Noncommutative q-photon-added coherent states

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We construct the photon-added coherent states of a noncommutative harmonic oscillator associated to a q-deformed oscillator algebra. Various nonclassical properties of the corresponding system are explored, first, by studying two different types of higher-order quadrature squeezing, namely, the Hillery type and the Hong-Mandel type, and second, by testing the sub-Poissonian nature of photon statistics in higher order with the help of the correlation function and the Mandel parameter. Also, we compare the behavior of different types of quadrature and photon number squeezing of our system with those of the ordinary harmonic oscillator by considering the same set of parameters.

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I. INTRODUCTION

It is familiar that a coherent state $|\alpha\rangle$ is an exceptional type of quantum state that resembles a classical light field very closely and exhibits Poisson photon number distribution with an average photon number $|\alpha|^2$. Coherent states have well-defined amplitude and phase, with minimal fluctuations permitted by the Heisenberg uncertainty principle. On the contrary, a Fock state $|n\rangle$ is entirely quantum mechanical, and in the language of Glauber and Sudarshan [1,2], it possesses purely *nonclassical* features.

The *photon-added coherent states* (PACS) are the intermediate states between the coherent state and the Fock state [3], which are obtained by the successive action (m times) of the canonical creation operator $a^\dagger$ on the standard coherent state $|\alpha\rangle$, as given by

$$|\alpha,m\rangle = \frac{1}{\mathcal{N}(\alpha,m)} a^{\dagger m}|\alpha\rangle, \qquad (1.1)$$

with the normalization constant being $\mathcal{N}^2(\alpha,m) = \langle\alpha|a^m a^{\dagger m}|\alpha\rangle$. The PACS (1.1) have been studied by many authors in different contexts [4–16]. In [5], it was shown that the states (1.1) can be realized as eigenstates of the annihilation operator, $f(\hat{n},m)a|\alpha,m\rangle = \alpha|\alpha,m\rangle$, with $f(\hat{n},m) = 1 - m/(1+\hat{n})$, such that $|\alpha,m\rangle$ can be considered as nonlinear coherent states. The overcompleteness of PACS was explicitly shown in [6] by using the Stieltjes power moment problem. The dynamical squeezing was investigated in [4]. Properties of nonlinear PACS have been explored in [14,16] in different contexts. PACS have various applications in quantum optics, quantum information, and computation, not only because they are nonclassical, but also because they can generate the entangled states [17,18]. Like many other nonclassical states, such as squeezed states [19–21], Schrödinger cat states [22,23], pair coherent states [24], and binomial states [25,26], the special features of PACS consist of quadrature squeezing, sub-Poissonian photon statistics, negativity in Wigner function, etc. PACS are produced in the interaction of a two-level atom, with a cavity field initially prepared in the coherent state [3]. Proper experiments behind the existence of a single PACS [27,28] and two PACS [29] have been performed successfully.

The goal of the present manuscript is to introduce the formalism for the construction of PACS for a noncommutative harmonic oscillator (NCHO) originating from a q-deformed algebra and to explore various nonclassical properties of the corresponding system. A detailed analysis of different types of squeezing properties in higher orders shows an improved degree of nonclassicality of the given system than that of an ordinary harmonic oscillator. This motivates us to explore the possibility that the noncommutative systems might be implemented for the purpose of quantum information processing with additional degrees of freedom, especially when we notice that the given NCHO might be realized physically.

Our manuscript is organized as follows: In Sec. II, we show how to build a NCHO out of a q-deformed oscillator algebra and, then, we construct the PACS of the corresponding system. In Sec. III, we analyze the properties of two types of higher-order quadrature squeezing, namely, the Hillery type and the Hong-Mandel type, and provide a comparative analysis of our results between the NCHO and the ordinary harmonic oscillator. In Sec. IV, we study the sub-Poissonian nature of photon statistics of our system by analyzing the behavior of the correlation function and the Mandel parameter in higher order. Our conclusions are stated in Sec. V.

II. q-DEFORMED PACS FOR NCHO

Following Refs. [30–34], we start by considering a one-dimensional q-deformed oscillator algebra for the deformed annihilation and creation operators A_q and $A_q^\dagger$ in the form

$$A_q A_q^\dagger - q^2 A_q^\dagger A_q = 1, \qquad \text{for} \quad |q| < 1. \qquad (2.1)$$

The Fock space of the corresponding algebra (2.1) can be defined by choosing q-deformed integers $[n]_q$ in such a way

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that the following relations hold:

$$|n\rangle_q := \frac{A_q^{\dagger n}}{\sqrt{[n]_q!}}|0\rangle_q, \qquad [n]_q! := \prod_{k=1}^{n}[k]_q, \qquad [0]_q! := 1,$$
$$[n]_q := \frac{1-q^{2n}}{1-q^2}, \qquad A_q|0\rangle_q = 0, \qquad {}_q\langle 0|0\rangle_q = 1. \tag{2.2}$$

It immediately follows that the operators A_q and $A_q^\dagger$ act as lowering and raising operators, respectively, in the deformed space

$$A_q|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q, \quad A_q^\dagger|n\rangle_q = \sqrt{[n+1]_q}|n+1\rangle_q. \tag{2.3}$$

This means that the states $|n\rangle_q$ form an orthonormal basis in the q-deformed Hilbert space $\mathcal{H}_q$ spanned by the vectors $|\psi\rangle := \sum_{n=0}^{\infty} c_n|n\rangle_q$ with $c_n \in \mathbb{C}$, such that $\langle\psi|\psi\rangle = \sum_{n=0}^{\infty}|c_n|^2 < \infty$. Therefore, the commutation relation between A_q and $A_q^\dagger$ is realized as follows:

$$[A_q, A_q^\dagger] = 1 + (q^2-1)A_q^\dagger A_q = 1 + (q^2-1)[n]_q. \tag{2.4}$$

The concept that the q-deformed algebras of type (2.1) can be implemented for the construction of q-deformed harmonic oscillators was given by many authors [30–34]. Here we recall Refs. [35–38] to construct a NCHO from the given algebra (2.1) instead. For this, we first express the deformed observables X and P in terms of the ladder operators $A_q, A_q^\dagger$ in the following form:

$$X = \gamma\left(A_q^\dagger + A_q\right) \qquad \text{and} \qquad P = i\delta(A_q^\dagger - A_q). \tag{2.5}$$

Thereafter, by choosing the appropriate constraints on the parameters, $\gamma = \sqrt{\hbar/(4m\omega)}, \delta = \sqrt{m\omega\hbar}$, and by using (2.1) we obtain the following commutation relation between the position and momentum variables:

$$[X,P] = \frac{2i\hbar}{1+q^2} + i\frac{q^2-1}{q^2+1}\left(2m\omega X^2 + \frac{P^2}{2m\omega}\right). \tag{2.6}$$

Notice that in the limit $q \to 1$, the commutator (2.6) reduces to the standard canonical commutation relation. The interesting feature of such a type of noncommutative space-time (2.6) is that it leads to the existence of a minimal length as well as a minimal momentum [35,36], which are the direct consequences of string theory. Furthermore, there exists a concrete self-adjoint representation of the ladder operators [38]

$$A_q = \frac{i}{\sqrt{1-q^2}}(e^{-i\hat{x}} - e^{-i\hat{x}/2}e^{2\tau\hat{p}}),$$
$$A_q^\dagger = \frac{-i}{\sqrt{1-q^2}}(e^{i\hat{x}} - e^{2\tau\hat{p}}e^{i\hat{x}/2}), \tag{2.7}$$

in terms of the canonical coordinates x,p satisfying $[x,p] = i\hbar$, with $\hat{x} = x\sqrt{m\omega/\hbar}$ and $\hat{p} = p/\sqrt{\hbar m\omega}$ being dimensionless observables, and the deformation parameter q being parametrized to $q = e^\tau$. It follows that the observables (2.5), which satisfy (2.6), are Hermitian with respect to the representation (2.7), i.e., $X^\dagger = X, P^\dagger = P$. Obviously, our representation (2.7) is not unique and with further investigations it may be possible to find other Hermitian representations. However, for our purpose it is important that there exists at least one such representation providing a self-consistent description of a physical system. In addition, we notice that our representation (2.7) is invariant under the simultaneous operation of parity $\mathcal{P}$ and time-reversal $\mathcal{T}$ operator, $\mathcal{PT}: x \to -x, p \to p, i \to -i$ [39,40]. It means that the $\mathcal{PT}$ symmetry is inherited in the canonical variables on the noncommutative space (2.5), $\mathcal{PT}: X \to -X, P \to P, i \to -i$, which indicates that any quantum model consisting of the observables X,P may possess entirely real eigenvalues. Indeed, in some of the recent articles by one of the authors [36,41], many similar types of noncommutative systems have been solved with a real energy spectrum. It should be mentioned that in recent days various non-Hermitian but $\mathcal{PT}$-symmetric systems have been utilized for experiments in optical laboratories in many different directions one might follow; for instance, see Refs. [42–45].

Similar types of systems that we are discussing here have been used for the purpose of construction of different types of quantum optical models, for instance, coherent states [38,46–48], cat states [49–53], and squeezed states [54]. Here we construct the PACS associated to the algebra (2.1) by following the original definition (1.1) as given by

$$|\alpha,m\rangle_q = \frac{1}{\mathcal{N}(\alpha,m,q)}A_q^{\dagger m}|\alpha\rangle_q$$
$$= \frac{1}{\mathcal{N}(\alpha,m,q)\mathcal{N}(\alpha,q)}\sum_{n=0}^{\infty}\frac{\alpha^n}{[n]_q!}\sqrt{[n+m]_q!}|n+m\rangle_q,$$
$$\alpha \in \mathbb{C}, \tag{2.8}$$

with the normalization constant

$$\mathcal{N}^2(\alpha,m,q) = {}_q\langle\alpha,m|A_q^m A_q^{\dagger m}|\alpha,m\rangle_q$$
$$= \frac{1}{\mathcal{N}^2(\alpha,q)}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{[n]_q!^2}[n+m]_q!, \tag{2.9}$$

where $|\alpha\rangle_q = [1/\mathcal{N}(\alpha,q)]\sum_{n=0}^{\infty}(\alpha^n/\sqrt{[n]_q!})|n\rangle_q$ is a standard q-deformed nonlinear coherent state [30,38,51], with $\mathcal{N}^2(\alpha,q) = \sum_{n=0}^{\infty}|\alpha|^{2n}/[n]_q!$. Note that the deformed PACS represented in (2.8) are absolutely general, which can be associated to any kind of q-deformed systems corresponding to the q-deformed integers $[n]_q$. In our case, we choose a particular form of $[n]_q$ as given in (2.2) and utilize it in the rest of our analysis. However, in the limit $q \to 1$, the entire structure reduces to the original PACS as proposed in [3]. q-deformed photon-added and photon-depleted coherent states have been studied before in [9], however, in a completely different context. They utilized a different type of q boson, called the inverse q boson, which originates from the q-deformed algebra introduced in [47].

III. HIGHER-ORDER SQUEEZING IN QUADRATURE COMPONENTS

We now analyze various nonclassical properties of the deformed PACS (2.8) in the following two sections. In this section, we check whether the quadrature components for deformed PACS are squeezed or not. There exist two different types of higher-order quadrature squeezing in the literature, Hillery type [55] and Hong-Mandel type [56]. These two types of higher-order squeezing have been studied for different quantum states in different contexts [11,12,57–64]. Let us discuss both of them in our case.

A. Hillery-type higher-order squeezing

According to Hillery [55], a quadrature $Y_N(\phi)$ of the form

$$Y_N(\phi) = \frac{1}{2}\left(A_q^N e^{-iN\phi} + A_q^{\dagger N} e^{iN\phi}\right) \tag{3.1}$$

is said to be squeezed for an arbitrary order N for the state $|\alpha,m\rangle_q$ for some values of ϕ if

$$_q\langle\alpha,m|[\Delta Y_N(\phi)]^2|\alpha,m\rangle_q < \frac{1}{4}{}_q\langle\alpha,m|\left[A_q^N, A_q^{\dagger N}\right]|\alpha,m\rangle_q, \tag{3.2}$$

where ϕ is the angle in the complex plane and ${}_q\langle[\Delta Y_N(\phi)]^2\rangle_q = {}_q\langle[Y_N(\phi)]^2\rangle_q - {}_q\langle[Y_N(\phi)]\rangle_q^2$ being the variance of the quadrature $Y_N(\phi)$. The left-hand side of (3.2) can be computed by utilizing the following equations:

$$_q\langle[Y_N(\phi)]^2\rangle_q = \frac{1}{4}\left[{}_q\left\langle A_q^{2N}\right\rangle_q e^{-2iN\phi} + {}_q\left\langle A_q^{\dagger 2N}\right\rangle_q e^{2iN\phi} + {}_q\left\langle A_q^N A_q^{\dagger N}\right\rangle_q + {}_q\left\langle A_q^{\dagger N} A_q^N\right\rangle_q\right], \tag{3.3}$$

$$_q\langle[Y_N(\phi)]\rangle_q^2 = \frac{1}{4}\left[{}_q\left\langle A_q^N\right\rangle_q^2 e^{-2iN\phi} + {}_q\langle A_q^{\dagger N}\rangle_q^2 e^{2iN\phi} + 2\left|{}_q\left\langle A_q^N\right\rangle_q\right|^2\right]. \tag{3.4}$$

A degree of Hillery-type squeezing can be obtained by computing the squeezing coefficient S_H as defined by

$$S_H = \frac{4{}_q\langle[\Delta Y_N(\phi)]^2\rangle_q - {}_q\left\langle\left[A_q^N, A_q^{\dagger N}\right]\right\rangle_q}{{}_q\left\langle\left[A_q^N, A_q^{\dagger N}\right]\right\rangle_q}, \tag{3.5}$$

where $S_H < 0$ corresponds to the existence of squeezing. The squeezing is guaranteed if the relation (3.2) holds (and $S_H < 0$) at least for the order $N = 1$; however, we are interested to present a fairly general expression of the squeezing coefficient for an arbitrary order $N \geqslant 1$. For this, we compute

$$_q\left\langle A_q^{\dagger N} A_q^L\right\rangle_q = \begin{cases} \frac{\alpha^{*(N-L)}}{\hat{\mathcal{N}}^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}[n+m]_q![n+m+N-L]_q!}{[n]_q![n+N-L]_q![n+m-L]_q!} & \text{if} \quad N > L \\ \frac{\alpha^{L-N}}{\hat{\mathcal{N}}^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}[n+m]_q![n+m+L-N]_q!}{[n]_q![n+L-N]_q![n+m-N]_q!} & \text{if} \quad L > N, \end{cases} \tag{3.6}$$

and

$$_q\left\langle A_q^N A_q^{\dagger L}\right\rangle_q = \begin{cases} \frac{\alpha^{N-L}}{\hat{\mathcal{N}}^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}[n+m+N]_q!}{[n]_q![n+N-L]_q!} & \text{if} \quad N > L \\ \frac{\alpha^{*(L-N)}}{\hat{\mathcal{N}}^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}[n+m+L]_q!}{[n]_q![n+L-N]_q!} & \text{if} \quad L > N, \end{cases} \tag{3.7}$$

with $\hat{\mathcal{N}} = \mathcal{N}(\alpha,m,q)\mathcal{N}(\alpha,q)$ and $N,L \in \mathbb{Z}$. Since we will always use the normal ordered form of the operators, we do not require (3.7) in our analysis; however, for the sake of completeness of our calculations we present it explicitly. Note that Eqs. (3.6) and (3.7) are valid for $|N-L| \geqslant 1$. For the case of $N = L$, we obtain

$$_q\left\langle A_q^{\dagger N} A_q^N\right\rangle_q = \frac{1}{\hat{\mathcal{N}}^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}([n+m]_q!)^2}{([n]_q!)^2[n+m-N]_q!}, \tag{3.8}$$

$$_q\left\langle A_q^N A_q^{\dagger N}\right\rangle_q = \frac{1}{\hat{\mathcal{N}}^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}[n+m+N]_q!}{([n]_q!)^2}. \tag{3.9}$$

In addition, we notice from (3.6) and (3.7) that ${}_q\langle A_q^{\dagger N} A_q^L\rangle_q = {}_q\langle A_q^{\dagger L} A_q^N\rangle_q^*$ and ${}_q\langle A_q^N A_q^{\dagger L}\rangle_q = {}_q\langle A_q^L A_q^{\dagger N}\rangle_q^*$ and, therefore, the expectation values of A_q^N and A_q^{2N} are obtained by taking the complex conjugates of the mean values of $A_q^{\dagger N}$ and $A_q^{\dagger 2N}$, respectively. Thus, the Nth-order squeezing coefficient acquires the form

$$S_H = 2\frac{\mathrm{Re}\left[\left({}_q\left\langle A_q^{2N}\right\rangle_q - {}_q\left\langle A_q^N\right\rangle_q{}^2\right)e^{-2iN\phi}\right] - \left|{}_q\left\langle A_q^N\right\rangle_q\right|^2 + {}_q\left\langle A_q^{\dagger N} A_q^N\right\rangle_q}{{}_q\left\langle A_q^N A_q^{\dagger N}\right\rangle_q - {}_q\left\langle A_q^{\dagger N} A_q^N\right\rangle_q}, \tag{3.10}$$

which can be evaluated easily with the help of (3.6), (3.8), and (3.9). In Fig. 1(a), we plot the first-order ($N = 1$) squeezing coefficient (3.10) as a function of angle ϕ for different values of the added photon numbers m and for a particular value of α. We observe that the squeezing ($S_H < 0$) depends on the angle ϕ and appears periodically. The degree of squeezing for a fixed angle ϕ (where the squeezing exists) becomes higher when m is increased, i.e., when more number of photons are added to the system. In Fig. 1(b), we show the nature of S_H in different orders ($N = 2,3,4$) for a particular value of ϕ and for a single PACS, where we notice that the squeezing occurs roughly for $|\alpha| > 1$. However, in this case the degree of squeezing becomes less when we increase the order N. Nevertheless, the most interesting observation follows from the fact that the squeezing coefficients of the NCHO (solid and dotted lines) are slightly more negative in comparison to those of the ordinary harmonic oscillator (scattered dots) in both of the figures for all cases corresponding to the same set of

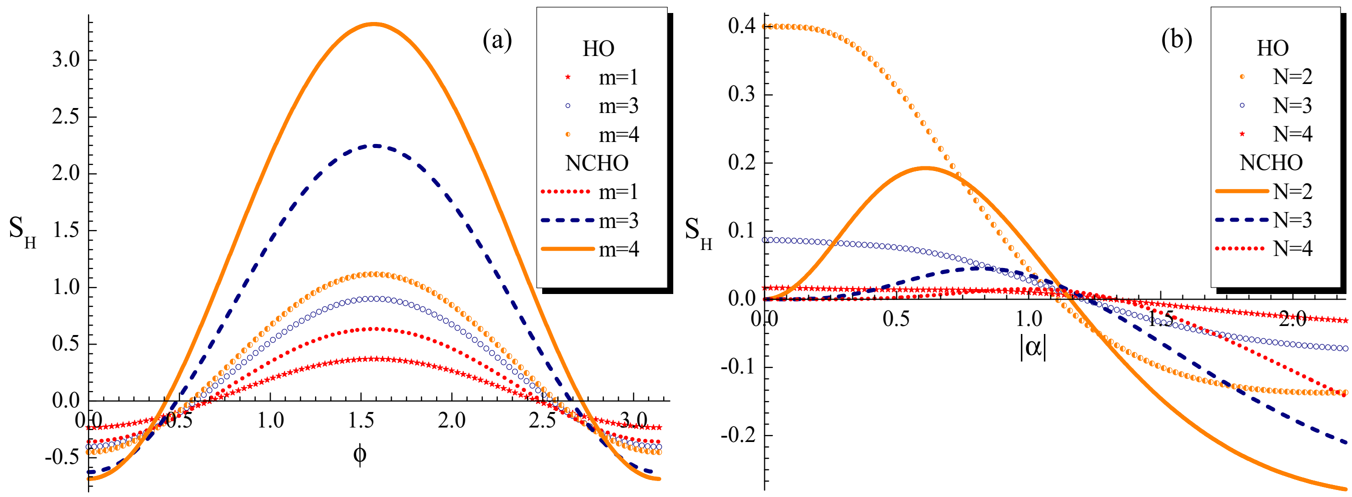

FIG. 1. Comparison of Hillery-type higher-order squeezing coefficient of a harmonic oscillator (scattered dots) versus a NCHO (for $q = 0.9$) (solid and dotted lines) as a function of (a) ϕ for different values of m, with $N = 1, \alpha = 2.1$, and (b) $|\alpha|$ for different values of N, with $\phi = 0.1, m = 1$.

parameters. This means that the quadratures of the NCHO are qualitatively more squeezed and, hence, NCHO is more nonclassical than that of the harmonic oscillator.

B. Hong-Mandel–type higher-order squeezing

Another type of higher-order squeezing was suggested by Hong and Mandel [56], who proposed that a quadrature of the form

$$Y(\phi) = \tfrac{1}{2}(A_q e^{-i\phi} + A_q^\dagger e^{i\phi}) \tag{3.11}$$

is said to be squeezed to the order $2N$ for the state $|\alpha,m\rangle_q$ for some values of ϕ if

$${}_q\langle\alpha,m|[\Delta Y(\phi)]^{2N}|\alpha,m\rangle_q < (2N-1)!!\frac{[A_q,A_q^\dagger]^N}{4^N}, \tag{3.12}$$

where $(2N-1)!! = 1.3.5....(2N-1)$, and $\Delta Y(\phi) = Y(\phi) -_q \langle Y(\phi)\rangle_q$. Naturally, the degree of the Hong-Mandel–type squeezing can be computed from the squeezing coefficient S_{HM}

$$S_{HM} = \frac{2^{2N}{}_q\langle[\Delta Y(\phi)]^{2N}\rangle_q - (2N-1)!![A_q,A_q^\dagger]^N}{(2N-1)!![A_q,A_q^\dagger]^N}, \tag{3.13}$$

where $S_{HM} < 0$ implies the presence of squeezing. We would like to mention that for the case of a usual harmonic oscillator satisfying $[a,a^\dagger] = 1$, the computation of the squeezing coefficient (3.13) becomes slightly easier. However, in our case, or more precisely, for any cases where $[A,A^\dagger] \neq 1$, one needs to consider the general form of the squeezing coefficient as given in (3.13). In [64], the authors attempted to compute the squeezing coefficient for a nonlinear PACS by considering the commutator of the nonlinear oscillator $[A,A^\dagger]$ to be the same as that of the harmonic oscillator, which we claim to be inappropriate. Nevertheless, in order to obtain a fairly general form of the squeezing coefficient S_{HM} up to order $2N$, we compute

$$[A_q,A_q^\dagger]^N = [1+(q^2-1)A_q^\dagger A_q]^N = \sum_{k=0}^{N}\binom{N}{k}(q^2-1)^k(A_q^\dagger A_q)^k, \tag{3.14}$$

$${}_q\langle[\Delta Y(\phi)]^{2N}\rangle_q = \sum_{k=0}^{2N}\binom{2N}{k}(-1)^k{}_q\langle Y(\phi)^{2N-k}\rangle_{q\,q}\langle Y(\phi)\rangle_q^k, \tag{3.15}$$

with $A_q^\dagger A_q = [n]_q$ being the number operator of the NCHO. Using the following relations,

$${}_q\langle Y(\phi)\rangle_q^k = \sum_{s=0}^{k}\binom{k}{s}2^{-k}e^{i\phi(2s-k)}{}_q\langle A_q\rangle_q^{k-s}{}_q\langle A_q^\dagger\rangle_q^s, \tag{3.16}$$

$${}_q\langle Y(\phi)^2\rangle_q = \tfrac{1}{4}\left[1+(1+q^2)_q\langle A_q^\dagger A_q\rangle_q + 2\mathrm{Re}\left\{{}_q\langle A_q^2\rangle_q e^{-2i\phi}\right\}\right], \tag{3.17}$$

$${}_q\langle Y(\phi)^3\rangle_q = \tfrac{1}{4}\mathrm{Re}\left[{}_q\langle A_q^3\rangle_q e^{-3i\phi} + (2+q^2)_q\langle A_q\rangle_q e^{-i\phi} + (1+q^2+q^4)_q\langle A_q^\dagger A_q^2\rangle_q e^{-i\phi}\right], \tag{3.18}$$

$$\begin{aligned}{}_q\langle Y(\phi)^4\rangle_q = \tfrac{1}{16}\big[&2+q^2+(3+5q^2+3q^4+q^6)_q\langle A_q^\dagger A_q\rangle_q + (\mu+q^4)_q\langle A_q^{\dagger 2}A_q^2\rangle_q \\ &+2\mathrm{Re}\left\{{}_q\langle A_q^4\rangle_q e^{-4i\phi} + (3+2q^2+q^4)_q\langle A_q^2\rangle_q e^{-2i\phi} + (\mu-q^8)_q\langle A_q^\dagger A_q^3\rangle_q e^{-2i\phi}\right\}\big],\end{aligned} \tag{3.19}$$

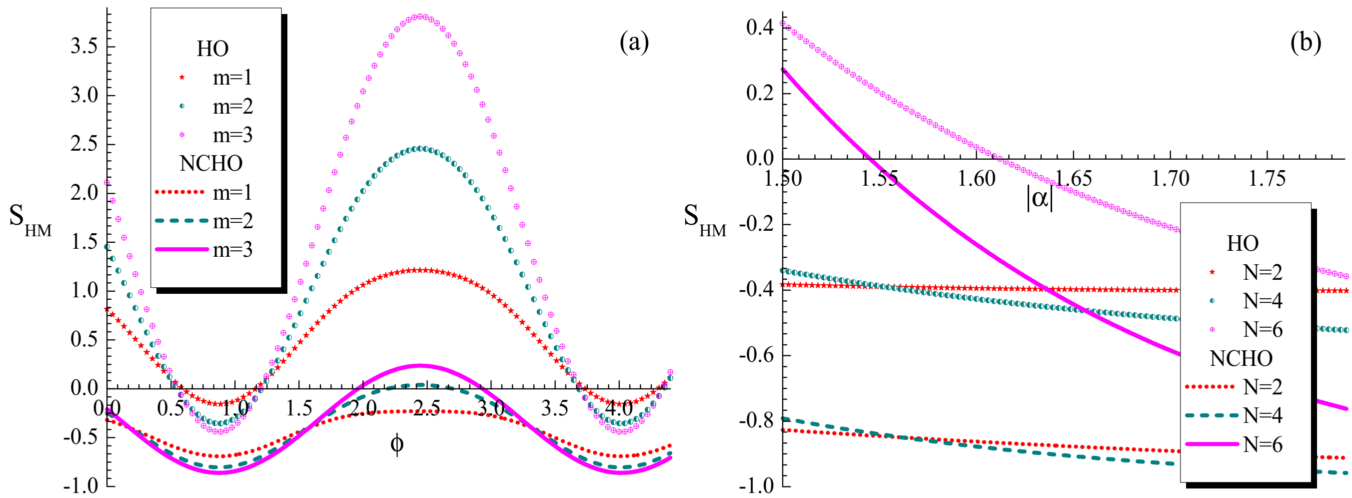

FIG. 2. Comparison of Hong-Mandel–type higher-order squeezing coefficient of a harmonic oscillator (scattered dots) versus a NCHO (for $q = 0.9$) (solid and dotted lines) as a function of (a) ϕ for different values of m, with $N = 4, \alpha = 1.0 + 1.2i$, and (b) $|\alpha|$ for different values of N, with $\phi = 0.1, m = 3$.

$$
\begin{aligned}
{}_q\langle Y(\phi)^5\rangle_q = \tfrac{1}{16}\mathrm{Re}\big[&{}_q\langle A_q^5\rangle_q e^{-5i\phi} + \{\mu_q\langle A_q^\dagger A_q^4\rangle_q + (4+3q^2+2q^4+q^6)_q\langle A_q^3\rangle_q\}e^{-3i\phi}\\
&+\{(5+6q^2+3q^4+q^6)_q\langle A_q\rangle_q + (1+3\mu+4q^2+6q^4+3q^6+q^{10})_q\langle A_q^\dagger A_q^2\rangle_q\\
&+(\mu+q^4+q^6+q^8+q^{10}+q^{12})_q\langle A_q^{\dagger 2}A_q^3\rangle_q\}e^{-i\phi}\big],
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
{}_q\langle Y(\phi)^6\rangle_q = \tfrac{1}{64}\big[&5+6q^2+3q^4+q^6+(9+22q^2+25q^4+19q^6+10q^8+4q^{10}+q^{12})_q\langle A_q^\dagger A_q\rangle_q\\
&+[5+9q^2+17q^4+18(q^6+q^8)+12q^{10}+7q^{12}+3q^{14}+q^{16}]_q\langle A_q^{\dagger 2}A_q^2\rangle_q+(\lambda+q^6+q^{10}\\
&+q^{12}+q^{14}+q^{18})_q\langle A_q^{\dagger 3}A_q^3\rangle_q+2\mathrm{Re}\{{}_q\langle A_q^6\rangle_q e^{-6i\phi}+[(5+4q^2+3q^4+2q^6+q^8)_q\langle A_q^4\rangle_q\\
&+(\mu+q^{10})_q\langle A_q^\dagger A_q^5\rangle_q]e^{-4i\phi}+[(9+13q^2+12q^4+7q^6+3q^8+q^{10})_q\langle A_q^2\rangle_q+(5+9q^2\\
&+12q^4+14q^6+10q^8+6q^{10}+3q^{12}+q^{14})_q\langle A_q^\dagger A_q^3\rangle_q+\lambda_q\langle A_q^{\dagger 2}A_q^4\rangle_q]e^{-2i\phi}\}\big],
\end{aligned}
\tag{3.21}
$$

and Eqs. (3.6) and (3.8) one can easily evaluate (3.15), which when replaced in (3.13) one obtains a complete expression of the Hong-Mandel squeezing coefficient S_{HM} explicitly up to order 6, where $\lambda = \mu + q^4 + q^6 + 2q^8 + 2q^{10} + 2q^{12} + q^{14} + q^{16}$ and $\mu = 1 + q^2 + q^4 + q^6 + q^8$. In Fig. 2, the nature of the Hong-Mandel–type squeezing coefficient (3.13) is shown as functions of ϕ and α in panels (a) and (b), respectively. The qualitative behavior of the plots in Fig. 2 remains similar to that of Fig. 1, such as the degree of squeezing is enhanced when we add more photons for a fixed angle ϕ in panel (a) and it is reduced when we increase the order N for a fixed value of α in panel (b). Also, the squeezing in quadrature of the NCHO stays higher than that of the harmonic oscillator.

IV. HIGHER-ORDER SUB-POISSONIAN PHOTON STATISTICS

Let us now study the higher-order photon statistics of the deformed PACS (2.8) and check whether it is sub-Poissonian or not. The concept of higher-order sub-Poissonian photon statistics was introduced in [65–67] in terms of factorial moment and was improved later by many authors; such as, [12,62,68]. Some of them studied the higher-order Mandel parameter, while some others used the higher-order correlation function for the purpose of testing the nature of the photon statistics. However, the method discussed in [12,62,65–68] does not work in our case, since we are not working on the usual quantum-mechanical systems satisfying $[a,a^\dagger] = 1$ but a different system obeying the q-deformed algebra (2.1). Indeed, in [64] the authors used those standard relations and attempted inappropriately to study the higher-order sub-Poissonian statistics for nonlinear systems. Instead, here we start with the original definition of the second-order correlation function (for zero delay time) $g^{(2)}(0)$ introduced in [1] and generalize it to an arbitrary order N as follows:

$$
g^{(N)}(0) = \frac{{}_q\langle(\Delta M)^N\rangle_q - {}_q\langle M\rangle_q}{{}_q\langle M\rangle_q^N} + 1. \tag{4.1}
$$

We also generalize the introductory definition of the Mandel parameter Q [69] to an arbitrary order N as given by

$$
Q_N = \frac{{}_q\langle(\Delta M)^N\rangle_q}{{}_q\langle M\rangle_q} - 1, \tag{4.2}
$$

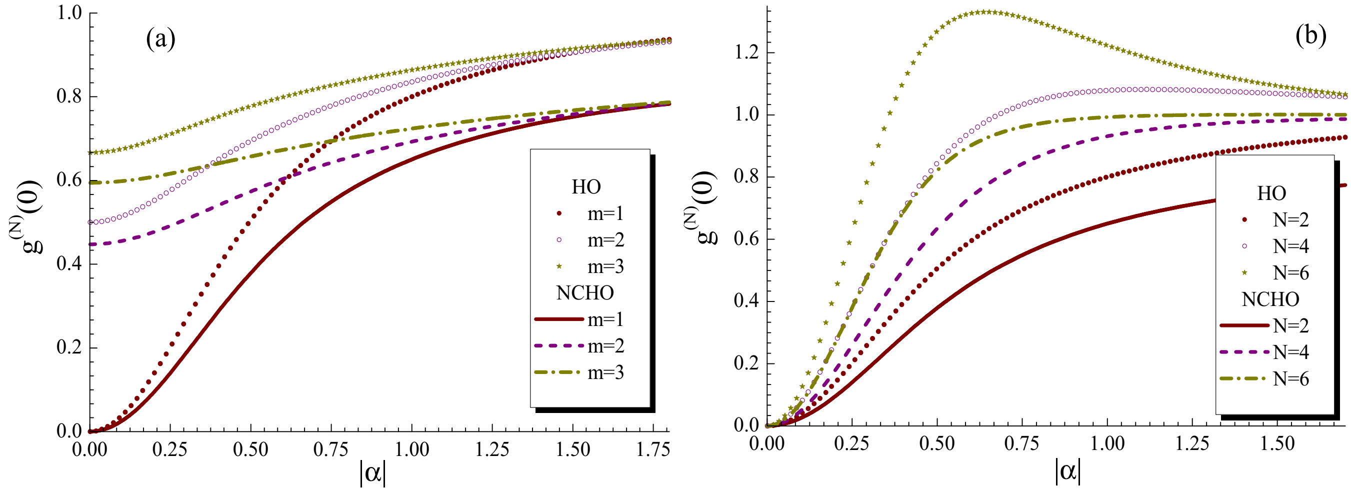

FIG. 3. Comparison of higher-order correlation function of a harmonic oscillator (scattered dots) versus a NCHO (for $q = 0.9$) (solid and dotted lines) as a function of $|\alpha|$ (a) for different values of m, with $N = 2$ and (b) for different values of N, with $m = 1$.

where $\Delta M = A_q^\dagger A_q - {}_q\langle A_q^\dagger A_q\rangle_q$ is the dispersion of the number operator $M = A_q^\dagger A_q$. It is well known that the definitions (4.1) and (4.2) are equivalent while testing the nature of photon statistics of the system. Or in other words, for all $N \geqslant 2$ the photon number distribution is Poissonian if $Q_N = 0$ (and $g^{(N)}(0) = 1$), whereas $Q_N > 0[g^{(N)}(0) > 1]$ and $Q_N < 0[g^{(N)}(0) < 1]$ correspond to the super-Poissonian (photon bunching) and sub-Poissonian (photon antibunching) cases, respectively. Although it seems that either the correlation function or the Mandel parameter would be sufficient to study for the purpose of testing the squeezing behavior of photon number, it is clear from Eqs. (4.1) and (4.2) that they are not trivially connected to each other for arbitrary orders. That is why we analyze both of them. Using the following identity,

$$ {}_q\langle(\Delta M)^N\rangle_q = \sum_{k=0}^{N} \binom{N}{k} (-1)^k {}_q\langle(A_q^\dagger A_q)^{N-k}\rangle_{q\,q}\langle A_q^\dagger A_q\rangle_q^k \tag{4.3} $$

with

$$ {}_q\langle(A_q^\dagger A_q)^N\rangle_q = \frac{1}{\hat{\mathcal{N}}^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[n+m]_q!}{([n]_q!)^2}[n+m]_q^N, \tag{4.4} $$

and the expectation value of the number operator ${}_q\langle A_q^\dagger A_q\rangle_q$, which is obtained by choosing $N = 1$ in (4.4), we can evaluate the higher-order correlation function $g^{(N)}(0)$ (4.1) and the higher-order Mandel parameter Q_N (4.2) explicitly up to order N. Figure 3(a) shows the behavior of the correlation function (4.1) with respect to $|\alpha|$ for different values of added photon numbers m. It is clear that with the increase of m the value of $g^{(N)}(0)$ increases, i.e., the photon number squeezing becomes less pronounced when m is increased. Figure 3(b) demonstrates the dependence of $g^{(N)}(0)$ for different orders of squeezing ($N = 2,4,6$) and a similar effect happens in this case as well, i.e., $g^{(N)}(0)$ increases with the increase of the order N, and for bigger values of $|\alpha|$ it eventually becomes higher than 1, for $N > 4$ roughly. Thus, the lower orders perform

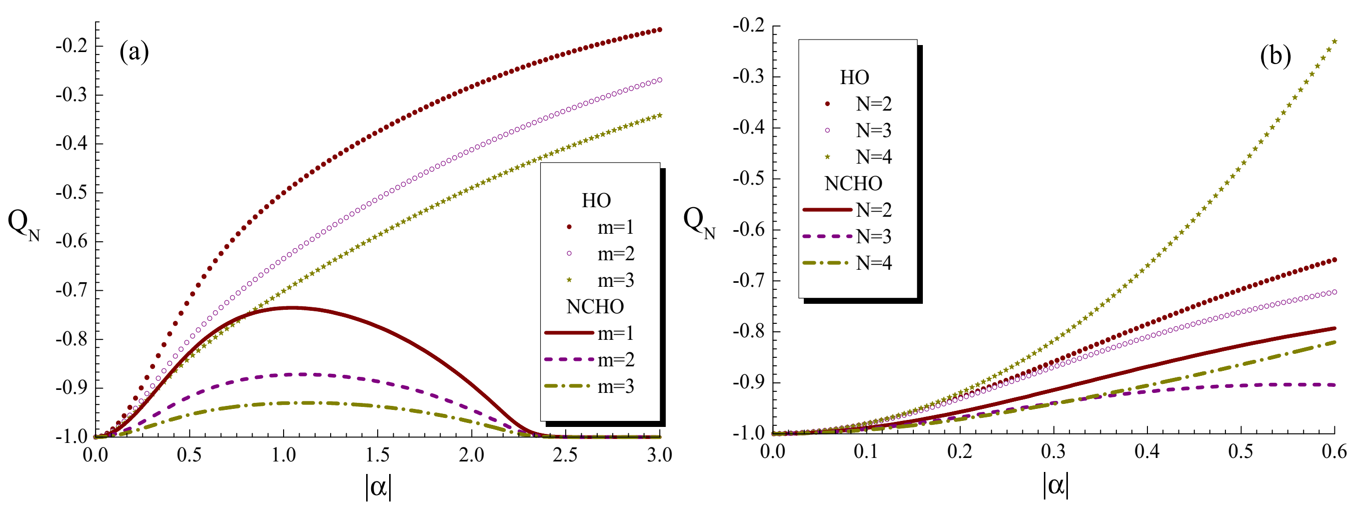

FIG. 4. Comparison of higher-order Mandel parameter of a harmonic oscillator (scattered dots) versus a NCHO (for $q = 0.9$) (solid and dotted lines) as a function of $|\alpha|$ (a) for different values of m, with $N = 2$ and (b) for different values of N, with $m = 1$.

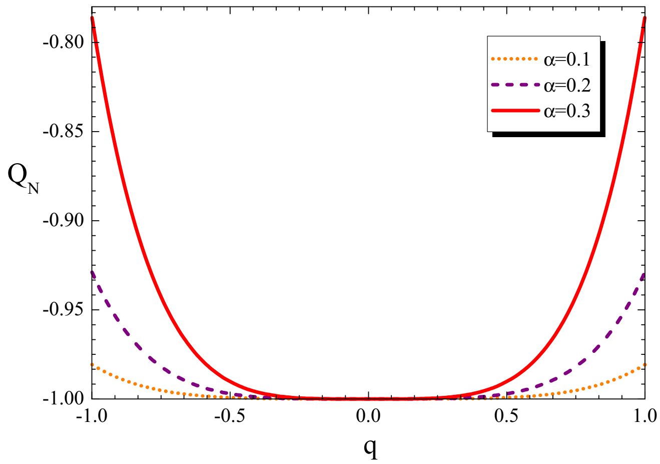

FIG. 5. Higher-order Mandel parameter of a NCHO as a function of q for different values of α, with $m = 1$ and $N = 2$.

better and provide better squeezing than the higher orders. However, we can still see the squeezing effect for higher orders when $|\alpha|$ remains small. The sub-Poissonian character with respect to the higher-order Mandel parameter Q_N is shown in Fig. 4, which shows that Q_N stays negative when we choose the parameters appropriately for different orders N and for different photon numbers m. Nevertheless, when we study the NCHO (solid and dotted lines), the whole analysis of higher-order sub-Poissonian behavior for the correlation function $g^{(N)}(0)$ and Mandel parameter Q_N becomes much improved. Moreover, the entire behavior can be enhanced by controlling the noncommutative parameter q, for instance, as Fig. 5 shows that the Mandel parameter becomes more and more negative when we go beyond the harmonic oscillator limit by increasing the noncommutativity, $|q| < 1$. However, this is just an example. Actually, it occurs for every case that we have discussed in our article, which we do not present here.

V. CONCLUSIONS

In summary, we have presented a formalism for the construction of PACS for a NCHO associated to a q-deformed oscillator algebra (2.8), which in a special limit ($q \to 1$) reduces to the PACS of the harmonic oscillator. Nonclassical properties of the corresponding system have been analyzed in detail by studying several types of quadrature (Hillery type and Hong-Mandel type) and photon number (Mandel parameter and correlation function) squeezing methods. In particular, we provide general expressions of the quadrature squeezing coefficients (Hillery and Hong-Mandel), Mandel parameter, and correlation function in arbitrary orders N, which can be utilized to study the nonclassical properties of similar types of systems. Qualitative comparisons of our results emerging out of the NCHO with that of the usual harmonic oscillator have been reported alongside. Throughout the analysis we observe an improved degree of squeezing for the noncommutative case in comparison to the usual system for the choice of the same set of parameters. Moreover, the entire behavior of squeezing can be enriched by increasing the noncommutativity of the underlying system by controlling the noncommutative parameter q. Since there exists a Hermitian representation of the corresponding system, it raises a natural question as to whether such types of systems are possible to implement in quantum optics. If so, one might obtain an extra degree of freedom for controlling the squeezing behavior of the PACS. However, the whole analysis remains a theoretical prediction only and, thus, it would be interesting to explore such types of models further in the context of quantum optics, quantum information, and computation.

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