Quantumness and the role of locality on quantum correlations

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Quantum correlations in a physical system are usually studied with respect to a unique and fixed decomposition of the system into subsystems, without fully exploiting the rich structure of the state space. Here, we show several examples in which the consideration of different ways to decompose a physical system enhances the quantum resources and accounts for a more flexible definition of quantumness measures. Furthermore, we give a different perspective regarding how to reassess the fact that local operations play a key role in general quantumness measures that go beyond entanglement—as discordlike ones. We propose a family of measures to quantify the maximum quantumness of a given state. For the discord-based case, we present some analytical results for $2 \times d$-dimensional states. Applying our definition to low-dimensional bipartite states, we show that different behaviors can be reported for separable and entangled states *vis-à-vis* those corresponding to the usual measures of quantum correlations. We show that there is a close link between our proposal and the criterion to witness quantum correlations based on the rank of the correlation matrix, proposed by Dakić, Vedral, and Brukner [Phys. Rev. Lett. **105**, 190502 (2010)].

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I. INTRODUCTION

Contemporary physics' recent technological and theoretical progress shows that quantum computation is a feasible and not-so-far-away perspective (see Refs. [1] and references therein). Novelties would bring significant improvement in the performance of information processing tasks, and the main ingredient involved resides in (quantum) correlations that cannot be implemented with classical systems. Hence, the study of quantum correlations have been one of the most pursued issues in quantum physics of the last decade (see, e.g., the excellent reviews of the Horodeckis [2] and Modi *et al.* [3]). Quantum entanglement and quantum discord (QD) are two of the main families of quantum correlation measures, which are closely related to the way in which a system can be decomposed as a mixture of product states. A nonentangled (or separable) state ρ_{sep}^{AB} over the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, with respect to the bipartition $A|B$, can be written as a convex combination of product states as $\rho_{\text{sep}}^{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B$, with $p_k \geqslant 0$ and $\sum_k p_k = 1$. In turn, a classically correlated (CC) state ρ_{clas}^{AB}, with respect to the same bipartition, can be expressed as a mixture of local orthogonal projectors as $\rho_{\text{clas}}^{AB} = \sum_{ij} p_{ij} |i^A\rangle\langle i^A| \otimes |j^B\rangle\langle j^B|$, where $p_{ij} \geqslant 0$, $\sum_{ij} p_{ij} = 1$ and $\langle i^{A(B)}|i'^{A(B)}\rangle = \delta_{ii'}$, with $0 \leqslant i \leqslant \dim(\mathcal{H}_{A(B)})$. That is, ρ_{clas}^{AB} is diagonal in a product basis $\{|i^A\rangle \otimes |j^B\rangle\}$. A state that is not CC is said to be quantum correlated (QC).

What is the main difference between a noncorrelated (or product) state, $\rho_{\text{prod}}^{AB} = \rho^A \otimes \rho^B$, and a CC one (as ρ_{clas}^{AB} given above), with regard to their quantum capabilities? One may suspect that a CC state is as useless as a product one when performing an information task that necessarily involves quantum resources. However, *this is not true*. Let us consider that one has ρ_{clas}^{AB} and also one has access to other local degrees of freedom, i.e., that our initial system + environment state is $\rho^{\text{ext}} = \eta^{\bar{A}} \otimes \rho_{\text{clas}}^{AB} \otimes \eta^{\bar{B}}$, where $\eta^{\bar{A}}$ ($\eta^{\bar{B}}$) depicts the state of the environmental degrees of freedom in A (B). *Now, it is easy to show that there are local observables with respect to whom the state is quantumly correlated.* It suffices to notice that $\rho^{A'B'} = \text{Tr}_{\text{env}} U \rho^{\text{ext}} U^\dagger$ is, in general, QC with respect to $A'|B'$, where "env" denotes the environment degrees of freedom and U denotes a local unitary operation (LU) that respects that local bipartition, i.e., $U = U^{\bar{A}A} \otimes U^{B\bar{B}}$, and accounts for the inspection of local observables.

Consideration of different observables of a quantum system leads to alternative descriptions, and quantum correlations are relative to such observables election. Zanardi [4] noticed the effect of this relative character *vis-à-vis* the quantum entanglement of multiqubit states and proposed a formalization under a general algebraic framework [5]. Later, Barnum *et al.* [6] gave a subsystem-independent notion of entanglement. Harshmann and Ranade [7] proved that all pure states of a finite-dimensional (and unstructured) Hilbert space are equivalent as entanglement resources in the ideal case that one has complete access and control of observables (see also [8] for alternative presentations of the problem). Given that CC implies separability and given that the question about separability becomes relative to the preferred observables (the ones that determine the local subsystems), the question about the correlations on CC states becomes relative too. It is worth noting that these ideas have been successfully applied, for example, to the investigation of quantum phase transitions [9–11] and to quantum entanglement in systems of indistinguishable particles [12].

In this work, we focus on the less studied situation of mixed states under a locality restriction: we allow only *local* unitary operations (over the enlarged Hilbert space) in order to explore the observables' subspaces of each local subsystem. In the pure state scenario, global unitary operations lead to equivalence regarding quantum correlations (in that case, entanglement).

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As expected, mixedness and locality impose some restrictions on the achievable quantum correlations when considering the mentioned relative character (see Appendix A for a discussion on the role of mixedness on discordlike measures under global unitaries, for states in $\mathbb{C}^4$).

We adopt here the distinction between *quantum correlations* and *quantumness* of correlations, previously discussed by Giorgi *et al.* in terms of genuine and nongenuine quantum correlations [13] and by Gessner *et al.* [14]. It is interesting to note that Ollivier and Zurek, in their seminal paper [15], have already coined the idea that QD accounts for the quantumness of correlations, and not to the amount of quantum correlations *per se*.

The paper is organized as follows. In Sec. II we discuss our main thesis, namely the way in which general quantum correlations depend on the subsystem decomposition of a given quantum system. In Sec. III we advance a proposal in order to quantify the mentioned effect from an information-theoretic viewpoint, which we call potential quantumness, and we prove several interesting properties of such a measure, showing its adequacy as a faithful quantum correlation measure. Finally, in Sec. IV we specialize our study to discordlike correlations. We present some analytical results for $2 \times d$ dimensional states and display some of their features when applied to simple low-dimensional models. Section V is devoted to a summary and conclusions.

II. QUANTUMNESS AND SUBSYSTEMS

Let us give a concrete example. We begin with a CC state of two qubits as $\rho^{AB}_{\text{clas}} = p\,|0^A\rangle\langle 0^A| \otimes |0^B\rangle\langle 0^B| + (1-p)\,|1^A\rangle\langle 1^A| \otimes |1^B\rangle\langle 1^B|$, with $0 \leqslant p \leqslant 1$, where $\{|i\rangle\}_{i=0,1}$ is the standard (computational) basis. If we have access to one auxiliary qubit on each location, we can set the extended state to be $\rho^{\text{ext}} = |0^{\bar{A}}\rangle\langle 0^{\bar{A}}| \otimes \rho^{AB}_{\text{clas}} \otimes |0^{\bar{B}}\rangle\langle 0^{\bar{B}}|$. Now, any LU operation $U^{\bar{A}A} \otimes U^{B\bar{B}}$ accounts for different partitions of the local subsystems $\bar{A}|A$ $(B|\bar{B})$ into new subsystems $\bar{A}'|A'$ $(B'|\bar{B}')$ (see Fig. 1). For example, if $U^{\bar{A}A} = U_{\text{cH}}U_{\text{S}} = (U^{B\bar{B}})^\dagger$, where U_{cH} is a controlled Hadamard gate

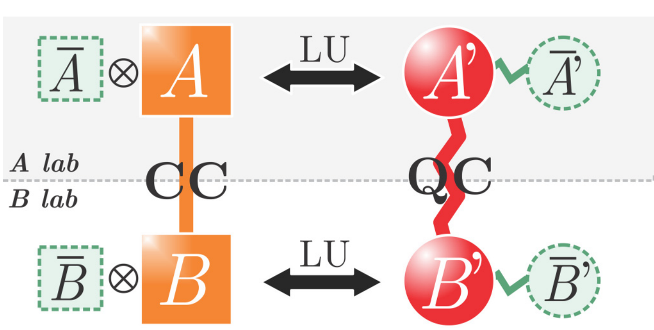

FIG. 1. For a composite system $A + B$, CC with respect to that bipartition, coupling auxiliary uncorrelated local systems ($\bar{A}$ and $\bar{B}$), and performing LU operations generally produces a QC state (see Proposition 1 for details). Here, the local unitaries have the form $U^{\bar{A}A} \otimes U^{B\bar{B}}$ and act by rearranging the local degrees of freedom, thus yielding "new" subsystems $A', \bar{A}', B', \bar{B}'$. In other words, the local unitaries induce new decompositions of $\bar{A}A$ and $B\bar{B}$ into different "primed" subsystems. (See text for details.)

and U_{S} is a swap gate, the transformed reduced state is $\rho^{A'B'} = p\,|0^{A'}\rangle\langle 0^{A'}| \otimes |0^{B'}\rangle\langle 0^{B'}| + (1-p)\,|+^{A'}\rangle\langle +^{A'}| \otimes |+^{B'}\rangle\langle +^{B'}|$, where $|\pm^{A'(B')}\rangle = \frac{1}{\sqrt{2}}(|0^{A'(B')}\rangle \pm |1^{A'(B')}\rangle)$. The new state, $\rho^{A'B'}$, is not CC anymore. Thus, we have revealed some hidden or "potential" quantumness of ρ^{AB}_{clas}, just by considering a transformation over local degrees of freedom. This cannot be done if the state is uncorrelated: for ρ^{AB}_{prod}, it is straightforward to show that the same procedure gives a new uncorrelated state $\rho^{A'B'}_{\text{prod}}$. *This result clearly distinguishes ρ^{AB}_{clas} from ρ^{AB}_{prod} with regards to quantum information processing capabilities.* Such feature holds for every nonproduct (i.e., correlated) state: *if ρ^{AB} is a bipartite correlated state and A and/or B has local access to auxiliary degrees of freedom, then it is possible to find quantum correlations between new subsystems A' and B' defining $\rho^{A'B'}$*. As we are going to discuss in Sec. III, this property is nothing but a reinterpretation of the already known fact that quantum correlations (others than entanglement) can be created by local operations [14,16–18]. We take this result to be our first proposition.

Proposition 1. Let ρ^{AB} be a nonproduct density operator over $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $\eta^{\bar{A}}$ and $\eta^{\bar{B}}$ be the "ready" states of two ancillary systems. Then, for the extended state $\eta^{\bar{A}} \otimes \rho^{AB} \otimes \eta^{\bar{B}}$ it is possible to find a subsystem decomposition that preserves the local bipartition and possesses quantum correlations.

Proof. The statement can be proved straightforwardly. If ρ^{AB} is nonproduct state then it can be CC or QC. If it is QC then there is no need to extend our system, it already possesses quantum correlations. If it is CC, we can choose the auxiliary states to be pure, $\eta^{\bar{A}} = \eta^{\bar{B}} = |0\rangle\langle 0|$. Then, over the extended state $|0\rangle\langle 0| \otimes \rho^{AB} \otimes |0\rangle\langle 0|$, we can apply LU operations $U_{\bar{A}A} \otimes U_{\bar{B}B}$ that correspond to different decompositions of each part (A and B) into subsystems. Finally, tracing out the auxiliary degrees of freedom results in a modified state $\rho^{A'B'}$. The action of the unitaries over the reduced state is equivalent to a local quantum trace-preserving operation (see, e.g., Ref. [19]). But, performance of arbitrary local channels can convert any CC state into a QC one [20]. This observation ends the proof. Note, however, that for product states there is not a local operation that correlates both parts, neither quantum nor even classically. ■

Another way of assessing this important difference (between CC and noncorrelated states) with regard to the quantum correlations arising from the possible reduction of CC states exists: when the A and B subsystems have nonprime dimensions, it is possible to find reductions of ρ^{AB} that respect the local bipartition $A|B$ and yet possess nonclassical correlations. For example, the two-qubits' QC state $\rho^{A_1B_1} = p\,|0^{A_1}\rangle\langle 0^{A_1}| \otimes |0^{B_1}\rangle\langle 0^{B_1}| + (1-p)\,|+^{A_1}\rangle\langle +^{A_1}| \otimes |+^{B_1}\rangle\langle +^{B_1}|$ (the same as in the previous example) can be regarded as a reduction of the four-qubits' one $\rho^{AB} = p\,|0^{A_2}\rangle\langle 0^{A_2}| \otimes |0^{A_1}\rangle\langle 0^{A_1}| \otimes |0^{B_1}\rangle\langle 0^{B_1}| \otimes |0^{B_2}\rangle\langle 0^{B_2}| + (1-p)\,|1^{A_2}\rangle\langle 1^{A_2}| \otimes |+^{A_1}\rangle\langle +^{A_1}| \otimes |+^{B_1}\rangle\langle +^{B_1}| \otimes |1^{B_2}\rangle\langle 1^{B_2}|$. The latter is clearly CC with respect to the bipartition $A_1A_2|B_1B_2 \cong A|B$. Thus, ρ^{AB} is a CC state with QC reductions, always preserving the same prescription for the local degrees of freedom. Again, this result is very general: *if ρ^{AB} is a bipartite correlated state for which A and B are composites, then it is possible to find a reduction that possesses quantum correlations:*

Proposition 2. Let ρ^{AB} be a nonproduct density operator over $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\dim \mathcal{H}_A$ and/or $\dim \mathcal{H}_B$ given by nonprime numbers. Then, it is possible to find a reduced state that preserves the local bipartition and possesses quantum correlations.

Proof. The proof is rather trivial from our first proposition. If ρ^{AB} has the properties of the statement, we can regard ρ^{AB} as the (already) extended state. Thus, applying local unitaries and tracing out some degrees of freedom yields the desired result. Being correlated is a necessary condition, since the reduction of any product state is trivially noncorrelated with respect to that fixed bipartition. ■

As pointed out in the Introduction, the relative character of quantum correlations with respect to the chosen partition of a system into subsystems has been carefully studied in the case of pure states [4–7]. In our presentation, *we focus on the case of mixed states under a locality restriction: only local unitary operations are allowed (over the enlarged Hilbert spaces) to explore the observables' subspaces of each local subsystem.*

Summing up, possession of CC states already implies some degree of quantumness in the correlations of both parts, in the following sense:

(i) CC states supplied with uncorrelated ancillas exhibit quantum correlations, in general, when alternative local observables are specified and,

(ii) reductions of CC states exhibit, in general, quantum correlations.

Consideration of these scenarios leads us, in the following section, to a notion of potential quantumness.

Remark. Local unitaries act over $\mathcal{H}_{\bar{A}A} \otimes \mathcal{H}_{B\bar{B}}$ by rearranging the local degrees of freedom to give an alternative decomposition into subsystems preserving the original local bipartition. For the reduced state ρ^{AB}, the transformation is equivalent to a local operation. The impact of local operations on quantum correlations has been seriously studied in the last years. Both properties of quantum correlations, stated in Propositions 1 and 2, rely on the more general one that quantum correlations can be created by local noise (i.e., local quantum channels) [16,21,22]. When one chooses this channel to be the trace operation, the above relations between classical and quantum correlations of composite systems arise.

We propose next a measure that attempts to quantify these facts from an information-theoretic point of view.

III. MEASURING THE POTENTIAL QUANTUMNESS

The two propositions discussed so far refer to closely related facts that can be quantified by consideration of appropriate information-theoretic measures of quantum correlations. Given the previous analysis, we give a straightforward operational definition for our potential quantumness (PQ) measure.

Definition 1. Let ρ^{AB} be a density operator over $\mathcal{H}_A \otimes \mathcal{H}_B$, and $\eta_0^{\bar{A}(\bar{B})} = |0\rangle\langle 0|$ the "ready" state over $\mathcal{H}_{\bar{A}(\bar{B})} = \mathbb{C}^d$ of an auxiliary system. The PQ of ρ^{AB}, of rank d, with respect to the bipartition $A|B$ is

$$\mathcal{P}_d^Q(\rho^{AB}) = \max_{U \in \mathrm{LU}} Q^{A|B}\big(U\eta_0^{\bar{A}} \otimes \rho^{AB} \otimes \eta_0^{\bar{B}} U^\dagger\big), \quad (1)$$

where $Q^{A|B}(\rho) := Q(\mathrm{Tr}_{\mathcal{H}_{\bar{A},\bar{B}}}\,\rho)$ implies tracing out the auxiliary systems, and $Q^{A|B}$ is any measure of bipartite quantum correlations between A and B.

Usually, a measure Q of quantum correlations is such that, for any bipartite state ρ^{AB},

(i) $Q(\rho^{AB}) \geqslant 0$,

(ii) $Q(\rho^{AB}) = 0$ if ρ^{AB} is a CC state (or in particular, a product state),

(iii) $Q(\rho^{AB})$ is maximal if and only if ρ^{AB} is a maximally entangled (pure) state,

(iv) $Q(\rho^{AB})$ is invariant under local unitary operations, and matches an entanglement monotone whenever ρ^{AB} is a pure state.

Those properties are fulfilled by every entanglement and discordlike measures. In those cases, the corresponding PQ measure satisfies some basic properties that make it suitable as a measure of quantum correlations:

(i) (positivity) $\mathcal{P}_d^Q(\rho^{AB}) \geqslant 0$ for every state ρ^{AB} and any dimension d of the auxiliary parts;

(ii) (minimum) for any value of d, $\mathcal{P}_d^Q(\rho^{AB}) = 0$ if and only if $\rho^{AB} = \rho^A \otimes \rho^B$;

(iii) (maximum) $\mathcal{P}_d^Q(\rho^{AB})$ is maximal if and only if ρ^{AB} is a maximally entangled state.

Positivity holds because Q itself is semidefinite positive. Indeed, from Definition 1 one deduces the stronger relation $\mathcal{P}_d^Q(\rho^{AB}) \geqslant Q(\rho^{AB}) \geqslant 0$. Regarding the second property, $\mathcal{P}_d^Q(\rho^A \otimes \rho^B) = 0$ holds because the unitaries involved do not mix $A\bar{A}$ with $B\bar{B}$ degrees of freedom, and there is no LU that can correlate them, not even in a classical sense. On the other hand, if ρ^{AB} is not a product state, then $\mathcal{P}_d^Q(\rho^{AB}) \neq 0$ [23]. The third property is fulfilled because Q itself saturates only for maximally entangled states, even when no extension or local operation is performed. Thus, $\mathcal{P}_d^Q(\rho^{AB})$ attains its maximum if and only if $\mathrm{Tr}_{\mathcal{H}_{\bar{A}},\mathcal{H}_{\bar{B}}}(U\eta_0^{\bar{A}} \otimes \rho^{AB} \otimes \eta_0^{\bar{B}} U^\dagger)$ is a maximally entangled state. Extension, local unitaries, and partial trace are equivalent to local operations. But local operations cannot create entanglement. Thus, $\mathcal{P}_d^Q(\rho^{AB})$ attains its maximum if and only if ρ^{AB} is already a maximally entangled (pure) state.

The defined measure exhibits many other interesting properties but, before presenting them, we prove the following proposition that provides an equivalent definition for $\mathcal{P}_d^Q$ without making any explicit reference to auxiliary systems.

Proposition 3. For every state ρ^{AB} and any dimension d as in Definition 1, the PQ of ρ^{AB}, of rank d, with respect to the bipartition $A|B$, is

$$\mathcal{P}_d^Q(\rho^{AB}) = \max_{E \in \mathrm{LO}(d)} Q(E[\rho^{AB}]), \quad (2)$$

where $E \in \mathrm{LO}(d)$ is any local operation of rank at most d, and $Q^{A|B}$ is any measure of bipartite quantum correlations between A and B.

Proof. The equivalency is straightforwardly proven remembering that, by Stinespring's dilation theorem [24], any quantum operation can be reproduced by adding an ancilla, performing a unitary operation over the enlarged Hilbert space, and finally tracing out the ancilla. In our case, the restriction regarding local unitaries imposes the corresponding locality condition on the quantum operations of Proposition 3. ■

Proposition 3 offers a concise interpretation for our measure: $\mathcal{P}_d^Q(\rho)$ quantifies the quantumness of the correlations of ρ attainable by local operations. Among the vast family of local operations, we can identify, for example, the local unitaries, the local unitals, and the local classical-quantum channels, etc. As we show below, while the local unitaries do not change the value of $\mathcal{P}_d^Q$, our measure is nonincreasing under arbitrary local operations.

Remark. The measure $\mathcal{P}_d^Q$ depends on the value of d, which can be regarded as a restriction on the class of accessible local operations (LOs) or a restriction on the dimension of the accessible auxiliary systems (see Definition 1). As $\mathrm{LO}(d' < d) \subset \mathrm{LO}(d)$, it is straightforward to show that $\mathcal{P}_{d'<d}^Q(\rho^{AB}) \leqslant \mathcal{P}_d^Q(\rho^{AB})$ for any ρ^{AB}. However, it is interesting to consider the following particular d-independent scenario. If ρ^{AB} is such that $\max\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\} = d_{\max}$, then any LO over that state can be implemented with auxiliary extensions of dimensions at most $d_{\max}^2$. Thus, $\mathcal{P}_d^Q(\rho^{AB}) \to \mathcal{P}_{d_{\max}^2}^Q(\rho^{AB})$ when $d \to \infty$. We shall refer to the later as the *maximum-PQ* or $\mathcal{P}_{\max}^Q$. It is clear that it accounts for the no-restriction-over-LO case. Finally, the case $d = 0$ trivially matches the corresponding Q measure, i.e., $\mathcal{P}_{d=0}^Q(\rho) = Q(\rho)$.

It is noteworthy that this measure, $\mathcal{P}_d^Q$, does not (necessarily) involve a dynamical interpretation. Instead, the unitaries appearing in Definition 1 attempt to capture the relative character of the correlations with respect to the partition of a given system into subsystems. That is, if ρ^{AB} is the state of a system (bi)partitioned according to $A|B$, and if there are auxiliary systems such that $\eta_0^{\bar{A}} \otimes \rho^{AB} \otimes \eta_0^{\bar{B}}$ is a possible joint state (as in Definition 1), then the operations $U \in \mathrm{LU}$ can be thought of as a resetting of the local subsystems (see Appendix B for a more detailed discussion).

A. Properties of the PQ measures

As expected, $\mathcal{P}_d^Q$ inherits some particular properties of the chosen Q measure. For example, if Q is the usual QD then $\mathcal{P}_d^Q$ is an asymmetric measure relying on one-partite measurements, while becoming symmetric if Q is the symmetric QD. But, even before specializing things for a certain Q, we can prove additional properties of the PQ measure.

First, it is interesting to note that $\mathcal{P}_d^Q$ matches an entanglement monotone for pure states, $\rho^{AB} = |\psi^{AB}\rangle\langle\psi^{AB}|$, and for every measure Q that complies with the previous characterization. Indeed, by Proposition 3, $\mathcal{P}_d^Q(|\psi^{AB}\rangle)$ is the maximum value of $Q(E[|\psi^{AB}\rangle])$ over $E \in \mathrm{LO}(d)$. Now, using that (i) for pure states, Q decreases monotonically under local operations and classical communication (LOCC), and (ii) that $\mathrm{LO} \subset \mathrm{LOCC}$, we have $\mathcal{P}_d^Q(|\psi^{AB}\rangle) = \max_{E\in\mathrm{LO}(d)} Q(E[|\psi^{AB}\rangle]) \leqslant \max_{E'\in\mathrm{LOCC}} Q(E'[|\psi^{AB}\rangle]) \leqslant Q(|\psi^{AB}\rangle)$. On the other hand, taking E to be the identity map, we have that $Q(E[|\psi^{AB}\rangle]) = Q(|\psi^{AB}\rangle)$, saturating the above inequality. Accordingly, $\mathcal{P}_d^Q(|\psi^{AB}\rangle) = Q(|\psi^{AB}\rangle)$ for every pure state $|\psi^{AB}\rangle$ and any value of d, proving that $\mathcal{P}_d^Q$ is, for pure states, no more than the corresponding entanglement monotone defined by Q.

Next, we prove certain properties concerning the behavior of any PQ measure under different classes of local operations. First, that $\mathcal{P}_d^Q$ is invariant under LU operations, as one would expect for any reasonable measure of quantum correlations. Second, that sufficiently low-ranked LOs cannot increase $\mathcal{P}_d^Q$, implied from its very definition by the optimization over LO operations. Third, that $\mathcal{P}_d^Q$ takes into account the quantum correlations from the possible reductions of a certain state. We sum up all this in the following proposition.

Proposition 4. For any PQ measure of rank d, $\mathcal{P}_d^Q$, the following properties are fulfilled:

(1) $\mathcal{P}_d^Q$ is invariant under LU operations;

(2) $\mathcal{P}_d^Q$ is nonincreasing under a LO of rank equal or lower than d;

(3) given a bipartite state ρ^{AB} where A and B are also composites, $A = \{A_i\}$ and $B = \{B_j\}$, $\mathcal{P}_d^Q(\rho^{AB})$ is lower bounded by the Q measure over all the possible (fine-grained) reductions $\rho^{A_iB_j} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{comp}}}\rho^{AB}$, with $\mathcal{H}_{\mathrm{comp}} = \bigotimes_{m\neq i,n\neq j} \mathcal{H}_{A_m} \otimes \mathcal{H}_{B_n}$, such that $\dim(\mathcal{H}_{A_i}) \times \dim(\mathcal{H}_{B_j}) \leqslant d$.

Proof. (1) Let $U^{A|B}$ be a unitary operation acting locally over $\mathcal{H}_A \otimes \mathcal{H}_B$ bipartition. For any state ρ over $\mathcal{H}_A \otimes \mathcal{H}_B$, the corresponding transformation is $\rho \mapsto \rho_1 = U^{A|B}\rho U^{A|B}$. From Definition 1,

$$
\begin{aligned}
\mathcal{P}_d^Q(\rho_1) &= \max_{V\in LU} Q^{A|B}\big(V\eta_0^{\bar{A}} \otimes \rho_1 \otimes \eta_0^{\bar{B}} V^\dagger\big) \\
&= \max_{V\in LU} Q^{A|B}\big(VU^{A|B}\eta_0^{\bar{A}} \otimes \rho \otimes \eta_0^{\bar{B}} U^{A|B} V^\dagger\big) \\
&= \max_{V'\in LU} Q^{A|B}\big(V'\eta_0^{\bar{A}} \otimes \rho_1 \otimes \eta_0^{\bar{B}} V'^\dagger\big),
\end{aligned}
$$

which is equal to $\mathcal{P}_d^Q(\rho)$. In the third line, we use the fact that the composition of a unitary operation V, that is local over $\bar{A}A|B\bar{B}$, and another unitary operation $U^{A|B}$ that is local on $A|B$, yields a unitary V' that is local over $\bar{A}A|B\bar{B}$.

(2) Let $E' \in \mathrm{LO}(d')$ and $E'[\rho] = \rho'$ be the corresponding transformations of ρ, with $d' \leqslant d$. Then, for ρ' it holds that

$$
\begin{aligned}
\mathcal{P}_d^Q(\rho') &= \max_{E\in\mathrm{LO}(d)} Q(E[\rho']) \\
&= \max_{E\in\mathrm{LO}(d)} Q(E\circ E'[\rho]) \\
&\leqslant \max_{E\in\mathrm{LO}(d)} Q(E[\rho]) = \mathcal{P}_d^Q(\rho),
\end{aligned}
$$

where "$\circ$" indicates composition of operations. We used, in the third line, the fact that operations of the form $E \circ E'$ span a subset of $\mathrm{LO}(d)$.

(3) Any reduction $\rho^{A_iB_j}$ is the result of a LO of the form $\mathrm{Tr}_{\mathcal{H}_{A_i}} \circ \mathrm{Tr}_{\mathcal{H}_{B_j}} \circ \mathbb{1}_{\mathcal{H}_{\mathrm{comp}}}$ over the state of the full system. Such a LO is thus included among those considered in the maximization procedure involved in the definition of $\mathcal{P}_d^Q$, if $\dim(\mathcal{H}_{A_i}) \times \dim(\mathcal{H}_{B_j}) \leqslant d$. ■

Regarding property (2) of Proposition 4, it is worth emphasizing that one does not expect $\mathcal{P}_d^Q$ to coincide with an entanglement monotone for general mixed states. Indeed, in most cases $Q \geqslant 0$ for separable states and then $\mathcal{P}_d^Q \geqslant 0$ for those states, while any entanglement monotone is, by definition, null for any separable state.

As stated before, Proposition 4 establishes that the PQ measure takes into account the second fact mentioned in Sec. III, namely that even CC states can have QC reductions, a phenomenon that is not captured by the usual discordlike measures. For a given CC state, however, any reduction is a separable state for the same bipartition [25–28]. As a consequence, the PQ of a given CC state is upper bounded by that of the separable states. From now on, we are going to concentrate efforts on the case of nonrestricted local capabilities, for which the situation is well described by $\mathcal{P}^{Q}_{\max}$. In that case, the propositions above assert that

(a) $\mathcal{P}^{Q}_{\max}$ matches an entanglement monotone for pure states,

(b) $\mathcal{P}^{Q}_{\max}$ is non-increasing under general LO and is invariant under LU,

(c) $\mathcal{P}^{Q}_{\max}$ takes into account the quantum correlations of every possible reduction of the considered state.

Further insight demands a specification of a particular class of functionals Q, as we are going to do with discordlike measures in Sec. IV. Before that, we recall some related works to remark on their similarities and differences with our proposal.

B. Relations to other measures of quantumness

Some quantum correlations' measures involving local unitary operations have recently appeared in the literature. They can be related, to some extent, to our above proposal.

Let us start by considering the interesting measure of quantumness advanced by Devi and Rajagopal (DR) [29]. Given a bipartite state ρ^{AB}, they consider (i) all the possible extensions $\rho^{\bar{A}AB}$ to a larger Hilbert space, such that $\mathrm{Tr}_{\bar{A}}\rho^{\bar{A}AB} = \rho^{AB}$, and (ii) the set of projective measurements over $\bar{A}A$. Hence, quantumness is defined as the minimum Kullback-Liebler relative entropy between the original state and the postmeasurement state. As in the potential discord (PD) case, this measure involves an enlarged Hilbert space. However, since DR's measure is computed via a minimization, this quantity is expected to be lower than, for example, QD. Indeed, the authors have shown that their measure is an upper bound to the relative entropy of entanglement.

As a second example, we refer again to the work of Dakić *et al.* [20], where the authors show that the rank of the correlation matrix of a bipartite state serves as a witness of quantum correlations. Any bipartite state can be written in terms of arbitrary bases $\{A_i\}$ and $\{B_j\}$ of Hermitian operators of the local Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$, as $\rho^{AB} = \sum_{ij} r_{ij} A_i \otimes B_j$. The number of nonzero singular values of the matrix (r_{ij}) is $L \leqslant d_{\min}^2$, with $d_{\min} = \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$. For CC states, $L \leqslant d_{\min}$. Hence, $L \geqslant d_{\min}$ implies quantum correlations. Nonetheless, there are states with nonzero QD and $L < d_{\min}$. In particular, any separable state that can be created by local operations on CC states has $L < d_{\min}$. In our treatment, this implies that any CC state displays the same degree of quantumness that the most QC separable state that can be locally created from it (the latter has the same L that the original CC state).

Another closely related work is the one by Guo and Wu [30], where the authors define a quantifier of quantum correlations computed via measurements over mutually unbiased bases. As in our case, product states are the only ones that exhibit zero quantumness within this framework.

Finally, we look at another related work, due to Gharibian [31], who defines a measure of nonclassicality as the minimal distance between the state and all its possible local unitary transformations. From our perspective, the LU operations accounts for a switch in local observables. Thus, Gharibian's measure captures the minimal disturbance suffered by a given state when changing the local observables. It turns out that Gharibian's measure is also a discordlike one, that is nonzero if and only if the state is not a CC state.

None of the above measures captures what our measure of potential quantumness does, namely the nonclassical correlations present in CC states. Next, so as to be able to present some numerical results and obtain deeper insight into these matters, we specialize things by regarding QD as our quantum correlation measure.

IV. QUANTUM POTENTIAL DISCORD

Definition (1) determines a family of correlation measures that depends on the particular functional $Q : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ that one chooses to quantify the quantum correlations, where $\mathcal{L}(\mathcal{H})$ denotes the corresponding space of density matrices. For example, we can take Q equal to the usual QD [15,31],

$$\delta(\rho) := \mathcal{I}(\rho) - \max_{\Pi} \mathcal{I}(\Pi[\rho]), \tag{3}$$

with $\mathcal{I}$ the quantum mutual information and $\Pi[\rho]$ the postlocal-measurement state. The measure δ attempts to capture the minimal disturbance suffered by the state under a local nonselective measurement, where $\max_{\Pi} \mathcal{I}(\Pi[\rho])$ is interpreted as the classical information accessible by local measurements. Also, QD is an essential resource in the performance of many quantum tasks such as, for example, quantum state merging [32,33], entanglement distribution [34,35], quantum measurements [36], and unambiguous quantum state discrimination [37].

Thus, potential discord (PD) should be defined as

$$\mathcal{P}^{\delta}(\rho) := \max_{E \in \mathrm{LO}} \delta(E[\rho]), \tag{4}$$

where $\delta(\rho)$ is the usual QD given by Eq. (3). We are going to consider local measurements over A, i.e., $\Pi[\rho] = \sum_i (\Pi_i^A \otimes \mathbb{1}^B)\rho(\Pi_i^A \otimes \mathbb{1}^B)$, for some local projective measurement $\{\Pi_i^A\}$. Analogous results can be found using bilocal measurements, or considering generalized (instead of projective) measurements.

A. Analytical bounds and results for PD

A careful observation of Eqs. (3) and (4) reveals a dual role of the local operations on the to-be-measured party. Indeed,

$$\mathcal{P}^{\delta}(\rho) = \max_{E}\{\mathcal{I}(E[\rho]) - \mathcal{I}(\Pi^E[\rho])\}. \tag{5}$$

Invoking the monotonicity of mutual information under quantum operations, one sees that both terms (the mutual information and the classical information) decrease under the action of E but, as they are subtracted, the net result could be greater or lower than the original discord of the state ρ. Moreover, from its definition, it is straightforward to observe that PD is an intermediate measure between QD and mutual information *for any state* ρ:

$$\delta(\rho) \leqslant \mathcal{P}^{\delta}(\rho) \leqslant \mathcal{I}(\rho). \tag{6}$$

In particular, *for pure states*, both QD and PD collapse to the entropy of entanglement (see Sec. III A) and one has

$$S(\rho^A) = \delta = \mathcal{P}^\delta \leqslant \mathcal{I} = 2S(\rho^A), \tag{7}$$

with $S(\rho^A) = S(\rho^B)$ the von Neumann entropy of the reduced state. On the other hand, *for CC states*, QD vanishes and one has $0 \leqslant \mathcal{P}^\delta \leqslant \mathcal{I}$. Finally, *all three measures are zero for product states*.

States with maximally mixed marginals. Let us consider the family of states with maximally mixed marginals in $d^A \times d^B$ dimensions, that is ρ^{AB} with $\mathrm{Tr}_{A}(B)\rho^{AB} \propto \mathbb{1}^{A(B)}$. Bell-diagonal states, including Werner and isotropic states, are particular examples of these. Their mutual information reads $\mathcal{I}(\rho^{AB}) = \ln(d^A d^B) - S(\rho^{AB})$. Moreover, local projective measurements leave the marginals invariant, yielding for the QD $\delta(\rho^{AB}) = \min_\Pi S(\Pi[\rho^{AB}]) - S(\rho^{AB}) = \min_\Pi S(\rho^{AB}||\Pi[\rho^{AB}])$, where $S(\cdot||\cdot)$ is the quantum relative entropy. Now, we prove that local unital channels cannot increase the quantum correlations of states included in this class.

Proposition 5. Discord for states with maximally mixed marginals in $2 \times d$ dimensions is nonincreasing under local unital operations.

Proof. Local unital operations preserve the maximally mixed marginals. Thus, for a local unital Λ one has $\delta(\Lambda[\rho]) = \min_\Pi S(\Lambda[\rho]||\Pi \circ \Lambda[\rho]) = \min_\Pi S(\Lambda[\rho]||\Lambda \circ \tilde{\Pi}[\rho]) \leqslant \min_\Pi S(\rho||\tilde{\Pi}[\rho]) \leqslant \delta(\rho)$, where $\tilde{\Pi}$ is the measurement defined by the original projectors transformed by the dual of Λ. That is, if $\{A_k\}$ are the Kraus operators defining Λ and $\{\Pi_i\}$ a set of orthogonal projectors, then $\tilde{\Pi}_i = \sum_k A^\dagger{}_k \Pi_i A_k$ determines $\tilde{\Pi}$. As unital channels in $\mathbb{C}^2$ preserve orthogonality [16], $\tilde{\Pi}$ is a well-defined projective measurement. Also, in the third step we have made use of the monotonicity of the relative entropy under quantum operations. ■

Hence, performing a local unital operation on a state with maximally mixed marginals in $2 \times d$ dimensions cannot enhance its quantumness. As a corollary, *PD restricted to a maximization over local unital operations coincides with the usual QD.* Besides, in a similar manner we can show that the same holds for CC states, namely that local unital channels cannot increase their quantumness as measured by QD.

Streltsov *et al.* [16] have proved that local unital channels cannot create quantum correlations for $2 \times d$ states as measured by the most of the distance-based measures of quantumness, as the relative entropy of discord or the geometric measure of quantumness defined in terms of the fidelity. Proposition 5 implies that the same holds for the usual QD when considering states with maximally mixed marginals.

Families with PD equal to QD. There are some special families of states whose symmetries and other particular properties suggest that PD must be equal to their QD. For example, for isotropic states in $d \times d$ dimensions, of the form $\rho^I_\eta = (1-\eta)(\mathbb{1}/d^2) + \eta\,|\beta\rangle\langle\beta|$, with $0 \leqslant \eta \leqslant 1$, QD is analytically calculated and the optimal measurement is universal, in the sense that any local measurement maximizes the postmeasurement mutual information. A local operation E over an isotropic state is such that

$$\delta(E[\rho^I]) = \mathcal{I}(E[\rho^I]) - \max_\Pi \mathcal{I}(\Pi \circ E[\rho^I])$$
$$= \min_\Pi \{\delta^\Pi(E[\rho]) - S(E[\rho^I_A]||\Pi \circ E[\rho^I_A])\},$$

where δ^Π denotes the nonoptimized version of δ. Now, suppose that we choose the measurement Π^E, given by the local eigenbasis of $E[\mathbb{1}/4]$. (In the particular case of E being unital, Π^E becomes arbitrary.) Using the joint convexity of relative entropy, one has

$$\mathcal{P}^\delta(\rho^I) = \max_E \delta(E[\rho^I])$$
$$\leqslant \max_E p\{\delta^{\Pi_E}(E[|\beta\rangle]) - pS(E[\mathbb{1}/d]||\Pi^E[\mathbb{1}/d])\}$$
$$\max_E p\delta^{\Pi^E}(E[|\beta\rangle]).$$

Although not tight—except in the trivial cases: $p = 0$ or $p = 1$—this bound suggests that maximizing PD for isotropic states involves quantum operations maximizing the QD of $E[|\beta\rangle]$. As β is a maximally discordant state, there is no local operation that can increase its QD and, in turn, provide a higher lower bound to the corresponding PD. Moreover, our numerical computations for random isotropic and Werner states of two qubits suggest that $S(E[\rho^I]||\Pi^E[E[\rho^I]]) \leqslant S(\rho^I||\Pi^E[\rho^I])$ for any LO, although we have been unable to find an analytical proof of this inequality.

Another interesting example is given by 2×2 dimensional states of the form $\rho_\alpha = \frac{\alpha}{2}\,|\beta\rangle\langle\beta| + \frac{1-\alpha}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|)$, with $0 \leqslant \alpha \leqslant 1$ (see Ref. [38]). As those are the ones that maximize discord for given values of entanglement of formation for a wide range of α (namely, α such that the entanglement is $\leqslant 0.620$), and observing that, for ρ_α states, discord is a monotonic nondecreasing function of the entanglement of formation, one concludes that any LO cannot enhance discord, since no LO can increase the entanglement of formation.

B. Numerical calculations for states of two qubits

In order to understand the role played by the maximization procedure involved in a PD computation, consider the one-parameter family of fully CC states $\rho^C_\eta = (1-\eta)\,|00\rangle\langle 00| + \eta\,|11\rangle\langle 11|$, with $0 \leqslant \eta \leqslant 1$. (Herein, in order to simplify notation, we use $|ij\rangle$ instead of $|i^A\rangle \otimes |j^B\rangle$.) For any value of η, the state is a mixture of product and mutually orthogonal (i.e., fully distinguishable) states. Thus, $Q(\rho^C_\eta) = 0$ for any reasonable measure of quantum correlations Q. However, for any $\eta \notin \{0,1\}$, a local operation will create quantum correlations with respect to the bipartition. As seen in Fig. 2, PD captures this idea, distinguishing the quantumness of the different members of the ρ^C_η family depending on the amount of quantum correlations that can be created under a LO. Highly symmetric families given by isotropic and Werner states have the same amount of PD than QD. We parametrize isotropic states as $\rho^I_\eta = (1-\eta)(\mathbb{1}/4) + \eta\,|\beta\rangle\langle\beta|$, with $|\beta\rangle$ a Bell-type state, and Werner states are given by $\rho^W_\eta = (\eta/3)P_+ + (1-\eta)P_-$, with $P_\pm = (\mathbb{1} \pm \mathbb{P})/2$ and $\mathbb{P} = \sum_{ij} |ij\rangle\langle ji|$.

In Fig. 3, the upper bound is given by mixtures of a rank-2 CC state, $\rho_{\text{class}} = (|00\rangle\langle 00| + |11\rangle\langle 11|)/2$, with the maximally entangled state $|\beta\rangle$, i.e., by the family $\rho^M_\gamma = (1-\gamma)\rho_{\text{class}} + \gamma\,|\beta\rangle\langle\beta|$, with $0 \leqslant \gamma \leqslant 1$. Separable states with QD $\geqslant 0.2018$ cannot be reached from CC states by LO, as a direct consequence of (i) the nature of the maximal QD of

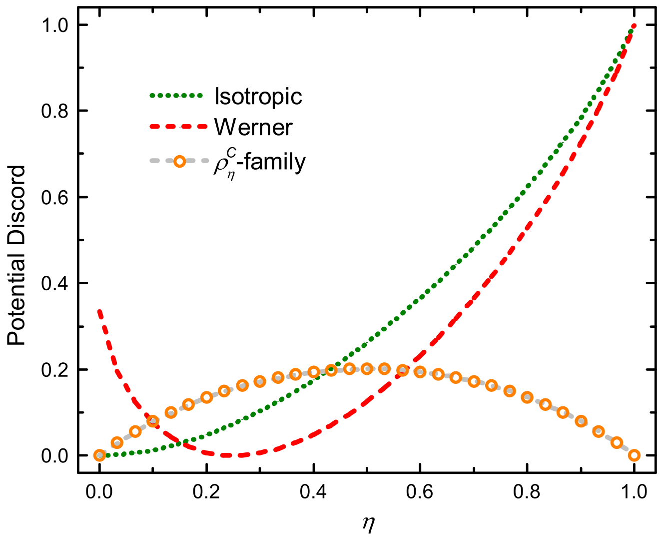

FIG. 2. Classically correlated states can exhibit some quantumness revealed by PD. That is the case for the ρ_η^C family of fully CC states (orange circles), which attains maximal PD when $\eta = 1/2$. Isotropic (green dotted line) and Werner (red dashed line) families have a PD that equals their QD, as no LO can increase quantum correlations for them (see Sec. IV A for a detailed discussion). PD for states of the ρ_η^C family fits very well with the function $\mathcal{P}^\delta(\eta) \approx 0.2018 - 0.6979(\eta - 0.5)^2 - 0.4204(\eta - 0.5)^4$.

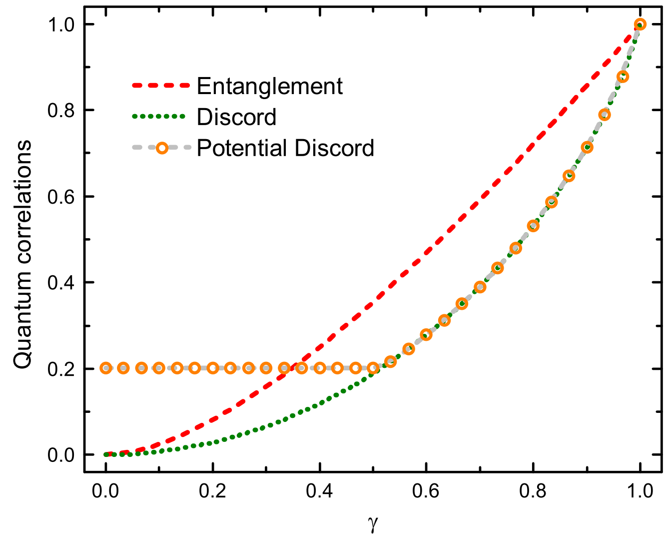

FIG. 4. Different measures of quantum correlations for the ρ_γ^M family. Entanglement of formation (red dashed line) is finite for any $\gamma > 0$. QD (green dotted line) behaves in the same qualitative way as entanglement, while PD (orange circles) exhibits a transition: below $\gamma = 1/2$ PD takes a constant minimum value and grows monotonically with γ for $\gamma \geqslant 1/2$.

separable states with fixed rank [27] and (ii) the local creation of quantum correlations from CC states [14]. Figure 4 shows how entanglement, QD, and PD behave differently for the states ρ_γ^M. The left border of Fig. 3 is reproduced by mixtures of rank-2 CC states with the maximally mixed state. Finally, the lower bound is given by isotropic states.

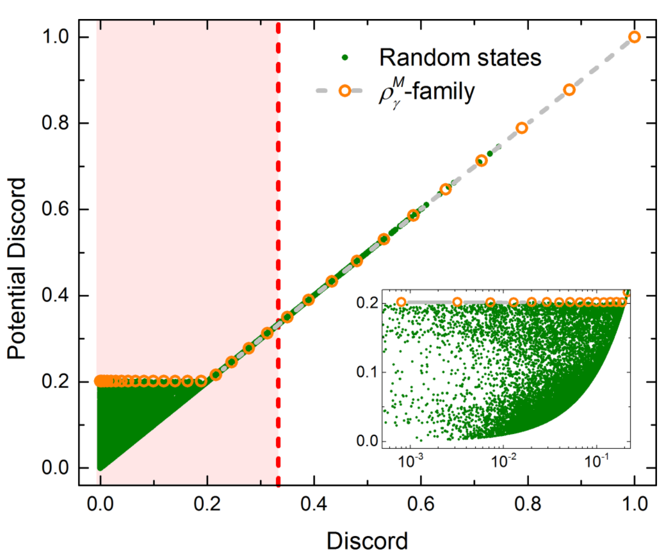

FIG. 3. Any separable state with QD $\leqslant 0.2018$ can be reached from a CC state of two qubits, performing local operations. Nonetheless, there are separable states with QD above 0.2018. The comparison between QD and PD displays this behavior. The upper bound is given by the ρ_γ^M family (orange circles; see text for details). Green data points correspond to $\sim 10^5$ randomly generated states. The (red shaded) region at the left of the vertical dashed line corresponds to the values of QD achievable by separable states. Inset: detail of the bottom leftmost region.

It is important to note that, as seen in Fig. 3, *for those states of two qubits with QD* $\geqslant$0.2018*, arbitrary local operations cannot increase their quantumness.* An open question is whether for arbitrary quantum states there is a lower bound for quantumness above which there would not exist a local operation that increases their quantumness.

Application: PD and local amplitude damping. In Ref. [18], the authors show how local noise—in particular, a local Markovian amplitude-damping channel (AD)—can enhance quantum correlations for a two-qubit system initially in the fully CC state, $\rho_0 = (|{+}0\rangle\langle{+}0| + |{-}1\rangle\langle{-}1|)/2$. Taking $E_0 = |0\rangle\langle 0| + \sqrt{1-p}\,|1\rangle\langle 1|$ and $E_1 = \sqrt{p}\,|0\rangle\langle 1|$ as the corresponding Kraus operators for the AD channel, with $p = 1 - e^{-\Gamma t}$ and Γ the relaxation rate, QD selects an intermediate value of p as the one that maximizes quantumness. Indeed, the authors find that, when the state is transformed under the local AD channel, QD reaches a maximal value of 0.07 when $p \approx 0.8$. On the other hand, PD sets $p = 0$ (when no operation is performed at all) as the one of maximum quantumness (see Fig. 5): the state ρ_0 has maximal PD among all CC states of two qubits, $\mathcal{P}^\delta(\rho_0) = 0.2018$, as it can be determined by a local unitary transformation of ρ_η^C with $\eta = 1/2$ (see Fig. 2).

The model can be interpreted in two different ways. First, we can assume that we are dealing with a system of two qubits, A and B, immersed in an environment of, at least, another two qubits $\bar{A}$ and $\bar{B}$. In that context, we can ask about the maximal quantumness we can obtain if we perform local operations in our laboratories with the aid of $\bar{A}$ and $\bar{B}$. Then, what the calculations show is that performing local AD channels is not the optimal election, since QD cannot reach its maximum in that case. Further, if we are restricted to that family of local operations, the best option is to set $p \approx 0.8$.

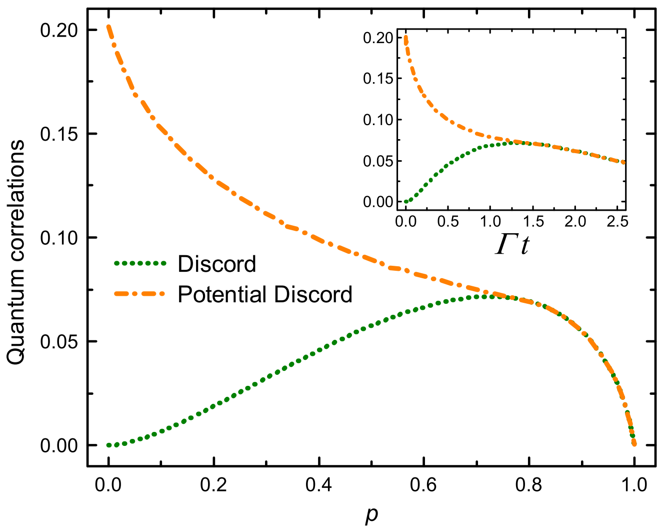

FIG. 5. Under an amplitude damping channel, QD (green dotted line) and PD (orange dashed-dotted line) exhibit different behaviors, indicating the fact that both measures reflect different aspects of the quantumness of a given state. Inset: QD and PD as a function of time (in units of the inverse of the relaxation rate Γ).

Alternatively, we can adopt a dynamic interpretation, where our system evolves in time according to the AD local channel. In that case, observing the behavior of the PD, one can conclude that the quantumness of our resources decrease monotonically with time, since the maximum of PD corresponds to $t = 0$.

In both cases, PD appears as a useful notion to understand the quantumness of the state of our composite system.

V. CONCLUSIONS

Summing up, we have investigated the relative character of quantum correlations in bipartite states with respect to the local observables of both parts, emphasizing that the question is closely related to that of the effect of local operations on quantum correlations. Those results have been summarized in Propositions 1–3. We have proposed a family of measures of what we call potential quantumness, which takes into account this relative character. These quantifiers involve a maximization procedure over any measure of bipartite quantum correlations that, in principle, makes it hard to compute it—considering that the usual measures of correlations involve an optimization procedure too [39]. However, in some low-dimensional or highly symmetrical situations, our quantumness measure can be simplified by taking advantage of known results. In particular, we have applied the measure to special families of states of $2 \times d$ dimensional states, showing that unital operations cannot enhance the quantumness of states with maximally mixed marginals for the discord-based version of our measure. Also, as an application to a typical situation, we have compared our quantumness measure's behavior with that of quantum discord for the case of two qubits evolving under the effect of a local amplitude-damping channel.

We stress the fact that our presentation is not in contradiction with the ones of Dakic *et al.* [20] and Gessner *et al.* [14], who state that quantum correlations of those separable states that can be produced from CC states by local operations are not genuinely quantum. Indeed, our results highlight this fact: we observe a collapse of the quantum correlations of those separable states to a constant value of what we call the potential quantumness. Moreover, we point out that this quantumness degree is already present in those states, without (and before) considering any kind of operation, as can be confirmed by measuring the appropriate local observables.

In conclusion, we have analyzed an alternative quantification of quantum correlations, based on the local capabilities of a given system, which may shed some light on the subject. Besides some specific analytical and numerical results that we have presented, it would be interesting to study if distance-based measures can provide less hard-to-compute versions.

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APPENDIX A: QUANTUM CORRELATIONS UNDER GLOBAL UNITARY OPERATIONS

When we remove the locality restriction, we are allowed to explore the whole observables' space. This situation was previously studied by Zanardi [4], where he distinguishes between *virtual* and real subsystems. Also, in Zanardi *et al.* [5], the authors studied the role of the relevant observables in the tensor product structure of the Hilbert space. Harshmann and

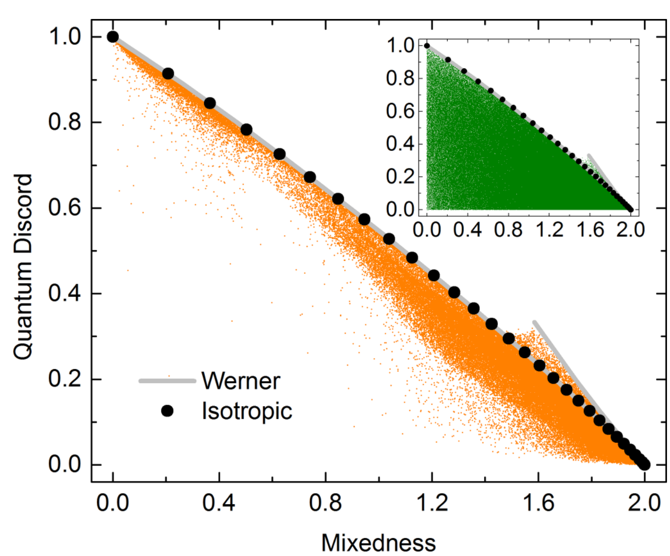

FIG. 6. Under global unitary operations, QD of a fixed state can be increased by an amount depending on the mixedness of the state. In particular, pseudopure states can be transformed into isotropic states (black dots), maximizing the QD for a certain value of entropy. Werner states (grey line) provide the upper bound for almost every value of entropy. Orange data points correspond to $\sim 10^5$ random states. Inset: original QD for the same states. The region between isotropic and Werner states is upper bounded by families of two parameters (see Ref. [41] for details).

Ranade [7] gave a formal proof of the fact that, in the pure states case, one can tailor the observables so as to go from a situation with no entanglement to a maximal entanglement one, for any fixed state. However, for mixed states the situation is radically different—as it can be seen, for example, taking the maximally mixed state, which is the same for any observables we choose—and the tailoring of observables cannot place all the states on an equal footing. Nonetheless, consideration of global unitary operations tends to accumulate the values of discord near the maximums for given values of entropy (Fig. 6).

Here, an important role is played by pseudopure states (weighted mixtures of a maximally mixed state with a pure state), namely $\rho_{a,\psi} = (1-a)(\mathbb{1}/d) + a\,|\psi\rangle\langle\psi|$, with $0 \leqslant a \leqslant 1$. For a fixed value of a, one can maximize the QD by taking $|\psi\rangle = |\beta\rangle$ as the maximally entangled state, so as to convert $\rho_{a,\psi}$ into an isotropic state. This can be accomplished applying a (global) unitary operation over $|\psi\rangle$, i.e., there is always a unitary U_ψ such that $U_\psi|\psi\rangle = |\beta\rangle$. Hence, $U_\psi \rho_{a,\psi} U_\psi^\dagger = (1-a)(\mathbb{1}/d) + a\,|\beta\rangle\langle\beta| = \rho_{a,\beta}$.

APPENDIX B: OBSERVABLES AND THE STRUCTURE OF THE HILBERT SPACE

The simplest case may be a four-qubit system, where A and B, with $\mathcal{H}_A \cong \mathbb{C}^4 \cong \mathcal{H}_B$, are subsystems of two qubits. The state space is determined by density operators over $\mathcal{H}_{AB} \cong (\mathbb{C}^2)^{\otimes 4} \cong (\mathbb{C}^4)^{\otimes 2} \cong \mathbb{C}^{16}$. If we impose a *locality* restriction for the $A|B$ partition, then any unitary operation $U^{A|B} = U^A \otimes U^B$ over $\mathbb{C}^4 \otimes \mathbb{C}^4$ acts locally over the A and B degrees of freedom. Thus, the action of $U^A \otimes U^B$ can be interpreted in two alternative ways:

(a) as a bilocal unitary transformation over the space of states;

(b) as a transformation over the spaces of local observable operators, that is a reconfiguration of the local degrees of freedom.

Indeed, for any state ρ^{AB} and any observable O, its expectation value is

$$\begin{aligned}\langle O\rangle_{U^{A|B}\rho^{AB}U^{A|B\dagger}} &= \mathrm{Tr}[(U^{A|B}\rho^{AB}U^{A|B^\dagger})O]\\ &= \mathrm{Tr}[\rho^{AB}(U^{A|B^\dagger}OU^{A|B})]\\ &= \langle U^{A|B^\dagger}OU^{A|B}\rangle_{\rho^{AB}}.\end{aligned}$$

What determines that ρ^{AB} has subsystems A and B? The determination of these subsystems is certainly not unique and usually relies on the accessible degrees of freedom of our joint system. A *qubit* is an abstract entity, well-suited for the description of quantum bistable systems, as, e.g., a spin one-half particle. For two independent particles for which the only relevant degrees of freedom are their one-half spins, the "natural" description is given by a density operator over $\mathbb{C}^2 \otimes \mathbb{C}^2$. The situation is better understood from the *observables perspective*. The natural observables are the spin operators in A and B, represented by the corresponding Pauli matrices σ_i^A and σ_i^B, with $i = x,y,z$. If they represent the relevant degrees of freedom, the description of our system can be given in terms of the algebra spanned by them, $\mathcal{O} = \mathrm{span}\{\mathbb{1}\otimes\mathbb{1}, \sigma_i\otimes\mathbb{1}, \mathbb{1}\otimes\sigma_j\}$, where $i,j = x,y,z$ and $\mathbb{1}$ is the identity operator for $\mathbb{C}^2$. We removed the A and B superscripts so as to simplify the notation. The algebra $\mathcal{O}$ induces a tensor product structure over the Hilbert space of the joint system, i.e., $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ [5]. But these "natural observables" are not necessarily natural at all: any unitary transformation $\mathcal{O} \mapsto \mathcal{O}_U := U\mathcal{O}U^\dagger$ could *a priori* be regarded, without any further physical assumption, on an equal footing with the presumed natural spin one.

For example, the pure noncorrelated state $|\psi\rangle = |00\rangle$ is unitarily equivalent to the maximally entangled state $|\beta_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ via a transformation U_ψ. Alternatively, one can assert that $|\beta_+\rangle$ is the representation of $|\psi\rangle$ in terms of the algebra $U_\psi \mathcal{O} U_\psi{}^\dagger$ of observables. Harshman and Ranade gave in Ref. [7] the formal proof that, for any pure state in $\mathbb{C}^N$, where $N \in \mathbb{N}$ is not a prime, the observables can be "tailored" to induce a subsystem decomposition for which the state has a desired level of entanglement, from a product state to a maximally entangled one. The situation is rather different for mixed states.

When dealing with mixed states, the unitaries do not allow one to surf the whole space of states. In particular, a unitary transformation cannot change the eigenvalues of a given state, thus preserving its original entropy. Let us suppose that we start with the mixed separable—indeed, CC—state $\rho_p^{AB} = p\,|00\rangle\langle 00| + (1-p)\,|11\rangle\langle 11|$. A unitary transformation $\rho_p^{AB} \mapsto U\rho_p^{AB}U^\dagger$ will preserve the purity and orthogonality of both components. For example, applying the U_ψ defined above, we have $U_\psi \rho_p^{AB} U_\psi{}^\dagger = p\,|\beta_+\rangle\langle\beta_+| + (1-p)\,|\beta_-\rangle\langle\beta_-|$, with $|\beta_\pm\rangle := (|00\rangle \pm |11\rangle)/\sqrt{2}$ two orthogonal maximally correlated states. For $p = 0$ or $p = 1$, ρ_p^{AB} is pure and its transformed version becomes a Bell-type state. When, $p \in (0,1)$, ρ_p^{AB} is never pure and its transformed version is not maximally entangled.

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[23] This can be proved by construction of a LU, U, such that $Q^{A|B}(U\rho^{\text{ext}}U^\dagger) \neq 0$. Let us divide the cases of ρ^{AB} being CC and QC. If it is QC, no extension is needed because ρ^{AB} is already such that $Q(\rho^{AB}) \neq 0$. If, on the contrary, ρ^{AB} is CC, we know that it has the form $\rho^{AB} = \sum_{ij} p_{ij}|i^A\rangle\langle i^A| \otimes |j^B\rangle\langle j^B|$, for $\{|i^A\rangle\}$ ($\{|j^B\rangle\}$) orthogonal states on $\mathcal{H}_A$ ($\mathcal{H}_B$). Consider the cases when only two of the probabilities are different from zero and, moreover, they involve different local projectors, i.e., $\rho^{AB} = p|0^A\rangle\langle 0^A| \otimes |0^B\rangle\langle 0^B| + (1-p)|1^A\rangle\langle 1^A| \otimes |1^B\rangle\langle 1^B|$. In this case, we can extend the state with ancillas of dimension 2 on both sides: $\rho^{\text{ext}} = |0^{\bar{A}}\rangle\langle 0^{\bar{A}}| \otimes \rho^{AB} \otimes |0^{\bar{B}}\rangle\langle 0^{\bar{B}}|$. Now, $|0^{\bar{A}}\rangle \otimes |0^A\rangle$ and $|0^{\bar{A}}\rangle \otimes |1^A\rangle$ are orthogonal states on $\mathcal{H}_{\bar{A}} \otimes \mathcal{H}_A$, and can be mapped via a unitary transformation to any other pair $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ of orthogonal states. The same happens on $\mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$, where a local unitary transformation takes $|0^B\rangle \otimes |0^{\bar{B}}\rangle$ and $|1^B\rangle \otimes |0^{\bar{B}}\rangle$ into $|\beta\rangle$ and $|\beta_\perp\rangle$, respectively. Tracing out the ancillas yields $\rho^{A'B'} = p\rho_\alpha^{A'} \otimes \rho_\beta^{B'} + (1-p)\rho_{\alpha_\perp}^{A'} \otimes \rho_{\beta_\perp}^{B'}$, which is in general a QC state, unless $\rho_\alpha^{A'}$ and $\rho_{\alpha_\perp}^{A'}$ have common eigenbasis (and the same for B'). If it is the case that both probabilities involve a common projector on one of the parties, the same procedure creates a QC state with respect to the other part. Generalization to higher rank states is straightforward.
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