Pfaffian states in coupled atom-cavity systems

Andrew L. C. Hayward and Andrew M. Martin
School of Physics, University of Melbourne, Victoria 3050, Australia
(Received 6 March 2016; published 19 May 2016)

Coupled atom-cavity arrays, such as those described by the Jaynes-Cummings-Hubbard model, have the potential to emulate a wide range of condensed-matter phenomena. In particular, the strongly correlated states of the fractional quantum Hall effect can be realized. At some filling fractions, the fraction quantum Hall effect has been shown to possess ground states with non-Abelian excitations. The most well studied of these states is the Pfaffian state of Moore and Read G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991), which is the ground state of a Hall liquid with a three-body interaction. We show how an effective three-body interaction can be generated within the cavity QED framework, and that a Pfaffian-like ground state of these systems exists.

DOI: 10.1103/PhysRevA.93.053614

Coupled atom-cavity systems (falling under the broad umbrella of cavity QED) are proving to be an excellent framework for the investigation of fundamental quantum phenomena, due to the versatility and control of system parameters. In particular, cavity QED is promising to be a powerful platform for quantum emulation, where a wide range of condensed-matter systems can be modeled through tailoring of atom-cavity interactions [1–6].

Recent work has shown how complex states of correlated light can be induced in cavity QED systems via nonlinearities in atom-light interactions. In particular, a class of states that corresponds to the quantum Hall effect has been predicted. In this paper, we extend this class to include Pfaffian-like states [7]. These states possess highly nontrivial topological properties that make them of great interest to the understanding of quantum Hall physics, as well as providing an opportunity to examine strongly correlated quantum states of light.

Since the unexpected discovery of fractional quantum Hall states [8], a small industry has grown up around the construction of complex states that might be found in two-dimensional quantum Hall fluids. These states (for the most part) lie in the massively degenerate lowest Landau level (LLL). Single-particle states in the lowest Landau level have a beautiful property that leads to the general structure found across all the quantum Hall effect (QHE) states. One can write the position of a single particle k in the two-dimensional (2D) plane as $z_k = (x_k + iy_k)/\ell_B$, with ℓ_B the magnetic length; then the single-particle states can all be written as a holomorphic function in z_k, multiplied by a Gaussian factor. As a consequence, all many-body states that lie in the LLL are functions of $\bar{z} = \{z_k | k \leqslant N_p\}$ and consist of a Gaussian term, $\exp[-\frac{1}{4}\sum_k |z_k|^2]$, multiplied by a Jastrow factor, an analytic function $F(\bar{z})$.

The Jastrow factor encodes the correlations between particles. The simplest correlated state was proposed by Laughlin [9] to account for the fractional QHE states found at $\nu = 1/q$. The Laughlin Jastrow factors are $F_L(\bar{z}) = \prod_{k>j}(z_k - z_j)^q$. These states have a wave function that vanishes as any two particles approach one another, minimizing the electron-electron repulsion within the lowest Landau level.

The Moore-Read Pfaffian state is defined by the Jastrow factor

$$\Psi^{\text{Pf}} \propto \text{Pf}\left(\frac{1}{z_k - z_j}\right)\prod_{k>j}(z_k - z_j)^q. \tag{1}$$

The terms in the Pfaffain factor cancel a factor in the Laughlin state, leading to a nonzero amplitude at the coincidence of just two particles.

This ground state has some interesting properties, most notably non-Abelian excitations and a (related) triply degenerate ground state in the toroidal geometry [7].

In this work, we consider a Jaynes-Cummings-Hubbard (JCH) system with synthetic magnetic field and demonstrate the existence of Pfaffian-like ground states at a filling factor $\nu = 1$. We first consider the conventional JCH system and find no evidence for Pfaffian-like states. We then consider a three-level Jaynes-Cummings (JC) cavity and show how there are regimes in which three-body interactions dominate. We then study the conditions under which Pfaffian-like states will arise when these three-level systems are Hubbard coupled.

Each cavity in the JCH lattice is described by

$$H^{\text{JC}} = \omega L + \Delta\sigma^+\sigma^- + \beta(\sigma^+ a + \sigma^- a^\dagger), \tag{2}$$

where a is the photonic annihilation operator, $\sigma^\pm$ are the atomic raising and lowering operators, L is the excitation number operator, Δ is the atom-photon detuning, β is the coupling energy, and $\hbar = 1$. The states $|g(e),n\rangle$, where n is the number of photons and $g(e)$ are the ground (excited) state of the atom, form the single-cavity basis. H^{JC} commutes with the total excitation number operator, L. Therefore, the total excitations in the cavity, ℓ, is a good quantum number. The eigenstates of Eq. (2) are termed polaritons, i.e., superpositions of atomic and photonic excitations, and are a function of ℓ and Δ/β.

The JCH model describes an array of individual Jaynes-Cummings cavities, which are coupled via a Hubbard-like photon tunneling term. In the case of a lossless system, the JCH model can be described by

$$H^{\text{JCH}} = \sum_i H_i^{\text{JC}} - \sum_{\langle i,j\rangle}\kappa_{ij}a_i^\dagger a_j, \tag{3}$$

where κ_{ij} is the tunneling rate between cavities i and j, and the sum over $\langle i,j\rangle$ is between nearest neighbors only.

The presence of the intracavity atom induces an intrinsic anharmonicity in the spectrum. This anharmonicity leads to an effective photon-photon repulsion which leads to the nontrivial dynamics found in JCH systems.

For large detuning ($|\Delta| \gg \beta$), eigenstates separate out into either atomic or photonic modes. In this limit, the photonic or atomic mode can be adiabatically eliminated. Eliminating the atomic modes, the photonic mode has a weak Kerr-type photon-photon repulsion [10] and the exchange of energy between atomic and photonic modes is strongly suppressed. However, virtual processes lead to effective interactions in the photonic and atomic submanifolds. Photons have an atomic mediated nonlinear on-site repulsion, making the JCH model equivalent to the Bose-Hubbard (BH) model [11]. Atomic modes are coupled with the effective hopping rate $\kappa_{ij}^{\mathrm{eff}} = \kappa_{ij}\beta^2/\Delta^2$ [12]. As the atomic modes are restricted to two levels, this is effectively a hard-core boson field for atomic states, in contrast to the weakly interacting photon field.

Investigation of quantum Hall physics in the JCH model requires the introduction of a synthetic magnetic field. An artificial magnetic field may be realized via the introduction of some time-reversal symmetry-breaking interactions. A number of techniques have been proposed to achieve this [6,13–17]. For example, one may exploit a time-dependent potential to induce magnetic flux across the lattice [16], thereby explicitly breaking the time symmetry of the system. A similar strategy is proposed in Ref. [6], where the authors utilize a time-dependent intersite coupling to induce a synthetic magnetic field. Alternatively, effective coupling to real magnetic fields can be used, such as in the proposal by Koch *et al.* [15]. Other means, such as the use of optically polarized media [13] or via atomically mediated intersite coupling [14], have also been proposed.

Inspired by results in ultracold-atom simulations [18–21], one might expect to find evidence of a Pfaffian ground state in the JCH model at $\nu = 1$. We numerically investigated the JCH model on a torus to this end. Existence of a Pfaffian-like state can be indicated by a number of properties of the ground state: a triply degenerate ground-state manifold, large overlap with the Pfaffian trial wave function, and a Chern number of 3 computed for the three ground states.

We conducted a comprehensive search over several parameters within the JCH model, but the tell-tale signature of a triple-degeneracy ground state proved elusive. Instead, simulations reveal, for some lattice configurations, a single separated ground state in the strongly interacting limit, with a transition to a gapless phase as the effective two-body interaction decreased. Other configurations possessed a gapless ground state extending all the way to the hard-core limit. While these results disagree with the Bose-Einstein condensate (BEC) findings, other lattice boson simulations have failed similarly [22].

Although the evidence from BECs suggests that a Pfaffian ground state should be preferred, there are other possible states at $\nu = 1$ filling for bosons which are in competition with the Pfaffian state. For example, Read [23] proposes a ground state in which, approximately, a single vortex is attached to each boson. This assignment exactly cancels out the external magnetic field, which reduces the problem to that of a Fermi liquid. Alternatively, it is conjectured [24] that a striped phase with charge density order may exist at $\nu = 1$, with some numerical simulations [25] finding evidence for this.

The relatively small size of the systems we have simulated makes it difficult to tease out the importance of different effects which determine the real nature of the ground state at this filling factor. However, the poor scaling of these systems means that significantly larger systems are impractical at this point in time. Of course, this problem is one of the primary motivators of work into quantum emulation.

For $\nu = 1$, the Pfaffian ground state is the highest-density ground state of the three-body δ potential Hamiltonian [26]. For each pair of particles, there is a number of terms in Eq. (1) for which each particle is in a different partition. However, there is no term for which three or more particles coincide that does not vanish.

The JCH in the limit of large detuning can be described by a Bose-Hubbard model with an effective two-body interaction, U_2. However, the atomic-cavity interaction induces interactions to all orders of n-photon interactions. These higher-order interactions are much smaller than the two-body interaction, and therefore the physics of the JCH very much mirrors that of the Bose-Hubbard model with two-body interaction. We now show that it is possible to eliminate the two-body interaction while retaining the higher-order interactions.

Three-body interactions (without corresponding two-body ones) are unnatural and do not arise in many physical systems. A number of schemes for creating effective three-body interactions have been proposed in the context of BECs [22,27–29] and in the circuit QED setting [30]. Below we show that an effective three-body interaction can be induced in an atom-cavity lattice by replacing the two-level atoms in the JCH model with appropriately tuned three-level atoms. Furthermore, it is demonstrated via simulation that an atom-cavity lattice consisting of these three-level atoms can possess a Pfaffian-like state as its ground state.

We consider the three-level system atom in the Ξ configuration, as shown in Fig. 1(a). This configuration consists of two evenly spaced excited states, with the atom-cavity system described by the Hamiltonian

$$\begin{aligned} H^{3L} = {}& \omega a^\dagger a + \epsilon_1 |e_1\rangle\langle e_1| + (\beta_1\sigma_1^+ a + \mathrm{H.c.}) \\ & + \epsilon_1 |e_2\rangle\langle e_2| + (\beta_2\sigma_2^+ a + \mathrm{H.c.}). \end{aligned} \tag{4}$$

Here, $\sigma_{1(2)}^+$ raises the atomic level from $g \leftrightarrow e_1$ ($e_1 \leftrightarrow e_2$), and levels 1 (2) have energies $\epsilon_{1(2)}$.

Choosing an atom with energy levels

$$\epsilon_1 = \omega - \Delta, \qquad \epsilon_2 = 2\omega - 2\Delta, \tag{5}$$

and transition strengths

$$g \leftrightarrow e_1 \equiv \beta_1, \qquad e_1 \leftrightarrow e_2 \equiv \beta_2 = \sqrt{2}\beta_1, \tag{6}$$

leads to an effective three-body interaction. This can be seen by considering the formulation of the JC system as a two-mode bosonic system with intermode tunneling. If one imposes a hard-core boson condition on one of the modes, then the system corresponds exactly to the JC cavity with a two-level atom. In the single-excitation subspace, the hard-core condition is

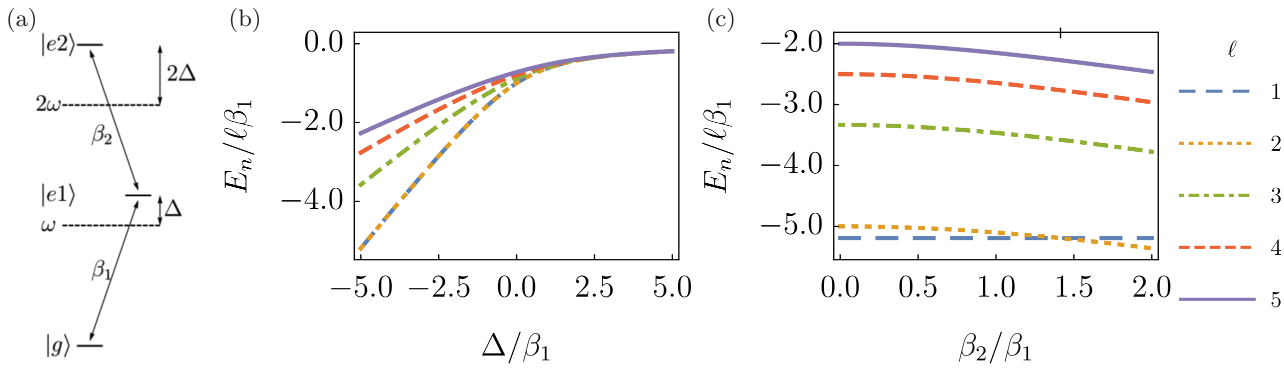

FIG. 1. (a) Three-level atom configuration for inducing a three-body interaction in the JCH lattice. (b) Energy per excitation in the three-level atom for $\ell = 1$–5 (blue, orange, green, red, and purple, respectively). Excitations in the $\ell = 1$ and $\ell = 2$ excitation subspace have the same energy cost. For higher excitation numbers, the energy per particle increases. (c) Energy per excitation as a function of β_2 [same colors as (b)] for $\Delta/\beta_1 = -5$.

automatically satisfied and the system is simply a free boson model.

If, instead of the hard-core condition, one imposes a three-particle hard-core condition ($U_3 \to \infty$), then a similar situation arises, except that for both the single and double excitation subspaces, there are no interactions. Mapping this model back to the atomic model, for one and two particles, the equivalent atom-cavity model corresponds exactly to the one presented previously, where the factor of $\sqrt{2}$ arises from the indistinguishably of the bosons.

The three-body nonlinearity is demonstrated in Fig. 1(b). Here, the energy cost per particle for the lower polariton branch is plotted as a function of the detuning, Δ. For one and two excitations, the energy per particle is the same. However, for $\ell = 3$ and above, there is an increased cost for adding additional particles. Figure 1(c) shows that at $\beta_2/\beta_1 = \sqrt{2}$, the two- and three-level atom-cavity systems share several properties. The nonlinearity is unbounded as the detuning is lowered, and disappears as the detuning is increased. Also, the nonlinearity does not grow quadratically with excitation number, as is the case for a pure three-body interaction, similar to the two-level atom case.

Figure 1(c) demonstrates how the two-body interaction is affected by the strength of β_2. As β_2 is tuned away from $\beta_2/\beta_1 = \sqrt{2}$, the effective two-body interaction becomes $+ve$ or $-ve$. However, at low detunings [$\Delta/\beta_1 = -5$ in Fig. 1(c)], the relative strength of this effective two-body interaction is much smaller than the three-body one.

This method for generating three-body interactions opens up possibilities for investigating the physics of topological quantum states that has proved elusive in traditional environments. Furthermore, this same technique can be extended to higher-order interactions. There is a hierarchy of states that generalize the Pfaffian [31] state, which are expected to be ground states of these higher-order interactions. The JCH model with this modification is a system in which such higher-order interactions might be achieved, outside of a fully functional quantum computer.

In practice, engineering the three-level system as described lies well within the capabilities of current cavity QED fabrication techniques. Engineering a system like this in circuit QED has been discussed in [30]. For cavity atoms, most Ξ configurations tend to be unstable, with fast relaxation rates that would preclude large-scale coherence in the system. This instability can be mitigated by instead using an M-like configuration (as in [32]), where classical driving can be used to create an effective three-level JCH system.

In our simulations, we restrict the system to a torus, to remove edge effects. The toroidal geometry permits twisted periodic boundary conditions, which reduces finite-size effects and allows for computation of the Chern number.

The Chern number [33], $\mathcal{C}$, is a measure of the topology of the ground state of the system and can provide evidence for the existence of a Pfaffian-like state. Here, we define a two-dimensional manifold over the two phases, $\theta_{x,y}$, that parametrizes the twisted periodic boundary conditions across the torus. For the Pfaffian state, the three ground states have a combined Chern number of 3.

We compute, by exact diagonalization, the low-energy band structure of the three-level JCH system for four excitations over the twisted boundary condition manifold [Fig. 2(a)]. For four particles, there is a large modulation of the energy as a function of the twist angles (which one expects to disappear in the many-particle limit [34]). We find that a quasigap can

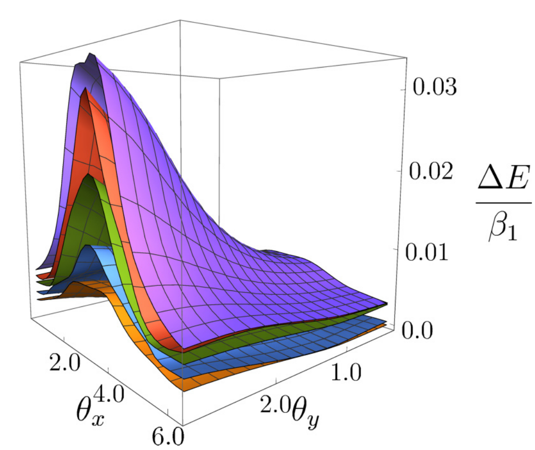

FIG. 2. Energy gap for the first five excited states in the three-level JCH model on a 4×4 lattice with four particles and four flux quanta ($\nu = 1$). There is a three-dimensional quasidegeneracy in the ground state, indicative of a Pfaffian-like state.

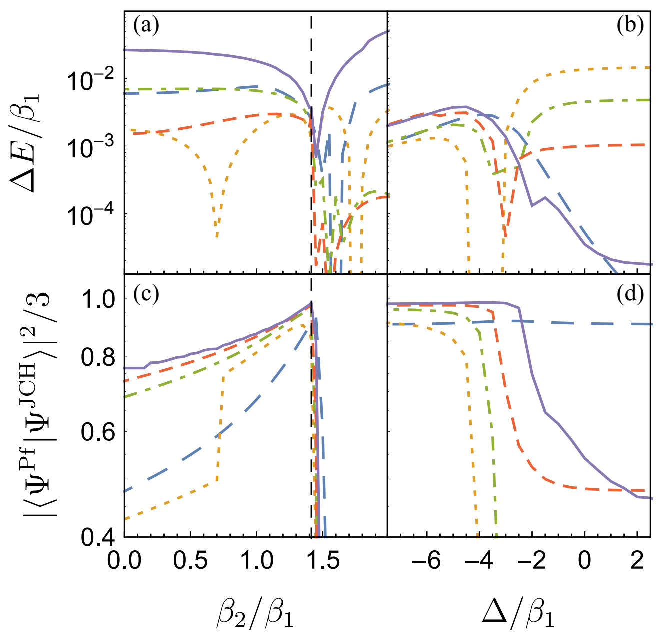

FIG. 3. Band gap and Pfaffian overlap as a function of β_2/β_1 (at $\Delta/\beta_1 = -5$) and Δ/β_1 (with $\beta_2/\beta_1 = \sqrt{2}$) for four particles on 4×4 (blue long-dashed line), 4×5 (orange dashed line), 5×5 (green dot-dashed line), 6×5 (purple solid line), and 6×6 (brown dotted line) lattice.

exist in both models, and computation of the Chern number ($\mathcal{C} = 3$) coupled with the threefold degeneracy provides strong evidence for Pfaffian physics.

In the case of Laughlin states on a lattice, the gap has been shown [33] to scale proportionally to the flux density per lattice plaquette. This does not seem to apply in the case of the Pfaffian state. We hypothesize that the first excited state of this system is not an excitation lying in a higher Landau level. Rather, it is some combination of quasiparticles and quasiholes. These excitations lie in the LLL, with no energy gap in the continuum limit, which would explain the rapid decrease in the gap as the flux density per lattice plaquette decreases.

With strong evidence for the Pfaffian state, we proceed to investigate the three-level JCH model in more detail, by exploring the properties of the ground state over a range of lattice sizes and system parameters. These investigations are presented in Fig. 3.

In Figs. 3(a) and 3(c), we show how the ground state changes as a function of β_2. The ground state experiences a transition from the Pfaffian state away from $\beta_2/\beta_1 = \sqrt{2}$. For $\beta_2/\beta_1 > \sqrt{2}$, where the effective two-body interaction becomes attractive, there is a very sharp transition to a collapsed state. On the other side, $\beta_2/\beta_1 < \sqrt{2}$, the gap and Pfaffian overlap remain fairly stable, although we observe a transition in the 4×5 lattice configuration.

In Figs. 3(b) and 3(d), we show how the ground state changes as a function of the detuning, Δ. We find that as in the Laughlin case [35], increasing the atomic detuning, which alters the effective interaction strength [Fig. 1(b)], can induce a transition from a Pfaffian state to an uncorrelated one. This transition is accompanied by a closing of the band gap and, for most cases, a drop off of the overlap with the trial wave function.

The Pfaffian states at $\nu = 1$ have a straightforward interpretation as the symmetrized product of two Laughlin states at $\nu = 1/2$ [36], assigning each particle to one of two Laughlin states. The wave function will vanish as two particles in the same Laughlin state approach each other, but not if those two particles are in different states. However, if any three particles coincide, then by construction the wave function will be zero at this point. The three degenerate states in the torus setting correspond to the singlet and two doublet states that one can construct from the doubly degenerate Laughlin states.

This reexpression of the Pfaffian wave function also allows one to translate findings from investigations into the Laughlin state in the JCH into the current work. For example, we find that the detuning for which the Pfaffian state undergoes a transition [Figs. 3(b) and 3(d)] is the same for the equivalent single Laughlin state [35]. Furthermore, the overlap with the trial Pfaffian wave function is very well approximated by the overlap with a single Laughlin function, to the power of two.

While we have shown that the ground state of the three-level JCH model possesses a Pfaffian-like ground state, preparing such a state in a real system presents significant challenges. In particular, the presence of dissipation and decoherence in photonic cavities will impede the system from finding its natural ground state. However, there has been much work towards combating these forces in atom-cavity systems, and several methods that drive a photonic system towards the desired equilibrium have been proposed [37–39]. In particular, Hafiz [40] and Kapit [6] consider this problem for fractional quantum Hall states.

In conclusion, we have described a method by which three-body interactions can be induced in Jaynes-Cummings-Hubbard systems. In the presence of synthetic magnetic fields, such interactions, i.e., strongly correlated states of light with Pfaffian-like topological properties, will exist. This opens up exciting possibilities for the exploration of exotic quantum states within the cavity QED framework, including states with non-Abelian quasiparticles pertaining to topological quantum computing.

The authors would like to acknowledge A. D. Greentree for helpful discussions.

[1] A. D. Greentree, C. Tahan, J. H. Cole, and L. C. L. Hollenberg, Nat. Phys. **2**, 856 (2006).
[2] D. Rossini and R. Fazio, Phys. Rev. Lett. **99**, 186401 (2007).
[3] J. Quach, M. I. Makin, C.-H. Su, A. D. Greentree, and L. C. L. Hollenberg, Phys. Rev. A **80**, 063838 (2009).
[4] D. Gerace, H. E. Türeci, A. Imamoglu, V. Giovannetti, and R. Fazio, Nat. Phys. **5**, 281 (2009).

[5] J. Q. Quach, C.-H. Su, A. M. Martin, A. D. Greentree, and L. C. L. Hollenberg, Opt. Express **19**, 11018 (2011).
[6] E. Kapit, M. Hafezi, and S. H. Simon, Phys. Rev. X **4**, 031039 (2014).
[7] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[8] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[9] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[10] N. Na, S. Utsunomiya, L. Tian, and Y. Yamamoto, Phys. Rev. A **77**, 031803 (2008).
[11] M. Hohenadler, M. Aichhorn, S. Schmidt, and L. Pollet, Phys. Rev. A **84**, 041608 (2011).
[12] M. I. Makin, Jared H. Cole, Charles D. Hill, Andrew D. Greentree, and Lloyd C. L. Hollenberg, Phys. Rev. A **80**, 043842 (2009).
[13] F. D. M. Haldane and S. Raghu, Phys. Rev. Lett. **100**, 013904 (2008).
[14] J. Cho, D. G. Angelakis, and S. Bose, Phys. Rev. Lett. **101**, 246809 (2008).
[15] J. Koch, A. A. Houck, K. L. Hur, and S. Girvin, Phys. Rev. A **82**, 043811 (2010).
[16] A. R. Kolovsky, Europhys. Lett. **93**, 20003 (2011).
[17] R. O. Umucalilar and I. Carusotto, Phys. Rev. A **84**, 043804 (2011).
[18] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Phys. Rev. Lett. **87**, 120405 (2001).
[19] N. Regnault and T. Jolicoeur, Phys. Rev. B **69**, 235309 (2004).
[20] C.-C. Chang, N. Regnault, T. Jolicoeur, and J. K. Jain, Phys. Rev. A **72**, 013611 (2005).
[21] M. R. Peterson, Th. Jolicoeur, and S. Das Sarma, Phys. Rev. Lett. **101**, 016807 (2008).
[22] L. Mazza, M. Rizzi, M. Lewenstein, and J. I. Cirac, Phys. Rev. A **82**, 043629 (2010).
[23] N. Read, Phys. Rev. B **58**, 16262 (1998).
[24] E. H. Rezayi, F. D. M. Haldane, and K. Yang, Phys. Rev. Lett. **83**, 1219 (1999).
[25] B. Chung and Th. Jolicœur, Phys. Rev. A **77**, 043608 (2008).
[26] M. Greiter, X. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[27] H. P. Büchler, A. Micheli, and P. Zoller, Nat. Phys. **3**, 726 (2007).
[28] A. J. Daley, J. M. Taylor, S. Diehl, M. Baranov, and P. Zoller, Phys. Rev. Lett. **102**, 040402 (2009).
[29] A. J. Daley and J. Simon, Phys. Rev. A **89**, 053619 (2014).
[30] M. Hafezi, P. Adhikari, and J. M. Taylor, Phys. Rev. B **90**, 060503 (2014).
[31] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[32] A. B. Matsko, I. Novikova, M. S. Zubairy, and G. R. Welch, Phys. Rev. A **67**, 043805 (2003).
[33] M. Hafezi, A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. A **76**, 023613 (2007).
[34] R. Varnhagen, Nucl. Phys. B **443**, 501 (1995).
[35] A. L. C. Hayward, A. M. Martin, and A. D. Greentree, Phys. Rev. Lett. **108**, 223602 (2012).
[36] N. Read and E. Rezayi, Phys. Rev. B **54**, 16864 (1996).
[37] F. Verstraete, M. M. Wolf, and J. Ignacio Cirac, Nat. Phys. **5**, 633 (2009).
[38] F. Pastawski, L. Clemente, and J. I. Cirac, Phys. Rev. A **83**, 012304 (2011).
[39] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, New J. Phys. **16**, 045014 (2014).
[40] M. Hafezi, M. D. Lukin, and J. M. Taylor, New J. Phys. **15**, 063001 (2013).