

Probabilistic quantum teleportation in the presence of noise

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We extend the research program initiated in [Phys. Rev. A **92**, 012338 (2015)] from noisy deterministic teleportation protocols to noisy probabilistic (conditional) protocols. Our main goal now is to study how we can increase the fidelity of the teleported state in the presence of noise by working with probabilistic protocols. We work with several scenarios involving the most common types of noise in realistic implementations of quantum communication tasks and find many cases where adding more noise to the probabilistic protocol increases considerably the fidelity of the teleported state, without decreasing the probability of a successful run of the protocol. Also, there are cases where the entanglement of the channel connecting Alice and Bob leading to the greatest fidelity is not maximal. Moreover, there exist cases where the optimal fidelity for the probabilistic protocols are greater than the maximal fidelity ($2/3$) achievable by using only classical resources, while the optimal ones for the deterministic protocols under the same conditions lie below this limit. This result clearly illustrates that in some cases we can only get a truly quantum teleportation if we use probabilistic instead of deterministic protocols.

DOI: [10.1103/PhysRevA.93.062330](https://doi.org/10.1103/PhysRevA.93.062330)**I. INTRODUCTION**

Quantum teleportation is a quantum communication task devised to transfer the quantum state of a physical system located at one place, say Alice's, to a different quantum system located at Bob's [1–6]. Two important aspects of the teleportation protocol are related to the fact that it works without the knowledge of the quantum state to be teleported and that the physical system originally described by this quantum state is not sent from Alice to Bob. For the perfect functioning of the teleportation protocol Alice and Bob need to share a maximally entangled state (maximally entangled quantum channel). In this case the protocol works deterministically and with unity fidelity, i.e., every run of the protocol ends up with Bob's system being described exactly by the original state teleported by Alice.

The requirement of a maximally entangled quantum channel connecting Alice and Bob is very difficult to achieve or maintain in practice since the inevitable presence of noise reduces the entanglement of the quantum state shared between them. In practical implementations of the teleportation protocol one can either adopt entanglement distillation techniques [7] or modify the original protocol in order to cope with the reduced level of entanglement [8–20]. In the first case Alice and Bob need to share several copies of partially entangled states before implementing an entanglement distillation protocol, whereby they obtain a maximally entangled state at the expense of many copies of partially entangled states. With this maximally entangled state, Alice and Bob are able to execute with success the original teleportation protocol. In the second case, the partially entangled state is used as is and the protocol is modified in order to achieve the greatest fidelity possible. In this last case, we can divide all strategies in two groups. In the first group we have the deterministic protocols and in the second group the probabilistic ones.

The deterministic protocols [11–20] do not postselect any measurement outcome at Alice's and therefore are always “successful” in the sense that any run of the protocol yields an output state to Bob, even if his state is not exactly described by the original (input) state with Alice. The probabilistic protocols, on the other hand, are not always successful as defined above since only certain measurement outcomes obtained by Alice are accepted. In the probabilistic protocols only those measurement outcomes leading to output states closest to the input are considered valid. In this way, by decreasing the success rate of the protocol one increases the fidelity of the state with Bob (output) with respect to the input state [8–10]. It is worth mentioning that the probabilistic protocols given in Refs. [8–10] assume the nonmaximally entangled state shared between Alice and Bob to be pure.

In this article we want to extensively study probabilistic teleportation protocols where the quantum channel connecting Alice and Bob are given by partially entangled mixed states. Our benchmarks are the optimal deterministic protocols given in Ref. [20], i.e., we want to find situations in which the reduction of the success rate (postselection) of the protocols in Ref. [20] gives considerable improvements in the fidelity of the teleported state.

Being more specific, here we deal with several scenarios involving the four most common types of noise one faces when implementing a quantum communication protocol: the bit-flip, the phase-flip or phase-damping, the depolarizing, and the amplitude-damping noises. We also study situations in which the state to be teleported is also subjected to noise. We show that several of the interesting results obtained in [20] for the deterministic protocols are also present in the probabilistic case. For example, we show scenarios where more noise increases the fidelity of the teleported state and where the entanglement of the quantum channel connecting Alice and Bob giving the greatest fidelity is not maximal. In addition to this, we show that there exist situations in which the probabilistic protocol outperforms the deterministic one in a very important aspect. Indeed, we show that there are scenarios where the optimal fidelity for the probabilistic protocol is not

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only greater than the optimal one for the deterministic protocol, but the only one surpassing the maximal value achievable by using only classical resources. This fact is a clear indication that in some scenarios a truly quantum teleportation can only be obtained by using probabilistic protocols.

II. TELEPORTATION IN THE DENSITY MATRIX FORMALISM

The mathematical concept needed to deal with noise and mixed states is the density matrix and thus the first thing we need to do is to recast the original teleportation protocol using density matrices. This was done in full detail in Ref. [20], and here we only give the key results necessary for the development of the ideas and concepts related to the probabilistic teleportation protocol.

The input qubit's density matrix, i.e., Alice's qubit to be teleported to Bob, $|\psi\rangle_{\text{in}} = a|0\rangle + b|1\rangle$, with $|a|^2 + |b|^2 = 1$, is

$$\rho_{\text{in}} = |\psi\rangle_{\text{in}} \langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}, \quad (1)$$

where the subscript *in* denotes ‘‘input’’ and * complex conjugation. The initially noiseless entangled state shared between Alice and Bob, $|B_1^\theta\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, has the following density matrix in the base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\rho_{\text{ch}} = |B_1^\theta\rangle \langle B_1^\theta| = \begin{pmatrix} \cos^2\theta & 0 & 0 & \sin\theta\cos\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin\theta\cos\theta & 0 & 0 & \sin^2\theta \end{pmatrix}. \quad (2)$$

Here ch means ‘‘channel’’ and the first and second qubits are with Alice and Bob, respectively. Note that θ is a free parameter that we can adjust to optimize the efficiency of the probabilistic teleportation. When $\theta = \pi/4$ we have the maximally entangled state $|\Phi^+\rangle$, one of the four Bell states. For any other value of $\theta \in [0, \pi/2]$ the entanglement of the state is not maximal, being zero for $\theta = 0$ and $\pi/2$ [21].

Using the above notation, the global state describing Alice's and Bob's qubits before the beginning of the protocol or the action of noise is

$$\rho = \rho_{\text{in}} \otimes \rho_{\text{ch}}. \quad (3)$$

The protocol begins by Alice making a projective measurement on her two qubits (the input state and her share of the entangled state). These qubits are projected onto one of the four states listed below that form a complete basis:

$$|B_1^\varphi\rangle = \cos\varphi|00\rangle + \sin\varphi|11\rangle, \quad (4)$$

$$|B_2^\varphi\rangle = \sin\varphi|00\rangle - \cos\varphi|11\rangle, \quad (5)$$

$$|B_3^\varphi\rangle = \cos\varphi|01\rangle + \sin\varphi|10\rangle, \quad (6)$$

$$|B_4^\varphi\rangle = \sin\varphi|01\rangle - \cos\varphi|10\rangle. \quad (7)$$

In the original protocol $\varphi = \pi/4$, with those states becoming the usual Bell states, $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle$, and $|\Psi^-\rangle$. Here φ is also a free parameter that is chosen to maximize the efficiency of the probabilistic teleportation. The projectors associated

with these four states are

$$P_j^\varphi = |B_j^\varphi\rangle \langle B_j^\varphi|, \quad j = 1, 2, 3, 4. \quad (8)$$

After this measurement the global state, Eq. (3), changes to

$$\tilde{\rho}_j = \frac{P_j^\varphi \rho P_j^\varphi}{\text{Tr}[P_j^\varphi \rho]} \quad (9)$$

with probability

$$Q_j(|\psi\rangle_{\text{in}}) = \text{Tr}[P_j^\varphi \rho], \quad (10)$$

where Tr is the trace operation. Note that we have explicitly written the dependence of Q_j on the input state $|\psi\rangle_{\text{in}}$, only for maximally entangled channels this probability is independent of the initial state [8–10].

In the second step of the protocol Alice tells Bob, using a classical communication channel, which $|B_j^\varphi\rangle$ she measured. After receiving this information, Bob knows that his state is now described by

$$\tilde{\rho}_{B_j} = \text{Tr}_{12}[\tilde{\rho}_j] = \frac{\text{Tr}_{12}[P_j^\varphi \rho P_j^\varphi]}{Q_j(|\psi\rangle_{\text{in}})}, \quad (11)$$

where Tr_{12} denotes the partial trace on qubits 1 and 2 (those with Alice).

In the third and last step of the protocol Bob implements a unitary operation U_j on his state in order to recover exactly the teleported state. After this unitary operation the final state with Bob is given by

$$\rho_{B_j} = U_j \tilde{\rho}_{B_j} U_j^\dagger = \frac{U_j \text{Tr}_{12}[P_j^\varphi \rho P_j^\varphi] U_j^\dagger}{Q_j(|\psi\rangle_{\text{in}})}. \quad (12)$$

It is worth noting that the unitary operation that Bob implements depends on Alice's measurement result and on the quantum channel used in the protocol. For ρ_{ch} given by Eq. (2), $U_1 = \mathbb{1}$, $U_2 = \sigma_z$, $U_3 = \sigma_x$, and $U_4 = \sigma_z \sigma_x$, where $\mathbb{1}$ is the identity matrix and σ_z and σ_x the standard Pauli matrices.

III. TELEPORTATION IN THE PRESENCE OF NOISE

The operator-sum representation formalism [22,23] is the mathematical concept we need to model in the simplest way the action of noise on a qubit. The key concept behind this formalism is that the noise can be described only by quantum operations belonging to the qubit's Hilbert space. The operators E_k representing a particular kind of noise are called Kraus operators, and for trace-preserving operations (conservation of probability) they must obey the condition

$$\sum_{j=1}^n E_j^\dagger E_j = \mathbb{1}, \quad (13)$$

where $\mathbb{1}$ is the identity operator acting on the qubit's Hilbert space and $1 \leq n \leq 4$. The action of the noise on the qubit k , described by the density matrix ρ_k , is

$$\rho_k \rightarrow Q_k = \sum_{j=1}^n E_j \rho_k E_j^\dagger. \quad (14)$$

Throughout this section we follow closely the notation and presentation of Ref. [20] and just list the most common

TABLE I. Here $p \in [0,1]$ is the probability that the noise has acted on the qubit, and $\sigma_j, j = x, y$, and z , are the standard Pauli matrices.

Bit flip	$E_1 = \sqrt{1-p} \mathbb{1}, \quad E_2 = \sqrt{p} \sigma_x.$
Phase flip (Phase damping)	$E_1 = \sqrt{1-p} \mathbb{1}, \quad E_2 = \sqrt{p} \sigma_z.$
Depolarizing	$E_1 = \sqrt{1-3p/4} \mathbb{1}, \quad E_2 = \sqrt{p/4} \sigma_x,$ $E_3 = \sqrt{p/4} \sigma_y, \quad E_4 = \sqrt{p/4} \sigma_z.$
Amplitude damping	$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$

types of noise we usually find in any realistic modeling of a qubit lying in a noisy environment. We consider four types of noise, namely, the bit-flip, the phase-flip or phase-damping, the depolarizing, and the amplitude-damping channels. The physical meanings of each one of these noise channels are extensively discussed in Refs. [22] and [23] and a brief discussion can be found in Ref. [20]. The Kraus operators representing the action of those noise channels are given in Table I.

Assuming that each qubit in the teleportation protocol is acted on by noise in an independent way, the global density matrix describing the initial state, Eq. (3), becomes [20]

$$\varrho = \sum_{i=1}^{n_I} \sum_{j=1}^{n_A} \sum_{k=1}^{n_B} E_{ijk}(p_I, p_A, p_B) \rho E_{ijk}^\dagger(p_I, p_A, p_B). \quad (15)$$

Equation (15) is obtained by applying Eq. (14) to each one of the qubits in Eq. (3). Here $E_{ijk}(p_I, p_A, p_B) = E_i(p_I) \otimes F_j(p_A) \otimes G_k(p_B)$, where $E_i(p_I) = E_i(p_I) \otimes \mathbb{1} \otimes \mathbb{1}$, $F_j(p_A) = \mathbb{1} \otimes F_j(p_A) \otimes \mathbb{1}$, and $G_k(p_B) = \mathbb{1} \otimes \mathbb{1} \otimes G_k(p_B)$ are, respectively, the Kraus operators related to the noise acting on the input qubit and Alice's and Bob's qubits of the quantum channel. In order to keep track that in general different types of noises can act during different times (probabilities), we explicitly show the dependence of the Kraus operators on those probabilities: p_I, p_A , and p_B . The density matrix ϱ , Eq. (15), should be used instead of ρ in Eqs. (9) to (12) to get the relevant quantities needed to analyze the probabilistic teleportation protocol in the presence of noise.

IV. RATE OF SUCCESS AND EFFICIENCY OF THE NOISY PROBABILISTIC TELEPORTATION

In the presence of noise [20], or when we deal with nonmaximally entangled channels [8–10], the probability Q_j of Alice measuring a determined generalized Bell state $|B_j^\varphi\rangle$ depends on the input state $|\psi\rangle_{\text{in}}$ to be teleported. Thus, in order to be as general as possible and to get results that are independent of a specific input state, we assume a uniform probability distribution

$$P_X(x) = \mathcal{P}(|\psi\rangle_{\text{in}}) \quad (16)$$

for those input states [20]. Here X is a continuous random variable whose possible values x are all pure qubits that define the sample space Ω . We will work with a probability

distribution $P_X(x)$ that is normalized,

$$\int_{\Omega} P_X(x) dx = \int_{\Omega} \mathcal{P}(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}} = 1, \quad (17)$$

and, as we said, uniform (Haar measure), i.e., $P_X(x)$ is the same (constant) for all x . With this choice for $P_X(x)$ all qubits have equal chances of being picked by Alice at each run of the protocol.

Being more specific, writing an arbitrary qubit as

$$|\psi\rangle = \alpha|0\rangle + \beta e^{i\gamma}|1\rangle, \quad (18)$$

with α, β , and γ positive real numbers such that $\alpha^2 + \beta^2 = 1$ and $\gamma \in [0, 2\pi]$, we can choose α^2 and γ as our independent variables. With this notation $\mathcal{P}(|\psi\rangle_{\text{in}}) = \mathcal{P}(\alpha^2, \gamma)$ and the normalization condition, Eq. (17), becomes

$$\int_0^{2\pi} \int_0^1 \mathcal{P}(\alpha^2, \gamma) d\alpha^2 d\gamma = 1. \quad (19)$$

For a uniform probability distribution [$\mathcal{P}(\alpha^2, \gamma)$ constant] Eq. (19) implies

$$\mathcal{P}(\alpha^2, \gamma) = \frac{1}{2\pi}. \quad (20)$$

We also have a discrete variable J whose values can be $j = 1, 2, 3$, and 4 (or $j = \Phi^+, \Phi^-, \Psi^+$, and Ψ^-), with each j representing one of the four possible generalized Bell states $|B_j^\varphi\rangle$. The probability to measure a given $|B_j^\varphi\rangle$ is written as $P_J(j)$. The conditional probability $P_{J|X}(j|x)$ is the chance of Alice measuring the Bell state j if she teleports the input state x and it is given by Eq. (10),

$$P_{J|X}(j|x) = Q_j(|\psi\rangle_{\text{in}}). \quad (21)$$

To determine $P_J(j)$ we first determine the joint probability distribution $P_{XJ}(x, j)$ by applying the well-known result of probability theory that says that

$$P_{XJ}(x, j) = P_{JX}(j, x) = P_X(x) P_{J|X}(j|x). \quad (22)$$

Thus, using Eq. (22) we get

$$P_{XJ}(x, j) = \mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}}). \quad (23)$$

Now, since the marginal probability distribution is $P_J(j) = \int_{\Omega} P_{XJ}(x, j) dx$, we have

$$P_J(j) = \int_{\Omega} \mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}}. \quad (24)$$

At last, using Eq. (24) and again Eq. (22) with the roles of X and J interchanged we get

$$\begin{aligned} P_{X|J}(x|j) &= \frac{P_{XJ}(x, j)}{P_J(j)} \\ &= \frac{\mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}})}{\int_{\Omega} \mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}}}. \end{aligned} \quad (25)$$

Equations (24) and (25) are the relevant probability distributions we need to quantitatively analyze the probabilistic teleportation protocol. Indeed, $P_J(j)$ is the probability to measure a given generalized Bell state j given a certain distribution for the input states and it can be interpreted as

the average chance of measuring $|B_j^\varphi\rangle$,

$$\overline{Q}^j = P_J(j) = \int_{\Omega} \mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}}. \quad (26)$$

This quantity is independent of $|\psi\rangle_{\text{in}}$ and it is referred to here as the success rate or probability of success of the probabilistic teleportation protocol when we postselect a particular measurement result j . $P_{X|J}(x|j)$, as we will show shortly, is the quantity we need to compute the input-state-independent efficiency of the protocol once we fix our attention to a given measurement outcome j . $P_{X|J}(x|j)$ is the probability distribution of the input states x when we consider only (postselect) those measurement results at Alice's yielding the same generalized Bell state j .

To quantify the efficiency of the probabilistic teleportation protocol we use the fidelity [24]. Since in our analysis the input state (our benchmark) is initially pure, the fidelity is

$$F_j = \text{Tr}[\rho_{\text{in}} \varrho_{B_j}] = {}_{\text{in}}\langle \psi | \varrho_{B_j} | \psi \rangle_{\text{in}}, \quad (27)$$

where ϱ_{B_j} is the state with Bob at the end of a run of the protocol, Eq. (12), with ρ changed to the noisy state ϱ , Eq. (15). Equation (27) ranges from zero to 1, being 1 whenever the output state (ϱ_{B_j}) is equal (up to an irrelevant global phase) to the input ($|\psi\rangle_{\text{in}}$) and zero whenever the two states are orthogonal.

Since F_j depends on the input state $|\psi\rangle_{\text{in}}$, we must average F_j over all possible input states to obtain a quantitative description of the efficiency of the protocol that is independent of $|\psi\rangle_{\text{in}}$. Since the probability distribution for $|\psi\rangle_{\text{in}}$ within a given fixed choice of measurement result j is $P_{X|J}(x|j)$ [Eq. (25)], we get

$$\begin{aligned} \overline{F}^j &= \int_{\Omega} F_j(x) P_{X|J}(x|j) dx \\ &= \frac{\int_{\Omega} F_j(|\psi\rangle_{\text{in}}) \mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}}}{\int_{\Omega} \mathcal{P}(|\psi\rangle_{\text{in}}) Q_j(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}}} \end{aligned} \quad (28)$$

for the efficiency of the probabilistic teleportation protocol when postselecting the measurement result j . Note that if we consider all measurement outcomes as acceptable we recover the deterministic protocol of Ref. [20]. In the present notation the quantity employed in Ref. [20] to quantify the efficiency of the deterministic protocol reads

$$\langle \overline{F} \rangle = \sum_{j=1}^4 P_J(j) \overline{F}^j = \int_{\Omega} \overline{F}(|\psi\rangle_{\text{in}}) \mathcal{P}(|\psi\rangle_{\text{in}}) d|\psi\rangle_{\text{in}}, \quad (29)$$

where $\overline{F}(|\psi\rangle_{\text{in}}) = \sum_j Q_j(|\psi\rangle_{\text{in}}) F_j(|\psi\rangle_{\text{in}})$. One of our goals in this work is to optimize Eq. (28) such that $\overline{F}^j > \langle \overline{F} \rangle$, with $\langle \overline{F} \rangle$ being the optimal efficiency of the deterministic protocol. In this case the probabilistic protocol outperforms the deterministic one in terms of efficiency, i.e., the teleported state with Bob is closer to the original one with Alice. The price we pay is a reduction of the probability of success since we have to discard measurement results different from j .

Summing up, Eqs. (24) and (28) are the relevant expressions employed here to quantify, respectively, the probability of success and the efficiency (fidelity) of the probabilistic teleportation protocol, and Eq. (29), the efficiency of the

deterministic protocol, is the benchmark we want to surpass by optimizing (28). With these equations and the ideas and concepts here developed, we are now ready to move on to the quantitative analysis of the interplay between probability of success and efficiency for several noise scenarios in the next section.

V. RESULTS

We study the efficiency of the probabilistic teleportation protocol in the three noise scenarios presented in Ref. [20] for the deterministic protocol. The first one assumes that only Bob's qubit is subjected to noise in addition to the input qubit, which can suffer the action of the same or of a different type of noise [see Fig. 1(a)]. Note that by choosing Alice's qubit of the quantum channel to be acted on by noise instead of Bob's leads to the same results [20]. The second scenario we investigate is the one in which the entangled states shared by Alice and Bob are subjected to the same kind of noise during the same time, while the input qubit can suffer the action of any one of the four types of noises explained in Sec. III [see Fig. 1(b)]. This situation occurs when the quantum channel is created by a third party symmetrically located between Alice and Bob such that both qubits of the channel find similar noisy environments during their flights to Alice and to Bob. In the notation of Sec. III this implies that $p_A = p_B = p$. The third scenario we investigate is the one in which all Alice's qubits are subjected to the same kind of noise while Bob's qubit can suffer the action of the same or of a different noise [see Fig. 1(c)]. This scenario is relevant when it is Alice that generates the entangled channel. In such a case the input qubit and her share of the entangled state lie in the same environment and therefore are acted on by the same noise and during the same time. In the notation of Sec. III it means that $p_i = p_A = p$.

Before we continue it is important to review and adapt the notation introduced in Ref. [20] designed to concisely label which qubits are subjected to a particular kind of noise in the expressions for the probability of success and efficiency that will follow. In this notation any quantity that depends on the arrangement of the types of noise acting on the three qubits of the teleportation protocol is written with three subindexes representing each type of noise. For example, the average probability (probability of success) of Alice obtaining a given

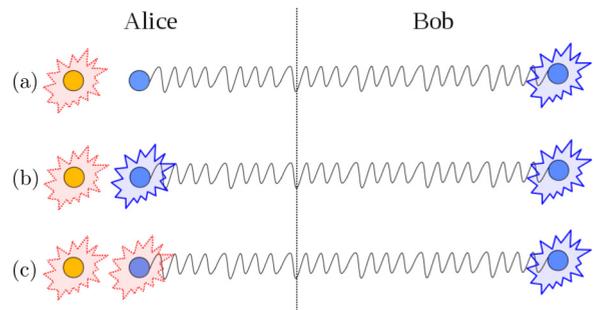


FIG. 1. The three noise scenarios studied here: (a) noise acting on Alice's input qubit and on Bob's output qubit; (b) noise acting on the input qubit and the same type of noise acting on the qubits of the channel; and (c) noise on Bob's qubit and the same type of noise on Alice's qubits. See text for details.

Bell state j for a given noise configuration is written as $\overline{Q}_{x,\varnothing,y}^j$, where the first subindex denotes that the input qubit is subjected to noise X , the second one represents that Alice's qubit of the quantum channel lies in a noiseless environment, and the third subindex tells us that Bob's qubit is subjected to noise Y .

A. Scenario 1

The first scenario we investigate is the one depicted in Fig. 1(a), where only the input and Bob's qubit are subjected to noise. The input qubit can suffer the action of any one of the four types of noise given in Sec. III as well as Bob's. We thus have 16 possible noise arrangements. For each one of these arrangements we have optimized all four \overline{F}^j [Eq. (28)] as a function of θ and φ , variables related, respectively, to the initial entanglement (prior to the action of noise) of the quantum channel and to the projective measurement implemented by Alice. See Eqs. (2) and (4)–(7). Comparing within a given noise arrangement the four optimal \overline{F}^j with the optimal $\langle \overline{F} \rangle$, the efficiency for the deterministic protocol [Eq. (29)], we noted that only four out of these 16 possibilities yielded at least one j such that $\overline{F}^j > \langle \overline{F} \rangle$ (see Fig. 2).

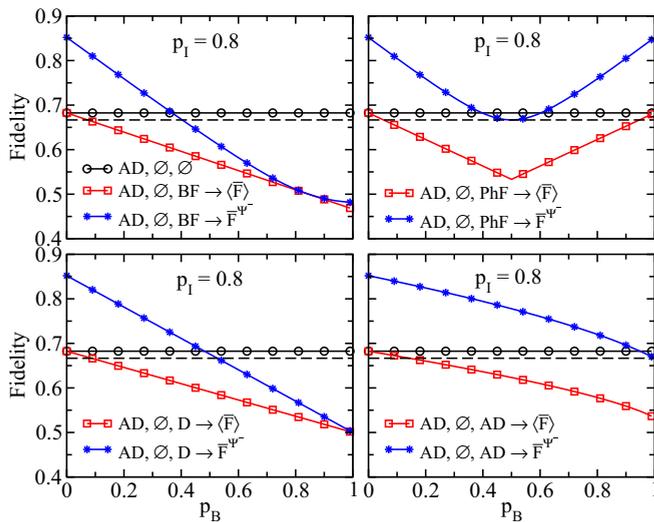


FIG. 2. Optimal efficiencies (average fidelities) as a function of p_B for the deterministic protocol when only the input qubit is subjected to noise (circle–black curves), for the deterministic protocol when both the input and Alice's qubit are subjected to noise (square–red curves), and for the probabilistic protocol with both input and Bob's qubit acted on by noise (star–blue curves). The several noise arrangements and the value of p_1 are given in the figure. The dashed-black curves mark the classical limit $2/3$ for the fidelity. The probabilistic optimal efficiencies are those obtained by postselecting $|\Psi^-\rangle$. The optimal \overline{F}^j are the same for all possible measurement results if Bob's qubit is subjected to the bit-flip, phase-flip, or depolarizing noise. When Bob's qubit is subjected to the amplitude-damping noise, the best results are obtained only if Alice postselects $|\Psi^+\rangle$ or $|\Psi^-\rangle$. For the range of p_B in which $\overline{F}^j > 2/3$, the rate of success \overline{Q}^j for all four cases are of the order of 10%, with the lowest values being around 5%. Here and in the following figures, all plotted quantities are dimensionless.

A common feature among these four cases is the action of the amplitude-damping noise on the input qubit. Indeed, if the input qubit is subjected to any other type of noise, the optimal probabilistic efficiency satisfies $\overline{F}^j = \langle \overline{F} \rangle$ for all j . Another feature shared by these four cases is the fact that the initial entanglement of the quantum channel connecting Alice and Bob giving the optimal efficiency is not maximal whenever $p_1 \neq 0$. This is a situation where *less* entanglement leads to *more* efficiency. This same feature is seen for the deterministic protocol when we also deal with the amplitude-damping noise [18,20]. For the other 12 cases in this scenario, the initial entanglement leading to the optimal efficiency is maximal for both the deterministic and probabilistic protocols.

In Fig. 2 we show for these four cases the optimal values for the efficiency of the probabilistic protocol versus the optimal one for the deterministic protocol when $p_1 = 0.8$ and for all values of p_B . For other values of p_1 we have the same qualitative behavior. Looking at Fig. 2 we notice another important feature shared by all four cases. We can always find a critical value for p_B below which $\overline{F}_{AD,\varnothing,Y}^j > \langle \overline{F} \rangle_{AD,\varnothing,Y}$ and at the same time $\overline{F}_{AD,\varnothing,Y}^j > \langle \overline{F} \rangle_{AD,\varnothing,\varnothing}$. In other words, the optimal probabilistic efficiency is not only greater than the optimal one for the deterministic protocol under the same noise conditions but also greater than the optimal deterministic protocol efficiency when Bob's qubit is not subjected to noise. This is an instance where *more* noise leads to *more* efficiency.

We also have two interesting results in those four noise arrangements. The first one occurs for high values of p_1 . Under this condition the fidelities of the deterministic protocols almost always lie below $2/3$, being slightly above this value only for very small values of p_B (see Fig. 2). Any fidelity below $2/3$ can be achieved using only classical resources (no need for entanglement) and the teleportation protocol is considered genuinely quantum for a uniform probability distribution (Haar measure) of input states only if we have fidelities greater than $2/3$ [25]. On the other hand, for the probabilistic protocol we can significantly surpass the classical limit for a considerable range of values for p_B . For the noise arrangements where Bob's qubit is subjected to either the phase-flip or the amplitude-damping noise, we obtain for almost all values of p_B fidelities greater than $2/3$, clearly illustrating that the probabilistic protocols are the only ones leading to a truly quantum teleportation. The second interesting result occurs when noise is unavoidable and Bob can choose in which noisy environment to keep his qubit. In such a case subjecting his qubit to a different kind of noise than that acting on the input qubit can be beneficial. This does not change the probability of success, since it only depends on what is happening at Alice's, but increases the efficiency of the protocol. For example, looking at Fig. 2 we see that $\overline{F}_{AD,\varnothing,PhF}^j > \overline{F}_{AD,\varnothing,AD}^j$ for high values of p_B . This is an illustration that *different* noises lead to *more* efficiency.

B. Scenario 2

Let us now move to the case where both qubits of the quantum channel are acted on by the same noise while Alice's input qubit is subjected to the same or a different type of noise [see Fig. 2(b)]. In this scenario $p_A = p_B = p$,

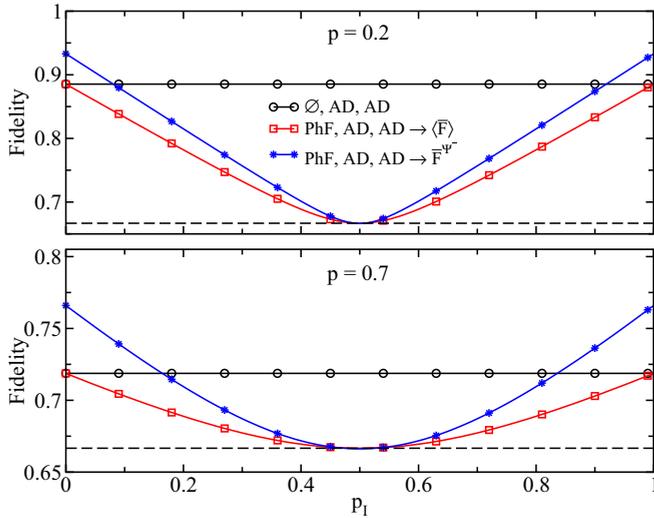


FIG. 3. Optimal efficiencies (average fidelities) as a function of p_I for the deterministic protocol when only the quantum channel is subjected to noise (circle–black curves) and for the deterministic (square–red curves) and probabilistic (star–blue curves) protocols when all three qubits are subjected to noise. The noise arrangements and the value of $p = p_A = p_B$ are in the figures. The dashed-black curves mark the classical limit $2/3$ for the fidelity. The optimal efficiencies for the probabilistic protocol are those obtained by postselecting $|\Psi^-\rangle$. Note that for almost all values of p_I the probabilistic protocol outperforms the deterministic one. Moreover, for small ($\lesssim 0.1$) and high ($\gtrsim 0.9$) values of p_I the probabilistic protocol gives a better result, even when compared to the deterministic protocol in which no noise acts on the input qubit (circle–black curves). The success rate \bar{Q}^j for this to happen is of the order of 0.5% for the two values of p shown above. This is another example where *more* noise leads to *more* efficiency.

and similarly to the previous scenario we have 16 possible combinations of noise. In order to optimize the efficiency of the probabilistic protocols we proceeded in the same way as explained in scenario 1. Out of these 16 arrangements, only those seven in which the amplitude-damping noise is present yield probabilistic protocols with optimal efficiencies greater than the optimal ones for the deterministic protocols. For these seven cases the initial entanglement of the entangled state shared by Alice and Bob giving the optimal efficiency is not maximal, similarly to what we have seen in scenario 1. This is again a situation where *less* entanglement leads to *more* efficiency.

In Fig. 3 we show the optimal fidelities for the deterministic and probabilistic protocols for the noise arrangement in which the quantum channel is subjected to the amplitude-damping noise and the input qubit to the phase-flip noise (PhF, AD, AD). But to one feature the same qualitative behaviors seen in this case are also present when the input qubit is subjected to the bit-flip (BF, AD, AD) and depolarizing (D, AD, AD) noises. The only notable qualitative difference is related to the fact that while for the PhF, AD, AD case the optimal efficiencies are symmetrical with respect to the line $p_I = 0.5$, this is not seen in the BF, AD, AD and D, AD, AD cases. For these last two cases, the greater p_I the lower the efficiency.

We have also noted an important fact concerning the numerical optimization of the efficiency for the probabilistic protocols whenever the two qubits of the quantum channel are acted on by the amplitude-damping noise. In this situation the tradeoff between efficiency and rate of success plays a crucial role in defining the range of values that θ (initial entanglement of the channel) and φ (measuring basis) can assume during the numerical search for the optimal efficiency. Indeed, if we allow θ and φ to run over all their possible values, i.e., from zero (no entanglement) to $\pi/4$ (maximal entanglement) and to $\pi/2$ (no entanglement), we are faced with solutions that give very high values for the efficiency ($\bar{F}^j \approx 1$) while the rate of success is zero to the precision adopted in the maximization algorithm (eight numerical figures). The optimal θ in this case is almost zero, which means a quantum channel with almost no entanglement. In order to avoid those unphysical solutions, we have restricted the ranges of θ and φ to be such that $\theta_{\min} \leq \theta, \varphi \leq \theta_{\max}$. We observed that the more we restricted the range of θ and φ , the greater the probability of success and the lower the efficiency; and when we set $\theta_{\min} = \theta_{\max} = \pi/4$, the probabilistic protocol gives the same efficiency of the deterministic protocol. The results presented in Fig. 3, in the circle–black curve of Fig. 5, and in Fig. 6 were obtained by setting $\theta_{\min} = 0.05\pi/2 = 0.07854$ and $\theta_{\max} = 0.95\pi/2 = 2.984$. For all the other optimal results reported here, we have assumed $0 \leq \theta, \varphi \leq \pi/2$.

In Fig. 4 we show the results obtained when we have the bit-flip noise acting on the qubits of the quantum channel and the amplitude-damping noise acting on the input qubit. The values of θ and φ employed to draw those curves are the ones optimizing $\bar{F}_{AD, BF, BF}^{\Phi^+}$. Now, contrary to the case where the amplitude-damping noise acted on the qubits

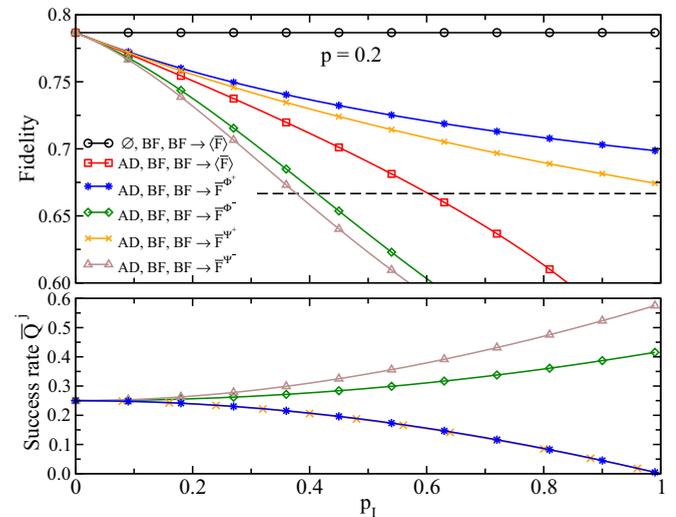


FIG. 4. Top panel: Optimal efficiencies (average fidelities) for the deterministic and probabilistic protocols as a function of p_I for the noise arrangement involving the action of the bit-flip noise on the quantum channel and the amplitude-damping noise on the input qubit. Bottom panel: The probability of success associated to each one of the four possible measurement results of Alice. The values of θ and φ used to plot all \bar{F}^j and \bar{Q}^j are those that maximize \bar{F}^{Φ^+} . The dashed-black curve marks the classical limit $2/3$ for the fidelity.

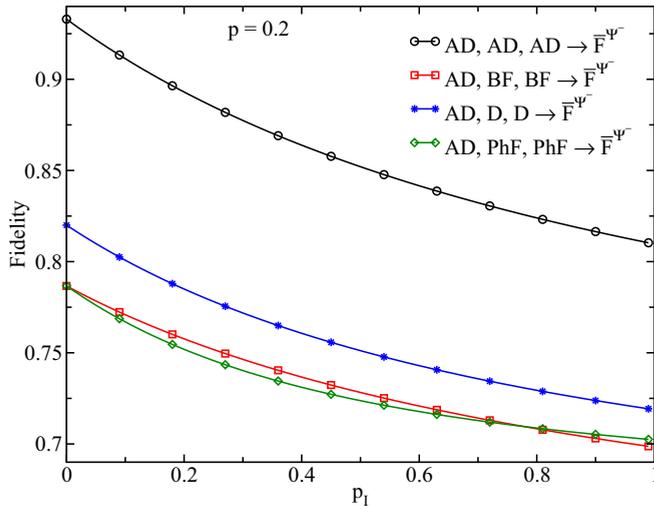


FIG. 5. Optimal efficiencies for the probabilistic protocol when different types of noise act on the qubits of the quantum channel.

of the channel, the optimal efficiency of the probabilistic protocol does not surpass the optimal efficiency of the deterministic protocol when no noise acts on the input qubit, i.e., $\bar{F}_{AD,BF,BF}^j < \langle \bar{F} \rangle_{\emptyset,BF,BF}$. However, we still get that $\bar{F}_{AD,BF,BF}^j > \langle \bar{F} \rangle_{AD,BF,BF}$, showing that the probabilistic protocol enhances the efficiency of the deterministic protocol under the same noise conditions. Furthermore, for high values of p_i only the probabilistic protocol yields fidelities greater than $2/3$, highlighting the importance of the probabilistic protocol in order to get a truly quantum teleportation.

In Fig. 5 we compare the optimal efficiency for a fixed type of noise acting on the input qubit among all possibilities of noise acting on the quantum channel. The greatest efficiency occurs when all qubits suffer the amplitude-damping noise. However, the probability of success in this case is the lowest one.

A very interesting noise arrangement is the one shown in Fig. 6, in which all qubits are acted on by the amplitude-damping noise (we continue to assume $p = p_A = p_B$). The first thing worth noticing is that we can always find a j such that $\bar{F}_{AD,AD,AD}^j > \langle \bar{F} \rangle_{AD,AD,AD}$ for any value of p_i and p , i.e., the probabilistic protocol always outperforms the deterministic protocol under the same noise arrangements. We have also noticed that the optimal $\bar{F}_{AD,AD,AD}^{\psi^\pm}$ is always greater than the optimal $\bar{F}_{AD,AD,AD}^{\phi^\pm}$. Another feature of this noise arrangement is related to the fact that $\bar{F}_{AD,AD,AD}^j > \langle \bar{F} \rangle_{\emptyset,AD,AD}$ for the whole range of p_i whenever $p \gtrsim 0.5$ (lower panel of Fig. 6). For values of $p \lesssim 0.5$ we can also have the probabilistic protocol beating the deterministic protocol with no noise acting on the input qubit. This only happens, however, when p_i is small (upper panel of Fig. 6). Last, for certain values of p and p_i , the optimal parameters θ and φ for a given postselected measurement result also yield high average fidelities for other two possible measurement outcomes, high enough to beat the optimal one of the deterministic protocol. In such cases Alice and Bob can considerably increase the rate of success of the probabilistic protocol, and still outperform the efficiency of

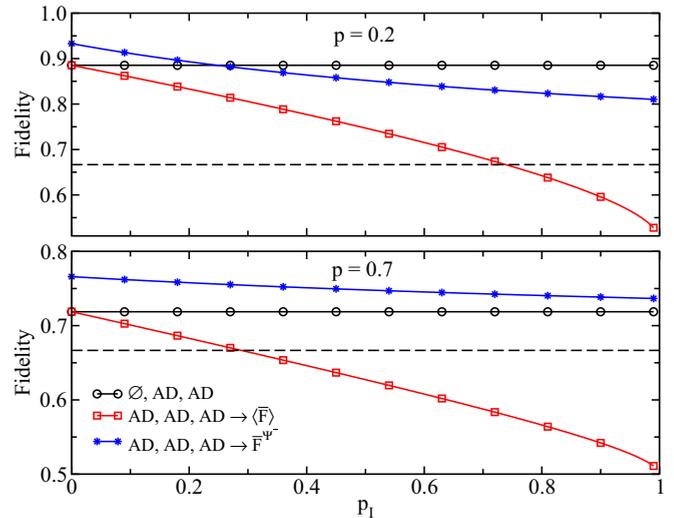


FIG. 6. Optimal efficiencies (average fidelities) as a function of p_i for the deterministic protocol when only the quantum channel is subjected to the amplitude-damping noise (circle–black curves) and for the deterministic (square–red curves) and probabilistic (star–blue curves) protocols when all three qubits are subjected to the amplitude-damping noise. The dashed-black curves mark the classical limit $2/3$ for the fidelity. The optimal efficiencies for the probabilistic protocol are those obtained by postselecting $|\Psi^- \rangle$. For the two panels above, the success rate \bar{Q}^{ψ^-} is never lower than 0.4% in the whole range of p_i .

the deterministic protocol, by postselecting three out of four measurement results.

C. Scenario 3

In this scenario the two qubits with Alice are acted on by the same type of noise during the same amount of time ($p_i = p_A = p$) and Bob's qubit is subjected to the same or a different type of noise [see Fig. 1(c)]. Again, we have 16 possible noise arrangements with only six out of these 16 possibilities yielding probabilistic protocols with greater optimal efficiencies than the ones for the deterministic protocols. Those six cases contain the amplitude-damping noise acting on Bob's qubit or on Alice's qubits. It is worth noticing that in this scenario there exists one case where the amplitude-damping noise acts on Bob's qubit without yielding a better performance for the probabilistic protocol. In this case, where Alice's qubits are acted on by the phase-flip noise (PhF, PhF, AD), both the probabilistic and deterministic optimal efficiencies coincide. For the other six cases in which the amplitude-damping noise is present the probabilistic protocol outperforms the deterministic one under the same noise conditions. The qualitative behavior of these six cases, as well as their most important features, are similar to the ones already reported in scenario 2. In particular, the optimal initial entanglement of the quantum channel connecting Alice and Bob is not maximal.

It is important to mention that for all scenarios shown in Fig. 1 and studied here we obtain nontrivial probabilistic protocols, in the sense that they outperform the efficiency of the corresponding deterministic protocols, if the amplitude-damping noise is present. Whenever the amplitude-damping

noise is absent, the efficiencies for the probabilistic and deterministic protocols coincide when optimizing the protocols as functions of θ and φ , i.e., as functions of the initial entanglement of the quantum channel and of the type of projective measurement implemented by Alice. In those cases where the coincidence occurs, the optimal θ is always the one leading to the greatest initial entanglement ($\theta = \pi/4$). However, if we work with a one-parameter optimization problem (φ) and fix θ such that $\theta \neq \pi/4$, we can obtain probabilistic protocols outperforming the deterministic ones for noise arrangements in which the amplitude-damping noise is not present. In other words, if we are constrained from the start to work with nonmaximally entangled quantum channels connecting Alice and Bob, other noise arrangements that do not include the amplitude-damping noise lead to nontrivial probabilistic protocols.

VI. CONCLUSION

We investigated the performance of the probabilistic (conditional) quantum teleportation protocol in the presence of noise. We have compared its optimal efficiency with the optimal one for the deterministic protocol under the same noise conditions. We analyzed several noise arrangements in which the qubits employed in the execution of the teleportation protocol are subjected to the most common types of noise encountered in the implementation of a quantum communication task, namely, the bit-flip, the phase-flip, the depolarizing, and the amplitude-damping noise.

For all noise arrangements here investigated, a total of 48 distinct cases, only 17 cases have a probabilistic protocol with an optimal efficiency (average fidelity) greater than the optimal efficiency of the deterministic protocol. We observed that a necessary condition for this to happen is that at least one of the qubits employed in the teleportation protocol must be subjected to the amplitude-damping noise. Moreover, and similarly to the deterministic case, for those 17 noise arrangements the initial entanglement (prior to the action of noise) of the quantum channel connecting Alice and Bob leading to the greatest efficiency is not maximal. This is an example where *less* entanglement means *more* efficiency, a feature already seen for deterministic protocols [18,20].

We also showed several noise arrangements where *more* noise means *more* efficiency. This happens whenever the efficiency of the probabilistic protocol, in which a certain number of qubits are subjected to noise, is greater than the efficiency of the corresponding deterministic protocol with a noise arrangement where fewer qubits are acted on by noise. In addition to this we also found situations in which *different* noises mean *more* efficiency. Indeed, under certain noise arrangements we showed that it is better to have the qubits subjected to different types of noise instead of the same noise in order to obtain the greatest efficiency.

We observed another important feature when comparing the optimal average fidelities of the probabilistic and deterministic protocols under the same noise arrangement. In this scenario we found noise arrangements where only the probabilistic protocol surpasses the classical threshold of $2/3$ for the average fidelity. This threshold means that a teleportation protocol yielding fidelities lower than $2/3$ can be simulated using only local operations and classical communication (LOCC). Teleportation protocols with fidelities lying below this limit are not considered truly quantum [25]. Therefore, for some noise arrangements we must employ the probabilistic instead of the deterministic protocol in order to obtain a quantum teleportation that is genuinely quantum.

Finally, for all the protocols here investigated we noted a tradeoff between the rate of success and the efficiency. Indeed, the optimal protocols here reported were obtained maximizing the average fidelity without any constraint on the value of the probability of success. We can increase the rate of success, however, if we decrease the efficiency of the protocol. This is achieved by imposing a constraint on the lowest acceptable value for the probability of success.

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