Tightening the entropic uncertainty bound in the presence of quantum memory

F. Adabi,[1] S. Salimi,[1,*] and S. Haseli[2]

[1]*Department of Physics, University of Kurdistan, P.O. Box 66177-15175, Sanandaj, Iran*

[2]*Department of Engineering Physics, Kermanshah University of Technology, Kermanshah, Iran*

(Received 23 February 2016; published 23 June 2016)

The uncertainty principle is a fundamental principle in quantum physics. It implies that the measurement outcomes of two incompatible observables cannot be predicted simultaneously. In quantum information theory, this principle can be expressed in terms of entropic measures. M. Berta *et al.* [Nat. Phys. **6**, 659 (2010)] have indicated that uncertainty bound can be altered by considering a particle as a quantum memory correlating with the primary particle. In this article, we obtain a lower bound for entropic uncertainty in the presence of a quantum memory by adding an additional term depending on the Holevo quantity and mutual information. We conclude that our lower bound will be tightened with respect to that of Berta *et al.* when the accessible information about measurements outcomes is less than the mutual information about the joint state. Some examples have been investigated for which our lower bound is tighter than Berta *et al.*'s lower bound. Using our lower bound, a lower bound for the entanglement of formation of bipartite quantum states has been obtained, as well as an upper bound for the regularized distillable common randomness.

DOI: 10.1103/PhysRevA.93.062123

I. INTRODUCTION

The uncertainty principle is the most basic feature of quantum mechanics and can be called the heart of quantum mechanics [1,2]. This principle bounds the uncertainties of measurement outcomes of two incompatible observables on a system in terms of the expectation value of their commutator. According to this principle, if measurement on a particle is selected from a set of two observables $\{X,Z\}$, then we have the following relation for quantum state $|\psi\rangle$ [3]:

$$\Delta X \Delta Z \geqslant \tfrac{1}{2}|\langle\psi|[X,Z]|\psi\rangle|, \tag{1}$$

where $\Delta X = \sqrt{\langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2}$, $\Delta Y = \sqrt{\langle\psi|Y^2|\psi\rangle - \langle\psi|Y|\psi\rangle^2}$ are the standard deviations and $[X,Z] = XZ - ZX$ is the commutator of observables X and Z. The uncertainty principle can be characterized in terms of Shannon entropies of the measurement outcome probability distributions of the two observables. The most famous version of the entropic uncertainty relation (EUR) was conjectured by Deutsch [4]. It was improved by Kraus [5] and then proved by Maassen and Uffink [6]. It states that, given two observables X and Z with eigenbases $\{|x_i\rangle\}$ and $\{|z_j\rangle\}$, for any state ρ_A,

$$H(X) + H(Z) \geqslant \log_2 \frac{1}{c} =: q_{MU}, \tag{2}$$

where q_{MU} is the incompatibility measure, $H(O) = -\Sigma_k p_k \log_2 p_k$ is the Shannon entropy of the measured observable $O \in \{X,Z\}$, p_k is the probability of the outcome k, $c = \max_{i,j} c_{ij}$, and $c_{ij} = |\langle x_i|z_j\rangle|^2$.

Various attempts have been made to improve and to generalize this relation [7–24]. In the following, we explain the generalization of EUR to the case in the presence of a memory particle [9]. One can describe the uncertainty principle by means of an interesting game between Alice and Bob. First, Bob prepares a particle in a quantum state and sends it to Alice. Then, Alice and Bob reach an agreement about measuring two observables X and Z on the particle by Alice. Alice does her measurement on the quantum state of the particle with one of the measurements and declares her choice of measurement to Bob. If Bob guesses the measurement outcome correctly, he will win the game. The minimum of Bob's uncertainty about Alice's measurement outcomes is bounded by Eq. (2). So far, it has been assumed that there is just one particle, but if Bob prepares a correlated bipartite state ρ_{AB} and sends just one of the particles to Alice and keeps the other particle as a quantum memory for himself, he can guess Alice's measurement outcomes with better accuracy. The uncertainty principle in the presence of quantum memory has been studied by Berta *et al.* [9], and they obtained the following relation:

$$S(X|B) + S(Z|B) \geqslant q_{MU} + S(A|B), \tag{3}$$

where $S(X|B) = S(\rho^{XB}) - S(\rho^B)$ and $S(Z|B) = S(\rho^{ZB}) - S(\rho^B)$ are the conditional von Neumann entropies of the postmeasurement states,

$$\rho^{XB} = \sum_i (|x_i\rangle\langle x_i| \otimes \mathbb{I})\rho^{AB}(|x_i\rangle\langle x_i| \otimes \mathbb{I}),$$

$$\rho^{ZB} = \sum_j (|z_j\rangle\langle z_j| \otimes \mathbb{I})\rho^{AB}(|z_j\rangle\langle z_j| \otimes \mathbb{I}),$$

and $S(A|B) = S(\rho^{AB}) - S(\rho^B)$ is the conditional von Neumann entropy. We discuss some special cases: first, if measured particle A and memory particle B are entangled, $S(A|B)$ is negative, and Bob's uncertainty about Alice's measurement outcomes can be reduced. Second, if A and B are maximally entangled, then $S(A|B) = -\log_2 d$ (d is the dimension of measured particle). As $\log_2 \frac{1}{c}$ cannot exceed $\log_2 d$, Bob can perfectly guess both X and Z. Third, if there is no quantum memory, Eq. (3) reduces to

$$H(X) + H(Z) \geqslant \log_2 \frac{1}{c} + S(A), \tag{4}$$

*shsalimi@uok.ac.ir

which is stronger than Maassen and Uffink's [6] uncertainty relation; since the measured particle is in the mixed state $S(A) \neq 0$, it tightens the lower bound of Eq. (2).

Pati *et al.* [10] proved that the uncertainties $S(X|B)$ and $S(Z|B)$ are lower bounded by an additional term compared to Eq. (3) as

$$S(X|B) + S(Z|B) \geqslant \log_2 \frac{1}{c} + S(A|B) + \max\{0, D_A(\rho^{AB}) - J_A(\rho^{AB})\}. \quad (5)$$

The classical correlation $J_A(\rho^{AB})$ is defined as

$$J_A(\rho^{AB}) = S(\rho^B) - \min_{\{\Pi_i^A\}} S\big(\rho^{B|\{\Pi_i^A\}}\big), \quad (6)$$

where the optimization is over all positive operator-valued measures $\{\Pi_i^A\}$ acting on measured particle A. Quantum discord is the difference between the total and the classical correlations,

$$D_A(\rho^{AB}) = I(A;B) - J_A(\rho^{AB}), \quad (7)$$

where total correlation is

$$I(A;B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}). \quad (8)$$

The lower bound in Eq. (5) tightens the bound in Eq. (3) if the discord $D_A(\rho^{AB})$ is larger than the classical correlation $J_A(\rho^{AB})$.

Coles and Piani [17] derived an improvement of the incompatibility measure q_{MU}, capturing the role of the second-largest entry of $[c_{ij}]$, denoted c_2, as

$$S(X|B) + S(Z|B) \geqslant q' + S(A|B), \quad (9)$$

where

$$q' = q_{MU} + \frac{1}{2}(1 - \sqrt{c}) \log_2 \frac{c}{c_2}; \quad (10)$$

when system A is a qubit, then $c = c_2$, and hence $q' = q_{MU}$.

In this paper, we introduce a lower bound for EUR by adding an additional term depending on the mutual information of the bipartite state and the Holevo quantities of the ensembles that Alice prepares for Bob by her measurements. We show that if Bob's accessible information about Alice's measurement outcomes is less than the mutual information, our lower bound is tighter than the lower bound proposed by both Berta *et al.*'s and Pati *et al.*'s lower bounds. We show that for complementary observables, there is a wide variety of quantum states for which our lower bound is stronger than the other lower bounds. We discuss four examples and show that our lower bound for pure states coincides with Berta *et al.*'s lower bound [9] and that for Werner states coincides with Pati *et al.*'s lower bound [10], but for Bell diagonal states and two-qubit X states our lower bounds are tighter than their lower bounds. It has been found that EUR has various applications, for example, in entanglement detection [25–28] and quantum cryptography [29,30]. As other applications, here we obtain a lower bound for the entanglement of formation of bipartite quantum states and an upper bound for the regularized distillable common randomness.

This paper is organized as follows. In Sec. II, we introduce the lower bound for EUR, and we show that for a wide variety of states, our EUR lower bound represents an improvement to Berta *et al.*'s uncertainty relation by raising the lower bound limits. In Sec. III, we examine our lower bound for four examples (pure, Werner, Bell diagonal, and two-qubit X states) and compare our lower bound with the other lower bounds. In Sec. IV, we discuss some of the applications of our lower bound. Section V includes the discussion and summary of our findings.

II. IMPROVED UNCERTAINTY RELATION WITH THE HOLEVO QUANTITY

In this section, we obtain a lower bound for EUR in the presence of a memory particle. Consider a bipartite state ρ^{AB} shared between Alice and Bob. Alice performs the X or Z measurement and announce her choice to Bob. Bob's uncertainty about both X and Z measurement outcomes is

$$\begin{aligned} S(X|B) + S(Z|B) &= H(X) - I(X;B) + H(Z) - I(Z;B) \\ &\geqslant q_{MU} + S(A) - [I(X;B) + I(Z;B)] \\ &= q_{MU} + S(A|B) \\ &\quad + \{I(A;B) - [I(X;B) + I(Z;B)]\}, \end{aligned}$$

where in the second line, we apply Eq. (4) and in the last line we use the identity $S(A) = S(A|B) + I(A;B)$. Therefore, the EUR is obtained as

$$S(X|B) + S(Z|B) \geqslant q_{MU} + S(A|B) + \max\{0,\delta\}, \quad (11)$$

where

$$\delta = I(A;B) - [I(X;B) + I(Z;B)]. \quad (12)$$

We note that when Alice measures observable P on her particle, she will obtain the ith outcome with probability $p_i = \mathrm{tr}_{AB}(\Pi_i^A \rho^{AB} \Pi_i^A)$ and Bob's particle will be left in the corresponding state $\rho_i^B = \frac{\mathrm{tr}_A(\Pi_i^A \rho^{AB} \Pi_i^A)}{p_i}$; then

$$I(P;B) = S(\rho^B) - \sum_i p_i S\big(\rho_i^B\big)$$

is the Holevo quantity, and it is equal to the upper bound of Bob's accessible information about Alice's measurement outcomes. Thus, one can see that if the sum of information that Alice sends to Bob by her measurements is less than the mutual information between A and B, the above EUR represents an improvement to Berta *et al.*'s uncertainty relation by raising the lower bound limit by the amount of δ. It is worth noting that the inequality (4) becomes an equality if observables X and Z are complementary and subsystem A is a maximally mixed state. Thus, our lower bound is perfectly tight for the class of states with maximally mixed subsystem A (including Werner states, Bell diagonal states, and isotropic states) and complementary observables. In other words, $S(X|B) + S(Z|B)$ coincides with our lower bound if X and Z are complementary and the subsystem A is maximally mixed.

It was conjectured [31] that the quantum mutual information is lower bounded by the sum of the classical mutual information in two mutually unbiased bases, namely,

$$I(A;B) \geqslant I(X;X') + I(Z;Z'), \quad (13)$$

where X' and Z' are two complementary observables measured on the memory particle. Although a stronger conjecture in which X' and Z' are replaced by the quantum memory B can be violated in general [32], we will show that when X and Z are complementary, there is a wide variety of states for which $\delta \geqslant 0$. We have

$$S(X|B) + S(Z|B) = H(X) + H(Z) - S(A) + S(A|B) + \delta \geqslant \log_2 d + S(A|B), \tag{14}$$

where in the last line we use Berta *et al.*'s inequality and d is the dimension of subsystem A. Here we see that

$$\delta \geqslant \log_2 d + S(A) - H(X) - H(Z); \tag{15}$$

hence, when the right-hand side (RHS) of the above inequality is zero, $\delta \geqslant 0$. When subsystem A is maximally mixed, $S(A), H(X)$, and $H(Z)$ are equal to $\log_2 d$, making the RHS of the above equation zero. Also, when X (Z) minimally disturbs subsystem A, $H(X)$ $[H(Z)]$ is equal to $S(A)$, and $H(Z)$ $[H(X)]$ is equal to $\log_2 d$, which, again, makes the RHS zero. So for all Bell diagonal states, Werner states, and maximally correlated mixed states we have $\delta \geqslant 0$, and for these states our inequality is tighter than Berta *et al.*'s uncertainty relation (3). Because Pati *et al.* in obtaining Eq. (5) used $J_A(\rho^{AB})$ instead of both $I(X;B)$ and $I(Z;B)$, we know that $J_A(\rho^{AB}) \geqslant I(X;B)$ and $I(Z;B)$. Thus, our lower bound is stronger than Eq. (5).

III. EXAMPLES

A. Pure bipartite state

First, we consider a pure bipartite state written in the Schmidt basis, $|\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|a_i\rangle|b_i\rangle$. For this state we have, $S(\rho^A) = S(\rho^B)$, $I(A;B) = 2S(\rho^B)$. Alice measures observable X or Z on her particle. Regardless of which observable Alice measures, whenever she obtains a particular outcome, the state of Bob's particle will be pure; then $S(\rho_i^B) = 0$ and $I(X;B) = I(Z;B) = S(\rho^B)$. Thus, $\delta = 0$, and our lower bound coincides with Berta *et al.*'s lower bound (3).

B. Werner state

As a second example, we consider a two-qubit Werner state,

$$\rho^{AB} = \frac{1-p}{4} I_A \otimes I_B + p|\Psi^-\rangle_{AB}\langle\Psi^-|, \tag{16}$$

where $0 \leqslant p \leqslant 1$ and $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the Bell state.

Because the Werner states are invariant under all unitary transformations of the form $U \otimes U$ so that $I(X;B) = I(Z;B) = J_A(\rho^{AB})$ and then $\delta = \{I(A;B) - [I(X;B) + I(Z;B)]\} = D_A(\rho^{AB}) - J_A(\rho^{AB})$, where we use Eq. (7), our lower bound equals the one which Pati *et al.* introduced.

C. Bell diagonal state

As the third example, we consider the set of two-qubit states with the maximally mixed marginal states. This state can be written as

$$\rho^{AB} = \frac{1}{4}\left(I \otimes I + \sum_{i,j=1}^{3} w_{ij}\sigma_i \otimes \sigma_j\right), \tag{17}$$

where $\sigma_i (i = 1,2,3)$ are the Pauli matrices. According to the singular-value-decomposition theorem, the matrix $W = \{w_{ij}\}$ can always be diagonalized by a local unitary transformation; then the above state transforms to the following form:

$$\rho^{AB} = \frac{1}{4}\left(I \otimes I + \sum_{i=1}^{3} r_i\sigma_i \otimes \sigma_i\right). \tag{18}$$

The above density matrix is positive if $\vec{r} = (r_1,r_2,r_3)$ belongs to a tetrahedron defined by the set of vertices $(-1,-1,-1),(-1,1,1),(1,-1,1)$, and $(1,1,-1)$. A projective measurement performed by Alice can be written by $P_\pm^A = \frac{1}{2}(I \pm \vec{n}\cdot\vec{\sigma})$, where $\vec{n}$ is a unit vector. If Alice measures observable P on her particle, Bob's qubit will be in the states $\rho_\pm^B = \frac{1}{2}(I \pm \sum_i n_i r_i \sigma_i)$ occurring with probability $\frac{1}{2}$. One can obtain the entropy as

$$S(\rho_\pm^B) = h\left(\frac{1 + \sqrt{(n_1r_1)^2 + (n_2r_2)^2 + (n_3r_3)^2}}{2}\right),$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy. From $\rho^B = p_+\rho_+^B + p_-\rho_-^B = \frac{1}{2}I$ and $S(\rho^B) = 1$, we conclude

$$I(P;B) = 1 - h\left(\frac{1 + \sqrt{(n_1r_1)^2 + (n_2r_2)^2 + (n_3r_3)^2}}{2}\right).$$

Now, we rearrange the three numbers $\{r_1,r_2,r_3\}$ according to their absolute values and denote the rearranged set as $\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}$ such that $|\bar{r}_1| \geqslant |\bar{r}_2| \geqslant |\bar{r}_3|$. When $\vec{\bar{n}} = (1,0,0)$ ($\vec{\bar{n}}$ is a rearranged unit vector corresponding to $\vec{\bar{r}}$), the Holevo quantity $I(P;B)$ reaches its maximum, $J_A(\rho^{AB}) = 1 - h(\frac{1+|\bar{r}_1|}{2})$. If Alice chooses X such that $I(X;B) = J_A(\rho^{AB})$ and Z is complementary to X, then $I(Z;B) = 1 - h(\frac{1+\sqrt{(\bar{n}_2\bar{r}_2)^2+(\bar{n}_3\bar{r}_3)^2}}{2})$, and one can see that $I(Z;B) \leqslant J_A(\rho^{AB})$; hence, $\delta \geqslant D_A(\rho^{AB}) - J_A(\rho^{AB})$, and our EUR is tighter than the EURs of Pati *et al.* [10].

Especially, when $r_1 = 1 - 2p$ and $r_2 = r_3 = -p$, with $0 \leqslant p \leqslant 1$, the state in Eq. (18) becomes

$$\rho = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{2}(|\Psi^+\rangle\langle\Psi^+| + |\Phi^+\rangle\langle\Phi^+|). \tag{19}$$

Now we consider three complementary observables, X, Y, and Z, corresponding to $\vec{\bar{n}} = (1,0,0)$, $\vec{\bar{n}} = (0,1,0)$, and $\vec{\bar{n}} = (0,0,1)$, respectively. One can see that

$$I(X;B) = J_A(\rho^{AB}) = \max\left\{1 - h(p), 1 - h\left(\frac{1+p}{2}\right)\right\},$$
$$I(Y;B) = 1 - h\left(\frac{1+p}{2}\right),$$
$$I(Z;B) = \min\left\{1 - h(p), 1 - h\left(\frac{1+p}{2}\right)\right\}. \tag{20}$$

Berta *et al.*'s lower bounds for two sets of complementary observables $\{X,Y\}$ and $\{X,Z\}$ are the same and are equal to

$$q_{MU} + S(A|B) = -p\log_2 p - (1-p)\log_2\left(\frac{1-p}{2}\right), \tag{21}$$

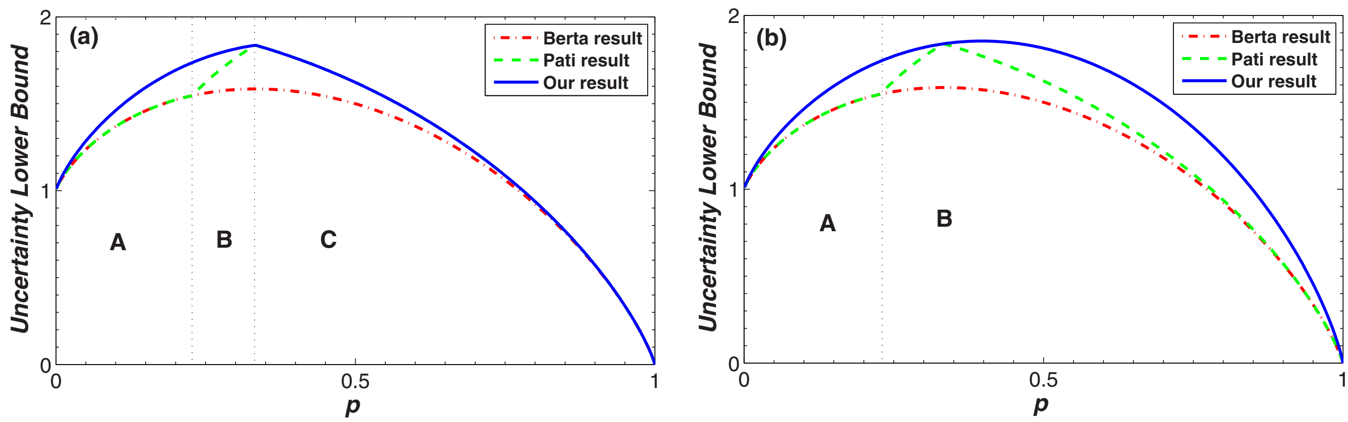

FIG. 1. Lower bounds of the entropic uncertainty relation of the two complementary observables in the presence of quantum memory when Bob prepares a correlated bipartite state in a special class of state: $\rho_{AB} = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{2}(|\Psi^+\rangle\langle\Psi^+| + |\Phi^+\rangle\langle\Phi^+|)$. The blue solid line shows our results, the green dashed line shows Pati *et al.*'s result, and the red dot-dashed line represents Berta *et al.*'s lower bound. (a) The uncertainty lower bound when one considers the two complementary observable X and Y, i.e., choosing $\vec{n} = (1,0,0)$, $\vec{n} = (0,1,0)$, respectively, and (b) the uncertainty lower bound when one considers the two complementary observable X and Z, i.e., choosing $\vec{n} = (1,0,0)$, $\vec{n} = (0,0,1)$, respectively.

and similarly, Pati *et al.*'s lower bound for two sets of complementary observables is the same:

$$\begin{aligned} &q_{MU} + S(A|B) + \max\{0, D_A(\rho^{AB}) - J_A(\rho^{AB})\} \\ &= -p\log_2 p - (1-p)\log_2\left(\frac{1-p}{2}\right) \\ &+ \max\left\{0, 2 + p\log_2 p + (1-p)\log_2\left(\frac{1-p}{2}\right)\right. \\ &\left. - 2\max\left[1 - h(p), 1 - h\left(\frac{1+p}{2}\right)\right]\right\}. \end{aligned} \quad (22)$$

The above discussion indicates that Berta *et al.*'s lower bound is not able to distinguish between any two observables in the set of the complementary observables. In other words the lower bound is observable independent for the complementary observables. Also, this argument is true for Pati *et al.*'s lower bound. But our lower bounds for two sets of complementary observables $\{X,Y\}$ and $\{X,Z\}$ are obtained as

$$\begin{aligned} &q_{MU} + S(A|B) + \delta \\ &= 2 - \max\left\{1 - h(p), 1 - h\left(\frac{1+p}{2}\right)\right\} - h\left(\frac{1+p}{2}\right) \end{aligned} \quad (23)$$

and

$$\begin{aligned} q_{MU} + S(A|B) + \delta = 2 &- \max\left\{1 - h(p), 1 - h\left(\frac{1+p}{2}\right)\right\} \\ &- \min\left\{1 - h(p), 1 - h\left(\frac{1+p}{2}\right)\right\}, \end{aligned} \quad (24)$$

respectively. As can be seen, our lower bounds of EUR for two sets of complementary of observables are different. In other words, the lower bound depends on the measured observables as well as correlations of quantum states. As can be seen from Figs. 1(a) and 1(b), in some intervals related to parameter p, the results obtained by Berta *et al.*, Pati *et al.*, and us overlap. In Fig. 1(a) we consider the two complementary observables X and Y, i.e., corresponding to $\vec{n} = (1,0,0)$, $\vec{n} = (0,1,0)$, respectively; in the A region Pati *et al.* and Berta *et al.* obtain the same results. However, if $p \in [1/3,1]$, then we face the situation that our result overlaps with Pati *et al.*'s result (we illustrate this by the C region). In Fig. 1(b) we consider the two complementary observables X and Z, i.e., corresponding to $\vec{n} = (1,0,0)$, $\vec{n} = (0,0,1)$, respectively. In this case, our result does not overlap with the results obtained by Berta *et al.* and Pati *et al.*; however, Berta *et al.*'s and Pati *et al.*'s results overlap in the A region.

D. Two-qubit X states

As the last example, we consider a special class of two-qubit X states,

$$\rho^{AB} = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|11\rangle\langle 11|,$$

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is a maximally entangled state and $0 \leqslant p \leqslant 1$. The density matrices of subsystems A and B are

$$\rho^A = \rho^B = \begin{pmatrix} \frac{p}{2} & 0 \\ 0 & 1 - \frac{p}{2} \end{pmatrix}.$$

One can obtain the conditional von Neumann entropy

$$\begin{aligned} S(A|B) = &-p\log_2 p - (1-p)\log_2(1-p) + \left(\frac{p}{2}\right) \\ &\times \log_2\left(\frac{p}{2}\right) + \left(1 - \frac{p}{2}\right)\log_2\left(1 - \frac{p}{2}\right), \end{aligned} \quad (25)$$

and the mutual information

$$\begin{aligned} I(A;B) = &p\log_2 p + (1-p)\log_2(1-p) - 2\left(\frac{p}{2}\right)\log_2\left(\frac{p}{2}\right) \\ &- 2\left(1 - \frac{p}{2}\right)\log_2\left(1 - \frac{p}{2}\right). \end{aligned} \quad (26)$$

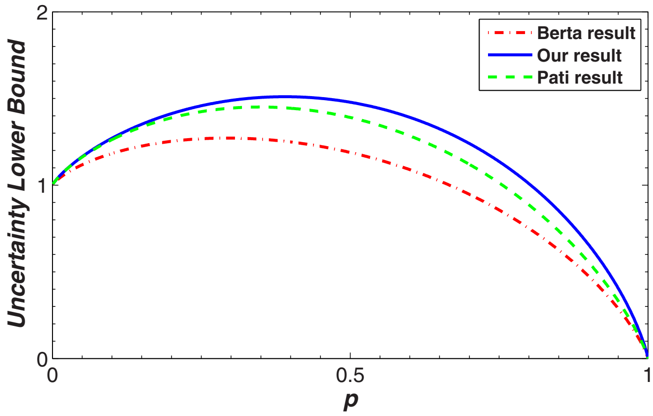

FIG. 2. Lower bounds of the entropic uncertainty relation of the two complementary observables σ_x and σ_z in the presence of quantum memory when Bob prepares a correlated bipartite state in a special class of state: $\rho^{AB} = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|11\rangle\langle 11|$. The blue solid line shows our results, the green dashed line shows Pati *et al.*'s result, and the red dot-dashed line represents Berta *et al.*'s lower bound.

If Alice measures observable $X = \sigma_x$ or $Z = \sigma_z$, where σ_x and σ_z are Pauli matrices, then one can see that

$$I(X;B) = -\frac{p}{2}\log_2\left(\frac{p}{2}\right) - \left(1-\frac{p}{2}\right)\log_2\left(1-\frac{p}{2}\right) + \frac{1}{2}(1-\sqrt{1-2p+2p^2}) \times \log_2\frac{1}{2}(1-\sqrt{1-2p+2p^2}) + \frac{1}{2}(1+\sqrt{1-2p+2p^2}) \times \log_2\frac{1}{2}(1+\sqrt{1-2p+2p^2}). \quad (27)$$

and

$$I(Z;B) = -\frac{p}{2}\log_2\left(\frac{p}{2}\right) - \left(1-\frac{p}{2}\right)\log_2\left(1-\frac{p}{2}\right) + \frac{p}{2}\log_2\left(\frac{p}{2-p}\right) + (1-p)\log_2\left(\frac{2(1-p)}{2-p}\right). \quad (28)$$

For this state the classical correlation equals $I(X;B)$; therefore, the quantum discord equals $D_A(\rho^{AB}) = I(A;B) - I(X;B)$ [33–35]. So we can obtain Berta *et al.*'s, Pati *et al.*'s, and our lower bounds. As can be seen from Fig. 2, our lower bound improves their results.

IV. APPLICATIONS

In addition to fundamental significance, the EUR has applications in various quantum information processing tasks [8,27]. In the following we mention some of the applications. According to Eq. (3), if $H(X|B) + H(Z|B) < \log_2\frac{1}{c}$ or if $I(X;B) + I(Z;B) > H(X) + H(Z) - log_2\frac{1}{c}$, then conditional entropy $S(A|B)$ is negative, and A and B must be entangled. According to our relation if $H(X|B) + H(Z|B) < \log_2\frac{1}{c} + \max\{0,\delta\}$, then the joint system is entangled. Furthermore, when $\delta \geqslant 0$, i.e., $I(A;B) \geqslant I(X;B) + I(Z;B)$, if $I(X;B) + I(Z;B) > S(A)$, then A and B are entangled [the conditional entropy $S(A|B)$ becomes negative], which is an improvement over using Berta *et al.*'s EUR.

Also, we can obtain a lower bound for the entanglement of formation $E_f(\rho_{AB})$ and its regularized form $E_f^\infty(\rho_{AB})$. Recall that

$$E_f(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i S(\mathrm{Tr}_B[|\psi_i\rangle\langle\psi_i|]),$$
$$E_f^\infty(\rho_{AB}) = \lim_{n\longrightarrow\infty}\frac{1}{n}E_f[(\rho_{AB})^{\otimes n}], \quad (29)$$

where the minimum is taken over all ensembles $\{p_i,|\psi_i\rangle\}$ satisfying $\sum_i p_i|\psi_i\rangle\langle\psi_i| = \rho_{AB}$. In Ref. [36] it was shown that $E_f(\rho_{AB}) \geqslant -S(A|B)$; by using the fact that entropies are additive for tensor-power states, we conclude that $E_f^\infty(\rho_{AB}) \geqslant -S(A|B)$. Suppose that Alice measures X or Z on her state, and corresponding to her measurement, Bob makes a measurement on his state to guess Alice's outcome. Let P_e^X and P_e^Z be the probabilities that Bob's guess about Alice's measurement outcomes is incorrect when she measures X and Z, respectively. According to the Fano inequality, $S(X|B) + S(Z|B) \leqslant b_F$, where $b_F \equiv h(P_e^X) + P_e^X\log_2(d-1) + h(P_e^Z) + P_e^Z\log_2(d-1)$. So we obtain a lower bound for the regularized entanglement of formation as follows:

$$E_f^\infty(\rho_{AB}) \geqslant \log_2\frac{1}{c} + \max\{0,\delta\} - b_F. \quad (30)$$

As another application, we obtain an upper bound for the regularized distillable common randomness [37]. Considering the n states $\rho_{CB}^{\otimes n}$ shared between Charlie and Bob, the optimum amount of classical correlation that they can share by means of classical communication from C to B is given by

$$C_D^{\rightarrow}(\rho_{CB}) = \lim_{n\longrightarrow\infty}\frac{1}{n}J(\rho_{CB})^{\otimes n}. \quad (31)$$

Koashi and Winter [38] show that

$$E_f^\infty(\rho_{AB}) + C_D^{\rightarrow}(\rho_{CB}) = S(\rho_B); \quad (32)$$

using this equality and Eq. (30), we obtain an upper bound for the distillable common randomness as follows:

$$C_D^{\rightarrow}(\rho_{CB}) \leqslant S(\rho_B) + b_F - \log_2\frac{1}{c} - \max\{0,\delta\}. \quad (33)$$

V. CONCLUSION

We have obtained a lower bound for the entropic uncertainty in the presence of quantum memory by adding an additional term depending on the Holevo quantity and mutual information. We have shown that our lower bound tightens that of Berta *et al.* whenever the mutual information between two particles is larger than the sum of two sources of classical information that Alice sends to Bob by her measurements. We have demonstrated that for the complementary observables, a wide variety of states, including Bell diagonal states and maximally correlated mixed states, fulfills this condition. We have compared our lower bound with the other lower bounds for some examples; especially, for a class of Bell diagonal states and two-qubit X states, the comparison of the lower bounds is depicted in Figs. 1 and 2, where it is clear that our lower bound (blue solid line) significantly improves the previously known results. We have discussed that our lower bound shows an improvement over the other lower bounds in entanglement detection. Using our lower bound, we have obtained a nontrivial lower bound for the entanglement of formation and an upper bound for the regularized common randomness.

ACKNOWLEDGMENTS

We thank the anonymous referee for helpful comments on our paper and M. Berta for useful discussions.

[1] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[2] *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983).
[3] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[4] D. Deutsch, Phys. Rev. Lett. **50**, 631 (1983).
[5] K. Kraus, Phys. Rev. D **35**, 3070 (1987).
[6] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988).
[7] I. Bialynicki-Birula, Phys. Rev. A **74**, 052101 (2006).
[8] S. Wehner and A. Winter, New J. Phys. **12**, 025009 (2010).
[9] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nat. Phys. **6**, 659 (2010).
[10] A. K. Pati, M. M. Wilde, A. R. U. Devi, A. K. Rajagopal, and Sudha, Phys. Rev. A **86**, 042105 (2012).
[11] M. A. Ballester and S. Wehner, Phys. Rev. A **75**, 022319 (2007).
[12] J. I de Vicente and J. Sánchez-Ruiz, Phys. Rev. A **77**, 042110 (2008).
[13] S. Wu, S. Yu, and K. Mølmer, Phys. Rev. A **79**, 022104 (2009).
[14] L. Rudnicki, S. P. Walborn, and F. Toscano, Phys. Rev. A **85**, 042115 (2012).
[15] T. Pramanik, P. Chowdhury, and A. S. Majumdar, Phys. Rev. Lett. **110**, 020402 (2013).
[16] L. Maccone and A. K. Pati, Phys. Rev. Lett. **113**, 260401 (2014).
[17] P. J. Coles and M. Piani, Phys. Rev. A **89**, 022112 (2014).
[18] S. Zozor, G. M. Bosyk, and M. Portesi, J. Phys. A **47**, 495302 (2014).
[19] L. Rudnicki, Z. Puchala, and K. Zyczkowski, Phys. Rev. A **89**, 052115 (2014).
[20] S. Liu, L.-Z. Mu, and H. Fan, Phys. Rev. A **91**, 042133 (2015).
[21] K. Korzekwa, M. Lostaglio, D. Jennings, and T. Rudolph, Phys. Rev. A **89**, 042122 (2014).
[22] L. Rudnicki, Phys. Rev. A **91**, 032123 (2015).
[23] J. Zhang, Y. Zhang, and C.-S. Yu, Sci. Rep. **5**, 11701 (2015).
[24] T. Pramanik, S. Mal, and A. S. Majumdar, Quantum Inf. Process. **15**, 981 (2016).
[25] M. H. Partovi, Phys. Rev. A **86**, 022309 (2012).
[26] Y. Huang, Phys. Rev. A **82**, 012335 (2010).
[27] R. Prevedel, D. R. Hamel, R. Colbeck, K. Fisher, and K. J. Resch, Nat. Phys. **7**, 757 (2011).
[28] L. Chuan-Feng, J.-S. Xu, X.-Y. Xu, K. Li, and G.-C. Guo, Nat. Phys. **7**, 752 (2011).
[29] M. Tomamichel, C. C. W. Lim, N. Gisin, and R. Renner, Nat. Commun. **3**, 634 (2012).
[30] N. H. Y. Ng, M. Berta, and S. Wehner, Phys. Rev. A **86**, 042315 (2012).
[31] J. Schneeloch, C. J. Broadbent, and J. C. Howell, Phys. Rev. A **90**, 062119 (2014).
[32] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, arXiv:1511.04857.
[33] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda, and A. O. Caldeira, Phys. Rev. A **81**, 052107 (2010).
[34] Q. Chen, C. Zhang, S. Yu, X. X. Yi, and C. H. Oh, Phys. Rev. A **84**, 042313 (2011).
[35] Y. Huang, Phys. Rev. A **88**, 014302 (2013).
[36] E. A. Carlen and E. H. Lieb, Lett. Math. Phys. **101**, 1 (2012).
[37] I. Devetak and A. Winter, Proc. R. Soc. A **461**, 207 (2005).
[38] M. Koashi and A. Winter, Phys. Rev. A **69**, 022309 (2004).