Quantum non-Gaussianity dynamics of two-mode single-photon squeezed Bell states based on cumulant theory

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(Received 31 December 2015; published 21 June 2016)

We characterize the non-Gaussianity of continuous-variable quantum states in terms of the cumulant theory and derive the exact formula of the cumulant of any order for such states. Exploiting the fourth-order cumulant method, we evaluate the quantum non-Gaussianity of two-mode single-photon squeezed Bell states and investigate their dynamics under the influence of two different types of decoherence models. It is shown that in a two-reservoir model, all the fourth-order cumulants of these states are very fragile, while in single-reservoir model, the fourth-order cumulants of one such state are insensitive to thermal noise, showing the time-invariant non-Gaussianity.

DOI: 10.1103/PhysRevA.93.062119

I. INTRODUCTION

It has been realized that non-Gaussian states play a more important role in the field of quantum information science than their Gaussian counterparts. With the help of non-Gaussian sources, the performance of quantum teleportation [1,2] and quantum error correction [3] can be significantly improved. Recent studies have highlighted the feasibility of using non-Gaussian sources to generate a larger entanglement [4,5], which is an essential requirement for universal quantum communication and computation. It was found that this outstanding role relies closely on the non-Gaussianity (nG) of a quantum state. One can then ask how to qualify and quantify the non-Gaussianity of any given state.

Much effort has been devoted to the quantification of the non-Gaussianity of a quantum state. Genoni *et al.* [6] first introduced the Hilbert-Schmidt distance to quantify the non-Gaussian character of a bosonic quantum state, evaluated the non-Gaussianity of some relevant states, and studied the evolution of non-Gaussianity of these states undergoing either Gaussification or de-Gaussification processes. Subsequently, they proposed the entropic measure of non-Gaussianity based on the quantum relative entropy [7], by which they evaluated the performance of conditional Gaussification toward twin-beam and de-Gaussification processes driven by Kerr interaction. In 2010, Genoni and Paris [8] investigated the relationships of these two non-Gaussianity measures. They pointed out that these measures have the same basic properties and share the same qualitative behavior for several families of non-Gaussian states. This entropic measure was also extended to the case of an arbitrary N-mode state [9]. The Bures metric was introduced as alternative measure of non-Gaussianity and the three non-Gaussianity measures were compared in Refs. [10] and [11]. It has been shown that there is a good consistency of these measures on the sets of damped states. Additionally, the non-Gaussianity was experimentally measured via relative entropy for single-photon-added coherent states [12] and phase-averaged coherent states [13], respectively. However, the intractable problems we face in their computations are choosing an appropriate reference Gaussian set of states and optimizing over such states. It is therefore worthwhile to study how to identify non-Gaussianity in a straightforward fashion.

In classical probability theory and statistics, the cumulants are widely believed to be a powerful tool in the description of the non-Gaussianity signatures of a probability distribution and have been extensively applied in many areas, including population biology [14], signal processing [15], finance [16], and classical physics [17,18]. More importantly, the cumulant method has been recently introduced into quantum optics and quantum information science. The quantum statistics of second-harmonic generation [19] and damped optical solitons [20] was investigated using cumulant-expansion techniques. The trace norm of the cumulant of the multiparty density matrix [21] and squared Frobenius norm of the cumulant parts of reduced density matrices [22] were utilized to measure genuine multiparty correlation. A scheme for a cumulant-based Bell test for entanglement was proposed in Ref. [23]. Teleportation of cumulants and principal momenta using squeezed-Bell-like states as resource was done in Ref. [24]. Additionally, recent works showed that the cumulants are particularly useful for testing the non-Gaussianity of a quantum state. Dubost *et al.* [25] used the third- and fourth-order cumulants to show the departure of a continuous-variable state from Gaussian behavior, in which the cumulant-based estimation was proven to be efficient, only requiring few preparations and measurements. Olsen and Corney [26] investigated the non-Gaussian statistic of the Kerr-squeezed state by calculating higher-order cumulants of quadrature variables. It was found that the nonlinear interaction can skew the distribution of the quadrature variables, giving rise to large third- and fourth-order cumulants for sufficiently long interaction times. Corney and Olsen [27] focused on non-Gaussian pure states generated by the anharmonic oscillator and used cumulants to characterize non-Gaussian behavior of the anharmonic oscillator. They showed that the truncated Wigner representation is a useful tool for calculating third- and fourth-order cumulants in nonclassical regimes, and such a system is predicted to be non-Gaussian over a wide range of particle numbers. However, to our best knowledge, relatively few studies have currently been made on the characterization of the non-Gaussianity of multimode non-Gaussian (entangled or separable) states in the language of cumulants. Hence, in this paper, we will deal with

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this question and derive a more general formula of cumulants of any order for any multimode bosonic quantum state. As an example, we compute the fourth-order cumulants, which are related to the negativity of the Wigner function [28], of a special class of two-mode non-Gaussian entangled states, namely, single-photon squeezed Bell states. Furthermore, we will study the dynamics of these fourth-order cumulants of such states in the two-reservoir (each mode coupled to its own thermal reservoir) and single-reservoir (two modes interacting with the same thermal reservoir) models.

Very recently, a new criterion has been introduced to detect the non-Gaussianity of a single-mode harmonic oscillator on the Wigner function and its effectiveness has been verified by considering the evolution of non-Gaussian pure states in a lossy Gaussian channel [29]. It has been shown that the criterion works well, detecting quantum non-Gaussianity in the nontrivial region of the noise parameters where no negativity of Wigner function can be observed. By following this line, this criteria can be extended to the case of s-parametrized quasiprobability distributions such as the Husimi Q function ($s=-1$). It has been proven that Husimi Q function–based quantum criterion is often more effective than a Wigner-function-based criteria in detecting quantum non-Gaussianity of various kinds of non-Gaussian states evolving in a lossy channel [30]. Now we briefly review the differences among these above-mentioned degrees of non-Gaussianity. An advantage of the Hilbert-Schmidt distance and the quantum relative entropy is that they are evaluated directly how different a non-Gaussian state is from its Gaussian counterpart with the same covariance matrix and the same vector. Hence they provide a useful avenue for distinction of Gaussian and non-Gaussian states and then are said to be measures of the state's non-Gaussianity, but unfortunately, these measures could not discriminate between quantum non-Gaussian states and mixtures of Gaussian states. For this purpose, another criterion of quantum non-Gaussianity is introduced, whose main idea is to construct a Gaussian convex hull: $\mathcal{G}=\{\rho\in\mathcal{B}(\mathcal{H})|\rho=\int d\boldsymbol{\lambda}\, p(\boldsymbol{\lambda})|\psi_G(\boldsymbol{\lambda})\rangle\langle\psi_G(\boldsymbol{\lambda})|\}$, where $\mathcal{B}(\mathcal{H})$ is the set of bounded operators, $p(\boldsymbol{\lambda})$ is a proper probability distribution, and $|\psi_G(\boldsymbol{\lambda})$ are pure Gaussian states. For a single-mode case, the most general pure Gaussian state can be written as $|\psi_G(\boldsymbol{\lambda})\rangle=D(\beta)S(\xi)|0\rangle$, where $D(\beta)$ and $S(\xi)$ are, respectively, the displacement and squeezing operators with the standard form presented in [31], $|0\rangle$ is the vacuum state, β and ξ are arbitrary complex numbers, and $\boldsymbol{\lambda}=\{\alpha,\xi\}$. According to Refs. [29] and [30], a quantum state ρ is defined quantum non-Gaussian if and only if $\rho\notin\mathcal{G}$. We shall stress that it essentially still belongs to a measure of the state's non-Gaussianity. However, in this paper our measure is a completely different one. We characterize the departure of the shape of a probability distribution of a quantum state from Gaussian from the viewpoint of the morphological statistics. It is thus a shape criterion.

The paper is organized as follows. In Sec. II we recall the definition of the cumulants for quantum states through the Wigner characteristic function and derive a formula of the cumulants for multimode non-Gaussian states. Section III gives the characteristic function of two-mode single-photon squeezed Bell states and the corresponding nonzero fourth-order cumulants. The dynamics of these cumulants in two different noise models is presented in Sec. IV, and a summary is given in Sec. V.

II. CUMULANTS OF QUANTUM STATES

Let us consider a system of N modes described by mode annihilation operators $\hat{a}_l$ and creation operators $\hat{a}_l^+$, $l=1,2,\ldots,N$, obeying the commutation relation $[\hat{a}_l,\hat{a}_m^+]=\delta_{lm}$. For a bosonic CV system, it is convenient to encode information in ρ by using the s-order characteristic function defined as [32]

$$\chi[\rho](\boldsymbol{\xi}):=\exp\left(\frac{s}{2}|\boldsymbol{\xi}|^2\right)\mathrm{Tr}[\rho\hat{D}(\boldsymbol{\xi})],\quad -1\leqslant s\leqslant 1,\tag{1}$$

where $\hat{D}(\boldsymbol{\xi})=\bigotimes_{l=1}^{N}\hat{D}(\xi_l)$, with $\hat{D}(\xi_l)=\exp(\xi_l\hat{a}_l^+-\xi_l^*\hat{a}_l)$ being the single-mode displacement operator and the complex vector $\boldsymbol{\xi}^T=(\xi_1,\xi_1^*,\ldots,\xi_N,\xi_N^*)\in\mathbb{C}^p$. The values $s=-1,0,1$ correspond, respectively, to the Q function, the Wigner function, and the P function. Here we make use of the Wigner characteristic function, i.e., $s=0$.

The characteristic function is often said to be the moment-generating function, by which we can evaluate the normally ordered moments of the field:

$$\begin{aligned}&\left\langle\hat{a}_1^{+m_1}\hat{a}_1^{m_2}\cdots\hat{a}_N^{+m_{2N-1}}\hat{a}_N^{m_{2N}}\right\rangle\\&\quad=\frac{\partial^{m_1}}{\partial(i\xi_1^*)^{m_1}}\frac{\partial^{m_2}}{\partial(i\xi_1)^{m_2}}\cdots\\&\qquad\times\frac{\partial^{m_{2N-1}}}{\partial(i\xi_N^*)^{m_{2N-1}}}\frac{\partial^{m_{2N}}}{\partial(i\xi_N)^{m_{2N}}}\chi[\rho](\boldsymbol{\xi})|_{\boldsymbol{\xi}=0}.\end{aligned}\tag{2}$$

Another approach to describe probability distribution is based on the cumulant-generating function, which is defined as the logarithm of the moment-generation function:

$$\Gamma(\boldsymbol{\xi}):=\ln\{\chi[\rho](\boldsymbol{\xi})\}.\tag{3}$$

Then the kth cumulant is defined by [33]

$$C^k=\frac{\partial^k}{\partial\boldsymbol{\xi}^k}\Gamma(\boldsymbol{\xi})|_{\boldsymbol{\xi}=0}.\tag{4}$$

Lemma 1. The cumulant of a CV quantum state is a sum of the cumulants of its Gaussian and non-Gaussian components. For the higher-order cumulant, it is uniquely determined by non-Gaussian components.

Proof. The characteristic function of any quantum state can be expressed as in the following form [8,34]:

$$\begin{aligned}\chi[\rho](\boldsymbol{\xi})&=f(\boldsymbol{\xi})\chi_G(\boldsymbol{\xi})\\&=f(\boldsymbol{\xi})\exp\left[-\frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\Omega}^T\boldsymbol{\sigma}\boldsymbol{\Omega}\boldsymbol{\xi}+i\mathbf{X}^T\boldsymbol{\Omega}\boldsymbol{\xi}\right],\end{aligned}\tag{5}$$

where the vector of mean values $\mathbf{X}\equiv\mathbf{X}[\rho]$ and the covariance matrix (CM) $\boldsymbol{\sigma}$ of a quantum state are, respectively, defined as $X_j=\mathrm{Tr}[\rho\hat{R}_j]=\langle\hat{R}_j\rangle$ and $\sigma_{kj}=\mathrm{Tr}[\rho(\hat{R}_k\hat{R}_j+\hat{R}_j\hat{R}_k)]/2-X_kX_j=\frac{1}{2}\langle\{\hat{R}_k,\hat{R}_j\}\rangle-\langle\hat{R}_k\rangle\langle\hat{R}_j\rangle$ with $\{A,B\}=AB+BA$, where the vector of field quadrature operators $\mathbf{R}$ is defined as $\mathbf{R}=(\hat{q}_1,\hat{p}_1,\ldots,\hat{q}_n,\hat{p}_n)^T=(R_1,R_2,\ldots,R_{2N})^T$ with $\hat{q}_k=\frac{1}{\sqrt{2}}(\hat{a}_k+\hat{a}_k^+)$, $\hat{p}_k=\frac{1}{i\sqrt{2}}(\hat{a}_k-\hat{a}_k^+)$, and the commutation relation given by $[\hat{R}_j,\hat{R}_k]=i\Omega_{jk}$ with Ω_{jk} being the elements of the symplectic matrix $\mathbf{\Omega} = i \bigoplus_{j=1}^{n} \boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_y$ being the y Pauli matrix. We see that Eq. (5) is a more general expression for continuous-variable states, especially for non-Gaussian states. For instance, the number state $|n\rangle$ has the characteristic function $\chi[|n\rangle\langle n|](\xi) = L_n(|\xi|^2)\exp(-\frac{1}{2}|\xi|^2)$, with L_n being the Laguerre polynomial of order n [31].

Then we can obtain the corresponding cumulant-generating function as

$$\Gamma(\boldsymbol{\xi}) = \ln f(\boldsymbol{\xi}) + \ln \chi_G(\boldsymbol{\xi}). \tag{6}$$

By the definition, the kth-order cumulant can be generated by evaluating the kth derivative of $\Gamma(\boldsymbol{\xi})$ at the origin,

$$\begin{aligned} C^k_{l_1 l_2 \ldots, l_{2N}} &= \left.\frac{\partial^k \Gamma(\mathbf{Z})}{\partial Z_1^{l_1} \partial Z_2^{l_2} \cdots \partial Z_{2N}^{l_{2N}}}\right|_{\mathbf{Z}=0} \\ &= \left.\frac{\partial^k \ln f(\mathbf{Z})}{\partial Z_1^{l_1} \partial Z_2^{l_2} \cdots \partial Z_{2N}^{l_{2N}}}\right|_{\mathbf{Z}=0} \\ &\quad + \left.\frac{\partial^k \ln \chi_G(\mathbf{Z})}{\partial Z_1^{l_1} \partial Z_2^{l_2} \cdots \partial Z_{2N}^{l_{2N}}}\right|_{\mathbf{Z}=0} \\ &= \left(C^k_{l_1 l_2 \ldots, l_{2N}}\right)_{NG} + \left(C^k_{l_1 l_2 \ldots, l_{2N}}\right)_G, \end{aligned} \tag{7}$$

where to keep the exposition simple, we have written the phase-space arguments in a compact vector form: $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_{2N-1}, Z_{2N}) = (\xi_1, \xi_1^*, \ldots, \xi_N, \xi_N^*)$, $k = l_1 + l_2 + \cdots + l_{2N}$, $l = (l_1, l_2, \ldots, l_{2N})$, and $m = (m_1, m_2, \ldots, m_{2N})$.

Since all cumulants of order greater than two are zero for the Gaussian function, the second term, $(C^k_{l_1 l_2 \ldots, l_{2N}})_G$, in Eq. (7) automatically vanishes. This ends the proof.

Lemma 2. To evaluate the cumulant of a CV quantum state, we can expand the function $f(\mathbf{Z})$ into a Taylor series about the origin:

$$\begin{aligned} f(\mathbf{Z}) = a_0 + \sum_l a_l Z_l + \sum_{l,m} a_{lm} Z_l Z_m + \sum_{l,m,n} a_{lmn} Z_l Z_m Z_n \\ + \sum_{l,m,n,h} a_{lmnh} Z_l Z_m Z_n Z_h + \cdots, \end{aligned} \tag{8}$$

where $a_{lmnh\ldots}$ are the coefficients in the Taylor series.

We perform the kth-order derivative of $f(\mathbf{Z})$ with respect to $\mathbf{Z}$ and obtain the cumulant of k order as

$$C^k = \sum_p (-1)^{|p|-1}(|p|-1)! \prod_{B\in p} \mathbb{G}\left(\prod_{j\in B} Z_j\right), \tag{9}$$

where the coefficients appearing in the function $f(\mathbf{Z})$ are denoted by $\mathbb{G}(\prod_{j\in B} Z_j)$, e.g., $a_l = \mathbb{G}(Z_l)$, $a_{lm} = \mathbb{G}(Z_l Z_m)$, $a_{lmn} = \mathbb{G}(Z_l Z_m Z_n)$, etc., and where p runs through the list of all partitions of $\{1,2,\ldots,2N-1,2N\}$, B runs through the list of all blocks of the partition p, and $|p|$ is the number of parts in the partition.

More precisely, we can write the kth-order cumulant in an explicit form,

$$C^k = (-1)^{k-1} \begin{vmatrix} a_l & 1 & 0 & 0 & \cdots & 0 \\ a_{lm} & \binom{1}{0} a_l & 1 & 0 & \cdots & 0 \\ a_{lmn} & \binom{2}{0} a_{lm} & \binom{2}{1} a_l & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ a_{lmn\ldots k} & \binom{k-1}{0} a_{lmn\ldots k-1} & \binom{k-1}{1} a_{lmn\ldots k-2} & \cdots & \cdots & \binom{k-1}{k-2} a_l \end{vmatrix}, \tag{10}$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ is Newton's binomial. ■

Lemma 2 provides a good way to calculate the cumulant of a CV quantum state, especially for the cumulant of higher order. Once the function $f(\mathbf{Z})$ is given, the high-order cumulants are easily derived according to formulas (9) and (10). For example, the third- and fourth-order cumulants of a CV quantum state can be readily obtained as

$$C^3_{lmn} = a_{lmn} - a_l a_{mn}[3] + 2a_l a_m a_n, \tag{11a}$$

$$\begin{aligned} C^4_{lmnh} &= a_{lmnh} - a_l a_{mnh}[4] - a_{lm} a_{nh}[3] + 2a_l a_m a_{nh}[6] \\ &\quad - x6 a_l a_m a_n a_h, \end{aligned} \tag{11b}$$

where the number in the square bracket represents the number of terms in the combination as the indices rotate, e.g., $l \to m$, $m \to n$, etc. In a single-mode non-Gaussian state, we obtain the third- and fourth-order cumulants by letting $(l,m,n,h) \in (1,2)$ and the fourth-order cumulants by $(l,m,n,h) \in (1,2,3,4)$, respectively.

Lemma 3. The cumulant is additive for a multipartite factorized product state.

Proof. We consider a multipartite factorized state in the product form of $\rho = \bigotimes_{j=1}^{M} \rho_j$, whose characteristic function is thus given by

$$\chi[\rho](\mathbf{Z}) = \mathrm{Tr}\left[\bigotimes_{j=1}^{M} \rho_j \hat{D}(Z_j)\right] = \prod_{j=1}^{M} \chi_j[\rho_j](Z_j). \tag{12}$$

From Eq. (3), we can obtain the generating function of the cumulant as

$$\Gamma(\mathbf{Z}) = \ln\left\{\prod_{j=1}^{M} \chi_j[\rho_j](Z_j)\right\} = \sum_{j=1}^{M} \ln\{\chi_j[\rho_j](Z_j)\}. \tag{13}$$

According to Lemma 1, we thus have

$$C^k_{l_1 l_2 \ldots, l_{2N}}[\rho] = \sum_{j=1}^{M} C^k_{l_1 l_2 \ldots, l_{2N}}[\rho_j], \tag{14}$$

which completed our proof. ■

It is well known that the cumulant satisfies homogeneity; this scaling property is also shared by the cumulant we define.

Lemma 4 (Homogeneity). Let $\mathbf{Z}$ be a $2l$-dimensional complex vector with the kth-order cumulant. Then for any constant λ, $\lambda\mathbf{Z}$ has an the kth-order cumulant, which is given by

$$C^k(\lambda\mathbf{Z}) = \lambda^k C^k(\mathbf{Z}). \tag{15}$$

Proof. To prove Lemma 4, we only need to modify the l-mode displacement operator as

$$\hat{D}(\lambda\mathbf{Z}) = \exp\left[\lambda\sum_l (Z_l\hat{a}_l^+ - Z_l^*\hat{a}_l)\right]. \tag{16}$$

Its action on the annihilation and creation operators produces

$$\hat{D}(\lambda\mathbf{Z})\hat{a}_l\hat{D}^+(\lambda\mathbf{Z}) = \hat{a}_l - \lambda Z_l, \tag{17a}$$

$$\hat{D}(\lambda\mathbf{Z})\hat{a}_l^+\hat{D}^+(\lambda\mathbf{Z}) = \hat{a}_l^+ - \lambda Z_l^*. \tag{17b}$$

Thus, we have

$$\chi[\rho](\lambda\mathbf{Z}) = f(\lambda\mathbf{Z})\exp\left[-\frac{1}{2}\lambda^2\mathbf{Z}^T\mathbf{\Omega}^T\boldsymbol{\sigma}\mathbf{\Omega}\mathbf{Z} + i\lambda\mathbf{X}^T\mathbf{\Omega}\mathbf{Z}\right], \tag{18}$$

with

$$\begin{aligned} f(\lambda\mathbf{Z}) = a_0 &+ \sum_l \lambda a_l Z_l + \sum_{l,m}\lambda^2 a_{lm}Z_lZ_m \\ &+ \sum_{l,m,n}\lambda^3 a_{lmn}Z_lZ_mZ_n \\ &+ \sum_{l,m,n,h}\lambda^4 a_{lmnh}Z_lR_mZ_nZ_h + \cdots. \end{aligned} \tag{19}$$

Repeating the same procedures as in Lemmas 1 and 2, we arrive at the kth-order cumulant as

$$C^k_{l_1l_2\ldots,l_{2N}}(\lambda\mathbf{Z}) = \lambda^k C^k_{l_1l_2\ldots,l_{2N}}(\mathbf{Z}). \tag{20}$$

One can see that it recovers Lemma 1 for the case of $\lambda = 1$. Thus, the lemma is proven. ■

It has been shown that the few lowest-order cumulants have certain geometric meanings and characterize the peak position, the width, the asymmetry, and the sharpness of a probability distribution, respectively. On the other hand, they can be used to judge whether a distribution is Gaussian or not. As stated above, a Gaussian distribution exists for only the first- and second-order cumulants, while the third-order cumulant vanishes for any symmetric distribution. Therefore, the fourth-order cumulant is a lowest-order indicator of characterizing non-Gaussianity. By means of the cumulant language, we may safely say that if a quantum state has a nonzero fourth-order cumulant, then it is non-Gaussian. More specifically, with the positive kurtosis it is called a super-Gaussian or platykurtotic, and that with negative kurtosis it is called sub-Gaussian or leptokurtotic.

III. CUMULANTS OF TWO-MODE SINGLE-PHOTON SQUEEZED BELL STATES

In this section, we will use the fourth-order cumulant to characterize the non-Gaussianity of a class of two-mode non-Gaussian entangled states whose density matrix can be written as

$$\rho_{\mathrm{SBS}} = \hat{S}_{12}(r)|\psi\rangle_{BSBS}\langle\psi|\hat{S}_{12}^+(r), \tag{21}$$

where $\hat{S}_{12}(r) = \exp\{-r\hat{a}_1^+\hat{a}_2^+ + r\hat{a}_1\hat{a}_2\}$, $r \in \mathbb{C}$ is the two-mode squeezed operator, and $|\psi\rangle_{BS} = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\varphi}|10\rangle)$ are the known Bell states. Such states are called two-mode single-photon squeezed Bell states, which have been used to implement continuous-variable quantum teleportation as a non-Gaussian resource [1].

Using some simple algebra, we can obtain the resulting characteristic function as

$$\chi_{\mathrm{SBS}}(\xi_1,\xi_2) = f(\xi_1,\xi_2)\exp\left[-\frac{1}{2}\boldsymbol{\xi}^T\mathbf{\Omega}^T\boldsymbol{\sigma}_{\mathrm{SBS}}\mathbf{\Omega}\boldsymbol{\xi}\right], \tag{22}$$

where $\boldsymbol{\xi}^T = (\xi_1,\xi_1^*,\xi_2,\xi_2^*)$ and the covariance matrix $\boldsymbol{\sigma}_{\mathrm{SBS}}$ is given by

$$\boldsymbol{\sigma}_{\mathrm{SBS}} = \frac{1}{2}\begin{pmatrix} \cosh(2r) & 0 & 0 & \sinh(2r) \\ 0 & \cosh(2r) & \sinh(2r) & 0 \\ 0 & \sinh(2r) & \cosh(2r) & 0 \\ \sinh(2r) & 0 & 0 & \cosh(2r) \end{pmatrix}, \tag{23}$$

and the function $f(\xi_1,\xi_2)$ reads

$$\begin{aligned} f(\xi_1,\xi_2) = 1 &- \tfrac{1}{2}\cosh(2r)|\xi_1|^2 - \tfrac{1}{2}\cosh(2r)|\xi_2|^2 \\ &- \tfrac{1}{4}\sinh(2r)e^{-i\varphi}\xi_1^2 - \tfrac{1}{4}\sinh(2r)e^{i\varphi}\xi_1^{*2} \\ &- \tfrac{1}{4}\sinh(2r)e^{i\varphi}\xi_2^2 - \tfrac{1}{4}\sinh(2r)e^{-i\varphi}\xi_2^{*2} \\ &- \tfrac{1}{2}\sinh(2r)\xi_1\xi_2 - \tfrac{1}{2}\sinh(2r)\xi_1^*\xi_2^* \\ &- \tfrac{1}{2}\cosh(2r)e^{-i\varphi}\xi_1\xi_2^* - \tfrac{1}{2}\cosh(2r)e^{i\varphi}\xi_1^*\xi_2. \end{aligned} \tag{24}$$

It is obvious to see that the function $f(\xi_1,\xi_2)$ is quadratic, and as a result, its moments of order greater than 2 vanish and all odd cumulants are nullified. But it has nonvanishing fourth-order cumulants, which are listed in Table I.

Let us now carefully analyze the non-Gaussianity of two-mode single-photon squeezed Bell states from Table I. In the limit of vanishing squeezing, $r = 0$, the two-mode single-photon squeezed Bell state reduces to the known Bell state. Thus the non-Gaussianity of the state is exactly equal to that of the Bell state, whose nonzero fourth-order cumulants are given by $C^4_{111} = -\frac{3}{4}$, $C^4_{2200} = C^4_{0022} = C^4_{2002} = C^4_{0220} = -\frac{1}{2}$, and $C^4_{2101} = C^4_{1012} = C^4_{1210} = C^4_{0121} = -\frac{1}{2}e^{-i\varphi}$. For the case of $\varphi = 0$, all the fourth-order cumulants of two-mode single-photon squeezed Bell states are negative, showing that the statistical distribution of measuring any physical observable is sub-Gaussian, while for a case of $\varphi = \pi$, the fourth-order cumulants may be positive or negative, depending on the linear combination of $\hat{a}_j^+$ and $\hat{a}_j$ and thus, $\hat{x}_j^+$ and $\hat{p}_j$. In the limit of very large squeezing, the fourth-order cumulants are proportional to e^{4r}, implying that squeezing the Bell state can result in a rapid increase of the non-Gaussianity, or the cumulant-based non-Gaussianity of the state can be enhanced by a squeezing operation. It should be stressed that this result is completely inconsistent with previous investigations [6–8,29] where the non-Gaussianity has been shown to be invariant

TABLE I. Fourth-order cumulants of two-mode one-photon squeezed Bell states. $A = \cosh^2(2r)$, $B = \sinh^2(2r)$, $D = \sinh(4r)$. The number m is the order of the derivative and the $\{4\}$ denotes the fourth order of the derivative with respective to one of four variables.

ξ_1	ξ_1^*	ξ_2	ξ_2^*	C^4_{lmnh}	ξ_1	ξ_1^*	ξ_2	ξ_2^*	C^4_{lmnh}
1	1	1	1	$-\frac{3}{4}A$		$\{4\}$			$-\frac{3}{16}B$
3	1	0	0	$-\frac{3}{16}De^{-i\varphi}$	2	1	1	0	$-\frac{5}{16}D$
3	0	1	0	$-\frac{3}{8}Be^{-i\varphi}$	2	1	0	1	$-\frac{4A+B}{8}e^{-i\varphi}$
3	0	0	1	$-\frac{3}{16}De^{-2i\varphi}$	2	0	1	1	$-\frac{5}{16}De^{-i\varphi}$
1	3	0	0	$-\frac{3}{16}De^{i\varphi}$	1	2	1	0	$-\frac{4A+B}{8}e^{i\varphi}$
0	3	1	0	$-\frac{3}{16}De^{2i\varphi}$	1	2	0	1	$-\frac{5}{16}D$
0	3	0	1	$-\frac{3}{8}Be^{i\varphi}$	1	1	2	0	$-\frac{5}{16}De^{i\varphi}$
1	0	3	0	$-\frac{3}{8}Be^{i\varphi}$	1	0	2	1	$-\frac{5}{16}D$
0	1	3	0	$-\frac{3}{16}De^{2i\varphi}$	1	1	0	2	$-\frac{5}{16}De^{-i\varphi}$
0	0	3	1	$-\frac{3}{16}De^{i\varphi}$	1	0	1	2	$-\frac{4A+B}{8}e^{-i\varphi}$
1	0	0	3	$-\frac{3}{16}De^{-2i\varphi}$	0	2	1	1	$-\frac{5}{16}De^{i\varphi}$
0	1	0	3	$-\frac{3}{8}Be^{-i\varphi}$	0	1	1	2	$-\frac{5}{16}D$
0	0	1	3	$-\frac{3}{16}De^{-i\varphi}$	0	1	2	1	$-\frac{4A+B}{8}e^{i\varphi}$
2	2	0	0	$-\frac{8A+B}{16}$	2	0	2	0	$-\frac{9}{16}B$
2	0	0	2	$-\frac{8A+B}{16}e^{-2i\varphi}$	0	2	2	0	$-\frac{8A+B}{16}e^{2i\varphi}$
0	2	0	2	$-\frac{9}{16}B$	0	0	2	2	$-\frac{8A+B}{16}$

under unitary Gaussian operations. It can be explained by the fact that their measures are closely related with the symplectic eigenvalues, which are invariant under the symplectic transformation associated with the Gaussian operations on the covariance matrix. So the squeezing operation acts as a Gaussian unitary operation that does not change the Gaussian character of a quantum state. However, in our case, we dealt with the non-Gaussianity of a quantum state in light of the statistical nature of its probability density function. In particular, we apply the fourth-order cumulant to characterize the departure of the shape of the probability distribution from Gaussian and then the squeezing operation modifies the shape of the probability distribution of the quantum states. Now we give discussions qualitatively on this topic. Let $\boldsymbol{U}_G$ be a Gaussian map. Then the density operation is transformed as $\rho \mapsto \widetilde{\rho} = \boldsymbol{U}_G \rho \boldsymbol{U}_G^+$. The characteristic function of the new state $\widetilde{\rho}$ is given by $\chi[\widetilde{\rho}](\boldsymbol{\xi}) = \mathrm{Tr}[\boldsymbol{U}_G \rho \boldsymbol{U}_G^+ \hat{D}(\boldsymbol{\xi})] = \mathrm{Tr}[\rho \hat{D}(S\boldsymbol{\xi})] = \chi(S\boldsymbol{\xi})$, where the symplectic transformation $S \in \mathrm{Sp}(2n,\mathbb{R}) = \{S | S\boldsymbol{\Omega}S^T = \boldsymbol{\Omega}\}$. It is obvious from Lemma 4 that the squeezing operations have an effect on the cumulants and then can alter the shape of the distribution. In addition, some schemes for detecting such types of moments have been proposed by balanced homodyne [35] or homodyne correlation techniques [36–38]. Since the cumulants are related closely to the moments, this renders it possible to experimentally check the cumulant-based non-Gaussianity of two-mode single-photon squeezed Bell states.

IV. DYNAMICS OF FOURTH-ORDER CUMULANTS UNDER NOISY ENVIRONMENTS

It is clear that the environment-induced decoherence is one of the main obstacles to any practical implementation of quantum information processing since it can destroy the quantum nature of a physical system and leads to an irreversible loss of information. Therefore, we will analyze the dynamics of fourth-order cumulants of two-mode single-photon squeezed Bell states under the influence of environment. Without loss of generality, we consider the two types of decoherence models. One is that each of the two modes is coupled to its own thermal environment, i.e., a two-reservoir model, and another is composed of the two modes interacting with a common thermal reservoir (model B), i.e., a common-reservoir model.

A. Two-reservoir model

In the first model, we consider a system of two noninteracting quantum bosonic fields (e.g., two cavities or two harmonic oscillators), each of them coupled to its own thermal environment. For simplicity, we assume that the energy decay rates γ of the two modes into their own environments are the same and that each environment is taken to be initially in a thermal state with the thermal mean-occupation number $\bar{n} = [\exp(\beta\omega_0) - 1]^{-1}$. Under the Born-Markov approximation and tracing over the bath variables, we obtain the master equation describing the evolution of the reduced density matrix operator of the system, in the interaction picture, as [39,40]

$$\frac{d}{dt}\rho = \sum_{j=1,2} \frac{\gamma}{2} \{\bar{n} L[\hat{a}_j^+]\rho + (\bar{n}+1) L[\hat{a}_j]\rho\}, \tag{25}$$

where the Lindblad superoperator is defined as $L[\hat{O}]\rho = 2\hat{O}\rho\hat{O}^+ - \hat{O}^+\hat{O}\rho - \rho\hat{O}^+\hat{O}$.

It is much more convenient to solve the master equation (25) by using the characteristic function method. With the help of the standard operator correspondence, we can transform the master equation (25) into a Fokker-Planck equation for a characteristic function, which is given by

$$\frac{\partial}{\partial t}\chi(\xi_1,\xi_2;t) = -\frac{\gamma}{2} \sum_{j=1,2} \left[(1+2\bar{n})|\xi_j|^2 + \xi_j \frac{\partial}{\partial \xi_j} + \xi_j^* \frac{\partial}{\partial \xi_j^*} \right] \chi(\xi_1,\xi_2;t), \tag{26}$$

whose solution is given by [41]

$$\chi(\xi_1,\xi_2;t) = \chi(\xi_1 e^{-\frac{\gamma}{2}t}, \xi_2 e^{-\frac{\gamma}{2}t};0) \times \exp\left[-\sum_{j=1,2} \left(\bar{n} + \frac{1}{2}\right)(1 - e^{-\gamma t})|\xi_j|^2 \right], \tag{27}$$

where $\chi(\xi_1,\xi_2;0)$ is the characteristic function at time $t = 0$.

Assuming the initial state of the system to be Eq. (21) and using Eqs. (6), (10), and (27), we can derive the input-output relationship of the fourth-order cumulants as

$$C^4_{lmnk}(t) = e^{-2\gamma t} C^4_{lmnk}(0), \tag{28}$$

where $C^4_{lmnk}(0)$ denotes the fourth-order cumulant at time $t = 0$. We see clearly from Eq. (28) that in this two-reservoir model, the fourth-order cumulant decreases exponentially and completely disappears in the long-time limit, implying that the corresponding non-Gaussianity is very fragile and difficult to maintain. The evolution dynamics of non-Gaussian cumulants under consideration is consistent with the Gaussian nature of such a noisy model, which leads in general to losses and decoherence of a quantum state. This is because in the independent noisy model, the local environments are uncorrelated and then do not induce the mutual interaction among the subsystems, thereby making a physical systems transition from quantum to classical behavior. For example, when transmitting through the Gaussian channel, the negativity of a Wigner function of the non-Gaussian states gradually reduces and even completely vanishes [42], while the entanglement [43] and the Gaussian quantum discord [44] of bipartite Gaussian states monotonically decrease. On the other hand, such a noisy Gaussian channel could be regarded as a bosonic Gaussian map that transforms Gaussian input states into Gaussian output ones. Namely, the Gaussian character of the initial states remains unchanged under the Gaussian channels. Nevertheless, our result shows that the local Gaussian map can transform the initial non-Gaussian states into Gaussian ones in the long-time limit. It is not difficult to understand from Eq. (27) together with Eq. (24) that in the long-time limit, the two-mode single-photon squeezed Bell state becomes a product of two single-mode thermal states [45], i.e., $\chi(\xi_1,\xi_2;\infty) = \exp[-\sum_{j=1,2}(\bar{n}+\frac{1}{2})|\xi_j|^2]$, which belongs to a typical kind of Gaussian state, and thus its non-Gaussianity completely vanishes.

Our approach can be easily generalized to the case of an n-mode system coupled to n identical local thermal environments. According to Lemma 4 and repeating the same procedure as earlier, we can obtain the input-output relationship of the kth-order cumulants for any quantum state as

$$C^k_{l_1l_2\ldots,l_{2N}}(t) = e^{-\frac{k\gamma t}{2}} C^k_{l_1l_2\ldots,l_{2N}}(0). \tag{29}$$

Obviously, in the long-time limit, we have $C^k_{l_1l_2\ldots,l_{2N}}(t) \to 0$, namely, driving any continuous-variable quantum state into a Gaussian one with the help of a local thermal environment. Therefore, the decoherence induced by the local thermal environment plays the role of a destructive mechanism that washes out quantum non-Gaussianity of a quantum state.

B. Common-reservoir model

We consider the dissipative evolution of two noninteracting single-mode bosonic fields coupled to a common finite-temperature reservoir modeled by a continuum of oscillators. Under the Born-Markov approximation and in the interaction picture, the master equation governing the dynamical process is given by

$$\frac{d}{dt}\rho = \sum_{l,m=1,2}\bigg[\frac{\gamma}{2}(1+\bar{n})(2\hat{a}_l\rho\hat{a}_m^+ - \hat{a}_l^+\hat{a}_m\rho - \rho\hat{a}_l^+\hat{a}_m) + \frac{\gamma}{2}\bar{n}(2\hat{a}_l^+\rho\hat{a}_m - \hat{a}_l\hat{a}_m^+\rho - \rho\hat{a}_l\hat{a}_m^+)\bigg], \tag{30}$$

where γ is the energy decay of the boson system and $\bar{n} = [\exp(\hbar\omega/k_BT)]$ is the mean thermal photon numbers of the thermal environment at temperature T. k_B is the Boltzmann's constant. We should note that the terms $l=m$ in Eq. (30) describe the individual dissipations of each mode due to the environment, while the other terms denote the couplings between the modes induced by the common bath.

By using the same strategy as above, we can obtain the corresponding Fokker-Planck equation for characteristic function as

$$\frac{\partial}{\partial t}\chi(\xi_1,\xi_2;t) = -\frac{\gamma}{2}\sum_{l,m=1,2}\bigg[(1+2\bar{n})\xi_l\xi_m^* + \xi_l\frac{\partial}{\partial\xi_m} + \xi_l^*\frac{\partial}{\partial\xi_m^*}\bigg]\chi(\xi_1,\xi_2;t). \tag{31}$$

To solve Eq. (30), we write the initial state of the physical system in the form of the following:

$$\chi(\xi_1,\xi_2;0) = Z\big[F_0 + F_1|\xi_1|^2 + F_2|\xi_2|^2 + F_3\xi_1^2 + F_3^*\xi_1^{*2} + F_4\xi_2^2 + F_4^*\xi_2^{*2} + F_5\xi_1\xi_2 + F_5^*\xi_1^*\xi_2^* + F_6\xi_1\xi_2^* + F_6^*\xi_1^*\xi_2\big] \times \exp\big[-\big(A_1|\xi_1|^2 + A_2|\xi_2|^2 + A_3\xi_1^2 + A_3^*\xi_1^{*2} + A_4\xi_2^2 + A_4^*\xi_2^{*2} + A_5\xi_1\xi_2 + A_5^*\xi_1^*\xi_2^* + A_6\xi_1\xi_2^* + A_6^*\xi_1^*\xi_2\big)\big], \tag{32}$$

where Z is a normalization constant and the coefficients F_j (or A_j) ($j = 0,1,2,\cdots,6$) are determined by the initial state.

Following the same line in Ref. [46] and using the identity

$$\exp\left(\gamma t x\frac{\partial}{\partial x}\right)(x^n e^{\alpha x}) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma t)^l\alpha^m}{l!m!}\left(x\frac{\partial}{\partial x}\right)^l x^{m+n} = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma t)^l\alpha^m}{l!m!}(m+n)^l x^{m+n}$$
$$= \sum_{m=0}^{\infty}\frac{\alpha^m}{m!}x^{m+n}\exp[(m+n)\gamma t] = (xe^{\gamma t})^n\exp(\alpha x e^{\gamma t}), \tag{33}$$

we can calculate the characteristic function for the evolved state to be

$$\chi(\xi_1,\xi_2;t) = Zf(\xi_1,\xi_2;t)\exp\big[-\tfrac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\Omega}^T\boldsymbol{\sigma}_{\rm SBS}(t)\boldsymbol{\Omega}\boldsymbol{\xi}\big], \tag{34}$$

where the time-dependent covariance matrix is given by

$$\boldsymbol{\sigma}_{\rm SBS}(t) = \frac{1}{2}\begin{pmatrix} 4A_1(t)+\bar{n}(t) & 8A_3(t) & 4A_6(t)+\bar{n}(t) & 4A_5(t) \\ 8A_3^*(t) & 4A_1(t)+\bar{n}(t) & 4A_5^*(t) & 4A_6^*(t)+\bar{n}(t) \\ 4A_6^*(t)+\bar{n}(t) & 4A_5(t) & 4A_2(t)+\bar{n}(t) & 8A_4(t) \\ 4A_5^*(t) & 4A_6(t)+\bar{n}(t) & 8A_4^*(t) & 4A_2(t)+\bar{n}(t) \end{pmatrix}, \tag{35}$$

with $\bar{n}(t) = (1+2\bar{n})(1-e^{-2\gamma t})$ and the function $f(\xi_1,\xi_2;t)$ is given by

$$f(\xi_1,\xi_2;t) = F_0 + F_1(t)|\xi_1|^2 + F_2(t)|\xi_2|^2 + F_3(t)\xi_1^2 + F_3^*(t)\xi_1^{*2} + F_4(t)\xi_2^2 + F_4^*(t)\xi_2^{*2} + F_5(t)\xi_1\xi_2 + F_5^*(t)\xi_1^*\xi_2^* + F_6(t)\xi_1\xi_2^* + F_6^*(t)\xi_1^*\xi_2, \tag{36}$$

with the time-dependent coefficients $F_j(t)$ given by

$$F_1(t) = \tfrac{1}{4}[F_1(1+e^{-\gamma t/2})^2 + F_2(1-e^{-\gamma t/2})^2 + 2\mathrm{Re}(F_6)(e^{-\gamma t}-1)], \tag{37a}$$

$$F_2(t) = \tfrac{1}{4}[F_1(1-e^{-\gamma t/2})^2 + F_2(1+e^{-\gamma t/2})^2 + 2\mathrm{Re}(F_6)(e^{-\gamma t}-1)], \tag{37b}$$

$$F_3(t) = \tfrac{1}{4}[F_3(1+e^{-\gamma t/2})^2 + F_4(1-e^{-\gamma t/2})^2 + F_5(e^{-\gamma t}-1)], \tag{37c}$$

$$F_4(t) = \tfrac{1}{4}[F_3(1-e^{-\gamma t/2})^2 + F_4(1+e^{-\gamma t/2})^2 + F_5(e^{-\gamma t}-1)], \tag{37d}$$

$$F_5(t) = \tfrac{1}{2}[(F_3+F_4)(e^{-\gamma t}-1) + F_5(e^{-\gamma t}+1)], \tag{37e}$$

$$F_6(t) = \tfrac{1}{4}[(F_1+F_2)(e^{-\gamma t}-1) + F_6(1+e^{-\gamma t/2})^2 + F_6^*(e^{-\gamma t/2}-1)^2], \tag{37f}$$

where the coefficients $F_j^*(t)$ can be obtained by conjugation of $F_j(t)$ and the other types of time-dependent coefficients $A_j(t)$ can be obtained from the coefficients $F_j(t)$ by substituting F_n with $A_n,(n=1,2,3,\ldots,6)$.

Now we analyze the time evolution the fourth-order cumulants of two-mode single-photon squeezed Bell states in the common thermal environment. First, considering the case of $\varphi = 0$, i.e., $|\psi\rangle_{\mathrm{SBS}} = \hat{S}_{12}(r)\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, we have

$$F_1(t) = F_2(t) = -\tfrac{1}{2}\cosh(2r)e^{-\gamma t}, \tag{38a}$$

$$F_3(t) = F_4(t) = -\tfrac{1}{4}\sinh(2r)e^{-\gamma t}, \tag{38b}$$

$$F_5(t) = -\tfrac{1}{2}\sinh(2r)e^{-\gamma t}, \tag{38c}$$

$$F_6(t) = \tfrac{1}{2}\cosh(2r)e^{-\gamma t}. \tag{38d}$$

After some calculation, we can obtain the same dynamic evolution of the fourth-order cumulants as in the previous case:

$$C^4_{lmnk}(t) = e^{-2\gamma t}C^4_{lmnk}(0), \tag{39}$$

which shows that interacting with the common thermal environment, the fourth-order cumulants of such states decrease exponentially and are close to zero in the long-time limit; moreover, the cumulant-based non-Gaussianity vanishes.

Next, we study the case of $\varphi = \pi$, i.e., $|\psi\rangle_{\mathrm{SBS}} = \hat{S}_{12}(r)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and call it a two-mode squeezed singlet Bell state. In this case, the coefficients in Eq. (37) become

$$F_1(t) = F_2(t) = -F_6(t) = -\tfrac{1}{2}\cosh(2r), \tag{40a}$$

$$F_3(t) = F_4(t) = -\tfrac{1}{2}F_5(t) = \tfrac{1}{4}\sinh(2r). \tag{40b}$$

From Eqs. (10) and (11), we can obtain a simple and important input-output relationship of the cumulants for such states as

$$C^4_{lmnk}(t) = C^4_{lmnk}(0). \tag{41}$$

It implies that the non-Gaussianity of such states, characterized by the fourth-order cumulant, is insensitive to the common thermal noise and is forever frozen at its initial value, showing the robustness of quantum non-Gaussianity. This fact can be understood from two aspects. One is that the interaction with the common environment will cause the cooperative decoherence for the noninteracting subsystems [47], thus allowing the existence of decoherence-free states [48] in which no decoherence occurs at all, even when these subsystems interact with the environment, while the independent reservoir is not, just as studied in Ref. [49]. Another is that as can be seen from Eq. (40) that the invariance of a two-mode squeezed singlet Bell state under a thermal channel takes root in its singlet Bell state, regardless of squeezing. It has been shown that such a singlet Bell state is invariant to any type of N-lateral unitary operation [50] and a common Markovian environment [51]. Therefore, in our case the non-Gaussianity of a two-mode squeezed singlet Bell state remains intact, in spite of the interaction with the environment.

As is well known, it is impossible to completely isolate a quantum signal from its surrounding environment during transmission, processing, and storage processes. So a lot of effort has been devoted to studying the input-output problems of an open quantum system, especially for the explicit input-output formalisms for some interesting physical quantities during the past decade years. For example, the input-output theories for lossy multilayer dielectric plates [52], optical cavity [53], and optomechanical systems [54] have been developed. The input-output formula of the covariance matrix of single-mode Gaussian state was proposed in the squeezed thermal environment with time-independent damping rate [45,55] and time-dependent damping rate [56]. The invariant quantum input-output transformations in phase space and in Hilbert space were introduced by Luis [57]. The approach can be applicable in some quantum processes such as amplification and losses. The dynamical equation of entanglement, quantified by the concurrence, was developed in a system of two two-level atoms located inside two spatially separated cavities caused by vacuum fluctuations [58] and of arbitrary bipartite systems with one part exposed to the generalized amplitude damping channel [59]. Therefore, the theory here developed should contribute to the future development of quantum computation and quantum communication using non-Gaussian continuous-variable entanglement.

V. CONCLUSIONS

In summary, we have exploited the cumulant theory to characterize the quantum non-Gaussian character of multipartite continuous-variable bosonic states and have presented analytical formula for the cumulant-based non-Gaussianity. We have investigated the non-Gaussianity of a special family of non-Gaussian two-mode entangled states: two-mode single-photon squeezed Bell states, based on the fourth-order cumulant, the lowest-order indicator of non-Gaussianity. It has been shown that the fourth-order cumulant, and thus the non-Gaussianity, is solely dominated by the squeezing parameter. Furthermore, we have examined the dynamics of fourth-order cumulants of such states in two different classes of decoherence models. Our analysis shows that the non-Gaussianity rapidly vanishes for two-mode single-photon squeezed Bell states considered here evolving in a local thermal bath, whereas in a common thermal bath, non-Gaussianity of some states remains invariant. We emphasize that unlike other measures of non-Gaussianity, the cumulant-based non-Gaussianity can be directly calculated from the non-Gaussian parts of Wigner characteristic function of a quantum state, thus greatly reducing the computation. We therefore anticipate that it provides an exciting possibility for investigating the dynamics of the non-Gaussianity of other continuous-variable n-mode nonclassical states in general Markov and non-Markov noisy channels using our approach.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grants No. 11174100, No. 11474120, and No. 11374264, the Key Project Foundation of Hunan Provincial Education Department, China, under Grant No. 14A114, the Science Research Fund of Education Department of Hunan Province under Grant No. 14C0896, and the Innovation Plan for Graduate Students at the University of Jiangsu Province under Grant No. CXLX13_80(133).

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