Towards a gauge-equivalent magnetic structure of the nonlocal nonlinear Schrödinger equation

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It is shown that the nonlocal nonlinear Schrödinger equation recently proposed by Ablowitz and Musslimani [Phys. Rev. Lett. **110**, 064105 (2013)] is gauge equivalent to the unconventional system of coupled Landau-Lifshitz equations. The first integrals of motion and one-soliton solution of an obtained model are given. The physical and geometrical aspects of model and their effect on expected metamagnetic structures are studied.

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I. INTRODUCTION

Reference [1] generalizes the classical representation according to which the necessary condition for the reality of the spectrum of Hamiltonian $H = p^2 + V(x)$ is Hermiticity ($H = H^\dagger$). It was shown that Hamiltonians which commutate with composition of the parity reflection operator ($\mathcal{P} : x \to -x,\, p \to -p$) and the time-reversal operator ($\mathcal{T} : x \to x,\, p \to -p,\, i \to -i$) have real eigenvalues. A necessary condition for a Hamiltonian H to be $\mathcal{PT}$ symmetric is that its potential $V(x)$ should satisfy the condition $V(x) = V^*(-x)$. This condition requires that the real part of the potential is even while its imaginary part is an odd function of position x.

This fact has given an impetus to new research in many diverse fields of physics, such as optics [2–5], quantum mechanics [6,7], magnetism [8–10], quantum field theory [11–14], and electric circuits [15,16]. See also Ref. [17] for a review of nonlinear phenomena in $\mathcal{PT}$-symmetric systems. The first experimental realization of $\mathcal{PT}$-symmetric structure has been achieved in optics [2,4], where the role of potential is played by the complex refractive-index $n(x)$. Such $\mathcal{PT}$-symmetric photonic crystals have been realized in heterogeneous multilayered structures, where the real part of the refractive index is the same in all layers, while the imaginary parts in neighboring layers differ in sign. It should be noted that such gain-loss balance principle is applied to many $\mathcal{PT}$-symmetrical models [2–5,8–10,15,16,18]. For definiteness, in Ref. [10] the $\mathcal{PT}$-symmetric macroscopic two-layered magnetic model with various magnetization vectors in nanolayers is considered. The mathematical model is presented by the system of coupled Landau-Lifshitz-Hilbert equations, where an additional dissipative Hilbert term is used for implementation of the gain-loss principle.

Simultaneously, nonlinear $\mathcal{PT}$-symmetric dynamic models closely related with the above-mentioned fields of physics are under active study [13,19–22]. The nonlocal nonlinear Schrödinger equation (NNLS), which was recently proposed by Ablowitz and Musslimani [19], is of special interest,

$$i\psi_t(x,t) + \psi_{xx}(x,t) + 2\alpha\psi(x,t)\psi^*(-x,t)\psi(x,t) = 0, \quad (1)$$

and is already applied to optics [2–5]. Here, the parameter $\alpha = \pm 1$ indicates the focusing (+) and defocusing (−) nonlinearity. It is obvious that *self-induced* potential $V(x,t) = \psi(x,t)\psi^*(-x,t)$ is $\mathcal{PT}$ symmetric: $V(x,t) = V^*(-x,t)$. The key point is that Eq. (1) has qualitative properties other than the standard nonlinear Schrödinger equation (NLS) and its classical generalizations [23–32]. For example, in the focusing case, the model (1) admits both bright (sech-type) and dark (tanh-type) soliton states [33], while the standard NLS model

$$i\psi_t(x,t) + \psi_{xx}(x,t) + 2|\psi(x,t)|^2\psi(x,t) = 0 \quad (2)$$

can support only bright solution. The NLS model is integrable by the inverse scattering method of Zakharov-Shabat's scheme [34]. A relation between the solutions of Eq. (2) and isotropic Heisenberg ferromagnetic model (Landau-Lifshitz equation)

$$\mathbf{s}_t = \mathbf{s} \times \mathbf{s}_{xx} \quad (3)$$

was first found in Ref. [35]. Here, $\mathbf{s}(x,t)$ is the unitary vector of a magnetization density (spin density). The Landau-Lifshitz equation (LL) is also completely integrable [36]. The models (2) and (3) and their generalizations belong to the same gauge equivalence class [37,38].

Hence, a quite relevant question arises: Which of magnetic structures is the NNLS model gauge equivalent to? The main purpose of our article is to resolve this issue. It permits extension of the class of possible new $\mathcal{PT}$-symmetric magnetic structures.

In Sec. II, the focusing nonlinearity NNLS model is shown to be gauge equivalent to the unconventional system of coupled LL equations and first integrals of motion are obtained. An exact one-soliton solution for the designed model is presented in Sec. III. In Sec. IV, we attempt to establish connection with real magnetic structures. The obtained model can be interpreted as the model for a two-sublattice pseudo-antiferromagnet. Finally, Sec. V contains the conclusion.

II. GAUGE EQUIVALENCE

The concept of the gauge equivalence for two spectral problems

$$\Psi_{1,x} = U_1(x,t,\lambda)\Psi_1, \quad \Psi_{1,t} = V_1(x,t,\lambda)\Psi_1, \quad (4a)$$

$$\Psi_{2,x} = U_2(x,t,\lambda)\Psi_2, \quad \Psi_{2,t} = V_2(x,t,\lambda)\Psi_2, \quad (4b)$$

was introduced in Ref. [37]. According to the definition, spectral problems (4a) and (4b) are gauge equivalent, if

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eigenfunctions Ψ_1 and Ψ_2 can be connected as $\Psi_1 = G\Psi_2$, where $G(x,t)$ is the solution of Eq. (4a) at zero spectral parameter λ. The Lax pair (U_1,V_1) is transformed to (U_2,V_2) under the above gauge transformation as

$$U_1 = GU_2G^{-1} + G_xG^{-1}, \quad V_1 = GV_2G^{-1} + G_tG^{-1}. \tag{5}$$

In Ref. [19] it was proved that the NNLS model (1) is integrable by the inverse scattering transform method. It was shown the NNLS model can be embedded into the 2×2 spectral problem of Zakharov-Shabat [34]. The corresponding Lax pair is given as follows:

$$U_1 = A_0 + \lambda A_1, \quad V_1 = B_0 + \lambda B_1 + \lambda^2 B_2. \tag{6}$$

In focusing case of nonlinearity ($\alpha = 1$) matrices A_j, B_k ($j = 0,1$; $k = \overline{0,2}$) are given by

$$\begin{aligned}
A_0 &= \begin{pmatrix} 0 & \psi^*(-x,t) \\ -\psi(x,t) & 0 \end{pmatrix}, \quad A_1 = i\sigma_3, \\
B_0 &= -i\begin{pmatrix} \psi^*(-x,t)\psi(x,t) & \psi_x^*(-x,t) \\ \psi_x(x,t) & -\psi^*(-x,t)\psi(x,t) \end{pmatrix}, \\
B_1 &= 2A_0, \quad B_2 = 2A_1.
\end{aligned} \tag{7}$$

Here and elsewhere, $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is the set of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

In view of relations $A_0\sigma_3 = -\sigma_3 A_0$ and $SS_x = -S_xS = 2G^{-1}A_0G$ the gauge transformation (5) yields, after some algebraic manipulation,

$$U_2(x,t) = i\lambda S, \quad V_2(x,t) = i\lambda SS_x + 2i\lambda^2 S, \tag{9}$$

where the matrix $S(x,t)$ is defined as $S = G^{-1}\sigma_3 G$, and G satisfies the system of equations

$$G_x = A_0G, \quad G_t = B_0G. \tag{10}$$

From zero-curvature condition $U_{2,t} - V_{2,x} + [U_2,V_2] = 0$ (here [,] is commutator) we can obtain the following matrix equation:

$$S_t = -\frac{i}{2}[S,S_{xx}]. \tag{11}$$

It is easy to check that $S^2 = \mathbf{I}$ and $\mathrm{Sp}\, S = 0$, where $\mathbf{I}$ is the unit matrix.

Note that a similar procedure is used for the NLS model [37]. However, in our case the sense of the matrix $S(x,t)$ is somewhat different—it cannot be associated with the classical vector of the Heisenberg ferromagnetic spin. Indeed, from Eq. (10) we find that in general the matrix $G(x,t)$ can be represented as

$$G = \begin{bmatrix} p(x,t) & q^*(-x,t) \\ q(x,t) & p^*(-x,t) \end{bmatrix}. \tag{12}$$

From definition $S = G^{-1}\sigma_3 G$ it follows that matrix S *is not Hermitian*, in contrast to the NLS model case. Matrix S is expressed explicitly as

$$S = \begin{pmatrix} s_3 & s_1 - is_2 \\ s_1 + is_2 & -s_3 \end{pmatrix}, \tag{13}$$

where (s_1,s_2,s_3) is *complex-valued* vector with components s_j

$$\begin{aligned}
s_1(x,t) &= \Delta[p^*(-x,t)q^*(-x,t) - p(x,t)q(x,t)], \\
s_2(x,t) &= \Delta i[p^*(-x,t)q^*(-x,t) + p(x,t)q(x,t)], \\
s_3(x,t) &= \Delta[p(x,t)p^*(-x,t) + q(x,t)q^*(-x,t)], \\
\Delta &= [p(x,t)p^*(-x,t) - q(x,t)q^*(-x,t)]^{-1}.
\end{aligned} \tag{14}$$

In our case the matrix S has $\mathcal{PT}$ symmetry

$$S(x,t) = \sigma_3 S^\dagger(-x,t)\sigma_3. \tag{15}$$

This follows directly from the identities

$$s_1(x,t) = -s_1^*(-x,t), \quad s_2(x,t) = -s_2^*(-x,t),$$
$$s_3(x,t) = s_3^*(-x,t).$$

Let us now represent Eq. (11) in the vector form.

Matrix S can be written as the sum of a Hermitian matrix and a skew-Hermitian matrix: $S = M + iL$, where $M = M^\dagger = (S^\dagger + S)/2$ and $L = L^\dagger = i(S^\dagger - S)/2$. Moreover, it is important to note that $\mathrm{Sp}\, M = 0$ and $\mathrm{Sp}\, L = 0$. Then, in the standard Pauli matrix representation $M = \mathbf{m}\cdot\boldsymbol{\sigma}$, $L = \mathbf{l}\cdot\boldsymbol{\sigma}$, the Eq. (11) is

$$\mathbf{m}_t = \mathbf{m}\times\mathbf{m}_{xx} - \mathbf{l}\times\mathbf{l}_{xx}, \tag{16a}$$

$$\mathbf{l}_t = \mathbf{m}\times\mathbf{l}_{xx} + \mathbf{l}\times\mathbf{m}_{xx}, \tag{16b}$$

where $\mathbf{m} = (m_1,m_2,m_3)$, $\mathbf{l} = (l_1,l_2,l_3)$ are real-valued vectors. Furthermore, from the invariance of the quadratic form $s_1^2 + s_2^2 + s_3^2$ the following invariants of system (16) can be obtained:

$$\mathbf{m}^2 - \mathbf{l}^2 = 1, \quad \mathbf{m}\cdot\mathbf{l} = 0. \tag{17}$$

Thus, NNLS model (1) is gauge equivalent to the coupled LL equations (CLL) (16) and (17).

In Ref. [19], an infinite series of conservation laws for Eq. (1) is also obtained (see also Refs. [39,40]). The first two of these integrals of motion can be written as

$$I_1 = \int_{-\infty}^{+\infty} \mathcal{I}_1 dx, \quad I_2 = \int_{-\infty}^{+\infty} \mathcal{I}_2 dx,$$

where $\mathcal{I}_1 = \psi^*(-x,t)\psi(x,t)$ and $\mathcal{I}_2 = i\psi(x,t)\psi_x^*(-x,t) - i\psi_x(x,t)\psi^*(-x,t)$ correspond to the pseudopower and the pseudomomentum, respectively. As is well known, in the standard case the integrals of motion of NLS and LL models are closely related [35,37]. In an analogous way, we can obtain the main integrals of motion of the CLL model and relate them to I_1 and I_2 integrals.

Let us introduce the density of energy and the current density as

$$\mathcal{E} = \frac{1}{2}\left(\mathbf{m}_x^2 - \mathbf{l}_x^2 + 2i\mathbf{m}_x\cdot\mathbf{l}_x\right), \tag{18a}$$

$$\mathcal{J} = \mathbf{m}_{xx}\cdot\mathbf{a} - \mathbf{l}_{xx}\cdot\mathbf{b} + i(\mathbf{m}_{xx}\cdot\mathbf{b} + \mathbf{l}_{xx}\cdot\mathbf{a}), \tag{18b}$$

where $\mathbf{a} = \mathbf{m}\times\mathbf{m}_x - \mathbf{l}\times\mathbf{l}_x$ and $\mathbf{b} = \mathbf{m}\times\mathbf{l}_x + \mathbf{l}\times\mathbf{m}_x$. It may easily be shown, using equations Eqs. (16) and (17), that the energy density $\mathcal{E}(x,t)$ and the current density $\mathcal{J}(x,t)$ satisfy the continuity equation $\mathcal{E}_t + \mathcal{J}_x = 0$. Furthermore, by using Eqs. (11), (16), and (17) we get for the energy density

$$\mathcal{E} = -\tfrac{1}{2}\det(M_x + iL_x) = 2\det A_0 = 2\mathcal{I}_1$$

and for the current density

$$\mathcal{J} = -\frac{i}{4}\,\mathrm{Sp}\,\{(M+iL)[M_x+iL_x, M_{xx}+iL_{xx}]\}$$
$$= -\tfrac{1}{2}\mathrm{Sp}\,\{S_x S_t\} = 2\mathrm{Sp}\,\{A_0 B_0\} = 2\mathcal{I}_2.$$

It is important to note the integral characteristics $W = \int_{-\infty}^{+\infty} \mathcal{E} dx$ and $J = \int_{-\infty}^{+\infty} \mathcal{J} dx$ can take real values only, although the densities $\mathcal{E}$ and $\mathcal{J}$ are complex-valued functions. This is due to the fact that the densities $\mathcal{E}$ and $\mathcal{J}$ can be expressed as the sum of a real-valued parity-even term and a parity-odd term that is purely imaginary:

$$\mathrm{Re}\,\mathcal{E}(-x,t) = \mathrm{Re}\,\mathcal{E}(x,t), \quad \mathrm{Im}\,\mathcal{E}(-x,t) = -\,\mathrm{Im}\,\mathcal{E}(x,t),$$
$$\mathrm{Re}\,\mathcal{J}(-x,t) = \mathrm{Re}\,\mathcal{J}(x,t), \quad \mathrm{Im}\,\mathcal{J}(-x,t) = -\,\mathrm{Im}\,\mathcal{J}(x,t).$$

In Ref. [40] this is presented in more detail.

III. SOLITON SOLUTION

The gauge equivalence allows us to construct solutions of CLL model (16)–(17) from the solutions of the original NNLS model (1).

On comparing the Galilean invariance of NLS and NNLS models,

$$\tilde{\psi}_{\mathrm{NLS}}(x,t) = \psi_{\mathrm{NLS}}(x-2vt+d,t)e^{ivx-iv^2t},$$
$$\tilde{\psi}_{\mathrm{NNLS}}(x,t) = \psi_{\mathrm{NNLS}}(x-2ivt+id,t)e^{-vx+iv^2t},$$

the solution of the NNLS model can be obtained from the well-known sech-type soliton solution of the NLS model by performing the mapping $v \to iv, d \to id$. As a result we obtain

$$\psi(x,t) = a \sec X e^{-i\Phi}, \tag{19}$$

where $X = -iax + 2avt + d$, $\Phi = ivx - (a^2+v^2)t - \phi$ and all parameters are real. The above solution satisfies zero boundary conditions $\lim_{|x|\to+\infty} \psi = 0$ at $|a/v| > 1$. Further, we will consider only this domain of parameters. In the strict sense, this solution is not solitonic, because its envelope oscillates in time with period $T = \pi/2av$. Moreover, solution (19) is singular at the point $(0, t_{\mathrm{sing}})$, where $t_{\mathrm{sing}} = (\pi - 2d)/4av + nT$ and n is integer (compare with the Eqs. (22) and (23) from Ref. [19]).

It should be noted that from invariance of the model (1) under the Galilean transformation with a pure imaginary velocity it follows the NNLS model cannot have propagating localized states. It is particularly true for solution (19).

It is evident that Eqs. (10) can be easily integrated, and the solution is given as

$$p(x,t) = -cv + aci \tan X,$$
$$q(x,t) = ac \sec X e^{-i\Phi},$$

where c is the constant of integration.

Then, by using Eqs. (14) the components of vectors $\mathbf{m}$ and $\mathbf{l}$ can be written as

$$m_k = \frac{s_k(x,t) - s_k(-x,t)}{2}, \quad m_3 = \frac{s_3(x,t) + s_3(-x,t)}{2},$$
$$l_k = \frac{s_k(x,t) + s_k(-x,t)}{2i}, \quad l_3 = \frac{s_3(x,t) - s_3(-x,t)}{2i}, \tag{20}$$

where $k = 1,2$ and $s_j(x,t)$ are given by

$$s_1 = \frac{2ia \sec X}{a^2 - v^2}(v \sin\Phi + a\cos\Phi \tan X), \tag{21a}$$
$$s_2 = \frac{2ia \sec X}{a^2 - v^2}(v \cos\Phi - a\sin\Phi \tan X), \tag{21b}$$
$$s_3 = 1 - \frac{2a^2 \sec^2 X}{a^2 - v^2}. \tag{21c}$$

As is easy to see, the NNLS model transforms to the canonical NLS model under the spatial parity symmetry $\psi(x,t) = \psi(-x,t)$. In this case, the matrix S in Eq. (11) is converted into the Hermitian and consequently $\mathbf{l} = 0$, whereas the system (16) and (17) become $\mathbf{m}_t = \mathbf{m} \times \mathbf{m}_{xx}$, where $\mathbf{m}^2 = 1$, which is the conventional LL equation. For the spatial parity symmetry of the considered solution (19) it is needed to put the restrictions $v = 0$ and $d = 0$. The dynamics projections of vector $\mathbf{m}(x,t)$ at such parameters is depicted in Figs. 1(a)–1(c). For the geometrical clarity in Fig. 1(d) the dynamics of vector $\mathbf{m}(x,t)$ in space $\mathbf{R}^3$ for the selected value $x = x_0$ is displayed. As was expected, vector $\mathbf{m}(x,t)$ describes a localized stationary nonlinear wave of a precession relative to the direction Oz.

The nonzero velocity case $v \neq 0$ is more interesting. Figure 2 shows the dynamic of the one-soliton solution for this case. We have depicted the dynamics of vectors $\mathbf{m}$ and $\mathbf{l}$ for the fixed value $x = x_0$ in the same manner as in the previous case. Now, the length of these vectors satisfies a hyperbolic condition (17), but individually they are not bounded. In this connection, $\mathbf{m}(x,t)$ and $\mathbf{l}(x,t)$ do not belong to the two-dimensional unit sphere, in contrast to the case of the LL equation. The corresponding two-dimensional surface

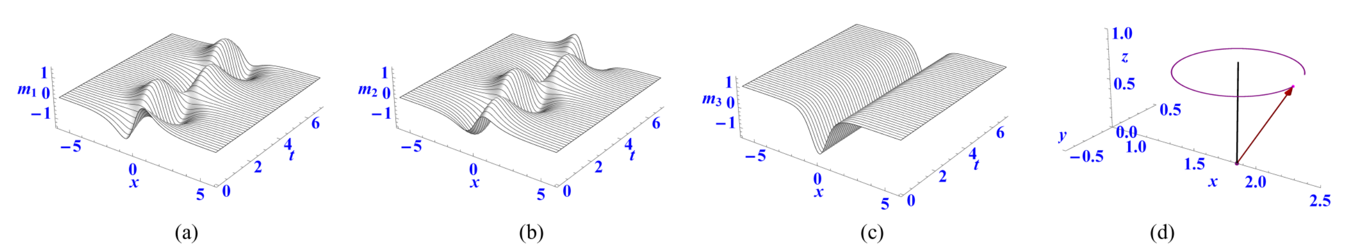

FIG. 1. Dynamics of the one-soliton solution of the CLL model. Panels (a)–(c) show the vector projections of $\mathbf{m}$; panel (d) shows dynamics of $\mathbf{m}(x_0,t)$ in space $\mathbf{R}^3$. The curve corresponds to the trajectory of vector. The parameters are chosen as $a = 1.2$, $v = 0.0$, $d = 0.0$, $\phi = 0.5$, and $x_0 = 1.85$.

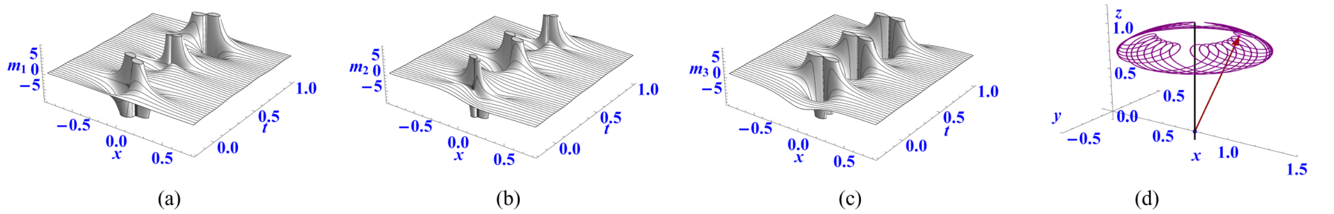

FIG. 2. Dynamics of the one-soliton solution of the CLL model. Panels (a)–(c) and (e)–(g) show the vector projections of $\mathbf{m}$ and $\mathbf{l}$ respectively; panels (d) and (h) show dynamics of $\mathbf{m}(x_0,t)$ and $\mathbf{l}(x_0,t)$ in space $\mathbf{R}^3$. The curves correspond to the trajectory of vectors. The parameters are chosen as $a = 3.1$, $v = 1.2$, $d = 1.7$, $\phi = 0.5$, and $x_0 = 0.75$.

for $\mathbf{m}(x,t)$ and $\mathbf{l}(x,t)$ are dependent on the space variable x. Figure 2(d) depicts that the dynamics of the vectors $\mathbf{m}(x,t)$ is more complex then just the precession. The additional term $\mathbf{l} \times \mathbf{l}_{xx}$ in Eq. (16a) outputs the system from the precessional steady state. A similar complex nonprecession dynamics also has the vector $\mathbf{l}(x,t)$.

The analysis of the obtained solution (20) and (21) implies some important properties:

(i) vectors $\mathbf{m}$ and $\mathbf{l}$ have finite boundary conditions

$$\lim_{|x|\to+\infty} \mathbf{m} = \mathbf{m}_0, \quad \lim_{|x|\to+\infty} \mathbf{l} = \mathbf{l}_0, \tag{22}$$

where $\mathbf{m}_0 = (0,0,1)$ and $\mathbf{l}_0 = (0,0,0)$;

(ii) the integrals

$$\int_{-\infty}^{+\infty} \mathbf{m} - \mathbf{m}_0\, dx = \mathbf{M}_{\text{sum}}, \quad \int_{-\infty}^{+\infty} \mathbf{l}\, dx = \mathbf{l}_0, \tag{23}$$

where $\mathbf{M}_{\text{sum}} = (0,0,4a/[v^2 - a^2])$ are constants of motion;

(iii) vectors $\mathbf{m}$ and $\mathbf{l}$ develop a singularity in the finite time $t = t_{\text{sing}}$ at $x = 0$.

IV. DISCUSSION

It is well known that in the phenomenological theory, the dynamics of multisublattice systems, in particular, the dynamics of Heisenberg antiferromagnetic spin systems at temperatures much below the critical, is described by the coupled LL equations [41–43]

$$\mathbf{m}_t = -\left\{\mathbf{m} \times \frac{\delta W}{\delta \mathbf{m}} + \mathbf{l} \times \frac{\delta W}{\delta \mathbf{l}}\right\}, \tag{24a}$$

$$\mathbf{l}_t = -\left\{\mathbf{m} \times \frac{\delta W}{\delta \mathbf{l}} + \mathbf{l} \times \frac{\delta W}{\delta \mathbf{m}}\right\}. \tag{24b}$$

Here $\mathbf{m}$ and $\mathbf{l}$ are normalized vectors of ferromagnetism and antiferromagnetism, respectively, which are related to the sublattices magnetization vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ in the two-sublattice model by the $\mathbf{m} = \alpha_1(\mathbf{M}_1 + \mathbf{M}_2)$ and $\mathbf{l} = \alpha_2(\mathbf{M}_1 - \mathbf{M}_2)$, where $\alpha_{1,2}$ are normalization coefficients. The functional W corresponds to the energy of a magnet. The equations of motion (24) in the nondissipative approximation are known to have two integrals of motion

$$\mathbf{m}^2 + \mathbf{l}^2 = 1, \quad \mathbf{m} \cdot \mathbf{l} = 0. \tag{25}$$

In our case, the energy density $\mathcal{E}$ is a complex quantity. In this regard, let us introduce two real energy characteristics W_m and W_l for CLL model:

$$W_m = \frac{1}{2} \int_{-\infty}^{+\infty} \mathcal{E}(x,t) + \mathcal{E}(-x,t)\, dx, \tag{26a}$$

$$W_l = \frac{1}{2i} \int_{-\infty}^{+\infty} \mathcal{E}(x,t) - \mathcal{E}(-x,t)\, dx. \tag{26b}$$

According to this definitions Eqs. (16) can be written as

$$\mathbf{m}_t = -\left\{\mathbf{m} \times \frac{\delta W_m}{\delta \mathbf{m}} + \mathbf{l} \times \frac{\delta W_m}{\delta \mathbf{l}}\right\}, \tag{27a}$$

$$\mathbf{l}_t = -\left\{\mathbf{m} \times \frac{\delta W_l}{\delta \mathbf{m}} + \mathbf{l} \times \frac{\delta W_l}{\delta \mathbf{l}}\right\}. \tag{27b}$$

Since in this case the invariants of motions $\mathbf{m}^2 - \mathbf{l}^2 = 1$ and $\mathbf{m} \cdot \mathbf{l} = 0$ have a pseudocharacter, we can speak here of an appropriate model of pseudo-antiferromagnetic. In the proposed interpretation, the vectors $\mathbf{m}_0$ and $\mathbf{l}_0$ in Eqs. (22) correspond to the ground state of pseudomagnetic. At the same time, the integrals (23) make sense that the total magnetization of excitation under the ground state $\mathbf{m}_0$ is preserved, and the total vector of antiferromagnetism is zero.

We should note that the model does not preserve lengths of sublattice magnetization vectors, $|\mathbf{M}_{1,2}| \neq \text{const}$. Such models were previously examined to describe the dynamics of relaxation processes in magnetic materials [42,43]. In our case, it is due solely to the pseudo-Euclidean nature of the model.

V. CONCLUSION

So, we showed that the $\mathcal{PT}$-symmetric nonlocal nonlinear Schrödinger equation is gauge equivalent to the unconventional model of coupled Landau-Lifshitz equations. This model, in particular cases, reduces to the classical LL equation. From the gauge connection with an initial NNLS model we derived the one-soliton solution and the first constants of motion of the model.

Physical aspects of the obtained model provide the extension of properties of traditional macroscopic magnetic systems. The geometrical aspect of the model, in spite of its pseudo- Euclidean nature, is not an artifact. The model discussed here as well as the initial NNLS model (1) in this regard can be useful in the physics of nanomagnetic artificial materials [44–48].

In the present paper, we have restricted ourselves only to the focusing NNLS model. The development of a gauge-equivalent magnetic model with the defocusing nonlinearity $\alpha = -1$ is also of profound interest. This and other issues will be considered elsewhere.

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