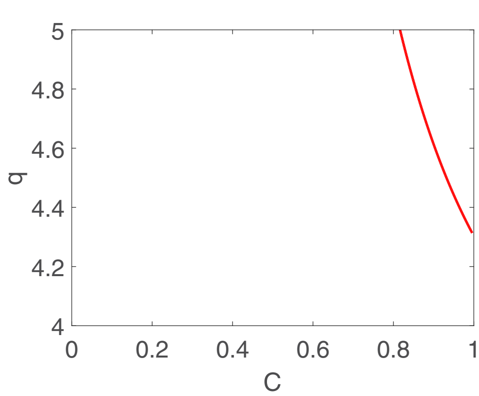

FIG. 8. The condition $\frac{\partial^2}{\partial \mathcal{C}^2} f_q(\mathcal{C}^2) = 0$ for $q \in [4,5]$.

APPENDIX B: $f_q^2(\mathcal{C}^2)$ IS AN INCREASING MONOTONIC AND CONVEX FUNCTION OF THE SQUARED CONCURRENCE $\mathcal{C}^2$

First, let's consider that the monotonicity of the function $f_q(x)$, $f_q(x)$ is defined as

$$f_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x}}{2}\right)^q - \left(\frac{1-\sqrt{1-x}}{2}\right)^q\right]. \tag{B1}$$

$f_q^2(\mathcal{C}^2)$ is an increasing monotonic function of the squared concurrence $\mathcal{C}^2$ and is equivalent to the first derivative $\frac{\partial}{\partial x} f_q^2(x) \geqslant 0$ with $q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]$ and $x = \mathcal{C}^2$. After some calculation, we have

$$\frac{\partial f_q^2(x)}{\partial x} = \frac{q f_q(x)}{2^q\sqrt{1-x}} \frac{A^{q-1} - B^{q-1}}{q-1}, \tag{B2}$$

where $A = 1 + \sqrt{1-x}$ and $B = 1 - \sqrt{1-x}$. It is easy to check that $\frac{\partial}{\partial x} f_q^2(x)$ is nonnegative for $q \geqslant 0$. Thus, $f_q^2(x)$ is an increasing monotonic function of x for $q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]$.

Second, the squared Tsallis q-entropy entanglement $f_q^2(\mathcal{C}^2)$ is a convex function of the squared concurrence $\mathcal{C}^2$ for $q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]$, which is equivalent to the second derivative $\frac{\partial^2}{\partial x^2} f_q^2(x) \geqslant 0$. Thus, we define the function,

$$l_q(x) = \frac{\partial^2 f_q^2(x)}{\partial x^2}, \tag{B3}$$

on the domain $D = \{(x,q) | x \in [0,1], q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]\}$. After a straightforward calculation, we have

$$\begin{aligned} l_q(x) = {} & \frac{q^2}{8(1-x)} \frac{(A^{q-1} - B^{q-1})^2}{2^{2(q-1)}(q-1)^2} \\ & + \frac{f_q(x)}{q-1}\left[\frac{q(1-q)}{8(1-x)} \frac{A^{q-2} + B^{q-2}}{2^{q-2}}\right. \\ & \left. + \frac{q}{4(1-x)^{3/2}} \frac{A^{q-1} - B^{q-1}}{2^{q-1}}\right]. \end{aligned}$$

The intermediate value theorem tells us if a continuous function on the domain has two values with opposite signs, there must exist a root on the domain. The function $l_q(x)$ is continuous on the domain D, and we plot the solution of $l_q(x) = 0$. As shown in Fig. 9, no point exists on the domain D such that $l_q(x) = 0$. Thus the value of $l_q(x)$ on the domain D have the same sign. When $q \to 1$, $f_q^2(\mathcal{C}^2)$ converges to squared entanglement of formation, in which the second derivative is positive [15]. Therefore, $l_q(x)$ is positive on the domain

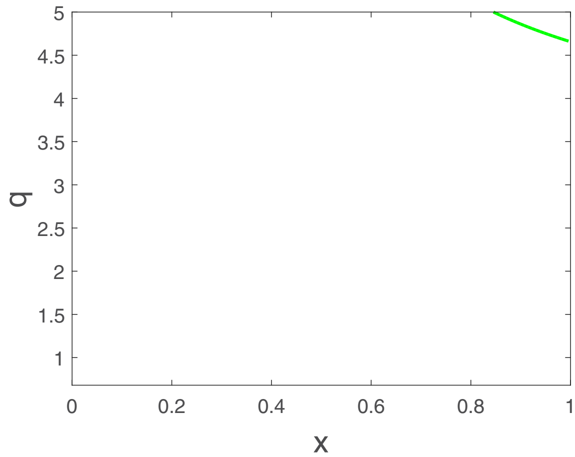

FIG. 9. The solution of $l_q(x) = 0$ on the domain D.

D. We have plotted the function $l_q(x)$ on the domain D in Fig. 10.

APPENDIX C: $f_q(\mathcal{C}^2)$ IS AN INCREASING MONOTONIC AND CONCAVE FUNCTION OF THE SQUARED CONCURRENCE $\mathcal{C}^2$

$f_q(\mathcal{C}^2)$ is an increasing monotonic function if the first derivative $\frac{\partial}{\partial x} f_q(x)$ is nonnegative.

$$\frac{\partial f_q(x)}{\partial x} = \frac{q}{2^{q+1}\sqrt{1-x}} \frac{A^{q-1} - B^{q-1}}{q-1}, \tag{C1}$$

which is nonnegative for $q \geqslant \frac{5-\sqrt{13}}{2}$ and $0 \leqslant x \leqslant 1$. Namely, $f_q(\mathcal{C}^2)$ is an increasing monotonic function of the squared concurrence $\mathcal{C}^2$.

The concavity of function $f_q(\mathcal{C}^2)$ is decided by the second derivative $\frac{\partial^2}{\partial x^2} f_q(x)$, and we define the function,

$$g_q(x) = \frac{\partial^2 f_q(x)}{\partial x^2}, \tag{C2}$$

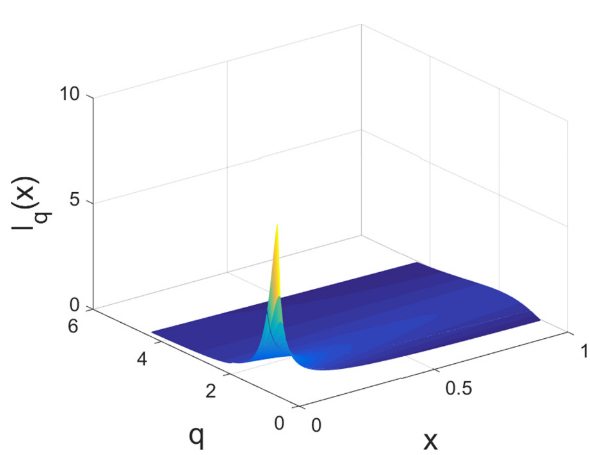

FIG. 10. The function $l_q(x)$ is positive on the domain D.

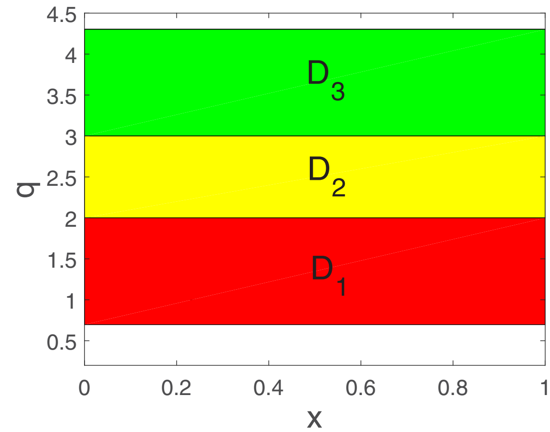

FIG. 11. The condition $g_q(x) = 0$, which holds on the domain only if $q = 2,3$ and cuts the domain D into three domains: D_1 (red), D_2 (yellow), and D_3 (green).

on the domain $D = \{(x,q)|x \in [0,1], q \in [\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}]\}$. We have

$$g_q(x) = \frac{q}{2^{q+2}(q-1)}\left[\frac{A^{q-2}}{1-x}\left(\frac{A}{\sqrt{1-x}} + (1-q)\right) - \frac{B^{q-2}}{1-x}\left(\frac{B}{\sqrt{1-x}} - (1-q)\right)\right]. \tag{C3}$$

In order to find the region of q such that $\frac{\partial^2}{\partial x^2} f_q(x) \leqslant 0$, we consider equality $\frac{\partial^2}{\partial x^2} f_q(x) = 0$ and plot the solution. As showed in Fig. 11, the equality holds on the domain only if $q = 2,3$, which cut the domain D into three domains: $D_1 = \{(x,q)|x \in [0,1], q \in [\frac{5-\sqrt{13}}{2}, 2]\}$, $D_2 = \{(x,q)|x \in [0,1], q \in (2,3]\}$, and $D_3 = \{(x,q)|x \in [0,1], q \in (3, \frac{5+\sqrt{13}}{2}]\}$. The corresponding functions for $q = 2,3$ are

$$f_2(x) = \frac{x}{2}, \quad f_3(x) = \frac{3x}{8}, \tag{C4}$$

where $0 \leqslant x \leqslant 1$. The intermediate value theorem tells us if a continuous function has two values on the domain with opposite signs, there must exist a root on the domain. The function $\frac{\partial^2}{\partial x^2} f_q(x)$ is a continuous function on the domain $D = D_1 \cup D_2 \cup D_3$. Therefore, we can consider the condition of $q = 1$, $q = \frac{5}{2}$, and $q = 4$ which are on the domain D_1, D_2, and D_3, respectively. When $q = 1$, the TEE converges to entanglement of formation, it has been proved in Ref. [16] that $g_1(x) < 0$ for $x \in [0,1]$. Thus, $g_q(x) < 0$ is nonpositive on the domain D_1 and equality holds only if $q = 2$. When $q = \frac{5}{2}$, we have

$$g_{\frac{5}{2}}(x) = -\frac{15}{64\sqrt{2}} \frac{A^{\frac{1}{2}} + B^{\frac{1}{2}}}{1-x} + \frac{5}{32\sqrt{2}} \frac{A^{\frac{3}{2}} - B^{\frac{3}{2}}}{(1-x)^{\frac{3}{2}}}. \tag{C5}$$

It's easy to check that $\lim_{x\to 0} g_{\frac{5}{2}}(x) = \frac{15}{128} > 0$ and $\lim_{x\to 1} g_{\frac{5}{2}}(x) = \frac{15}{256\sqrt{2}} > 0$. Thanks to the continuous $g_{\frac{5}{2}}(x)$ and the intermediate value theorem, we can obtain $g_{\frac{5}{2}}(x) > 0$ for $x \in [0,1]$. Thus, $g_q(x)$ is nonnegativity on the domain D_2 and equality holds only if $q = 3$. As showed in Fig. 12, the function $g_q(x)$ is nonnegativity on the domain D_2. When

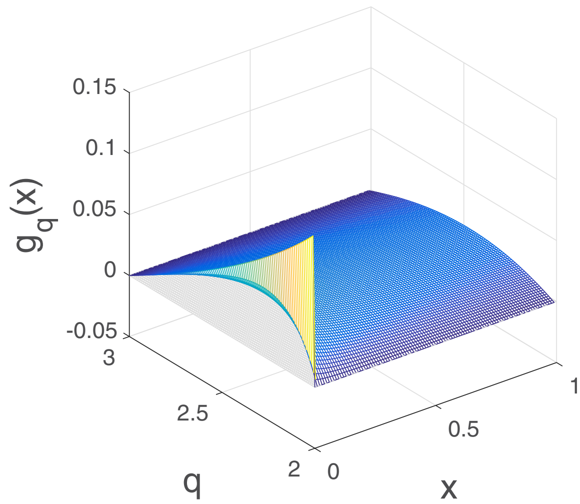

FIG. 12. $g_q(x)$ is nonnegativity on the domain D_2.

$q=4$, we have

$$f_4(x)=\frac{8x-x^2}{24}, \tag{C6}$$

and $g_4(x)=-\frac{1}{12}<0$ for $x\in[0,1]$. Thus, $g_q(x)<0$ is negativity on the domain D_3. Therefore, the function $f_q(x)$ is concave on the domain $D'=\{(x,q)|x\in[0,1],q\in[\frac{5-\sqrt{13}}{2},2]\cup[3,\frac{5+\sqrt{13}}{2}]\}$.

APPENDIX D: MONOGAMY OF THE αth POWER OF TEE

Assuming $\sum_{i=2}^{N-1}\mathcal{T}_q^2(\rho_{A_1A_i})\geqslant\mathcal{T}_q^2(\rho_{A_1A_N})$, from Eq. (16) we have

$$\begin{aligned}
&\mathcal{T}_q^\alpha(\rho_{A_1|A_2\ldots A_N})\\
&\geqslant\left(\mathcal{T}_q^2(\rho_{A_1A_2})+\cdots+\mathcal{T}_q^2(\rho_{A_1A_N})\right)^{\frac{\alpha}{2}}\\
&=\left(\sum_{i=2}^{N-1}\mathcal{T}_q^2(\rho_{A_1A_i})\right)^{\frac{\alpha}{2}}\left(1+\frac{\mathcal{T}_q^2(\rho_{A_1A_N})}{\sum_{i=2}^{N-1}\mathcal{T}_q^2(\rho_{A_1A_i})}\right)^{\frac{\alpha}{2}}\\
&\geqslant\left(\sum_{i=2}^{N-1}\mathcal{T}_q^2(\rho_{A_1A_i})\right)^{\frac{\alpha}{2}}\left(1+\left(\frac{\mathcal{T}_q^2(\rho_{A_1A_N})}{\sum_{i=2}^{N-1}\mathcal{T}_q^2(\rho_{A_1A_i})}\right)^{\frac{\alpha}{2}}\right)\\
&=\left(\sum_{i=2}^{N-1}\mathcal{T}_q^2(\rho_{A_1A_i})\right)^{\frac{\alpha}{2}}+\mathcal{T}_q^\alpha(\rho_{A_1A_N})\\
&\geqslant\mathcal{T}_q^\alpha(\rho_{A_1A_2})+\cdots+\mathcal{T}_q^\alpha(\rho_{A_1A_N}),
\end{aligned}$$

where the second inequality holds because the property $(1+x)^t\geqslant1+x^t$, where $0\leqslant x\leqslant1$ and $t\geqslant1$, and the third inequality holds because the property $(\sum x_i^2)^{\frac{\alpha}{2}}\geqslant\sum x_i^\alpha$, where $0\leqslant x_i\leqslant1$ and $\alpha\geqslant2$.

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