Strong-field atomic ionization in an elliptically polarized laser field and a constant magnetic field

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Within the framework of the quasistationary quasienergy state (QQES) formalism, the tunneling and multiphoton ionization of atoms and ions subjected to a perturbation by a high intense laser radiation field of an arbitrary polarization and a constant magnetic field are considered. On the basis of the exact solution of the Schrödinger equation and the Green's function for the electron moving in an arbitrary laser field and crossed constant electric and magnetic fields, the integral equation for the complex quasienergy and the energy spectrum of the ejected electron are derived. Using the "imaginary-time" method, the extremal subbarrier trajectory of the photoelectron moving in a nonstationary laser field and a constant magnetic field are considered. Within the framework of the QQES formalism and the quasiclassical perturbation theory, ionization rates when the Coulomb interaction of the photoelectron with the parent ion is taken into account at arbitrary values of the Keldysh parameter are derived. The high accuracy of rates is confirmed by comparison with the results of numerical calculations. Simple analytical expressions for the ionization rate with the Coulomb correction in the tunneling and multiphoton regimes in the case of an elliptically polarized laser beam propagating at an arbitrary angle to the constant magnetic field are derived and discussed. The limits of small and large magnetic fields and low and high frequency of a laser field are considered in details. It is shown that in the presence of a nonstationary laser field perturbation, the constant magnetic field may either decrease or increase the ionization rate. The analytical consideration and numerical calculations also showed that the difference between the ionization rates for an s electron in the case of right- and left-elliptically polarized laser fields is especially significant in the multiphoton regime for not-too-high magnetic fields and decreases as the magnetic field increases. The paper generalizes the results obtained earlier [V. M. Rylyuk, Phys. Rev. A **86**, 013402 (2012)].

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I. INTRODUCTION

The interaction of an intense laser radiation field with matter has been investigated extensively during the past 30 years [1–11]. This was particularly supported by the rapid development of lasers producing high intensities (10^{14}–10^{15} W/cm^2) that generate forces comparable to intra-atomic forces and ultrashort pulse durations of the order of femtoseconds (10^{-15} s) down to attoseconds (10^{-18} s) [12,13]. Ionization is one of the effects which characterizes the interaction of light with matter. The basic concepts of the theory of ionization were developed by Keldysh [14], who demonstrated that the tunneling effect and the multiphoton ionization are two limiting cases of the common process of nonlinear photoionization, whose character depends on the magnitude of the adiabatic parameter (Keldysh parameter)

$$\gamma = \frac{\omega\sqrt{2m|E_0|}}{eF}, \tag{1}$$

where ω is the laser frequency, $|E_0|$ is the binding energy of the level, and F is the magnitude of the perturbing field. The adiabatic Keldysh parameter can be interpreted as a ratio of the time $\tau = b/v_{at}$ of flight of the electron under the barrier of the width $b = |E_0|/eF$, moving with the atomic velocity $v_{at} = \sqrt{2|E_0|/m}$, to a half period of the laser field (the electron ejected from under the barrier with highest probability near the maximum of the electric field): $\gamma = 2\omega\tau$. The dimensionless Keldysh parameter (1) also can be defined as

$$\gamma = \sqrt{\frac{|E_0|}{2U_p}}, \tag{2}$$

where $U_p = e^2F^2/(4m\omega^2)$ is the ponderomotive (quiver) energy for a linearly polarized laser field. Tunneling ionization of atomic states takes place when $\gamma \ll 1$ (so-called adiabatic regime). This inequality means that the electron overcomes the barrier for the time which is less than a period of the laser field, i.e., the ionization proceeds similarly to the case of a constant electric field. In the opposite case, i.e., for $\gamma \gg 1$ the ionization is a multiphoton process, which is realized at a relatively high-frequency and low-intensity laser field. In Ref. [14] was used a tunneling approximation requiring $U_p \gg |E_0|$ and $|E_0| \gg \hbar\omega$.

In the case of a low-intensity laser field, the ionization process can be described within the perturbation theory [15]. But, for a high-intensity laser field it is better to use nonperturbative methods to calculate the ionization rates. Such is the method of zero-range potential [16–19], which is an approximate model for a negative ion. The method of zero-range potential consists of replacing a deep potential well of a small radius by a boundary condition at the point of the center of the well. However, the results of numerical calculations indicate that the zero-range potential model in which the Coulomb interaction of the photoelectron with the atomic or ionic core is disregarded provides only a qualitative description of the nonlinear ionization even in the simplest case of the hydrogen atom. Therefore, in the case of ionization of neutral atoms and positive ions, the long-range Coulomb interaction of the photoelectron with the parent ion has to be taken into

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consideration. Impressive progress in strong-field laser physics in recent years has led to the emergence of powerful sources of coherent radiation in the UV and x-ray wavelength ranges. Such sources are based on free-electron lasers [20–22]. Typical values of the Keldysh parameter in experiments [23,24] amount to $\gamma \approx 30$–100, which correspond to the multiphoton ionization regime. In this regard, arises the question about the development of a multiphoton ionization theory in this range of parameters. The questions considered above were also expounded in reviews [25–30].

In this paper, we consider the ionization of atoms and ions in the wide range of the Keldysh parameter subjected to a perturbation by a high intense elliptical laser beam, propagating at an arbitrary angle to the constant magnetic field, with taking into account the Coulomb interaction of the photoelectron with the parent ion. For this purpose, we find the exact solution and the Green's function of the Schrödinger equation for an electron moving in these fields. On their basis, using the QQES formalism [31,32] and the method of imaginary time, we derive the ionization rate with Coulomb corrections in the tunneling and the multiphoton regimes. In order to find the Coulomb factor in the tunneling regime as the first-order correction to the classical action, we employ the quasiclassical perturbation theory and the method of subbarrier trajectories, generalized to the case of the presence of a nonstationary laser field and a constant magnetic field. In order to correctly describe the Coulomb correction in the multiphoton regime must be taken into account the distortion of the subbarrier electron trajectory by the Coulomb field. With this aim we are making an attempt to generalize the approach developed in Refs. [33–36] for the case of a linearly polarized laser field, to the case of an elliptic laser field and a constant magnetic field. We analyze in details analytical expressions for the exponential and preexponential factors in the ionization rate with the Coulomb correction in the tunneling and the multiphoton limits for small and large magnetic fields. These expressions show that contrary to the widespread view, the effect of a constant magnetic field does not always lead to the suppression of the ionization rate of a weakly bound level. Our consideration is also completed by numerical calculations of ionization rates and so-called "stabilization factor" in the field of UV and x-ray lasers and comparison with the results from *ab initio* numerical solutions, and with the experiment where the interaction of a high-intensity x-ray laser radiation field with rare gas atoms was investigated. Note also that the effect of the magnetic field on the ionization probability has been studied in [37–42].

This work generalizes the results obtained in Ref. [42], where the tunneling ionization of atoms and ions in an elliptical electromagnetic wave propagating along a constant magnetic field was considered, to the case of the multiphoton ionization and arbitrary angles between a laser beam and the magnetic field. On the other hand, the ionization probability and the Coulomb corrections in the tunneling and the multiphoton regimes obtained in this paper, in the limit $H \to 0$ and for the special case of a linearly polarized laser field ($g = 0$), coincide with the corresponding results obtained in Refs. [34,35]. Besides, the ionization rates obtained in this paper, in the static limit ($\omega \to 0$), coincide with the ionization rates obtained in Ref. [39] for the case of constant electric and magnetic fields. Thus, the unified approach adopted in this paper, basing on the QQES formalism and the exact solution of the Schrödinger equation for the electron moving in an arbitrary laser field and constant magnetic field, allows (i) to investigate peculiarities of the strong-field atomic ionization in the wide range of the Keldysh parameter in the case, when an elliptically polarized laser field and a constant magnetic field are present simultaneously, (ii) to investigate the energy spectrum of photoelectrons in the presence of the constant magnetic field, (iii) in the limit when the constant magnetic field vanishes ($H \to 0$) to get the saddle-point equation and all the known PPT expansions for ionization rates in the case of an elliptical laser field [27], and (iv) in the static limit ($\omega \to 0$) to get the known results for the ionization rate in the case of constant electric and small and large magnetic fields, obtained in the framework of the "imaginary-time" method and the method of zero-range potential.

Here, we note the ionization problem induced by an electromagnetic wave and a magnetic field is also of great practical importance. Weaker magnetic fields may have a great effect in semiconductors [43]. Strong magnetic fields of some mega-Gauss may also be observed in laser-produced plasmas [44]. The method of magnetic cumulation (explosion-assisted compression of an axial magnetic field), proposed by Sakharov in [45,46], made it possible to obtain the record-high magnitudes $H = 15$–25 MG. Further progress in this area gives the hope of achieving 30–100 MG fields [47]. Ultrastrong magnetic fields can been countered in astrophysics. For instance, at the surface of white dwarf stars may also be observed magnetic fields ranging from 2 MG to roughly 1000 MG [48]. White dwarf stars are remarkable in that it is possible to observe their optical spectra [48,49] which allows us to study the effect of large electric and magnetic fields on the atomic levels, primarily for the atoms of hydrogen and helium. The approach developed in this paper also can be used for the investigation of the effect of a magnetic field on the thermally stimulated ionization of impurity centers in semiconductors by submillimeter radiation [40,50,51] and the Poole-Frenkel effect [52]. The special case of calculating the ionization probability in the magnetic field is so-called the Lorentz ionization, i.e., ionization of atoms and ions as a result of their motion in a constant or quasistationary magnetic field [53]. Another interesting area of application of this approach, based on knowledge of the exact Green's function for the electron moving in an arbitrary laser and the magnetic fields, can be investigation of the high-order harmonic generation (HHG) in the magnetic field. In particular, in Refs. [54,55] was shown that a properly chosen combination of the static electric and magnetic fields can increase both the harmonic intensity and the harmonic order. Also, the approach adopted here, on the basis of the numerical analysis of subbarrier Coulomb-corrected trajectories considered in Refs. [56,57], can be used for the explanation of the too large width of the electron momentum distribution during the tunneling under the influence of a circularly polarized light pulse [58], an extra longitudinal momentum spread of the electron arising from strong-field ionization of helium [59,60], and significant deviations for the location of the center of the electron momenta distribution at low values of ellipticity of laser light [61].

The paper is organized as follows. In Sec. II, we derive the equation for the complex quasienergy. Section III represents analytical results in the case of an elliptically polarized laser beam propagating at an arbitrary angle to the constant magnetic field. These include (i) energy spectrum of photoelectrons, (ii) tunneling regime, (iii) multiphoton regime, (iv) the limit of low frequencies ($\omega \ll \omega_H$), and (v) the limit of large magnetic fields ($\omega_H \gtrsim \omega$). Section IV represents the method of imaginary time to calculate the Coulomb correction. These include (i) Coulomb correction in the tunneling regime and (ii) Coulomb correction in the multiphoton regime. Section V concludes the work. Appendix A presents the exact solution of the Schrödinger equation and the Green's function for an electron in an electromagnetic wave and crossed constant electric and magnetic fields. Appendix B presents two approaches to the determination of ionization rate. Appendix C presents Kapitza's method for the calculation of the imaginary part of the classical action. In the Supplemental Material [63], the following are considered: (A) energy spectrum of photoelectrons in the quasiclassical case; (B) some limiting cases of the ionization rate; (C) some limiting cases of Coulomb factors; and (D) width of the barrier and the emission angle of photoelectrons.

II. EQUATION FOR THE COMPLEX QUASIENERGY

Let us consider the decay of a weakly bound level induced by a monochromatic laser field having an arbitrary elliptical polarization with the electron initially bound by a zero-range potential and the constant magnetic field $\boldsymbol{H}$. In the integral form, the Schrödinger equation for the wave function $\Psi_\epsilon(\boldsymbol{r},t) = \exp(-i\epsilon t/\hbar)\ \Phi_\epsilon(\boldsymbol{r},t)$ of an electron in the field of potential $U(\boldsymbol{r})$ reads as

$$\Phi_\epsilon(\boldsymbol{r},t) = \int_{-\infty}^{t} dt' e^{i\epsilon(t-t')/\hbar} \int d\boldsymbol{r}' G(\boldsymbol{r},t;\boldsymbol{r}',t')U(\boldsymbol{r}')\Phi_\epsilon(\boldsymbol{r}',t'), \tag{3}$$

where ϵ is the complex quasienergy and $G(\boldsymbol{r},t;\boldsymbol{r}',t')$ is the retarded Green's function which obeys the equation

$$\left(i\hbar\frac{\partial}{\partial t} - \frac{1}{2m}\left\{-i\hbar\nabla + \frac{e}{c}[\boldsymbol{A}(t) + \boldsymbol{A}_H]\right\}^2\right)G(\boldsymbol{r},t;\boldsymbol{r}',t') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'). \tag{4}$$

In Eq. (4), $\boldsymbol{A}(t)$ and $\boldsymbol{A}_H$ are the vector potentials of the laser field and the constant magnetic field $\boldsymbol{H}$. Here and below, we neglect the electron spin and indicate explicitly that the electron charge is negative. The retarded Green's function $G(\boldsymbol{r},t;\boldsymbol{r}',t')$ can be represented in the form (see Appendix A)

$$G(\boldsymbol{r},t;\boldsymbol{r}',t') = -\frac{i}{\hbar}\theta(t-t')\left(\frac{m}{2\pi i\hbar}\right)^{3/2}\frac{1}{\sqrt{(t-t')}} \times \frac{\omega_H}{2\sin\frac{\omega_H(t-t')}{2}}\ \exp\left\{\frac{i}{\hbar}S(\boldsymbol{r},t;\boldsymbol{r}',t')\right\}, \tag{5}$$

where $\theta(t-t')$ is the Heaviside function (step function), $\omega_H = |e|H/mc$ is the cyclotron frequency, and $S(\boldsymbol{r},t;\boldsymbol{r}',t')$ is the classical action (A19) for an electron moving in the net electromagnetic field of the above-mentioned configuration.

For calculating the decay rate we use the QQES formalism [31,32] (see also Refs. [62,64–68]). In this approach, the real and imaginary parts of the complex quasienergy ϵ determine the position and the total decay rate of the bound level. In the zero-range potential model, the wave function $\Phi_\epsilon(\boldsymbol{r},t)$ from Eq. (3), for the electron in s state, satisfies the boundary condition at $r \to 0$ [16,17,31,32]:

$$\Phi_\epsilon(\boldsymbol{r},t) \simeq \left(\frac{1}{r} - \frac{1}{a} - 2i\frac{e}{\hbar c}\frac{\boldsymbol{r}\boldsymbol{A}_f}{r}\right)f_\epsilon(t) + O(r), \quad f_\epsilon(t) = f_\epsilon\left(t + \frac{2\pi}{\omega}\right) \tag{6}$$

and the following relation:

$$\mathrm{U}(\boldsymbol{r})\Phi_\epsilon(\boldsymbol{r},t) = -2\pi\delta(\boldsymbol{r})f_\epsilon(t), \tag{7}$$

where $a = \hbar/\sqrt{2m|E_0|}$ is the so-called scattering length, E_0 is the energy of an electron bound by the zero-range potential alone, $\boldsymbol{A}_f$ is the sum of the vector potentials of laser and constant magnetic fields, and $f_\epsilon(t)$ is the new unknown function. Using Eqs. (6) and (7), we get from Eq. (3) the integral equation for the unknown complex quasienergy $\epsilon = E - i\Gamma$ and the function $f_\epsilon(t)$:

$$(\sqrt{-\epsilon} - \sqrt{|E_0|})f_\epsilon(t) = \frac{1}{2}\sqrt{\frac{\hbar}{\pi i}}\int_0^\infty dt' \exp\left(\frac{i}{\hbar}\epsilon t'\right) \times \left\{\mathcal{G}(\boldsymbol{0},t;\boldsymbol{0}',t-t')f_\epsilon(t-t') - \frac{f_\epsilon(t)}{t'^{3/2}}\right\}, \tag{8}$$

where the function $\mathcal{G}(\boldsymbol{0},t;\boldsymbol{0}',t-t')$ is given by Eq. (A18) in the limit $\boldsymbol{r} \to \boldsymbol{0}$ and $\boldsymbol{r}' \to \boldsymbol{0}$. This equation has not only the solution $\{\epsilon, f_\epsilon(t)\}$, but also the set of solutions

$$\{\epsilon + \hbar l\omega, f_\epsilon(t)\exp(il\omega t)\}, \quad l = \pm 1, \pm 2, \ldots \tag{9}$$

corresponding quasienergies. Equation (8) is the starting point for the two basic analytical approaches to the determination of ionization rates. In the next section, we consider each of these two approaches in details for the case of an elliptically polarized laser beam propagating at an arbitrary angle to the constant magnetic field.

III. ANALYTICAL RESULTS FOR THE CASE OF AN ELLIPTICALLY POLARIZED LASER BEAM PROPAGATING AT AN ARBITRARY ANGLE TO THE CONSTANT MAGNETIC FIELD

Now, consider an elliptically polarized monochromatic laser beam, of frequency ω, which propagates at an angle θ to the constant magnetic field $\boldsymbol{H}$ (we choose the direction of $\boldsymbol{H}$ as the z axis). The components of its vector potential are

$$A_x(t) = -\frac{F}{\omega}\cos(\theta)\sin\omega t, \quad A_y(t) = g\frac{F}{\omega}\cos\omega t, \quad A_z(t) = -\frac{F}{\omega}\sin(\theta)\sin\omega t, \tag{10}$$

where F is the laser field amplitude and g is the ellipticity of the laser field ($-1 \leqslant g \leqslant +1$).

A. Energy spectrum of photoelectrons

Within the framework of the first approach (see Appendix B) from Eq. (8) we arrive at the expression for the ionization rate Γ in the form of the standard PPT theory [69]:

$$\Gamma \simeq \sum_{n=0}^{\infty} \sum_{m>m_n}^{\infty} \Gamma_m^{(n)}(F,H,\omega),$$
$$m_n = \omega_0\left(n+\frac{1}{2}\right) + \lambda\left(1+\frac{1+g^2}{2\gamma^2}\right), \tag{11}$$

where $\omega_0 = \omega_H/\omega$ is the ratio of the cyclotron frequency to the laser frequency, $\lambda = |E_0|/\hbar\omega$ is the multiquantum parameter, m_n is the photoionization threshold, and

$$\Gamma_m^{(n)}(F,H,\omega) = \frac{|E_0|}{2}\left(\frac{\hbar}{2\pi}\right)^{3/2} \int_{-\infty}^{\infty} \frac{L_z}{2\pi\hbar} dp_z \times \int_{-\infty}^{\infty} \frac{L_x}{2\pi\hbar} dp_x \left|F_m^{(n)}(p_x,p_z)\right|^2 \times \delta\left[E_n + \frac{\kappa^2}{4\gamma^2}(1+g^2) + |E_0| - m\hbar\omega\right] \tag{12}$$

are the ionization probabilities for the Landau energy level with the number n with the absorption of m photons. At the same time, we note that the functions $F_m^{(n)}(p_x,p_z)$ describe the probability of each multiphoton process and can be represented as the product of an oscillating function and $\exp\{-\lambda F_q(v)\}$ ($v = \mathrm{Im}\, t$), where

$$F_q(v) = \left[1 + 2\varepsilon_n + \frac{1-\omega_0^2+\mathcal{G}^2(g,\omega_0,\theta)}{2\gamma^2(1-\omega_0^2)}\right]v - \frac{(1-\omega_0^2)^2 - (1+\omega_0^2)\mathcal{G}^2(g,\omega_0,\theta)}{4\gamma^2(1-\omega_0^2)^2} \times \sinh(2v) - \frac{\omega_0\mathcal{G}^2(g,\omega_0,\theta)}{\gamma^2(1-\omega_0^2)^2}\sinh^2 v \tag{13}$$

and $2\varepsilon_n = k_z^2 + (n+\frac{1}{2})/\lambda_H$, $k_z = p_z/\kappa$, $\lambda_H = |E_0|/\hbar\omega_H$, and $\mathcal{G}(g,\omega_0,\theta) = g + \omega_0\cos(\theta)$. The maximum of this function is defined by the condition $F_q'(v_0) = 0$, i.e., by the saddle-point equation

$$\left[1 - \frac{1+\omega_0^2}{(1-\omega_0^2)^2}\mathcal{G}^2(g,\omega_0,\theta)\right]\sinh^2(v_0) - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{(1-\omega_0^2)^2} \times [1-\omega_0\sinh(2v_0)] = (1+2\varepsilon_n)\gamma^2. \tag{14}$$

Let us consider some special cases. In the case of a linearly polarized laser field ($g = 0$), in the tunneling regime ($\gamma \ll 1$) for small magnetic fields ($\omega_0 \ll 1$), the saddle point reads as

$$v_0 \simeq \gamma(1+2\varepsilon_n)\left[1 - \frac{\sqrt{1+2\varepsilon_n}}{6}\gamma^2 + \frac{3}{40}(1+2\varepsilon_n)^{3/2}\gamma^4\right] + \frac{\cos^2(\theta)}{2}\left[\frac{1}{\gamma\sqrt{1+2\varepsilon_n}} + \frac{\gamma}{2}\sqrt{1+2\varepsilon_n} - \frac{\gamma^3}{4}(1+2\varepsilon_n)^{3/2}\right] \times \omega_0^2. \tag{15}$$

The function $F_m^{(n)}(p_z) = (L_x/2\pi)\int_{-\infty}^{\infty}|F_m^{(n)}(p_x,p_z)|^2 dp_x$ gives the distribution over Landau energy levels. With taking into account Eq. (13) in the first order of ε_n we derive that

$$F_m^{(n)}(p_z) \simeq \frac{\omega_H}{L_z}\frac{\gamma}{\lambda}\exp\left\{-\frac{2F_0}{3F}f_q(\gamma,\theta,\omega_0)\right\}\exp\left\{-\frac{F_0}{F}c_0\varepsilon_n\right\} \times [e^{\beta_0}L_n(-\beta_0) + \cos(\phi_0)], \tag{16}$$

where F_0 is the magnitude of the inner-atomic field, $L_n(x)$ are the Laguerre polynomials, and

$$f_q(\gamma,\theta,\omega_0) \simeq 1 - \frac{\gamma^2}{10} + \frac{9}{280}\gamma^4 + \frac{3}{2}\frac{\cos^2(\theta)}{\gamma^2}\left(1+\frac{\gamma^2}{6}-\frac{\gamma^4}{40}\right)\omega_0^2,$$
$$c_0 \simeq 1 - \frac{\gamma^2}{6} + \frac{3}{40}\gamma^4 + \frac{\cos^2(\theta)}{2\gamma^2}\left(1+\frac{\gamma^2}{2}-\frac{\gamma^4}{8}\right)\omega_0^2,$$
$$\beta_0 = 2\lambda\cos^2(\theta)\omega_0^3,$$
$$\phi_0 \simeq \frac{2k_z}{\gamma}\sin(\theta)[2+\cos^2(\theta)\omega_0^2] - \pi k_z^2. \tag{17}$$

Note that the function $F_q(v)$ in Eq. (13) and the saddle-point equation (14) depend on the ellipticity g and the angle θ only via the factor $\mathcal{G}(g,\omega_0,\theta)$. The case $\mathcal{G}(g,\omega_0,\theta) = 0$ is of special interest. The condition $g + \omega_0\cos(\theta) = 0$ is satisfied for a linearly polarized laser beam ($g = 0$) propagating perpendicularly ($\theta = \pi/2$) to the constant magnetic field $\boldsymbol{H}$. Another more interesting case, where $\mathcal{G}(g,\omega_0,\theta) = 0$ for a left-elliptically polarized laser field ($g < 0$) if the inequality $|g| \leqslant \omega_H/\omega$ holds. In this case the saddle point for $\gamma \ll 1$ takes the simple form

$$v_0 \simeq \gamma(1+2\varepsilon_n)\left[1 - \frac{1+2\varepsilon_n}{6}\gamma^2 + \frac{3}{40}(1+2\varepsilon_n)^2\gamma^4\right] \tag{18}$$

and

$$F_m^{(n)}(p_z) \simeq \frac{\omega_H}{L_z}\frac{\gamma}{\lambda}\exp\left\{-\frac{2F_0}{3F}(1+2\varepsilon_n)^{3/2} \times \left[1 - \frac{\gamma^2}{10}(1+2\varepsilon_n) + \frac{9}{280}(1+2\varepsilon_n)^2\gamma^4\right]\right\}. \tag{19}$$

In the first order of ε_n, we have

$$F_m^{(n)}(p_z) \simeq \frac{\omega_H}{L_z}\frac{\gamma}{\lambda}\exp\left\{-\frac{2F_0}{3F}\left[1-\frac{\gamma^2}{10}+\frac{9}{280}\gamma^4\right]\right\} \times \exp\left\{-\frac{F_0}{F}\left(1-\frac{\gamma^2}{6}+\frac{3}{40}\gamma^4\right)\varepsilon_n\right\}. \tag{20}$$

Note that the exponential factor in Eq. (20) coincides with the corresponding expression in Eq. (36) in Ref. [69].

Let us consider the case of constant electric and magnetic fields. Then, Eq. (13) can be written as

$$F_q(v,\varphi) = \left[1 + 2\varepsilon_n + \frac{\sin^2(\varphi)}{\gamma_H^2}\right]v - \frac{v^2}{\gamma_H^2}\sin^2(\varphi) - \frac{v^3}{3\gamma_H^2}\cos^2(\varphi), \tag{21}$$

where $\varphi = \pi/2 - \theta$ is the angle between the constant electric and magnetic fields and $\gamma_H = \sqrt{2|E_0|}\omega_H/F$ is the magnetic Keldysh parameter which is the ratio of the cyclotron frequency to the inverse tunneling time. The Keldysh parameter γ_H also can be represented as the ratio $2b_0/R_L$, where $b_0 = F_0^{2/3}/(2F)$ is the width of the barrier in the adiabatic limit and $R_L = F_0^{1/3}/H$ is the Larmor radius. In turn, the saddle-point equation (14) takes the form

$$(1-v_0)^2\sin^2(\varphi) = -(1+2\varepsilon_n)\gamma_H^2 + v_0^2 \tag{22}$$

and can be solved exactly

$$v_0 = -\tan^2(\varphi) + \sec(\varphi)\sqrt{(1+2\varepsilon_n)\gamma_H^2 + \tan^2(\varphi)}. \tag{23}$$

For large magnetic fields, i.e., for $\gamma_H \gg 1$, and at $\varphi \neq \pi/2$ we have

$$F_m^{(n)}(p_z) \simeq \frac{\omega}{L_z}\frac{\gamma_H}{\lambda_H\cos(\varphi)}\exp\left\{-\frac{2F_0}{3F}f_s(\gamma_H,\varphi)\right\} \times [e^{\beta_s}L_n(-\beta_s)+\cos(\phi_s)], \tag{24}$$

where

$$f_s(\gamma_H,\varphi) \simeq \frac{(1+2\varepsilon_n)^{3/2}}{\cos(\varphi)} - \frac{3\tan^2(\varphi)}{2\gamma_H}\left[1+2\varepsilon_n - \frac{\sqrt{1+2\varepsilon_n}}{\gamma_H\cos(\varphi)}\right]. \tag{25}$$

In the first order of ε_n we obtain

$$F_m^{(n)}(p_z) \simeq \frac{\omega}{L_z}\frac{\gamma_H}{\lambda_H\cos(\varphi)}\exp\left\{-\frac{2F_0}{3F}f_s^{(0)}(\gamma_H,\varphi)\right\} \times \exp\left\{-\frac{F_0}{F}c\varepsilon_n\right\}[e^{\beta_s}L_n(-\beta_s)+\cos(\phi_s)], \tag{26}$$

where

$$f_s^{(0)}(\gamma_H,\varphi) \simeq \frac{1}{\cos(\varphi)} - \frac{3\tan^2(\varphi)}{2\gamma_H}\left[1-\frac{1}{\gamma_H\cos(\varphi)}\right],$$
$$\beta_s \simeq 2\lambda_H\tan^2(\varphi),$$
$$c \simeq \frac{2}{\cos(\varphi)}\left[1-\frac{\tan(\varphi)}{\gamma_H}\sin(\varphi)+\frac{\tan^2(\varphi)}{2\gamma_H^2}\right],$$
$$\phi_s \simeq \frac{4k_z}{\gamma_H}\omega_0\cos(\varphi) - \pi k_z^2. \tag{27}$$

For crossed constant electric and magnetic fields, i.e., at $\varphi = \pi/2$, the saddle point for $\gamma_H \gg 1$ is

$$v_0 \simeq \frac{1+(1+2\varepsilon_n)\gamma_H^2}{2} \tag{28}$$

and

$$F_m^{(n)}(p_z) \simeq \frac{\omega}{L_z}\frac{\gamma_H^2}{\lambda_H}\exp\left\{-\frac{F_0}{4F}f_s^{(0)}(\gamma_H,\pi/2)\right\} \times \exp\left\{-\frac{F_0}{F}\tilde{c}\varepsilon_n\right\}[e^{\tilde{\beta}_s}L_n(-\tilde{\beta}_s)+\cos(\tilde{\phi}_s)], \tag{29}$$

where

$$f_s^{(0)}(\gamma_H,\pi/2) \simeq \gamma_H + \frac{1}{\gamma_H}\left(1+\frac{1}{\gamma_H^2}\right), \quad \tilde{c} \simeq \gamma_H + \frac{1}{\gamma_H},$$
$$\tilde{\beta}_s \simeq \frac{F_0}{4F}\gamma_H, \quad \tilde{\phi}_s \simeq \pi k_z^2. \tag{30}$$

Here and above, we use the atomic units $e = m = c = \hbar = 1$. Equations (16), (20), (26), and (29) give the distribution over the Landau energy levels, i.e., the energy spectrum of photoelectrons. The rapidly oscillating factors $\sim\cos(\phi)$ in these equations occur due to the contribution of the two saddle points at Re $t = 0$ and Re $t = \pi$ and describe the so-called tunneling interference [30,69]. According to Eqs. (16), (17), and (20), the width of photoelectrons energy spectrum in the tunneling regime is

$$\Delta\varepsilon_n \simeq \frac{F}{\sqrt{2|E_0|}} \tag{31}$$

and it coincides with the typical value of the transverse energy of photoelectrons in the quasiclassical case in the absence of a magnetic field [see Eq. (14) in Ref. [25]]. As can be seen from Eqs. (26) and (27), the width of photoelectrons energy spectrum in the case of constant electric and magnetic fields at $\varphi \neq \pi/2$ is

$$\Delta\varepsilon_n(\varphi) \simeq \frac{F\cos(\varphi)}{2\sqrt{2|E_0|}}. \tag{32}$$

At the same time, as follows from Eqs. (29) and (30), in the case of crossed constant electric and magnetic fields ($\varphi = \pi/2$) the typical value of the energy of photoelectrons for $\gamma_H \gg 1$ is the order of

$$\Delta\varepsilon_n(\pi/2) \simeq \frac{F}{\gamma_H\sqrt{2|E_0|}} \ll \Delta\varepsilon_n(\varphi), \tag{33}$$

i.e., it is small compared to the case $\varphi \neq \pi/2$. In other words, in the case of crossed electric and magnetic fields, the subbarrier trajectory of the electron is strongly "pressed" to the magnetic field. The energy spectrum of photoelectrons in the quasiclassical case is considered in the Supplemental Material [63] (see Sec. A).

Neglecting by the oscillating factor and integrating the expressions in Eqs. (16), (20), (26), and (29) over p_z and summing their over the Landau energy levels n and over all multiphotons processes (over m), we arrive from Eq. (11) at the total ionization rate for an s electron in the form

$$\Gamma \simeq P\frac{\exp\left\{-\frac{2F_0}{3F}f\right\}}{1-\exp(-2\gamma_H c)}\exp\left\{-c(2\gamma+\gamma_H) + \frac{\beta}{1-\exp(-2\gamma_H c)}\right\}\Phi\left[\exp(-2\gamma c),\frac{1}{2},1-\delta\right], \tag{34}$$

where P is the preexponential factor and f, c, and β are defined by Eqs. (17), (27), and (30) and

$$\Phi(z,s,v) = \sum_{m=0}^{\infty}\frac{z^m}{(v+m)^s} \tag{35}$$

is the generalized ζ function [70], $0 < \delta < 1$. In the following two subsections we consider analytical expressions for the total ionization rate in the tunneling and the multiphoton regimes.

B. Tunneling regime

Within the framework of the second approach, using the expressions for the classical action from Eqs. (A19) and (A20) far from the resonance $\omega = \omega_H$, we transform the integral over t in Eq. (B8) into one in the complex plane z. Applying the modified saddle-point method [71] we obtain the width of the bound s electron level:

$$\mathrm{Im}\beta = \frac{\sqrt{\pi}\omega_H}{2\sqrt{2z_0}|E_0|\sinh(\omega_0 z_0)}\mathrm{Re}\left\{\frac{[3\lambda F(z_0)]^{1/6}}{\sqrt{|F''(z_0)|}}\,Ai(\nu^2)\right\}, \tag{36}$$

where

$$Ai(\nu^2) = \frac{1}{2\pi i}\int_L \exp\left(\frac{u^3}{3} - \nu^2 u\right)du \tag{37}$$

is the Airy function depending on $\nu^2 = [3\lambda F(z_0)]^{2/3}$,

$$F(z_0) = \left[1 + \frac{1-\omega_0^2 + \mathcal{G}^2(g,\omega_0,\theta)}{2\gamma^2(1-\omega_0^2)}\right]z_0 - \frac{(1-\omega_0^2)^2 - (1+\omega_0^2)\mathcal{G}^2(g,\omega_0,\theta)}{4\gamma^2(1-\omega_0^2)^2}\sinh(2z_0) - \frac{\omega_0\mathcal{G}^2(g,\omega_0,\theta)}{\gamma^2(1-\omega_0^2)^2}\coth(\omega_0 z_0)\sinh^2 z_0 \tag{38}$$

and the saddle point z_0 is determined by the condition $F'(z_0) = 0$, i.e., by the equation

$$\sinh z_0\left\{1 - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{(1-\omega_0^2)^2}[\coth z_0 - \omega_0\coth(\omega_0 z_0)]^2\right\}^{1/2} = \gamma. \tag{39}$$

Note that the saddle point z_0 has a simple physical interpretation: $t_0 = -iz_0/\omega$ is the time of the subbarrier motion of the electron [72–74]. It should be noted that the function $F(z_0)$ in Eq. (38) differs from the function $F_q(v)$ in Eq. (13) by $\coth(\omega_0 z_0)$ in the last term, which arises due to the summation over Landau energy levels. At the same time, we note that the condition $\coth(\omega_0 z_0) \sim 1$ is equivalent to the limit of strong magnetic fields. Equation (36) with the Airy function can be used to calculate the ionization rate beyond the quasiclassical approach.

For further simplifications, consider the case of weak electric fields $F \ll F_0$ and use the well-known asymptotic of the Airy function [75] for large arguments ($\nu^2 \gg 1$) to obtain the quasiclassical ionization rate (B9). In the adiabatic limit, where the inequality $\gamma \ll 1$ holds, the saddle point z_0 in Eq. (39) can be represented in the following analytical form:

$$z_0 \simeq \gamma - \frac{\gamma^3}{6}\left[1 - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{3}\right] + \frac{3}{40}\left\{1 - \frac{58}{81}\left[1 + \frac{4}{29}\omega_0^2 - \frac{35}{174}\mathcal{G}^2(g,\omega_0,\theta)\right]\mathcal{G}^2(g,\omega_0,\theta)\right\}\gamma^5. \tag{40}$$

Then, from Eq. (B9) for the ionization rate we obtain

$$\Gamma \simeq |E_0|\frac{F}{2F_0}P(\gamma_H,\gamma,g,\theta)\exp\left\{-\frac{2F_0}{3F}f(\gamma,g,\omega_0,\theta)\right\}, \tag{41}$$

where the exponential factor is

$$f(\gamma,g,\omega_0,\theta) \simeq 1 - \frac{\gamma^2}{10}\left[1 - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{3}\right] + \frac{9}{280}\left\{1 - \frac{58}{81}\left[1 + \frac{4}{29}\omega_0^2 - \frac{35}{174}\mathcal{G}^2(g,\omega_0,\theta)\right]\mathcal{G}^2(g,\omega_0,\theta)\right\}\gamma^4 \tag{42}$$

and the preexponential factor reads as

$$P(\gamma_H,\gamma,g,\theta) \simeq 1 - \frac{\gamma_H^2}{6}\left(1 - \frac{7}{60}\gamma_H^2\right) + \frac{\gamma_H^2}{18}\left[1 - \frac{7}{30}\gamma_H^2 - \frac{7}{15}\left(1 - \frac{563}{1470}\gamma_H^2\right)\mathcal{G}^2(g,\omega_0,\theta)\right]\gamma^2 + \frac{1}{30}\left\{1 - \frac{19}{18}\gamma_H^2 - \frac{7}{9}\left[1 - \frac{2131}{1470}\gamma_H^2 - \frac{5}{21}\left(1 - \frac{49}{30}\gamma_H^2\right)\mathcal{G}^2(g,\omega_0,\theta)\right]\mathcal{G}^2(g,\omega_0,\theta)\right\}\gamma^4. \tag{43}$$

Equations (40)–(43) show that the ionization rate (41) depends on the ellipticity g and the angle θ only via the factor $\mathcal{G}(g,\omega_0,\theta) = g + \omega_0\cos(\theta)$. Note that, replacing in Eqs. (40) and (42) the factor $\mathcal{G}(g,\omega_0,\theta)$ to g and dropping the magnetic term $\sim 4/29\omega_0^2$, we obtain the well-known expansions for the case of an elliptically polarized electromagnetic wave in the tunneling limit in the absence of a magnetic field [see, e.g., Eq. (3.7) in Ref. [27]]. Since for an elliptical laser beam propagating perpendicularly ($\theta = \pi/2$) to the constant magnetic field $\boldsymbol{H}$ the factor $\mathcal{G}(g,\omega_0,\pi/2) = g$, the influence of the constant magnetic field in the tunneling limit on the saddle point and the exponential factor, in this case, is reduced to the term $\sim 4/29\omega_0^2$. In the next orders of $\gamma^{n>4}$ in the tunneling expansion of $f(\gamma,g,\omega_0,\theta)$ the constant magnetic field gives the terms $\sim\omega_0^{n>2}$. In contrast, the preexponential factor (43) strongly depends on the constant magnetic field. In particular, $P(\gamma_H,\gamma \to 0,g,\theta) \simeq 1 - \gamma_H^2(1 - 7\gamma_H^2/60)/6$ describes the diamagnetic shift in the ionization rate. Some limiting expressions of Eqs. (38)–(43) are considered in the Supplemental Material [63] (see Sec. B).

As follows from Eqs. (41) and (42), in the case of a low-frequency monochromatic laser field at $\theta \neq \pi/2$, due to the terms proportional to g, the constant magnetic field causes the reduction or enhancement of the ionization probability. This effect can be explained by the distortion of the subbarrier trajectory due to the screwlike electron motion. As a result, the subbarrier motion of the electron in the right-elliptically polarized laser field ($g > 0$) becomes longer and the ionization rate decreases. But since a left-elliptically polarized laser field ($g < 0$) rotates the electron against the constant magnetic field, the subbarrier electron trajectory becomes shorter and the ionization probability increases. Let us define the factor $R(\gamma,g,\theta)$ as follows: $R(\gamma,g,\theta) = \Gamma^-/\Gamma^+$. Here, Γ^- and Γ^+ are the ionization rates for a left- and right-elliptically polarized laser field, respectively. This factor shows the difference between the ionization probabilities for the cases of left- and right-polarized laser fields. In the adiabatic limit ($\gamma \ll 1$) for small magnetic fields ($\gamma_H \ll 1$) the factor $R(\gamma,g,\theta)$ can be represented in the form

$$R(\gamma,g,\theta) \simeq 1+|g|\frac{4F_0}{45F}\left[1+|g|\frac{2F_0}{45F}\cos(\theta)\gamma_H\gamma\right]\cos(\theta)\gamma_H\gamma + \frac{14}{135}|g|\left[1-\frac{10}{21}g^2-\frac{29F_0}{49F}\left(1-\frac{35}{87}g^2\right)\right] \times \cos(\theta)\gamma_H\gamma^3. \quad (44)$$

In Refs. [8,76] has been shown that during the tunneling under the influence of a circularly polarized laser field (in the absence of a constant magnetic field) for the electron with the angular momentum $m \neq 0$, $\Gamma^- > \Gamma^+$. This means that the counter-rotating electron with the angular momentum $m \neq 0$ has significantly better chances to penetrate the barrier than the corotating electron. As follows from Eq. (44), for small electric fields $F \ll F_0$, $R(\gamma,g,\theta) > 1$ even for s electrons ($m = 0$). But in the adiabatic limit for small magnetic fields ($\gamma_H \ll 1$) the factor $R(\gamma,g,\theta)$ weakly deviates from unity. However, in Sec. IV, we show that in the multiphoton regime the difference between Γ^- and Γ^+ may be significant.

Some results of numerical calculations for the ionization probability of the negative ion H^- (the weakly bound level with the binding energy $|E_0| = 0.7542$ eV) in the field of a CO_2 laser ($\omega \approx 0.117$ eV, $\lambda_0 \approx 10.6$ μm) are shown in Figs. 1 and 2. The ionization rates Γ_{ai} and Γ are calculated according to Eqs. (36) and (B9), respectively. As can be seen from Figs. 1 and 2, the ionization probabilities Γ_{ai} and Γ in the tunneling regime ($\gamma \ll 1$) weakly decrease with increasing the magnetic field H. The magnitude of the Keldysh parameter $\gamma = 0.0776$ corresponds to the boundary between the weak and strong electric fields: $F = F_0$. Figures 1 and 2 show that for $F = F_0$ and $0.5F_0$ the quasiclassical ionization probability Γ in Eq. (B9) calculated within the framework of the standard saddle-point method with a quadratic expansion is very close approximation to the ionization rate Γ_{ai} calculated beyond the quasiclassical picture according to Eq. (36).

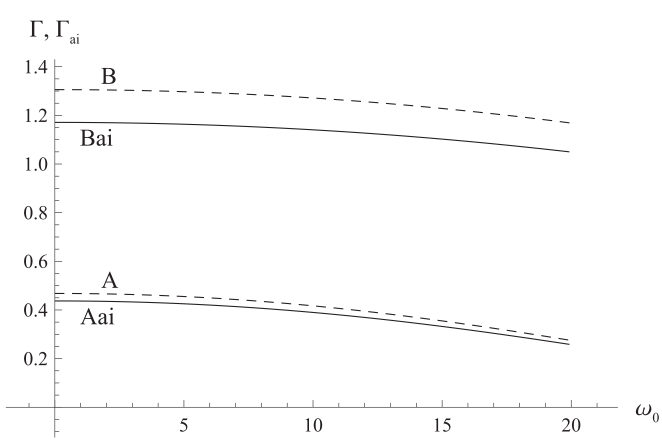

FIG. 1. Ionization rates Γ_{ai} [Eq. (36)] (solid line) and Γ [Eq. (B9)] (dashed line) vs ω_0 for the negative ion H^- at $\theta = 0$ and $g = -1$, $\lambda \approx 6.45$; $\gamma_A = 0.0776$ ($F = F_0$) and $\gamma_B = 0.0388$ ($F = 2F_0$). Arbitrary units are used for Γ_{ai} and Γ.

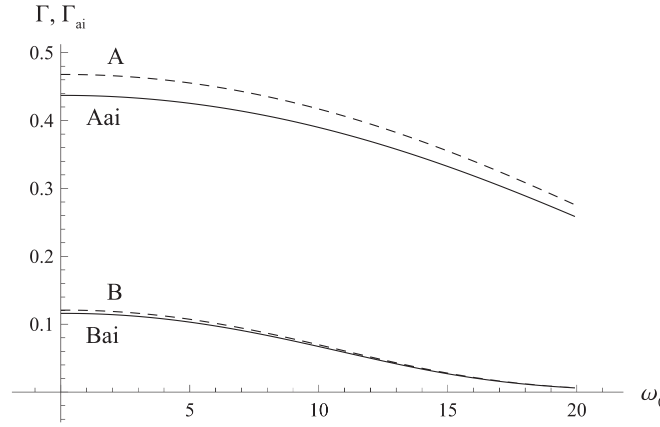

FIG. 2. Ionization rates Γ_{ai} [Eq. (36)] (solid line) and Γ [Eq. (B9)] (dashed line) vs ω_0 for the negative ion H^- at $\theta = \pi/4$ and $g = +1$, $\lambda \approx 6.45$; $\gamma_A = 0.0776$ ($F = F_0$) and $\gamma_B = 0.155$ ($F = 0.5F_0$). Arbitrary units are used for Γ_{ai} and Γ.

C. Multiphoton regime

In the multiphoton regime, i.e., for $\gamma \gg 1$ the ionization rate (B9) far from the resonance $\omega_0 = 1$ ($\omega = \omega_H$) can be represented in the form

$$\Gamma_M \simeq \frac{\omega_H \exp\{-2\lambda F_M(\gamma,g,\omega_0,\theta)\}}{4\sqrt{z_M(\gamma,g,\omega_0,\theta)}\sinh[\omega_0 z_M(\gamma,g,\omega_0,\theta)]}, \quad (45)$$

where

$$F_M(\gamma,g,\omega_0,\theta) \simeq \left[1+\frac{1+\mathcal{G}^2(g,\omega_0,\theta)\left(1-\omega_0^2\right)}{2\gamma^2}\right] \times z_M(\gamma,g,\omega_0,\theta) + \frac{S_M(g,\omega_0,\theta)}{2} \quad (46)$$

and

$$S_M(g,\omega_0,\theta) = -[1-\mathcal{G}^2(g,\omega_0,\theta)(1-\omega_0)^2]\mathcal{A}^{-1}, \quad \mathcal{A} = 1-\frac{\mathcal{G}^2(g,\omega_0,\theta)}{(1+\omega_0)^2}, \quad (47)$$

$z_M(\gamma,g,\omega_0,\theta) \simeq \ln(2\gamma/\sqrt{\mathcal{A}})$ is the saddle point in the multiphoton limit ($\gamma \gg 1$). Equations (45)–(47) are generally valid for $\omega_0 > 1$, but nonetheless they have the right limit at $\omega_0 \to 0$ ($H \to 0$). Indeed, from Eqs. (45)–(47), in this limit we obtain

$$\Gamma_{0,M} \simeq \frac{\omega}{4z_M^{3/2}(\gamma,g,0,0)}\exp\{-2\lambda F_M(\gamma,g,0,0)\}, \quad (48)$$

where

$$F_M(\gamma,g,0,0) \simeq \left(1+\frac{1-g^2}{2\gamma^2}\right)z_M(\gamma,g,0,0) - \frac{1}{2} \quad (49)$$

and $z_M(\gamma,g,0,0) \simeq \ln(2\gamma/\sqrt{1-g^2})$ is the saddle point for $\gamma \gg 1$ [under the condition $1-g^2 \gg 1/\ln(2\gamma)$] at $\omega_0 = 0$ [see, e.g., Eq. (3.8) in Ref. [27]]. The multiphoton ionization rate (48) differs from the corresponding result obtained in Ref. [77] [see Eq. (33a)] only by the factor $|C_{\kappa 0}|^2/\sqrt{2\pi\lambda}$. Equation (48) reproduces a power law typical of perturbation theory of order $N^0_{\min}$ in the multiphoton regime [25]

$$\Gamma_{0,M} \sim F^{2N^0_{\min}}, \quad (50)$$

where $N^0_{\min} = [\lambda] + 1$ is the minimal number of photons required for ionization in the absence of a magnetic field ($[\lambda]$ is the integer part of number λ). Note that Eqs. (46) and (47) are valid if $\mathcal{A} \neq 0$. The parameter $\mathcal{A} = 0$ only in the case of a right-circularly polarized laser beam ($g = +1$), propagating along ($\theta = 0$) the constant magnetic field $\boldsymbol{H}$. As was proved in Ref. [42] the saddle point for the case $g = +1$ at $\theta = 0$ and for $\omega_0 > 1$ exists only for $\gamma < \gamma_{cr}$, where $\gamma_{cr} = 1/\sqrt{\omega_0 - 1}$. For $\gamma \geqslant \gamma_{cr}$ the saddle-point equation (39) has no roots. This means that this case can not be considered within the framework of the quasiclassical approximation.

For large magnetic fields ($\omega_0 \gg 1$) at $\theta \neq 0,\pi/2$ the ionization rate in the multiphoton limit takes the form

$$\Gamma_M \simeq \frac{\omega_H}{2\sqrt{\ln[2\gamma\csc(\theta)]}}\left[\frac{\sin(\theta)}{2\gamma}\right]^{\omega_0+2\lambda} \times \exp\{-\lambda S_M(g,\omega_0 \gg 1,\theta)\}, \tag{51}$$

where

$$S_M(g,\omega_0 \gg 1,\theta) \simeq \cot^2(\theta)\left[\omega_0 + 4\frac{g-\cos(\theta)}{\sin(\theta)\sin(2\theta)}\right]\omega_0^3. \tag{52}$$

For an elliptical laser beam propagating perpendicularly to the constant magnetic field, i.e., at $\theta = \pi/2$ the ionization probability for $\omega_0 \gg 1$ is

$$\Gamma_M \simeq \frac{\omega_H}{2\sqrt{\ln(2\gamma)}}\frac{\exp\{-\lambda(g^2\omega_0^2 - 1)\}}{(2\gamma)^{\omega_0+2\lambda}}. \tag{53}$$

For an elliptically polarized laser beam ($g \neq \pm 1$) propagating along the constant magnetic field, i.e., at $\theta = 0$ (crossed electric and magnetic fields) the ionization rate for $\omega_0 \gg 1$ can be represented in the form

$$\Gamma_M \simeq \frac{\omega_H}{2\sqrt{z_M(\gamma,g,\omega_0,0)}}\left(\frac{1-g}{2\gamma\gamma_H}\right)^{\lambda+\omega_0/2} \times \exp\{-\lambda S_M(g,\omega_0 \gg 1,0)\}, \tag{54}$$

where

$$z_M(\gamma,g,\omega_0,0) \simeq \ln\left(\gamma\sqrt{\frac{2\omega_0}{1-g}}\right),$$

$$S_M(g,\omega_0 \gg 1,0) \simeq \frac{1}{2(1-g)}\left(\omega_0 + \frac{3g-1}{2}\right)\omega_0^4. \tag{55}$$

Note that Eqs. (51), (53), and (54) represent a power law of perturbation theory in the multiphoton regime in the presence of large magnetic fields:

$$\Gamma_M \sim F^{2N_{\min}} e^{-a\lambda\omega_0^n}, \tag{56}$$

where $N_{\min} = [\lambda + \omega_0/2] + 1$. Equation (54) for the multiphoton ionization rate is incorrect for the cases $g = \pm 1$ at $\theta = 0$ since these cases are the peculiar ones. As mentioned above, in the case of a right-circularly polarized laser beam ($g = +1$) propagating along ($\theta = 0$) the constant magnetic field, the saddle-point equation (39) in the multiphoton regime for $\omega_0 \gg 1$ has no roots. In contrast, in the case of a left-circularly polarized laser beam ($g = -1$) propagating along the constant magnetic field for $\omega_0 \gg 1$ the saddle point exists for each γ. However, due to the term $\sim\coth(\omega_0 z_0)$ in Eq. (39), analytical consideration of this case is rather difficult. Besides, here we note that the preexponential factors in the multiphoton regime in Eqs. (45)–(54) have only semiquantitative accuracy.

D. Limit of low frequencies $\omega_0 \gg 1$

Let us consider the case of constant electric and magnetic fields with a small frequency-dependent correction due to the laser field. Then, the saddle-point equation (39) takes the form

$$(1 - x_0\coth x_0)^2\left\{(\sin(\varphi) + g\overline{\omega}_0)^2 + \left(2 + \frac{x_0^2}{3}\right)\sin^2(\varphi)\,\overline{\omega}_0^2\right\} = -\gamma_H^2 + x_0^2\left\{1 + \frac{\overline{\omega}_0^2}{3}\left[x_0^2 - 2\sin^2(\varphi)(1 - x_0\coth x_0)\right]\right\}, \tag{57}$$

where $x_0 = \omega_H t_0$, $\varphi = \pi/2 - \theta$ and $\overline{\omega}_0 = 1/\omega_0 = \omega/\omega_H \ll 1$. In the static limit $\omega \to 0\,(\overline{\omega}_0 \to 0)$ Eq. (57) gives the saddle-point equation (1.6) obtained in Ref. [39] for the case of constant electric and magnetic fields. The ionization probability in this limit can be represented in the form

$$\Gamma_s = P_s(x_0,\varphi)\ \exp[-2\lambda_H F_s(x_0,\varphi)], \tag{58}$$

where

$$F_s(x,\varphi) \simeq F_s^{(0)}(x,\varphi) + F_s^{(1)}(x,\varphi)\,\overline{\omega}_0 + F_s^{(2)}(x,\varphi)\overline{\omega}_0^2 + O\left(\overline{\omega}_0^3\right) \tag{59}$$

and

$$F_s^{(0)}(x,\varphi) = \left[1 + \frac{\sin^2(\varphi)}{\gamma_H^2}\right]x - \frac{x^3}{3\gamma_H^2}\cos^2(\varphi) - \sin^2(\varphi)\frac{x^2}{\gamma_H^2}\coth x, \quad F_s^{(1)}(x,\varphi) = 2g\sin(\varphi)\frac{x}{\gamma_H^2}\left(1 + \frac{x^2}{3} - x\coth x\right),$$
$$F_s^{(2)}(x,\varphi) = \frac{x}{\gamma_H^2}\left\{g^2 + (2 + x^2)\sin^2(\varphi) + \frac{x^2}{3}\left[g^2 - \frac{x^2}{5}\cos^2(\varphi)\right] - \left[g^2 + \left(2 + \frac{x^2}{3}\right)\sin^2(\varphi)\right]x\coth x\right\}. \tag{60}$$

In the adiabatic limit, i.e., for small magnetic fields ($\gamma_H \ll 1$) for the ionization rate we obtain

$$\Gamma_s \simeq |E_0|\frac{F}{2F_0}P_s(\gamma_H,g,\varphi,\gamma)\exp\left\{-\frac{2F_0}{3F}f_s(\gamma_H,g,\varphi,\overline{\omega}_0)\right\}, \tag{61}$$

where

$$f_s(\gamma_H,g,\varphi,\overline{\omega}_0) \simeq 1 + \frac{\gamma_H^2}{30}\sin^2(\varphi)\left(1 - \frac{2}{21}\left[1 - \frac{35}{24}\sin^2(\varphi)\right]\gamma_H^2 + \frac{1}{105}\left\{1 - \frac{35}{9}\sin^2(\varphi)\left[1 - \frac{55}{72}\sin^2(\varphi)\right]\right\}\gamma_H^4\right) + g\frac{\gamma_H^2}{15}\sin(\varphi)$$
$$\times\left\{1 - \frac{2}{21}\left[1 - \frac{35}{12}\sin^2(\varphi)\right]\gamma_H^2\right\}\overline{\omega}_0 - \frac{\gamma_H^2}{30}\left(3 - g^2 + \frac{29}{42}\gamma_H^2\left\{\sin^2(\varphi) + \frac{g^2}{29}[4 - 35\sin^2(\varphi)]\right\}\right)\overline{\omega}_0^2 \quad (62)$$

and the preexponential factor is given by Eq. (17) in the Supplemental Material [63]. In particular, in the case of a circularly polarized laser field propagating along ($\varphi = \pi/2$) the constant magnetic field (crossed constant electric and magnetic fields), from Eq. (62) and Eq. (17) in the Supplemental Material [63], we obtain

$$f_s(\gamma_H,g=\pm 1,\pi/2,\overline{\omega}_0) \simeq 1 + \frac{\gamma_H^2}{30}\left(1 + \frac{11}{252}\gamma_H^2 + \frac{53}{68040}\gamma_H^4\right) + g\frac{\gamma_H^2}{15}\left(1 + \frac{23}{126}\gamma_H^2\right)\overline{\omega}_0 - \frac{\gamma_H^2}{15}\left[1 - \frac{\gamma_H^2}{42}\right]\overline{\omega}_0^2,$$
$$P_s(\gamma_H,g=\pm 1,\pi/2,\gamma) \simeq 1 - \frac{\gamma_H^2}{6}\left[1 + \frac{1}{540}\gamma_H^2 + \frac{13}{14175}\gamma_H^4\right] - \frac{11}{405}g\gamma_H^3\left(1 + \frac{1741}{2310}\gamma_H^2\right)\gamma + \frac{11}{270}\gamma_H^2\left(1 - \frac{1471}{2310}\gamma_H^2\right)\gamma^2$$
$$- \frac{11}{405}g\gamma_H\left(1 - \frac{2963}{2310}\gamma_H^2\right)\gamma^3 + \frac{11}{810}\left(1 - \frac{1307}{2310}\gamma_H^2\right)\gamma^4. \quad (63)$$

In the static limit ($\gamma \to 0$), the exponential and preexponential factors in Eq. (63) coincide with Eqs. (2.3) and (2.4) in Ref. [39].

E. Limit of large magnetic fields $\gamma_H \gtrsim 1$

The limit $\gamma_H \gtrsim 1$ is analogous to the antiadiabatic approximation in the theory of multiphoton ionization. At $\varphi \neq \pi/2$, the saddle point can be represented in the form

$$x_0 \simeq x_0^+ + x_1\overline{\omega}_0 + x_2\overline{\omega}_0^2 + O(\overline{\omega}_0^3), \quad (64)$$

where

$$x_0^\pm = -\tan^2(\varphi) \pm \sec(\varphi)\sqrt{\gamma_H^2 + \tan^2(\varphi)}, \quad x_1 = 2g\tan(\varphi)\sec(\varphi)\frac{(1 - x_0^+)^2}{x_0^+ - x_0^-},$$
$$x_2 = \frac{\sec^2(\varphi)}{x_0^+ - x_0^-}\left[g^2 + 2\sin^2(\varphi) - x_0^+\left(2g^2 + 4\sin^2(\varphi) - x_0^+\left\{g^2 + 3\sin^2(\varphi) - \frac{1}{3}x_0^+[4\sin^2(\varphi) + x_0^+\cos^2(\varphi)]\right\}\right)\right.$$
$$\left. + x_1[4g\sin(\varphi)(x_0^+ - 1) - x_1\cos^2(\varphi)]\right]. \quad (65)$$

For large magnetic fields, i.e., for $\gamma_H \gg 1$ the saddle point reads as

$$x_0 \simeq \frac{\gamma_H}{\cos(\varphi)} - \tan^2(\varphi)\left[1 - \frac{1}{2\gamma_H\cos(\varphi)}\right] + g\frac{\tan(\varphi)}{\cos^2(\varphi)}\left[1 - \frac{2}{\gamma_H\cos(\varphi)} + \frac{5 - \cos(2\varphi)}{4\gamma_H^2\cos^2(\varphi)}\right]\gamma$$
$$- \frac{\gamma_H}{4\cos^3(\varphi)}\left(\frac{2}{3} - \frac{1}{\gamma_H^2\cos^2(\varphi)}\{[4 + \cos(2\varphi)]\sin^2(\varphi) + 2g^2[2 - \cos(2\varphi)]\}\right)\gamma^2, \quad (66)$$

which in the limit $\gamma \to 0$ coincides with Eq. (B6) in Ref. [39]. Then, the ionization rate in this limit is given by

$$\Gamma_s \simeq |E_0|\frac{F}{2F_0}P_s(\gamma_H,g,\varphi,\gamma)\,\exp\left\{-\frac{2F_0}{3F}f_s(\gamma_H,g,\varphi,\gamma)\right\} \quad (67)$$

and for $\gamma_H \gtrsim 1$ the function

$$f_s(\gamma_H,g,\varphi,\gamma) \simeq \frac{x_0^+}{\gamma_H}\left\{1 + \frac{1}{\gamma_H^2}\left[(1 - x_0^+)\sin^2(\varphi) - \frac{x_0^{+2}}{3}\cos^2(\varphi)\right]\right\} + O(\gamma) \quad (68)$$

coincides with the function $F_q(v,\varphi)$ in Eq. (21) (at $\varepsilon_n = 0$) and with the corresponding result obtained in Ref. [39] [see Eq. (3.3)]. In the limit $\gamma_H \gg 1$,

$$f_s(\gamma_H,g,\varphi,\gamma) \simeq \frac{1}{\cos(\varphi)} - \frac{3\tan^2(\varphi)}{2\gamma_H}\left[1 - \frac{1}{\gamma_H\cos(\varphi)}\right] + \frac{g}{\gamma_H}\frac{\tan(\varphi)}{\cos^2(\varphi)}\left[1 - \frac{3}{\gamma_H\cos(\varphi)}\right]\gamma$$
$$- \frac{1}{10\cos^3(\varphi)}\left(1 - \frac{5}{2\gamma_H^2\cos^2(\varphi)}\{[4 + \cos(2\varphi)]\sin^2(\varphi) + 2g^2[2 - \cos(2\varphi)]\}\right)\gamma^2, \quad (69)$$

which in the static limit ($\gamma \to 0$) coincides with the function $f_s^{(0)}(\gamma_H,\varphi)$ in Eq. (27) and with Eq. (B7) in Ref. [39]. In turn, the preexponential factor is

$$P_s(\gamma_H,g,\varphi,\gamma) \simeq \left(2\gamma_H + \frac{\tan(\varphi)}{\cos(\varphi)}\left\{1 - \frac{2g}{\cos(\varphi)}\left[\gamma_H - \frac{7+\cos(2\varphi)}{4\cos(\varphi)}\right]\gamma\right\} + \frac{\gamma_H}{\cos^3(\varphi)}\left[3\gamma_H - \frac{1+3\cos(2\varphi)}{12\cos(\varphi)} + g^2\frac{\tan^2(\varphi)}{\cos(\varphi)}\right]\gamma^2\right)e^{-x_0^+}, \tag{70}$$

which in the static limit gives Eq. (B8) in Ref. [39]. In particular, in the case of crossed constant electric and magnetic fields ($\varphi = \pi/2$), the saddle point in Eq. (57) reads as

$$x_0 \simeq \frac{1+\gamma_H^2}{2} + \frac{g}{4}\gamma_H^3\left[1 - \frac{1}{\gamma_H^2}\left(2 - \frac{1}{\gamma_H^2}\right)\right]\gamma + \frac{\gamma_H^4}{4}\left[g^2 - \frac{1}{3} + \frac{1}{2\gamma_H^2}\left(1 - 5g^2 + 4\frac{g^2-1}{\gamma_H^2}\right)\right]\gamma^2. \tag{71}$$

Then, the ionization rate for large magnetic fields ($\gamma_H \gg 1$) at $\varphi = \pi/2$ can be represented in the form

$$\Gamma_s \simeq |E_0|\frac{F}{2F_0}P_s(\gamma_H,g,\pi/2,\gamma)\ \exp\left\{-\frac{F_0}{4F}f_s(\gamma_H,g,\pi/2,\gamma)\right\}, \tag{72}$$

where the exponential and preexponential factors are given by

$$f_s(\gamma_H,g,\pi/2,\gamma) \simeq \gamma_H + \frac{1}{\gamma_H}\left(2 + \frac{1}{\gamma_H^2}\right) + g\gamma_H^2\left[\frac{1}{3} - \frac{1}{\gamma_H^2}\left(1 - \frac{1}{\gamma_H^2}\right)\right]\gamma + \frac{\gamma_H^3}{4}\left[g^2 - \frac{1}{3} + \frac{2}{3\gamma_H^2}(1-5g^2) + \frac{4}{\gamma_H^4}(g^2-1)\right]\gamma^2,$$

$$P_s(\gamma_H,g,\pi/2,\gamma) \simeq \left\{2\gamma_H - \frac{1}{\gamma_H} - \frac{\gamma_H^3}{2}g\left(1 - \frac{7}{2\gamma_H^2}\right)\gamma + \frac{\gamma_H^6}{2}\left[\frac{g^2}{8} + \frac{1}{\gamma_H^2}\left(\frac{1}{3} - \frac{29}{16}g^2\right)\right]\gamma^2\right\}e^{-x_0}. \tag{73}$$

In the static limit, the exponential and preexponential factors in Eq. (73) coincide with Eqs. (2.5) and (2.6) obtained in Ref. [39] within the framework of the "imaginary-time" method for the case of crossed constant electric and magnetic fields. Equations (61) and (67) for ionization rates are valid for an s electron bound in a zero-range potential subjected to an influence of a strong constant magnetic field and additionally they take into account the frequency-dependent corrections due to an elliptical laser field.

IV. COULOMB CORRECTION AND METHOD OF IMAGINARY TIME

A. Coulomb correction in the tunneling regime

The zero-range potential is an adequate model for negative ions H^-, Na^-, etc. In this model, a deep potential well of a small radius is replaced by a boundary condition at the point of the center of the well. But, in the case of ionization of neutral atoms and positive ions must be taken into account the long-range Coulomb interaction of the ejected electron with the parent ion. For this purpose, in Refs. [76,78–82] were proposed the so-called eikonal-like approximation and the R-matrix method. The influence of the Coulomb potential of the atomic core on the ionized electron dynamics in the continuum on the basis a fully quantum mechanical analysis also was taken into account in Refs. [83–89].

To account for the effect of the Coulomb force on the photoelectron we can employ the quasiclassical perturbation theory and the method of imaginary time [90,91]. Being a generalization of the well-known method of complex classical trajectories considered by Landau [75,92], to the case of time-dependent fields, the method of imaginary time describes the tunneling transition of an electron from a bound state to the continuum using the classical equations of motion but with an imaginary "time." In Ref. [93], within the framework of the method of imaginary time, was calculated the Coulomb correction to the atomic ionization rate in the tunneling limit $\gamma \ll 1$ in the case of an elliptical wave. The imaginary part of the abbreviated classical action S calculated along such trajectories determines within the exponential accuracy the transition probability of an electron from the bound state with the energy E_0 to the continuum:

$$\Gamma \propto \exp\left\{-\frac{2}{\hbar}\mathrm{Im}[S_0(t_0) + E_0 t_0]\right\} \tag{74}$$

with the minimum value of $\mathrm{Im}S_0$. In Eq. (74), t_0 is the initial time in the complex half-plane for the subbarrier motion which can be determined from the equation [35,69,77,91]

$$E(t_0) = E_0, \tag{75}$$

where $E(t) = -\partial S_0(t)/\partial t$. The quasiclassical approximation is applicable if both conditions $\lambda \gg 1$ and $(2/\hbar)\mathrm{Im}[S_0(t_0) + E_0 t_0] \gg 1$ are satisfied. The second inequality is equivalent to the condition that the electric field of the laser beam with the frequency ω is much smaller than the inner-atomic field. The abbreviated classical action in Eq. (74) for an electron moving under the influence of a laser field and a constant magnetic field reads as

$$S_0(t_0) = \int_{t_0}^{\infty}\left\{\frac{\boldsymbol{v}^2(t)}{2m} - \frac{e}{c}\boldsymbol{A}_H\boldsymbol{v}(t) - e\boldsymbol{E}(t)\boldsymbol{r}(t)\right\}dt - [\boldsymbol{v}(t)\boldsymbol{r}(t)]|_{t=t_0}^{t\to\infty}, \tag{76}$$

where $\boldsymbol{A}_H = (-Hy,0,0)$ is the vector potential of the constant magnetic field $\boldsymbol{H}$ in the Landau gauge (the z axis being taken in the direction of the magnetic field $\boldsymbol{H}$), $\boldsymbol{E}(t)$ is the electric field of an elliptically polarized monochromatic laser beam, of frequency ω, with the laser field amplitude F and the ellipticity g ($-1 \leqslant g \leqslant +1$) with the components

$$E_x(t) = F\cos(\theta)\cos\omega t,\quad E_y(t) = gF\sin\omega t,$$
$$E_z(t) = F\sin(\theta)\cos\omega t, \tag{77}$$

propagating at an angle θ to the constant magnetic field $\boldsymbol{H}$. Here and above, we neglect by an electron spin. The Coulomb interaction between an ejected electron and the atomic core can be taken into account assuming the classical action (76) calculated for a zero-range potential as the zeroth approximation and calculating the corrections to this action. We can consider the two types of such corrections. First of all, in the region $\kappa r \gg 1$ it is possible to take into account the influence of the Coulomb interaction within the perturbation theory which gives the first-order correction to the classical action (76):

$$\delta S_I = -\int_{t_0}^{0} \delta V[\boldsymbol{r}_0(t)]dt, \qquad (78)$$

where $\delta V[\boldsymbol{r}_0(t)] = -Z/|\boldsymbol{r}_0(t)|$ and $\boldsymbol{r}_0(t)$ is the electron trajectory unperturbed by the Coulomb interaction. We note that this formula can be used if the perturbation $\delta V[\boldsymbol{r}_0(t)]$ is small along the subbarrier trajectory $\boldsymbol{r}_0(t)$. In our case, the atomic potential δV has a singularity at $r = 0$ and hence the integral in Eq. (78) diverges logarithmically as $r \to 0$ (or $t \to t_0$) and requires regularization. In addition, the quasiclassical approach is still not valid in the region $\kappa r \lesssim 1$. In order to overcome this difficulty, the so-called matching procedure can be used (see details in Refs. [57,90,91,93,94]). With this aim one can introduce the matching point r_0 such that

$$1 \ll \kappa r_0 \ll \kappa b, \qquad (79)$$

where b is the "dynamic" width of the barrier [see Eqs. (45)–(48) in the Supplemental Material [63]]. At distances $r > r_0$, the Coulomb interaction weakly distorts the subbarrier trajectory of the photoelectron, so that Eq. (78) still works, while for $r < r_0$ the external fields can still be neglected and therefore necessary to make the result continuous with the wave function of the free atom:

$$\psi_0(r) \simeq \frac{C_\kappa}{\sqrt{\pi}} \kappa^{3/2}(\kappa r)^{\eta-1} e^{-\kappa r} \sim r^{-1} \times \exp\{-\mathrm{Im}[S_0(r) + \delta S_I(r)]\}, \quad r \gg \kappa^{-1} \qquad (80)$$

where C_κ is the asymptotic coefficient in the atomic wave function for the $1s$ state at large distances from a potential well. In particular, $C_\kappa = 1$ for the ground state of the H atom [35,77]. The arbitrary matching point r_0 drops out of the final answer and ultimately this approach gives the Coulomb factor $Q_I(z_0,\theta)$:

$$Q_I(z_0,\theta) = 2\lambda z_0 \exp\{J(z_0,\theta)\},$$
$$J(z_0,\theta) = \int_0^1 \left[\frac{\gamma z_0}{|\boldsymbol{r}_0([1-s]z_0)|} - \frac{1}{s}\right] ds \qquad (81)$$

in the expression for the ionization probability Γ_η for the $1s$ state:

$$\Gamma_\eta = 4|C_\kappa|^2 Q_I^{2\eta}(z_0,\theta)\Gamma, \qquad (82)$$

where Γ is the ionization rate of a weakly bound level (B9), the saddle point z_0 is determined by Eq. (39), and $\eta = Z(|E_0|/|E_H|)^{-1/2}$ is the Sommerfeld parameter ($Z = 1$ for neutral atoms and $Z = 0$ for negative ions, $|E_H| = 13.6$ eV). The parameter η is usually close to unity and, in particular, for the H atom, $\eta = 1$. Besides, in Eq. (81), $|\boldsymbol{r}_0(z)| = \sqrt{r_{0x}^2(z) + r_{0y}^2(z) + r_{0z}^2(z)}$, where $\boldsymbol{r}_0 = (r_{0x}, r_{0y}, r_{0z})$ is the subbarrier zero-order (unperturbed by the Coulomb interaction) trajectory (5) in the Supplemental Material [63] of the electron in the atom. For small electric fields $F \ll F_0$, the Coulomb correction in Eq. (81) increases the tunneling ionization probability by several orders of magnitude, which is an experimentally determined effect [95]. In the general case, the integral in Eq. (81) can be calculated only numerically.

The ionization rate (82) in the adiabatic limit can be represented in the form

$$\Gamma_\eta \simeq 4|C_\kappa|^2|E_0|\left(\frac{2F_0}{F}\right)^{2\eta-1} P_\eta(\gamma_H,\gamma,g,\theta) \times \exp\left\{-\frac{2F_0}{3F} f_\eta(\gamma,g,\omega_0,\theta)\right\}, \qquad (83)$$

where

$$f_\eta(\gamma,g,\omega_0,\theta) \simeq 1 - \frac{\gamma^2}{10}\left(1 - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{3} + 5\eta\frac{F}{F_0}\left\{1 + \left[\frac{\mathcal{G}(g,\omega_0,\theta)}{3} - g\right]\mathcal{G}(g,\omega_0,\theta)\right\}\right) + \left(\frac{9}{280}\left\{1 - \frac{58}{81}\left[1 + \frac{4}{29}\omega_0^2 - \frac{35}{174}\mathcal{G}^2(g,\omega_0,\theta)\right]\mathcal{G}^2(g,\omega_0,\theta)\right\} + \frac{11}{60}\eta\frac{F}{F_0}\left\{1 - \frac{28}{33}g\left(1 + \frac{2}{7}\omega_0^2\right)\mathcal{G}(g,\omega_0,\theta) - \frac{5}{22}\left[g^2 + \frac{1}{15}(16 - 9\omega_0^2)\right]\mathcal{G}^2(g,\omega_0,\theta) - \frac{5}{22}\left[\mathcal{G}(g,\omega_0,\theta) - \frac{10}{3}g\right]\mathcal{G}^3(g,\omega_0,\theta)\right\}\right)\gamma^4 \qquad (84)$$

and the preexponential factor is

$$P_\eta(\gamma_H,\gamma,g,\theta) \simeq 1 - \frac{\gamma_H^2}{6}\left(1 - \frac{7}{60}\gamma_H^2\right) - \frac{\gamma^2}{3}\left(\left[1 - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{3}\right]\eta - \frac{\gamma_H^2}{6}\left\{1 - \frac{7}{15}\mathcal{G}^2(g,\omega_0,\theta) + \left[1 - \frac{3}{5}\mathcal{G}^2(g,\omega_0,\theta)\right]\eta\right\} + \frac{7}{180}\left\{1 - \frac{563}{735}\mathcal{G}^2(g,\omega_0,\theta) + \frac{\eta}{2}\left[1 - \frac{5}{7}\mathcal{G}^2(g,\omega_0,\theta)\right]\right\}\gamma_H^4 + \frac{7}{8100}\eta\mathcal{G}^2(g,\omega_0,\theta)\gamma_H^6\right) + O(\gamma^4). \qquad (85)$$

At $\eta = 0$, the ionization rate (83) coincides with the ionization probability (41) for the case of a weakly bound level. As follows, from Eqs. (83) and (84) the ionization probability in the adiabatic limit with taking into account the Coulomb correction, as in the case of a weakly bound level, may increase with the magnetic field if a neutral atom or a positive ion subjected to a left-elliptically polarized laser field perturbation ($g < 0$) and the ionization rate decreases with the magnetic field in the case of a right-elliptically polarized laser field ($g > 0$). Equations (83)–(85) generalize the corresponding expressions for ionization rates obtained in Ref. [39] within the framework of the imaginary-time method for the case of constant electric and magnetic fields. Our result additionally takes into account the frequency-dependent corrections due to an elliptically polarized laser field and the Coulomb interaction of the photoelectron with the atomic or ionic core. In the limit $H \to 0$ ($\gamma_H \to 0$) from Eqs. (83) and (84) one can obtain the expression for the ionization rate obtained within the exponential accuracy in Ref. [96] [see Eq. (7.7)].

In the static limit ($\gamma \to 0$) for small magnetic fields, i.e., for $\gamma_H \ll 1$ we obtain

$$\Gamma_\eta \simeq 4|C_\kappa|^2|E_0|\left(\frac{2F_0}{F}\right)^{2\eta-1} P_\eta(0,\gamma_H,\varphi)\exp\left\{-\frac{2F_0}{3F} f_\eta(0,\gamma_H,\varphi)\right\}, \tag{86}$$

where

$$f_\eta(0,\gamma_H,\varphi) \simeq 1+\frac{\gamma_H^2}{30}\sin^2(\varphi)\Bigg(1-5\eta\frac{F}{F_0}-\frac{2}{21}\left\{1-\frac{35}{24}\sin^2(\varphi)-\frac{21F}{16F_0}\eta[1+5\cos(2\varphi)]\right\}\gamma_H^2 + \left[\frac{1}{105}\left(1-\frac{207F}{20F_0}\eta\right)-\frac{1}{27}\left(1-\frac{119F}{10F_0}\eta\right)\sin^2(\varphi)+\frac{55}{1944}\left(1-\frac{146F}{11F_0}\eta\right)\sin^4(\varphi)\right]\gamma_H^4\Bigg)+O\left(\gamma_H^8\right) \tag{87}$$

and

$$P_\eta(0,\gamma_H,\varphi) \simeq 1-\frac{\gamma_H^2}{6}\left[1-\frac{2}{3}\eta\sin^2(\varphi)\right]+\frac{7}{360}\left[1-\frac{4}{3}\left(1+\frac{9}{7}\eta\right)\sin^2(\varphi)+\frac{20}{63}(1+3\eta+\eta^2)\sin^4(\varphi)\right]\gamma_H^4 + \frac{563}{56700}\sin^2(\varphi)\left\{1-\frac{1715}{1689}\sin^2(\varphi)+\frac{525}{1126}\eta\left[1-\frac{10}{9}\sin^2(\varphi)\left(1+\frac{13}{25}\eta\right)\right]\right\}\gamma_H^6+O\left(\gamma_H^8\right). \tag{88}$$

The case of the H atom ($\eta = 1$) is of special interest. In view of Eqs. (86), (87), and (88), the ionization probability for the 1s state of the H atom is

$$\Gamma_H \simeq 8|E_H|\frac{F_H}{F} P_1(0,\gamma_H,\varphi)\exp\left\{-\frac{2F_H}{3F} f_1(0,\gamma_H,\varphi)\right\}, \tag{89}$$

where

$$P_1(0,\gamma_H,\varphi) \simeq 1-\frac{\gamma_H^2}{18}[2+\cos(2\varphi)]+\frac{7}{360}\left\{1-\frac{64}{21}\sin^2(\varphi)\left[1-\frac{25}{48}\sin^2(\varphi)\right]\right\}\gamma_H^4 + \frac{1}{113\,400}[1651-2030\sin^2(\varphi)]\sin^2(\varphi)\gamma_H^6 \tag{90}$$

and $f_1(0,\gamma_H,\varphi) = f_\eta(0,\gamma_H,\varphi)$ at $\eta = 1$, $F_H = 5.142\times 10^9$ V/cm is the magnitude of the electric field at the first Bohr orbit. As follows from Eqs. (87), (88), and (90) for parallel constant electric and magnetic fields, i.e., at $\varphi = 0$ the ionization rate weakly depends on the magnetic field via the diamagnetic shift in the preexponential factor

$$P_1(0,\gamma_H,0) \simeq 1-\frac{\gamma_H^2}{6}+\frac{7}{360}\gamma_H^4+O\left(\gamma_H^8\right). \tag{91}$$

For crossed constant electric and magnetic fields, i.e., at $\varphi = \pi/2$, the ionization rate for the 1s state of the H atom in the static limit is given by

$$\Gamma_H \simeq 8|E_H|\frac{F_H}{F} P_1(0,\gamma_H,\pi/2)\exp\left\{-\frac{2F_H}{3F} f_1(0,\gamma_H,\pi/2)\right\}, \tag{92}$$

where

$$f_1(0,\gamma_H,\pi/2) \simeq 1+\frac{\gamma_H^2}{30}\left\{1-5\frac{F}{F_H}+\frac{\gamma_H^2}{252}\left(11-126\frac{F}{F_H}\right)+\frac{\gamma_H^4}{340\,200}\left(265-11\,344\frac{F}{F_H}\right)\right\} \tag{93}$$

and the preexponential factor is

$$P_1(0,\gamma_H,\pi/2) \simeq 1-\frac{\gamma_H^2}{18}-\frac{29}{3240}\gamma_H^4-\frac{379}{113\,400}\gamma_H^6. \tag{94}$$

Neglecting in Eqs. (93) and (94) the terms $\sim\gamma_H^{n>2}$ we obtain that Eq. (92) coincides with the ionization rate (45) in Ref. [42] [at $\delta(F,H) = 0$]. From Eqs. (92)–(94) follows that in contrast to the case of parallel constant electric and magnetic fields, the ionization probability in the case of crossed fields strongly depends on the magnetic field. In the limit $H \to 0$ from Eqs. (92)–(94) we arrive at the well-known Landau-Lifshitz formula (see in Ref. [75]) for the ionization probability from the ground state of the H atom due to a constant electric field,

$$\Gamma_H = 8|E_H|\frac{F_H}{F}\exp\left\{-\frac{2F_H}{3F}\right\}. \tag{95}$$

In the case of constant electric and magnetic fields, the Coulomb factor in Eq. (81) can be represented in the form

$$Q_I(x_0,\varphi) = 2\lambda_H x_0 \exp\{J(x_0,\varphi)\}, \quad J(x_0,\varphi) = \int_0^1 \left[\frac{\gamma_H x_0}{|\boldsymbol{r}_0([1-s]x_0)|} - \frac{1}{s}\right] ds. \tag{96}$$

For large magnetic fields, i.e., for $\gamma_H \gg 1$ and at $\varphi \neq \pi/2$, the integral in Eq. (96) with taking into account Eq. (65) can be calculated analytically: $J(x_0,\varphi) \simeq \ln(2)$ and the Coulomb factor $Q_I(x_0,\varphi) \simeq 2F_0/F\cos(\varphi)$. Then, the ionization rate in the static limit with taking into account the Coulomb interaction, the photoelectron with the parent ion for large magnetic fields ($\gamma_H \gg 1$) at $\varphi \neq \pi/2$, in view of Eqs. (69) and (70), is given by

$$\Gamma_\eta \simeq 8|C_\kappa|^2|E_0|\left(\frac{2F_0}{F}\right)^{2\eta-1} \frac{\gamma_H e^{-\gamma_H/\cos(\varphi)}}{[\cos(\varphi)]^{2\eta}} \exp\left\{-\frac{2F_0}{3F\cos(\varphi)}\right\}, \tag{97}$$

or with taking into account the relation $\gamma_H = (F_0/F)(H/H_0)$ it can be rewritten as

$$\Gamma_\eta \simeq 8|C_\kappa|^2|E_0|\left(\frac{2F_0}{F}\right)^{2\eta-1} \frac{\gamma_H}{[\cos(\varphi)]^{2\eta}} \exp\left\{-\frac{F_0}{F\cos(\varphi)}\left(\frac{2}{3}+\frac{H}{H_0}\right)\right\}. \tag{98}$$

The expression for the ionization rate in Eq. (98) is consistent with the result obtained in Ref. [39] [see Eqs. (B7)–(B9)]. In the case of crossed constant electric and magnetic fields, i.e., at $\varphi = \pi/2$ the ionization rate in the static limit with the Coulomb correction for large magnetic fields, in view of Eq. (73), can be represented in the form

$$\Gamma_\eta \simeq 4|C_\kappa|^2|E_0|(2\lambda_H)^{2\eta-1} \exp\{2\eta J(\gamma_H)\} e^{-\gamma_H^2/2} \exp\left\{-\frac{F_0}{4F}\left(\gamma_H + \frac{2}{\gamma_H}\right)\right\}, \tag{99}$$

where $J(\gamma_H) \simeq 0.42545 + \gamma_H \arcsin(1 - 2/\gamma_H^2)$ [see Eq. (28) in the Supplemental Material [63]].

Let us define the "stabilization factor" $S(\gamma_H,\gamma,g,\theta)$: $S(\gamma_H,\gamma,g,\theta) = \Gamma_\eta/\Gamma_{\eta 0}$, $\Gamma_{\eta 0}$ being here the ionization probability for an elliptical laser field at $H = 0$. This factor determines the extent of the influence of a magnetic field on the ionization rate with taking into account of the Coulomb correction. In some simple cases, the factor $S(\gamma_H,\gamma,g,\theta)$ can be found analytically. In the adiabatic limit when the inequalities $\gamma \ll 1$, $\gamma_H \ll 1$ hold and at $\theta = \pi/2$, it can be written as

$$S(\gamma_H,\gamma,g,\pi/2) \simeq 1 - \frac{\gamma_H^2}{6} + \frac{7}{360}\gamma_H^4 + \frac{\gamma_H^2}{18}\left(\left[1 - \frac{7}{15}g^2\left(1 + \frac{\eta}{14} - \frac{4F_0}{49F}\right)\right]\gamma^2 \right.$$
$$\left. - \frac{8}{15}\left\{1 + \frac{g^2}{12}\left[\eta + \frac{g^2}{3}\left(11 - \eta - \frac{F_0}{F}\right) - \frac{1}{70}\left(943 - \frac{62F_0}{F}\right)\right]\right\}\gamma^4\right) \tag{100}$$

and for crossed electric and magnetic fields ($\theta = 0$) the factor $S(\gamma_H,\gamma,g,\theta)$ in this limit can be represented in the form

$$S(\gamma_H,\gamma,g,0) \simeq 1 - \frac{\gamma_H^2}{6}\left(1 - \frac{4}{3}\eta + \frac{2F_0}{15F}\right) + \frac{g}{9}\left(\eta - \frac{2F_0}{5F}\right)\gamma_H\gamma + \frac{4}{135}\gamma_H^2$$
$$\times\left(1 - 2\eta + \frac{29F_0}{56F} + \frac{3}{8}g^2\left\{1 - \frac{31F_0}{21F}\left(1 - \frac{28F_0}{465F}\right) + \frac{7}{3}\eta\left[1 + \frac{1}{21}\left(5\eta - \frac{4F_0}{F}\right)\right]\right\}\right)\gamma^2. \tag{101}$$

Note that in the static limit ($\gamma \to 0$), the expression in Eq. (100) coincides with the "stabilization factor" for parallel electric and magnetic fields obtained in Ref. [39] [see Eq. (1.19)].

B. Coulomb correction in the multiphoton regime

Another Coulomb correction appears due to distortion of the electron trajectory by the Coulomb field [33–35]. The trajectory of the electron in the Coulomb field can be represented in the form

$$\boldsymbol{r}(t) \simeq \boldsymbol{r}_0(t) + \boldsymbol{r}_1(t) + \cdots, \tag{102}$$

where $\boldsymbol{r}_0(t)$ is the trajectory disregarding the Coulomb interaction, $\boldsymbol{r}_1(t) \sim \sqrt{\mu}$, and $\mu = Z\omega/\kappa^3$ is the parameter determining the contribution of the Coulomb field to the distortion of the electron trajectory. Here, we note that the parameter μ is small since for example in the field of a titanium-sapphire laser ($\hbar\omega \approx 1.55$ eV) for the H atom $\mu = 0.057$, for the He atom $\mu = 0.023$, and for the ion Xe^+ $\mu = 0.059$ [33]. Hence, the effect of Coulomb interaction on the electron trajectory can be taken into account within the framework of a perturbation theory. If correction $\boldsymbol{r}_1(t)$ to the electron trajectory has been determined, then the corresponding correction to the classical action (76) is

$$\delta S_{II} = \int_{t_0}^{\infty}\left[\boldsymbol{v}_0\boldsymbol{v}_1 + \frac{1}{2}\boldsymbol{v}_1^2 - \boldsymbol{E}(t)\boldsymbol{r}_1 + y_0 H\boldsymbol{e}_x\boldsymbol{v}_1 + y_1 H\boldsymbol{e}_x\boldsymbol{v}_0\right]dt$$
$$- (\boldsymbol{v}_0\boldsymbol{r}_1 + \boldsymbol{v}_1\boldsymbol{r}_0 + \boldsymbol{v}_1\boldsymbol{r}_1)|_{t=t_0}^{t\to\infty}, \tag{103}$$

where $\boldsymbol{e}_x$ is the unit vector along the x axis. The trajectory of the electron in the field $V = -Z/r$ of the atomic core, the laser field (77), and the constant magnetic field $\boldsymbol{H}$ satisfy the equation

$$\ddot{\boldsymbol{r}} = -Z\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} - \boldsymbol{E}(t) - [\dot{\boldsymbol{r}},\boldsymbol{H}]. \tag{104}$$

The solution of Eq. (104) can be found using Kapitza's method (see Appendix C) and, as a result, the imaginary part of the classical action (103) can be represented in the form

$$\mathrm{Im}\delta S_{II} = \mu\frac{\kappa^2}{\omega}\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta)z_M(\gamma,g,\omega_0,\theta). \tag{105}$$

For small magnetic fields $[\omega_0 z_M(\gamma,g,\omega_0,\theta) \ll 1]$ and for $g^2 \neq 1$ we can put in Eq. (105)

$$z_M(\gamma,g,\omega_0,\theta) \simeq \ln\left(\frac{2\gamma}{\sqrt{1-g^2}}\right). \tag{106}$$

If for $\omega_0 \lesssim 1$ the inequality $\omega_0 z_M(\gamma,g,\omega_0,\theta) \gg 1$ holds, we have [see Eq. (47)]

$$z_M(\gamma,g,\omega_0,\theta) \simeq \ln\left(\frac{2\gamma}{\sqrt{\mathcal{A}}}\right), \quad \mathcal{A} = 1 - \frac{\mathcal{G}^2(g,\omega_0,\theta)}{(1+\omega_0)^2}. \tag{107}$$

For the case of a circular laser field ($g^2 = 1$) and for small magnetic fields $[\omega_0 z_M(\gamma,g=\pm 1,\omega_0,\theta) \ll 1]$, the saddle point in Eq. (105) is

$$z_M(\gamma,g=\pm 1,\omega_0,\theta) \simeq \ln\left[\gamma\sqrt{\frac{2\ln(\gamma)}{(\mathcal{B}_\pm/2)\ln(\gamma)+1-\mathcal{B}_\pm}}\right]. \tag{108}$$

Then, the Coulomb correction due to distortion of the electron trajectory by the Coulomb field, under the condition $\omega_0 z_M(\gamma,g,\omega_0,\theta) \ll 1$ for $g^2 \neq 1$, in view of Eqs. (74), (105), and (106), can be represented in the form

$$Q_{II}(z_0,\theta) = \exp\{-2\,\mathrm{Im}(\delta S_{II})\} \simeq (2\gamma/\sqrt{1-g^2})^{-2\eta\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta)}, \tag{109}$$

where $\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta)$ is given by Eqs. (C3), (C4), and (C5). If the opposite inequality $\omega_0 z_M(\gamma,g,\omega_0,\theta) \gg 1$ holds, the multiphoton Coulomb correction for $g^2 \neq 1$, in view of Eqs. (74), (105), and (107), reads as

$$Q_{II}(z_0,\theta) \simeq (2\gamma/\sqrt{\mathcal{A}})^{-2\eta\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta)}, \tag{110}$$

where $\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta)$ is given by Eqs. (C3), (C4), and (C6). For a circularly polarized laser field, the Coulomb correction in the multiphoton regime for small magnetic fields $[\omega_0 z_M(\gamma,g=\pm 1,\omega_0,\theta) \ll 1]$, in view of Eqs. (74), (105), and (108), is

$$Q_{II}(z_0,\theta) \simeq \left[\gamma\sqrt{\frac{2\ln(\gamma)}{(\mathcal{B}_\pm/2)\ln(\gamma)+1-\mathcal{B}_\pm}}\right]^{-2\eta\mathcal{J}(\boldsymbol{\xi},g=\pm 1,\omega_0,\theta)}, \tag{111}$$

where $\mathcal{J}(\boldsymbol{\xi},g=\pm 1,\omega_0,\theta)$ is given by Eqs. (C3), (C4), and (C7).

Thus, the ionization rate (82) now can be rewritten in the form

$$\Gamma_\eta = 4|C_\kappa|^2 Q_I^{2\eta}(z_0,\theta)Q_{II}(z_0,\theta)\Gamma, \tag{112}$$

which is valid both for the tunneling and for multiphoton ionization, i.e., for an arbitrary values of the Keldysh parameter γ. In Eq. (112), the Coulomb factor $Q_I(z_0,\theta)$ is given by Eq. (81) and the Coulomb factor $Q_{II}(z_0,\theta)$ for $\gamma \gg 1$ is defined by Eqs. (109)–(111). For $\gamma \ll 1$, we put that $Q_{II}(z_0,\theta) \to 1$. In other words, the Coulomb factor $Q_{II}(z_0,\theta)$ gives the contribution to the ionization probability only in the multiphoton regime. On the other hand, as follows from Eqs. (109)–(111), $Q_{II}(z_0,\theta) \ll 1$ for $\gamma \gg 1$, i.e., this Coulomb factor effectively decreases the ionization rate. Nonetheless, the total Coulomb correction in Eq. (112) remains numerically large. In the limit $H \to 0$ for a linearly polarized laser field ($g = 0$) in the multiphoton regime ($\gamma \gg 1$), from Eq. (C8), we have $\mathcal{J}(\boldsymbol{\xi},0,0,0) \approx 1$. Then, from Eq. (109), in this limit, we obtain

$$Q_{II}(z_0,\theta) \simeq (2\gamma)^{-2\eta}. \tag{113}$$

In Refs. [34,35] was also proposed the interpolation formula for the Coulomb factor $Q_{II}(z_0,\theta)$, for the case of a linearly polarized laser field ($g = 0$) in the absence of a magnetic field, which is valid in the intermediate region of the Keldysh parameter $\gamma \approx 1$:

$$Q_{II}(z_0,\theta) \simeq (1 + 2e^{-1}\gamma)^{-2\eta}. \tag{114}$$

Note that similar interpolation for Eqs. (109)–(111) when $g \neq 0$ and $\omega_0 \neq 0$ would be much less substantiated. The Coulomb factor (113) consistent with the results obtained in Refs. [34] [see Eq. (5)] and in [35] [see Eq. (23)].

The results of numerical calculations for the ionization probability Γ_η of the H atom (the binding energy $|E_0| = 13.6$ eV, $\mu = 0.057$) in the field of the second harmonic of a titanium-sapphire laser ($\hbar\omega \approx 2.94$ eV, $\lambda_0 \approx 422$ nm for intensity $J = 10^{13}$ W/cm^2), obtained according to Eqs. (B9), (82), and (112), are shown in Figs. 3 and 4. Besides, in these figures also are shown the numerical results which were obtained for the case of a linearly polarized laser field ($g = 0$) by two different methods [34]: the complex quasienergy (Floquet) method (dark circles) and the solution of the time-dependent Schrödinger equation (triangles) in the field of a short pulse with a duration of 10 field periods. As can be seen from Figs. 3 and 4, accounting for the Coulomb correction (81) leads to an increase in ionization rates (solid lines,

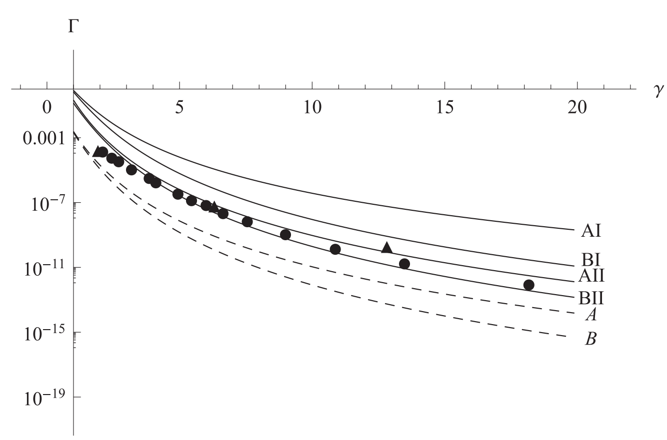

FIG. 3. Ionization rate Γ when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored [see Eq. (B9)] (dashed line) vs γ for the 1s state of the H atom ($\eta = 1$) in the field of titanium-sapphire laser at $\theta = 0$ and $\omega_0 = 0$, $\lambda \approx 4.63$; $g_A = 0$ and $g_B = \pm 1$. Numerical results for the ionization rate for the 1s state of the H atom in a linearly polarized laser field ($g = 0$), obtained by the Floquet method (filled circle) and by the solution of the time-dependent Schrödinger equation (filled triangle) [34]. Arbitrary units are used for Γ.

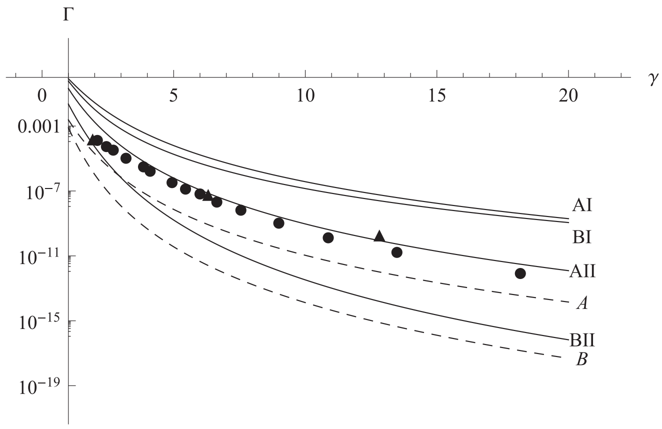

FIG. 4. Ionization rate Γ when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored [see Eq. (B9)] (dashed line) vs γ for the 1s state of the H atom ($\eta = 1$) in the field of titanium-sapphire laser at $\theta = 0$ and $g = 0$, $\lambda \approx 4.63$; $\omega_{0A} = 0$ and $\omega_{0B} = 2$. Numerical results for the ionization rate for the 1s state of the H atom in a linearly polarized laser field ($g = 0$), obtained by the Floquet method (filled circle) and by the solution of the time-dependent Schrödinger equation (filled triangle) [34]. Arbitrary units are used for Γ.

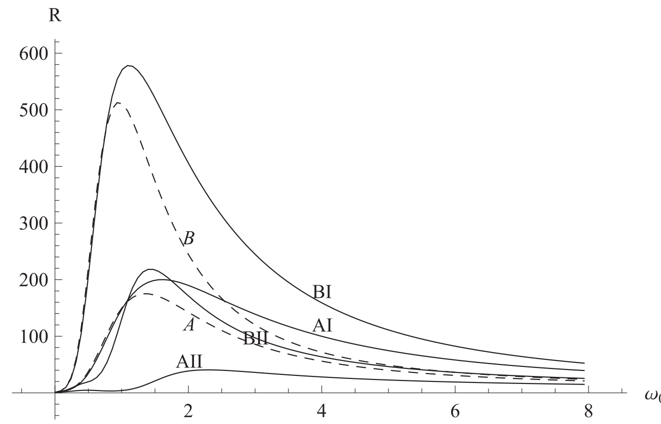

FIG. 5. The factor R when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the 1s state of the H atom ($\eta = 1$) at $\theta = \pi/3$ and $|g| = 1$, $\lambda \approx 8.774$; $\gamma_A = 5$ and $\gamma_B = 10$.

AI, BI) compared with the case of a weakly bound level (B9) with the same binding energy. Both ionization rates (solid line, AI and dashed line) significantly deviate from the numerical results. On the other hand, as can be seen from Figs. 3 and 4, the Coulomb factor $Q_{II}(z_0,\theta)$ in Eq. (112) effectively decreases ionization rates (solid lines, AII, BII) but they are still greater than ionization probabilities in the case of a zero-range potential. Figures 3 and 4 show, in general, a good, quantitative agreement of the analytical ionization rate (112) for $\omega_0 = 0$ and $g = 0$ (solid line, AII) with the numerical results. As can be seen from Fig. 3, the ionization rate (112) for a circular laser field (solid line, BII) less than the ionization rate for a linearly polarized laser field, but still close to the numerical results. On the the other hand, Fig. 4 shows that the magnetic field significantly reduces the ionization probability (solid line, BII) and as a result it significantly deviates from the numerical results, which were obtained in the absence of a magnetic field.

In Ref. [23] was investigated the interaction of a high-intensity x-ray laser radiation field with rare gas atoms. In particular, in the multiphoton regime were observed charge states of xenon up to Xe^{6+}. Comparison of the results for the ionization rate obtained according to Eq. (112) (for $\omega_0 = 0$ and $g = 0$) with the probabilities calculated from the results of Ref. [97], under the conditions of experiment [23], for ionization of xenon by the linearly polarized laser field ($g = 0$) with a photon energy of $\hbar\omega = 12.7$ eV and an intensity of 10^{13} W/cm^2 demonstrate satisfactory agreement. So, for Xe^{3+} ($\eta = 2.15, \gamma \simeq 51.36, \lambda \simeq 3.7, \omega_0 = 0$), Eq. (112) gives $\Gamma_\eta(g = 0) \simeq 10^{-9}$ a.u. and for Xe^{4+} ($\eta = 2.38$, $\gamma \simeq 58$, $\lambda \simeq 4.73$, $\omega_0 = 0$), correspondingly, we obtain $\Gamma_\eta(g = 0) \simeq 10^{-12}$ a.u. At the same time, these quantities determined in Ref. [97] are $w(Xe^{3+}) \simeq 2.7\times10^{-9}$ a.u. and $w(Xe^{4+}) \simeq 8.3\times10^{-12}$ a.u. The probabilities calculated from Eq. (112) are lower than the numerical results obtained in Ref. [97]. This discrepancy may be related with the method of determining cross sections (see details in Ref. [35]). For comparison, we present here the data for ionization rates for elliptical laser field. For Xe^{3+}, Eq. (112) gives $\Gamma_\eta(g = \pm0.5) \simeq 1.55\times10^{-9}$ a.u. and $\Gamma_\eta(g = \pm1) \simeq 7\times10^{-10}$ a.u. For Xe^{4+}, from Eq. (112) we obtain $\Gamma_\eta(g = \pm0.5) \simeq 1.67\times10^{-12}$ a.u. and $\Gamma_\eta(g = \pm1) \simeq 3.86\times10^{-13}$ a.u. It must be emphasized that accounting for the second Coulomb correction is very important for the interpretation of experimental data for the multiphoton ionization. Indeed, under the conditions of experiment [23], for Xe^{3+} according to Eq. (113) we have $Q_{II}(z_0,\theta) \simeq 2.2\times10^{-9}$ and for Xe^{4+} we obtain $Q_{II}(z_0,\theta) \simeq 1.5\times10^{-10}$.

Some results of numerical calculations for the factor $R(\gamma,g,\theta)$ and for the "stabilization factor" $S(\gamma_H,\gamma,g,\theta)$ in the case of the H atom and the positive ion Xe^+ (the binding energy $|E_0| = 20.98$ eV, $\mu = 0.059$) in the field of titanium-sapphire laser ($\hbar\omega \approx 1.55$ eV, $\lambda_0 \approx 800$ nm for intensity $J = 10^{13}$ W/cm^2) and in the case of the positive ion Xe^{5+} (the binding energy $|E_0| = 72$ eV, $\mu = 0.23$) in the field of x-ray laser ($\hbar\omega \approx 12.7$ eV, $\lambda_0 \approx 98$ nm for intensity $J = 10^{14}$ W/cm^2) are shown in Figs. 5–13. Figures 5 and 6 show that for not very large magnitudes ω_0 the factor $R(\gamma,g,\theta) \gg 1$, i.e., in contrast to the tunneling case [see Eq. (44)] the difference between multiphoton ionization rates for the counter-rotating and corotating electrons is significant: $\Gamma_\eta^- \gg \Gamma_\eta^+$. These figures also show that the maximum of the factor $R(\gamma,g,\theta)$ is strongly dependent on the angle θ and the ellipticity g. It should be noted that the difference between ionization rates for left- and right-elliptically polarized laser fields decreases as the magnetic field increases, i.e., for $\omega_0 \gg 1$. Besides, as can be seen from Figs. 5 and 6, the Coulomb correction (81) overestimates the magnitude of the factor $R(\gamma,g,\theta)$. Accounting for the second Coulomb correction $Q_{II}(z_0,\theta)$ in the ionization probability in Eq. (112) reduces the difference between ionization rates for left- and right-elliptically polarized laser fields. It is seen from Fig. 7 that in the case of a right-elliptically polarized laser field the factor $S(\gamma_H,\gamma,g,\theta)$ decreases as the magnetic field increases, i.e., the constant magnetic field stabilizes the bound level. This

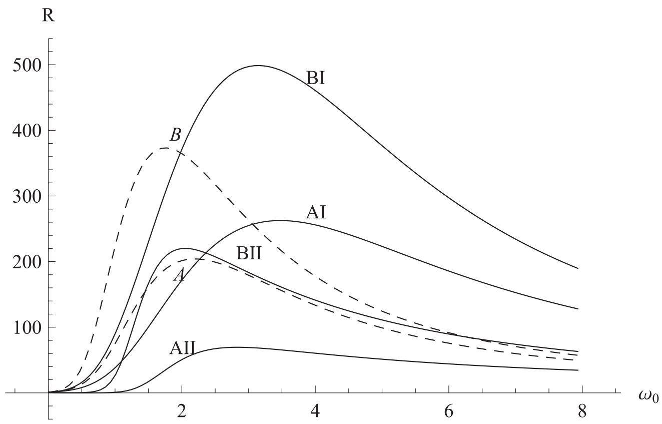

FIG. 6. The factor R when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the positive ion Xe^+ ($\eta = 1.61$) at $\theta = \pi/4$ and $|g| = 0.5$, $\lambda \approx 13.536$; $\gamma_A = 5$ and $\gamma_B = 10$.

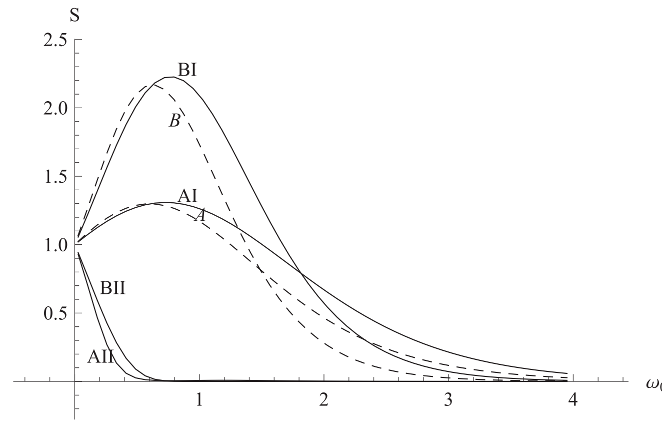

FIG. 8. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the the positive ion Xe^+ ($\eta = 1.61$) in the field of titanium-sapphire laser at $\theta = \pi/3$ and $g = -0.6$, $\lambda \approx 13.536$; $\gamma_A = 2$ and $\gamma_B = 5$.

stabilization effect is greater when the Coulomb corrections are ignored. Figure 7 also shows that the accounting for the second Coulomb correction in the ionization rate (112) tends to increase the stabilization factor for $\omega_0 < 1$. On the other hand, Figs. 8–13 show that in the case of a left-elliptically polarized laser field the ionization rate with the Coulomb correction (81) can grow with the magnetic field and has the maximum near $\omega_0 \sim 1$. For the H atom or the positive ion Xe^+ in the field of a titanium-sapphire laser with $\hbar\omega \approx 1.55$ eV the position of this maximum corresponds to the magnetic field $H \sim 100$ MG. As mentioned above, this dynamic effect is due to the fact that a constant magnetic field and a left-elliptically polarized laser field rotate the electron into opposite directions and, thus, in this case the laser field partially compensates the effect of the magnetic field. At the same time, a right-elliptically polarized laser field and a constant magnetic field rotate the electron at the same direction and thereby the laser field enhances the effect of the ionization suppression. Note that this maximum is greater for the positive ions Xe^+ and Xe^{5+} than for the H atom, i.e., for an atom or ion with the higher binding energy. As follows from these figures, the ionization probability grows at a greater rate for small angles θ and large magnitudes ellipticity g. This growth is maximal when a left circularly polarized laser beam ($g = -1$) propagates along ($\theta = 0$) the constant magnetic field (crossed electric and magnetic fields). On the other hand, one sees from Figs. 8–13 that accounting for the second Coulomb correction $Q_{II}(z_0,\theta)$ in Eq. (112) leads to sharp stabilization of a bound level.

Above were derived simple analytical expressions for the ionization rate of a weakly bound level for large magnetic fields in the multiphoton regime. With taking into account the limiting expressions for the Coulomb factors [see Eqs. (30)–(35)

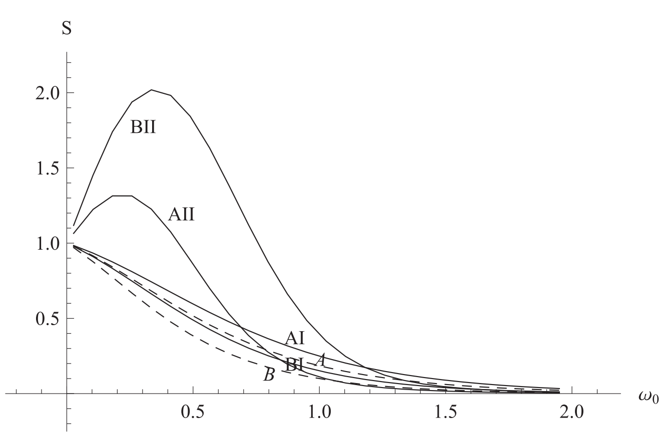

FIG. 7. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the $1s$ state of the H atom ($\eta = 1$) in the field of titanium-sapphire laser at $\theta = \pi/3$ and $g = +0.3$, $\lambda \approx 8.774$; $\gamma_A = 5$ and $\gamma_B = 10$.

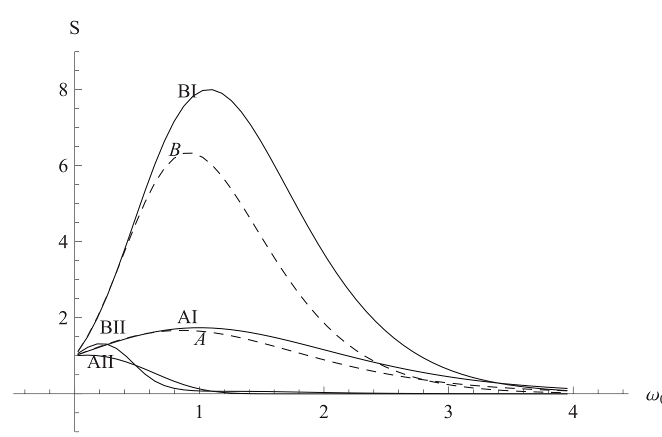

FIG. 9. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the $1s$ state of the H atom ($\eta = 1$) in the field of titanium-sapphire laser at $\theta = \pi/3$ and $g = -1$, $\lambda \approx 8.774$; $\gamma_A = 2$ and $\gamma_B = 5$.

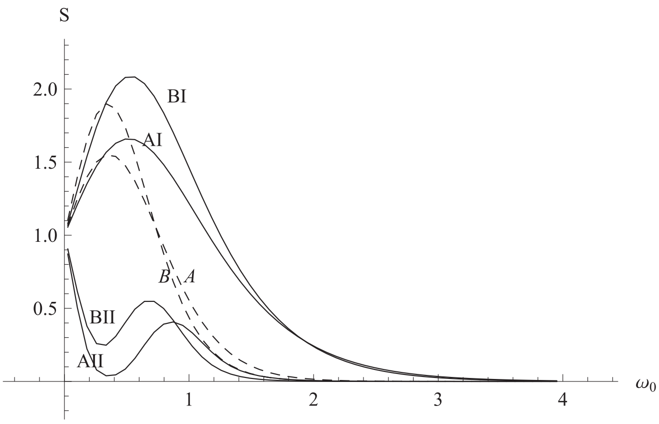

FIG. 10. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the $1s$ state of the H atom ($\eta = 1$) in the field of titanium-sapphire laser at $\theta = 0$ and $g = -0.5$, $\lambda \approx 8.774$; $\gamma_A = 5$ and $\gamma_B = 10$.

in the Supplemental Material [63]], these ionization rates can be generalized for the case of atoms and ions. So, for large magnetic fields ($\omega_0 \gg 1$) at $\theta \neq 0,\pi/2$ and for $g \neq +1$, in view of Eqs. (51), (52), (30), and (31) in the Supplemental Material [63], the ionization rate (112) in the multiphoton regime reads as

$$\Gamma_{\eta,M} \simeq 4|C_\kappa|^2|E_0|\frac{F}{F_0}\left(\frac{F_0}{F}\sqrt{\frac{1-g}{2\omega_0}}\right)^{2\eta}\left[\frac{\sin(\theta)}{2}\right]^{\omega_0+2(\lambda-\eta)} \times \frac{\gamma_H}{\gamma^{\omega_0+2(\lambda+\eta)}\sqrt{\ln[2\gamma\csc(\theta)]}} \times \exp\left\{-\lambda\cot^2(\theta)\left[\omega_0+4\frac{g-\cos(\theta)}{\sin(\theta)\sin(2\theta)}\right]\omega_0^3\right\}. \quad (115)$$

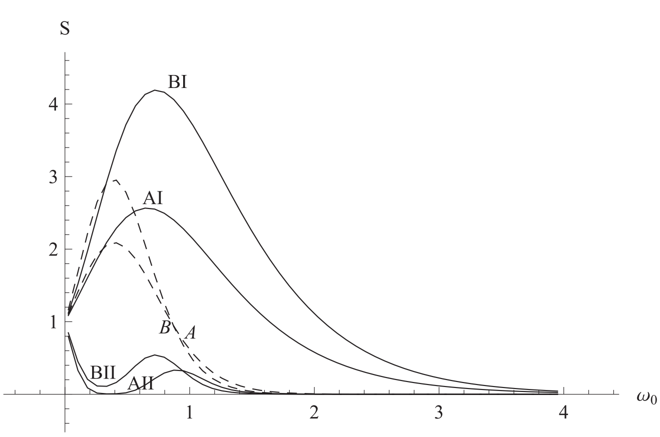

FIG. 11. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the positive ion Xe^+ ($\eta = 1.61$) in the field of titanium-sapphire laser at $\theta = 0$ and $g = -0.5$, $\lambda \approx 13.536$; $\gamma_A = 5$ and $\gamma_B = 10$.

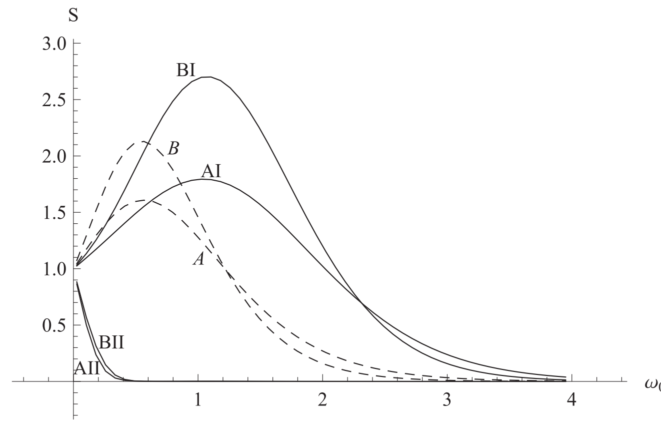

FIG. 12. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the positive ion Xe^{5+} ($\eta = 2.61$) in the field of x-ray laser at $\theta = \pi/3$ and $g = -0.8$, $\lambda \approx 5.67$; $\gamma_A = 5$ and $\gamma_B = 10$.

In particular, the ionization probability for large magnetic fields at $\theta = \pi/2$ in the multiphoton regime is

$$\Gamma_{\eta,M} \simeq 4|C_\kappa|^2|E_0|\frac{F}{F_0}\left[\frac{2F_0}{F}\sqrt{\frac{2(1-g)}{\omega_0}}\right]^{2\eta} \times \frac{\gamma_H}{(2\gamma)^{\omega_0+2(\lambda+\eta)}\sqrt{\ln(2\gamma)}} \times \exp\left\{-\lambda\left(g^2\omega_0^2-1\right)\right\}. \quad (116)$$

The ionization probability (112) for large magnetic fields ($\omega_0 \gg 1$) for $g \neq +1$ at $\theta = 0$ (crossed electric and magnetic fields), in view of Eqs. (54) and (55), and Eqs. (33) and (35) in the Supplemental Material [63], in the multiphoton regime

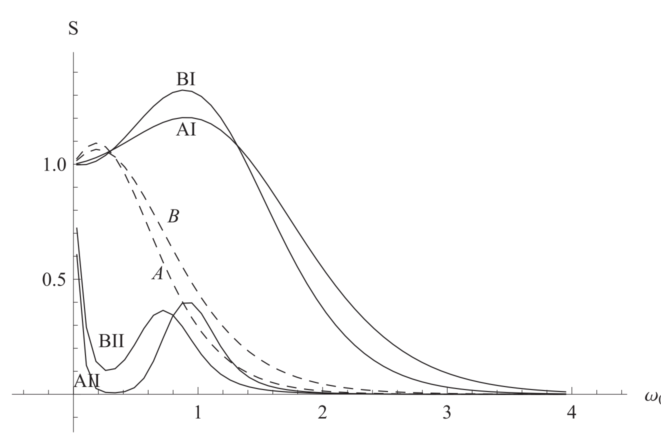

FIG. 13. The function S when the Coulomb correction (81) (solid line, AI, BI) and the Coulomb correction Q_{II} in Eq. (112) (solid line, AII, BII) are accounted for and when they are ignored (dashed line) vs ω_0 for the positive ion Xe^{5+} ($\eta = 2.61$) in the field of x-ray laser at $\theta = \pi/4$ and $g = -0.3$, $\lambda \approx 5.67$; $\gamma_A = 5$ and $\gamma_B = 10$.

reads as

$$\Gamma_{\eta,M} \simeq 4|C_\kappa|^2|E_0|\left[\frac{F_0\ln(a)}{\gamma F}\right]^{2\eta-1}\left(\frac{1-g}{2\gamma\gamma_H}\right)^{\lambda+\omega_0/2} \times\left(\sqrt{\frac{2\omega_0}{1-g}}\,\gamma\right)^{-2\eta\sqrt{\frac{2\omega_0}{1-g}}}\omega_0\sqrt{\ln(a)} \times\exp\left\{2\eta J(z_0,0)-\frac{\lambda}{2(1-g)}\left(\omega_0+\frac{3g-1}{2}\right)\omega_0^4\right\}, \quad (117)$$

where $a=\gamma\sqrt{2\omega_0/(1-g)}$ and $J(z_0,0)=0.254\,33$.

V. CONCLUSIONS

We have considered the tunneling and multiphoton ionization of atoms and ions subjected to a perturbation by an elliptically polarized monochromatic laser field and a constant uniform magnetic field, with taking into account the Coulomb interaction of the photoelectron with the parent ion. With this aim we found the exact wave function and Green's function for an electron in these fields. On this basis and using the quasistationary quasienergy state formalism, we obtained a transcendental equation for the complex quasienergy of a weakly bound level and the photoelectron energy spectrum. Besides, within the framework of the imaginary-time method, the extremal subbarrier trajectory of the electron ejected from under the barrier under the influence of a constant magnetic field and nonstationary laser field was derived. The general solution was analyzed in the case when the elliptically polarized monochromatic laser beam propagates at an arbitrary angle to the constant magnetic field. Simple analytical expressions for the ionization rate of a weakly bound level in the tunneling and multiphoton regimes were obtained and compared with the results known in the literature. The limits of small and large magnetic fields and low and high frequency of a laser field were considered in details. It was shown that in contrast to the case of constant electric and magnetic fields, in the presence of an electromagnetic wave the magnetic field may either decrease or increase the ionization rate. If the electron is subjected to a nonstationary laser field perturbation, the constant magnetic field slows down the ionization rate in the case of a right-polarized laser field (the laser field and the magnetic field rotate the electron in the same direction) and speeds it up for a left-polarized laser field (the laser field and the magnetic field rotate the electron into opposite directions). This dynamic effect is due to the distortion of the electron subbarrier trajectory in the magnetic field and it is greater in the multiphoton regime at small angles between the laser beam and the constant magnetic field and for large magnitudes of ellipticity of the laser field. This effect is maximal for the case of a circularly polarized laser beam propagating along the constant magnetic field (crossed electric and magnetic fields).

In the framework of the QQES formalism completed by the method of imaginary time we have derived the ionization rate with the Coulomb correction in the cases of the tunneling and the multiphoton regimes. We have considered the Coulomb correction of the two types. In order to find the Coulomb factor in the tunneling regime, we employed the quasiclassical perturbation theory and the method of subbarrier trajectories. In order to describe the Coulomb correction in the multiphoton regime, we have taken into account the distortion of the subbarrier electron trajectory by the Coulomb field. We have analyzed in details analytical expressions for the exponential and preexponential factors in the ionization rate with the Coulomb correction in the tunneling and the multiphoton limits for small and large magnetic fields. In particular, within this approach, we have calculated the ionization rate for the ground state of the H atom in an elliptically polarized laser beam propagating at an arbitrary angle to a constant magnetic field. Passing to the static limit, we obtained the formulas for the ionization probability of the H atom in crossed constant electric and magnetic fields. It was shown that accounting for the first Coulomb correction (tunneling regime) leads to an increase of the ionization rates of neutral atoms and positive ions in comparison with those for negative ions. At the same time, as it was clarified, the second Coulomb correction effectively decreases the ionization rate in the multiphoton regime. Nonetheless, the total Coulomb correction remains numerically large. Using the imaginary-time method, we also calculated the barrier width and the emission angle of the ejected electron for small and large magnetic fields (see Supplemental Material [63], Sec. D). It was shown that for small magnetic fields the ejected electron moves along the electric field, and for large magnetic fields it moves along the magnetic field during the subbarrier motion.

To support the above statements, we performed numerical calculations of the factor R (ratio of the ionization rates for left- and right-polarized laser fields), ionization rates, and the stabilization factor S in the cases of the H atom and positive ions Xe^+ and Xe^{5+} in the fields of titanium-sapphire and x-ray lasers. These calculations showed that the difference between ionization rates for the cases of a left- and right-elliptically polarized laser field, for s electrons in the constant magnetic field, sharply increases at the transition from the adiabatic regime to the multiphoton regime. However, accounting for the second Coulomb correction reduces this difference. This difference also decreases as the magnetic field increases and becomes vanishingly small for rather large magnetic fields. The numerical calculations also showed that in the case of a right-elliptically polarized laser field, a constant magnetic field suppresses the ionization probability, i.e., it stabilizes a bound electron level. In contrast, a left-elliptically polarized laser field counteracts the constant magnetic field and increases the ionization rate of a bound level. As a result, the ionization rate in the tunneling regime ($\gamma \ll 1$) can grow. But, accounting for the second Coulomb correction in the multiphoton regime ($\gamma \gg 1$) leads to a decrease in the ionization rate and as a consequence to sharp stabilization of a bound level. Besides, the high accuracy of the expressions for ionization rates in the wide range of the Keldysh parameter, obtained in this paper, is confirmed by comparison with the numerical results obtained by the Floquet method and the solution of the time-dependent Schrödinger equation and with the experiment where was investigated the interaction of a high-intensity x-ray laser radiation field with rare gas atoms.

The formulas obtained in this paper allow one to obtain simple estimates for the ionization rate for not only negative ions, but also for neutral atoms and positive ions in an elliptically polarized laser beam propagating at an arbitrary angle to the constant magnetic field.

ACKNOWLEDGMENT

I am deeply grateful to A. L. Mitler for performing numerical calculations.

APPENDIX A: EXACT SOLUTION OF THE SCHRÖDINGER EQUATION AND THE GREEN'S FUNCTION FOR AN ELECTRON IN AN ELECTROMAGNETIC WAVE AND CROSSED CONSTANT ELECTRIC AND MAGNETIC FIELDS

Let us choose the direction of the constant magnetic field $\boldsymbol{H}$ as the z axis and the direction of the constant electric field $\boldsymbol{F}_d$ as the y axis. The vector potential of the constant magnetic field $\boldsymbol{H}$ is conveniently taken here as (Landau gauge)

$$A_{H,x} = -Hy,\ \ A_{H,y} = 0,\ \ A_{H,z} = 0$$

and the Schrödinger equation for an electron in these fields and the electromagnetic wave with the vector potential $\boldsymbol{A}(t)$, propagating in an arbitrary direction, reads as

$$i\hbar\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t} = \frac{1}{2m}\left[-i\hbar\nabla - \boldsymbol{e}_x\frac{e}{c}Hy + \frac{e}{c}\boldsymbol{A}(t)\right]^2\Psi(\boldsymbol{r},t) + eF_d y\Psi(\boldsymbol{r},t), \tag{A1}$$

where $\boldsymbol{e}_x$ is the unit vector along the x axis. To separate the variables in Eq. (A1), we seek the wave function $\Psi(\boldsymbol{r},t)$ in the form

$$\Psi(\boldsymbol{r},t) = \exp\left[\frac{i}{\hbar}\left(p_x x + p_z z - \frac{1}{2m}\int_0^t\left\{\left[p_z + \frac{e}{c}A_z(\tau)\right]^2 + \frac{2e}{c}p_x A_x(\tau) + \frac{e^2}{c^2}\left[A_x^2(\tau) + A_y^2(\tau)\right]\right\}d\tau\right)\right]\tilde{\Psi}(\boldsymbol{r},t). \tag{A2}$$

Substituting (A2) into (A1) gives the equation for the new wave function $\tilde{\Psi}(\boldsymbol{r},t)$:

$$i\hbar\frac{\partial\tilde{\Psi}(\boldsymbol{r},t)}{\partial t} = \hat{H}\tilde{\Psi}(\boldsymbol{r},t), \tag{A3}$$

where the Hamiltonian

$$\hat{H} = \frac{1}{2m}\left\{\hat{p}_y^2 + p_x^2 - \frac{2e}{c}Hp_x y + \frac{e^2H^2}{c^2}y^2 + \frac{2e}{c}\left[\hat{p}_y A_y(t) - \frac{e}{c}HyA_x(t)\right]\right\} + eF_d y \tag{A4}$$

and $\hat{p}_y = -i\hbar\frac{\partial}{\partial y}$ is the y component of the momentum operator. Let us represent $\tilde{\Psi}(\boldsymbol{r},t)$ in the form (see also Ref. [98])

$$\tilde{\Psi}(\boldsymbol{r},t) = \exp\left\{-\frac{i}{\hbar}\phi(t)\right\}\hat{\mathcal{D}}(p(t),q(t))\ \chi(\boldsymbol{r},t), \tag{A5}$$

where $\hat{\mathcal{D}}(p(t),q(t))$ is the unitary operator

$$\hat{\mathcal{D}}(p(t),q(t)) = \exp\left\{\frac{i}{\hbar}[p(t)y - q(t)\hat{p}_y]\right\}, \tag{A6}$$

$p(t) = \omega_H f_2(t)$, $q(t) = f_1(t)/m$, and $f_2(t)$ and $f_1(t)$ are new unknown functions. The major property of $\hat{\mathcal{D}}(p(t),q(t))$ is

$$\hat{\mathcal{D}}^+(p(t),q(t))(\alpha\hat{p}_y + \beta y)\hat{\mathcal{D}}(p(t),q(t)) = \alpha[\hat{p}_y + p(t)] + \beta[y + q(t)], \tag{A7}$$

where α and β are the arbitrary constants and $\hat{\mathcal{D}}^+$ is the conjugate operator. The operator $\hat{\mathcal{D}}(p(t),q(t))$ is often met in theoretical problems. In particular, $\exp(i\tau)\hat{\mathcal{D}}$ (τ is the arbitrary constant) gives the representation of the Heisenberg-Weil group. Substituting the function (A5) into (A3), we get the equation for the unknown function $\chi(\boldsymbol{r},t)$:

$$i\hbar\frac{\partial\chi(\boldsymbol{r},t)}{\partial t} = \{\hat{\mathcal{D}}^+(p(t),q(t))\hat{H}\hat{\mathcal{D}}(p(t),q(t)) + i\hbar\hat{\mathcal{D}}^+(p(t),q(t))\hat{C}\hat{\mathcal{D}}(p(t),q(t)) - \dot{\phi}(t)\}\chi(\boldsymbol{r},t). \tag{A8}$$

Here and above, $\dot{\phi}(t) = \frac{\partial\phi(t)}{\partial t}$ and

$$\hat{C} = i(f_1\dot{f}_2 - f_2\dot{f}_1) + i\sqrt{2}\left(\frac{a_H}{\hbar}\dot{f}_1\ \hat{p}_y - \frac{1}{a_H}\dot{f}_2\ y\right).$$

When deriving $\hat{\mathcal{D}}(p(t),q(t))$ in Eq. (A8), we made use of the relation

$$\frac{\partial}{\partial t}e^{\hat{A}} = e^{\hat{A}}\left\{\frac{\partial\hat{A}}{\partial t} - \frac{1}{2}\left[\hat{A},\frac{\partial\hat{A}}{\partial t}\right]\right\},$$

correct for any operator $\hat{A}$ provided $\hat{A}$ and $\frac{\partial\hat{A}}{\partial t}$ commute with $[\hat{A},\frac{\partial\hat{A}}{\partial t}]$; here, $[\hat{X},\hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}$.

The relation (A7) makes it possible to disjoint the terms in the curly brackets in Eq. (A8) into several parts. To separate the variables, we require that the time-dependent parts, which are proportional to $\hat{p}_y$ and y, be equal to zero. This gives two equations for $f_1(t)$ and $f_2(t)$:

$$\dot{f}_1(t) = \omega_H f_2(t) + \frac{e}{c}A_y(t), \quad \dot{f}_2(t) = -\omega_H f_1(t) + \frac{e}{c}A_x(t), \tag{A9}$$

which can be united into a single equation for the complex function $f(t) = f_1(t) + i f_2(t)$:

$$\dot{f}(t) + i\omega_H f(t) = \frac{e}{c}[A_y(t) + iA_x(t)]. \tag{A10}$$

We need only a partial solution of Eq. (A10), which reads as

$$f(t) = \frac{e}{c}\int_{-\infty}^{t}[A_y(\tau) + iA_x(\tau)]e^{i\omega_H(\tau-t)}d\tau. \tag{A11}$$

Separating the real and imaginary parts in Eq. (A11), we obtain $f_1(t) = \mathcal{P}_x(t)/\omega_H$ and $f_2(t) = \mathcal{P}_y(t)/\omega_H$, where

$$\mathcal{P}_x(t) = \frac{e}{c}\omega_H\int_{-\infty}^{t}\{A_y(\tau)\cos\omega_H(t-\tau) + A_x(\tau)\sin\omega_H(t-\tau)\}d\tau,$$
$$\mathcal{P}_y(t) = \frac{e}{c}\omega_H\int_{-\infty}^{t}\{A_x(\tau)\cos\omega_H(t-\tau) - A_y(\tau)\sin\omega_H(t-\tau)\}d\tau. \tag{A12}$$

The time-independent terms in the curly brackets in Eq. (A8) give the stationary equation for the harmonic oscillator displaced by the electric field F_d:

$$i\hbar\frac{\partial\chi(y,t)}{\partial t} = \frac{1}{2m}\{\hat{p}_y^2 + p_x^2 + m^2\omega_H^2 y^2 + 2my(eF_d - p_x\omega_H)\}\chi(y,t). \tag{A13}$$

The normalized solutions of Eq. (A13) are expressed in terms of the Hermite polynomials H_n by

$$\chi_n(y,t) = A_n\exp\left\{\frac{i}{\hbar}(p_x + p_z - E_n t)\right\}\exp\left\{-\frac{(y-y_0)^2}{2a_H^2}\right\}H_n\left(\frac{y-y_0}{a_H}\right), \tag{A14}$$

where $E_n = p_z^2/2m + \hbar\omega_H(n+\frac{1}{2}) + p_x v_d - mv_d^2/2$ are the Landau energy levels in the constant electric field, $v_d = cF_d/H$ is the drift velocity, $y_0 = (a_H^2/\hbar)(p_x - mv_d)$, and $a_H = (\hbar c/eH)^{1/2}$ is the magnetic length. The time-dependent terms in the curly brackets in Eq. (A8) determine the function $\phi(t)$ in the exponent in Eq. (A5) in the following form:

$$\phi(t) = \frac{\mathcal{P}_x(t)\,\mathcal{P}_y(t)}{2m\omega_H} - \frac{v_d}{\omega_H}\left[\mathcal{P}_y(t) - \frac{e}{c}\omega_H\int_{-\infty}^{t}A_x(\tau)d\tau\right]$$
$$+ \frac{e}{mc}\int_{-\infty}^{t}\left\{p_z A_z(\tau) + \frac{e}{2c}\boldsymbol{A}^2(\tau) + \frac{1}{2}[\mathcal{P}_y(\tau)A_y(\tau) - \mathcal{P}_x(\tau)A_x(\tau)]\right\}d\tau. \tag{A15}$$

Substituting the solution (A14) into (A5) and using for the operator (A6), the Beiker-Hausdorf identity

$$\exp\{\hat{X} + \hat{Y}\} = \exp\left\{-\frac{1}{2}[\hat{X},\hat{Y}]\right\}\exp\{\hat{X}\}\exp\{\hat{Y}\},$$

we get the solution of Eq. (A1) in the form

$$\Psi_n(\boldsymbol{r},t) = A_n\exp\left\{\frac{i}{\hbar}\left[p_x x + p_z z + \mathcal{P}_y(t)y + \frac{p_x}{m\omega_H}\mathcal{P}_y(t) - E_n t - \phi(t)\right]\right\}\exp\left\{-\frac{[y-y_0(t)]^2}{2a_H^2}\right\}H_n\left[\frac{y-y_0(t)}{a_H}\right], \tag{A16}$$

where $A_n = 1/(\sqrt{L_x L_z a_H}\sqrt{2^n n!\sqrt{\pi}})$ and $y_0(t) = y_0 + \mathcal{P}_x(t)/(m\omega_H)$ is the coordinate of the center of classical circular motion of the electron, depending on time via the function $\mathcal{P}_x(t)$. The latter result is obtained from the identity (see in Ref. [75])

$$\exp\left\{-\frac{i}{\hbar}\frac{\mathcal{P}_x(t)}{m\omega_H}\,\hat{p}_y\right\}F(y) = F\left[y - \frac{\mathcal{P}_x(t)}{m\omega_H}\right]$$

by applying the operator (A6). The wave function (A16) describes the states of an electron in laser and constant magnetic fields, which act simultaneously. The exponent in it differs from that in the well-known Volkov wave function by the additional contributions related to the functions $\mathcal{P}_x(t)$ and $\mathcal{P}_y(t)$. It is also seen that the wave function (A16) has the preexponential factor similar to that in the standard Landau wave function for an electron in a constant magnetic field [75], but with the argument of the Hermite polynomial being dependent on time via the function $\mathcal{P}_x(t)$. Thus, the functions $\mathcal{P}_x(t)$ and $\mathcal{P}_y(t)$, which are dependent of both the laser and constant magnetic fields, describe the combined effect of the two fields on the electron.

Based on the exact solution (A16), the retarded Green's function from Eq. (4) with using of the spectral expansion

$$G(\boldsymbol{r},t;\boldsymbol{r}',t') = -\frac{i}{\hbar}\theta(t-t')\mathcal{G}(\boldsymbol{r},t;\boldsymbol{r}',t'), \tag{A17}$$

where

$$\mathcal{G}(\boldsymbol{r},t;\boldsymbol{r}',t') = \int_{-\infty}^{\infty} \frac{L_x}{2\pi\hbar}dp_x \int_{-\infty}^{\infty} \frac{L_z}{2\pi\hbar}dp_z \sum_{n=0}^{\infty} \Psi_n^*(\boldsymbol{r}',t')\,\Psi_n(\boldsymbol{r},t) \tag{A18}$$

after some algebra, can be represented in the form (5), where

$$S(\boldsymbol{r},t;\boldsymbol{r}',t') = \frac{m}{2}\frac{\mathcal{Z}^2(t,t')}{t-t'} + \frac{m\omega_H}{2}\left\{\frac{1}{2}\cot\frac{\omega_H}{2}(t-t')[\mathcal{X}^2(t,t') + \mathcal{Y}_-^2(t,t')] + \mathcal{X}(t,t')\,\mathcal{Y}_+(t,t')\right\}$$
$$+\frac{mv_d^2}{2}(t-t') + \Phi(t,t') + \mathcal{P}_y(t)y - \mathcal{P}_y(t')y' \tag{A19}$$

is the corresponding classical action. Here,

$$\mathcal{Z}(t,t') = z - z' + \frac{e}{mc}\int_t^{t'} A_z(\tau)d\tau,\ \ \mathcal{X}(t,t') = x - x' - v_d(t-t') + \frac{1}{m\omega_H}[\mathcal{P}_y(t') - \mathcal{P}_y(t)],$$
$$\mathcal{Y}_-(t,t') = y - y' + \frac{1}{m\omega_H}[\mathcal{P}_x(t') - \mathcal{P}_x(t)],\ \ \mathcal{Y}_+(t,t') = y + y' + 2\frac{v_d}{\omega_H} - \frac{1}{m\omega_H}[\mathcal{P}_x(t') + \mathcal{P}_x(t)],$$
$$\Phi(t,t') = \frac{1}{2m\omega_H}[\mathcal{P}_x(t')\mathcal{P}_y(t') - \mathcal{P}_x(t)\mathcal{P}_y(t)] + \frac{v_d}{\omega_H}\left[\mathcal{P}_y(t) - \mathcal{P}_y(t') + \frac{e}{c}\omega_H\int_t^{t'} A_x(\tau)d\tau\right]$$
$$+\frac{e^2}{2mc^2}\int_t^{t'} \boldsymbol{A}^2(\tau)d\tau + \frac{e}{2mc}\int_t^{t'}[\mathcal{P}_y(\tau)A_y(\tau) - \mathcal{P}_x(\tau)A_x(\tau)]d\tau. \tag{A20}$$

It is easy to show that in the limit $H \to 0$, the Green's function (5) coincides with the corresponding Volkov Green's function obtained in Ref. [32]. On the other hand, at $\boldsymbol{F}_d = \boldsymbol{0}$ and in the static limit $\omega \to 0$, we arrive at the expression for the Green's function obtained in Ref. [19] for the case of crossed constant electric and magnetic fields. All expressions in this paper can be obtained by setting $\boldsymbol{F}_d = \boldsymbol{0}$ ($v_d = 0$) in Eq. (A20).

APPENDIX B: TWO APPROACHES TO THE DETERMINATION OF IONIZATION RATE

The first approach to the determination of ionization rates relates to the possibility of the expansion of the wave functions (A16), included in the function $\mathcal{G}(\boldsymbol{0},t;\boldsymbol{0}',t-t')$, in Fourier series

$$\Psi_n(\boldsymbol{0},t) = \frac{1}{2\pi}\exp\left\{-\frac{i}{\hbar}\left[E_n + \frac{\kappa^2}{4\gamma^2}(1+g^2)\right]t\right\}\sum_{m=-\infty}^{\infty} F_m^{(n)}(p_x,p_z)\,e^{-im\omega t}, \tag{B1}$$

where $E_n = p_z^2/2m + \hbar\omega_H(n+\frac{1}{2})$ are the Landau energy levels, $\kappa = \sqrt{2m|E_0|}$ is the inner-atomic momentum. Then, Eq. (8) for the complex quasienergy can be written as

$$(\sqrt{-\epsilon} - \sqrt{|E_0|})f_\epsilon(t) = \frac{1}{2}\sqrt{\frac{\hbar}{\pi i}}\int_0^{\infty} dt'\exp\left(\frac{i}{\hbar}\epsilon t'\right)\left\{\sum_{m,m'=-\infty}^{\infty} \Phi_{m',m}(t')\,e^{i(m'-m)\omega t} f_\epsilon(t-t') - \frac{f_\epsilon(t)}{t'^{3/2}}\right\}, \tag{B2}$$

where the functions $\Phi_{m',m}(t')$ are

$$\Phi_{m',m}(t') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{L_x}{2\pi\hbar}dp_x\int_{-\infty}^{\infty}\frac{L_z}{2\pi\hbar}dp_z\sum_{n=0}^{\infty} F_{m'}^{(n)*}(p_x,p_z)F_m^{(n)}(p_x,p_z)$$
$$\times\exp\left\{-\frac{i}{\hbar}\left[E_n + \frac{\kappa^2}{4\gamma^2}(1+g^2) + m'\hbar\omega\right]t'\right\} \tag{B3}$$

and the superscript $*$ denotes the complex conjugate. Following Ref. [32] we also expand the function $f_\epsilon(t)$ into Fourier series

$$f_\epsilon(t) = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} f_l e^{-il\omega t} \tag{B4}$$

and represent Eq. (B2) in the form

$$(\sqrt{-\epsilon}-\sqrt{|E_0|})f_l=\frac{1}{2}\sqrt{\frac{\hbar}{\pi i}}\int_0^\infty dt'\exp\left(\frac{i}{\hbar}\epsilon t'\right)\left\{\sum_{m,m'=-\infty}^{\infty}f_{m'-m+l}\Phi_{m',m}(t')\,e^{i(m'-m+l)\omega t'}-\frac{f_l}{t'^{3/2}}\right\}. \tag{B5}$$

Let us restrict ourselves to the case $l=0$ and consider only the diagonal terms $m=m'$ in the sum under the integral in Eq. (B5). Then, neglecting the Stark shift ($E\simeq E_0$) for small laser field-induced width $\hbar\Gamma\ll E$ of the bound level, with taking into account the relation

$$\int_0^\infty e^{ix\tau}d\tau=i\frac{P}{x}+\pi\delta(x), \tag{B6}$$

we find the ionization rate (11) as a sum of probabilities of many-photon processes.

The second approach uses a direct calculation of the function $\mathcal{G}(\mathbf{0},t;\mathbf{0}',t-t')$, i.e., the retarded Green's function (5). In this case, Eq. (8) can be represented in the form

$$(\sqrt{-\epsilon}-\sqrt{|E_0|})f_\epsilon(t)=\frac{1}{2}\sqrt{\frac{\hbar}{\pi i}}\int_0^\infty\frac{dt'}{t'^{3/2}}\exp\left(\frac{i}{\hbar}\epsilon t'\right)\left\{\frac{\omega_H t'}{2\sin(\omega_H t'/2)}f_\epsilon(t-t')\exp\left[\frac{i}{\hbar}S(t,t')\right]-f_\epsilon(t)\right\}, \tag{B7}$$

where $S(t,t')=\lim_{\boldsymbol{r},\boldsymbol{r}'\to\mathbf{0}}S(\boldsymbol{r},t;\boldsymbol{r}',t-t')$ and the classical action $S(\boldsymbol{r},t;\boldsymbol{r}',t')$ is defined by Eqs. (A19) and (A20). The real and imaginary parts of the complex quasienergy ϵ determine the Stark shift $\Delta\epsilon=E-E_0$ and the laser field-induced width $\hbar\Gamma$ of the bound level. Equation (B7) is valid for a laser beam with an arbitrary strength and direction of propagation and a magnetic field of arbitrary strength. It opens up the possibility for the calculation of ionization rates in the area of strong electric and magnetic fields in the wide range values of the Keldysh parameter γ. For large multiquantum parameter $\lambda\gg1$, in order to derive basic analytical results, we set $f_\epsilon(t)\approx$ const and average the right-hand side of Eq. (B7) with respect to t over the period π/ω. This approximation corresponds to Eqs. (16a) and (16b) in Ref. [32]. Then, Eq. (B7) transforms to the closed equation for the complex quasienergy

$$\beta\simeq1+\frac{1}{2\sqrt{\pi i\lambda}}\frac{1}{\pi}\int_0^\pi d\tau\int_0^\infty\frac{dt}{t^{3/2}}\exp(-i\lambda\beta^2t)\left\{\frac{\omega_0 t}{2\sin(\omega_0 t/2)}\exp[i\lambda S(\tau,t)]-1\right\}, \tag{B8}$$

where $\beta=\sqrt{-\epsilon/|E_0|}$ describes the alteration of the electron energy by the laser and magnetic fields. The relative accuracy of Eq. (B8) can be estimated as $1/(16\lambda)$ [32]. For example, according to this criterion, the relative accuracy of calculation of ionization rates on Figs. 3 and 4 ($\lambda=4.63$) reaches $\sim1.5\%$. For $\lambda\gg1$ and for small electric fields $F\ll F_0$ from Eq. (B8) one can obtain the quasiclassical ionization rate

$$\Gamma=\frac{\omega_H}{2\sqrt{2z_0}\sinh(\omega_0 z_0)}\frac{1}{\sqrt{|F''(z_0)|}}\exp[-2\lambda F(z_0)] \tag{B9}$$

for an electron in a zero-range potential under the influence of an elliptically polarized laser beam, propagating at an angle θ to a constant magnetic field.

APPENDIX C: KAPITZA'S METHOD AND CALCULATION OF IMAGINARY PART OF THE CLASSICAL ACTION

In a rapidly oscillating field, in accordance with Kapitza's method [99,100], the trajectory of the electron can be represented in the form

$$\boldsymbol{r}(t)=\boldsymbol{R}(t)+\boldsymbol{r}_0(t), \tag{C1}$$

where $\boldsymbol{R}(t)$ is the trajectory describing the "smooth" motion of the electron averaged over the small rapid oscillations at frequency ω and the zero-order trajectory $\boldsymbol{r}_0(t)$ corresponds to these small oscillations. Passing in Eq. (104) to the dimensionless variables $\boldsymbol{\xi}=\boldsymbol{r}/b$ ($b=\kappa/\omega$) and $\phi=\omega t$ and averaging over the period $2\pi/\omega$, we obtain

$$\frac{d^2\boldsymbol{\xi}}{d\phi^2}=-\nabla_{\boldsymbol{\xi}}\mathcal{U}(\boldsymbol{\xi},g,\omega_0,\theta),\quad \mathcal{U}(\boldsymbol{\xi},g,\omega_0,\theta)=-\mu\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta), \tag{C2}$$

where

$$\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{[\xi_x+\zeta_{x,0}(\varphi)]^2+[\xi_y+\zeta_{y,0}(\varphi)]^2+[\xi_z+\zeta_{z,0}(\varphi)]^2}} \tag{C3}$$

and

$$\zeta_{x,0}(\varphi)=-\frac{1}{\gamma}\frac{1}{1-\omega_0^2}\left\{[g\omega_0+\cos(\theta)][1-\cos(\varphi)]+\mathcal{G}(g,\omega_0,\theta)\frac{\sinh(z_0)}{\sinh(\omega_0 z_0)}[\cos(\omega_0\varphi)-1]\right\},$$

$$\zeta_{y,0}(\varphi)=\frac{1}{\gamma}\frac{\mathcal{G}(g,\omega_0,\theta)}{1-\omega_0^2}\left[\sin(\varphi)-\sin(\omega_0\varphi)\frac{\sinh(z_0)}{\sinh(\omega_0 z_0)}\right],\quad \zeta_{z,0}(\varphi)=-\frac{1}{\gamma}\sin(\theta)[1-\cos(\varphi)]. \tag{C4}$$

In Eq. (C3), ξ_x,ξ_y,ξ_z can be estimated as the dimensionless initial points $\xi_{x,0},\xi_{y,0},\xi_{z,0}$ in the initial instant $t=0$ of the electron motion in real time without taking into account the Coulomb interaction ($\mu=0$). From Eq. (5) in the Supplemental Material [63] at $z=0$ in the multiphoton regime ($\gamma\gg 1$) at $g^2\neq 1$ for small magnetic fields ($\omega_0 z_0\ll 1$) we have

$$\xi_x\sim\xi_{x,0}\simeq\frac{\cos(\theta)}{\sqrt{(1-g^2}},\quad \xi_y\sim\xi_{y,0}=0,\quad \xi_z\sim\xi_{z,0}\simeq\frac{\sin(\theta)}{\sqrt{1-g^2}}. \tag{C5}$$

If the opposite inequality $\omega_0 z_0\gg 1$ holds,

$$\xi_x\simeq\frac{g-\cos(\theta)}{\sqrt{(1+\omega_0)^2-\mathcal{G}^2(g,\omega_0,\theta)}},\quad \xi_y=0,\quad \xi_z\simeq\frac{\sin(\theta)}{\sqrt{1-\mathcal{G}^2(g,\omega_0,\theta)/(1+\omega_0)^2}}. \tag{C6}$$

In the case of a circular laser beam propagating at an angle θ to the constant magnetic field $\boldsymbol{H}$ for $\omega_0 z_0\ll 1$, we have

$$\xi_x\simeq\frac{1}{\sqrt{2}}\frac{\omega_0+\cos(\theta)}{1-\omega_0^2}\sqrt{\frac{\ln(\gamma)}{(\mathcal{B}_\pm/2)\ln(\gamma)+1-\mathcal{B}_\pm}},\quad \xi_z\simeq\frac{\sin(\theta)}{\sqrt{2}}\sqrt{\frac{\ln(\gamma)}{(\mathcal{B}_\pm/2)\ln(\gamma)+1-\mathcal{B}_\pm}}, \tag{C7}$$

$\xi_y=0$ and $\mathcal{B}_\pm=1-\mathcal{G}^2(g=\pm1,\omega_0,\theta)/(1-\omega_0^2)^2$. If in the case of a circular laser field the opposite inequality $\omega_0 z_0\gg 1$ holds, then the expressions for ξ_x, ξ_y, and ξ_z in Eq. (C6) are valid. It should be noted that ξ_x and ξ_z in Eq. (C6) in the case of a right-circularly polarized laser beam ($g=+1$), propagating along the constant magnetic field $\boldsymbol{H}$ ($\theta=0$) have a singularity. In this case, as was mentioned above, the saddle-point equation (39) has no roots. In the absence of the constant magnetic field ($\omega_0\to 0$) in the case of an elliptically polarized laser field, with taking into account Eq. (C5), the integral in Eq. (C3) is simplified as follows:

$$\mathcal{J}(\boldsymbol{\xi},g,0,0)\simeq\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{\{1/\sqrt{1-g^2}-\gamma^{-1}[1-\cos(\varphi)]\}^2+g^2\gamma^{-2}[\sin(\varphi)-\varphi\sinh(z_0)/z_0]^2}} \tag{C8}$$

and for a circularly polarized laser field this integral reads as

$$\mathcal{J}(\boldsymbol{\xi},g=\pm1,0,0)\simeq\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{\{\sqrt{\ln(\gamma)/2}-\gamma^{-1}[1-\cos(\varphi)]\}^2+\gamma^{-2}[\sin(\varphi)-\varphi\sinh(z_0)/z_0]^2}}, \tag{C9}$$

where the saddle point z_0 here is determined by Eq. (39) for $\omega_0=0$.

The motion corresponding to smooth trajectory $\boldsymbol{\xi}(t)$ is conservative with an energy integral

$$\frac{1}{2}\left(\frac{d\boldsymbol{\xi}}{d\phi}\right)^2+\mathcal{U}(\boldsymbol{\xi},g,\omega_0,\theta)=\mathcal{E}=\text{const}, \tag{C10}$$

where the constant energy $\mathcal{E}$ corresponding the ionization threshold defines the extremal trajectory along which the ionization probability attains its maximal value, i.e., zero electron velocity at infinity ($\mathcal{E}=0$). Kapitza's method is applicable under the multiphoton ionization condition since

$$\omega T=2\pi\omega b^{3/2}Z^{-1/2}=2\pi\sqrt{\frac{2\lambda}{\eta}}\simeq\mu^{-1/2}\gg 1, \tag{C11}$$

where T is the characteristic time of classical motion of the photoelectron in the Coulomb field [100], $b=\kappa/\omega$ is the "dynamic" width of the barrier (width of the barrier in the multiphoton regime), and $\eta=Z/\kappa$ is the Sommerfeld parameter.

In the absence of the constant magnetic field and for a linearly polarized laser field ($g=0$), the integral in Eq. (C3) can be calculated analytically and for the potential $\mathcal{U}(\boldsymbol{\xi},g,\omega_0,\theta)$ in Eq. (C2) we have [35]

$$\mathcal{U}(\xi,g=0,0,0)=-\frac{\mu}{\sqrt{\xi^2-2\xi/\gamma}}. \tag{C12}$$

In the multiphoton regime, i.e., for $\gamma\gg 1$, the potential $\mathcal{U}(\xi,g=0,0,0)$ can be reduced to the Coulomb potential and the relation (C10) gives the drift momentum for the electron at the moment $t=0$ when it overcomes the barrier [see Eq. (A10) in Ref. [35]]

$$p_z\simeq\kappa\sqrt{2\mu}. \tag{C13}$$

Thus, accounting for the Coulomb interaction leads to the point of emergence from under the barrier is not the point at which the electron comes to a halt even in the one-dimensional case. In the general case ($g\neq 0$, $H\neq 0$), passing in Eq. (C10) to the imaginary time $\phi\to i\tau$ we obtain the boundary condition for the subbarrier energy (for the subbarrier trajectory $\boldsymbol{\xi}$):

$$\frac{1}{2}\left(\frac{d\boldsymbol{\xi}}{d\tau}\right)^2=-\mu\mathcal{J}(\boldsymbol{\xi},g,\omega_0,\theta),\quad \tau=0. \tag{C14}$$

The Coulomb correction to the subbarrier trajectory has to be determined from the following equation:

$$\frac{d^2\boldsymbol{\xi}}{d\tau^2} = i\omega_0\left[\boldsymbol{e}_z,\frac{d\boldsymbol{\xi}}{d\tau}\right] + \mu\frac{\boldsymbol{\xi}_0}{|\boldsymbol{\xi}_0|^3} \tag{C15}$$

with the initial condition (C14) and for $\boldsymbol{\xi}(\tau=0)=\mathbf{0}$. In Eq. (C15), $\boldsymbol{e}_z$ is the unit vector along the z axis and $\boldsymbol{\xi}_0 = \boldsymbol{r}_0/b$, where $\boldsymbol{r}_0$ is the trajectory (5) in the Supplemental Material [63]. The solution of Eq. (C15) corresponding to these initial conditions is

$$\boldsymbol{\xi}(\tau) = i\boldsymbol{n}\tau\sqrt{2\mu\mathcal{J}(\xi,g,\omega_0,\theta)} + \int_0^{\tau}\boldsymbol{\Phi}(\tau')d\tau', \tag{C16}$$

where $\boldsymbol{n}$ is the unit vector ($\boldsymbol{n}^2=1$) and $\boldsymbol{\Phi}=(\Phi_x,\Phi_y,\Phi_z)$:

$$\Phi_{x,y}(\tau') = \mu\int_0^{\tau'}\exp\{\omega_0(\tilde{\tau}-\tau')\}\frac{\xi_{0,x,y}(\tilde{\tau})}{|\boldsymbol{\xi}_0(\tilde{\tau})|^3}d\tilde{\tau},\quad \Phi_z(\tau') = \mu\int_0^{\tau'}\frac{\xi_{0,z}(\tilde{\tau})}{|\boldsymbol{\xi}_0(\tilde{\tau})|^3}d\tilde{\tau}. \tag{C17}$$

In the general case, the integral in Eq. (103) can be evaluated by parts with using the equation of motion $\dot{\boldsymbol{v}}_0 = -\boldsymbol{E}(t) - [\boldsymbol{v}_0,\boldsymbol{H}]$. Let us consider the magnetic part of the classical action

$$\delta S_{II,H} = \int_{t_0}^{\infty}[y_0H\boldsymbol{e}_x\boldsymbol{v}_1 + y_1H\boldsymbol{e}_x\boldsymbol{v}_0]dt + \int_{t_0}^{\infty}[\boldsymbol{v}_0,\boldsymbol{H}]\boldsymbol{r}_1dt = Hy_0x_1|_{t=t_0}^{t\to\infty}. \tag{C18}$$

Since the ionization probability depends only on the imaginary part of the action, the contribution of the upper limit in Eq. (C18) is insignificant. In turn, taking into account Eq. (5) [$y_0 \sim r_{0y}(z_0)=0$], we obtain that $\delta S_{II,H}=0$. Further, retaining contributions on an order no higher than the first in parameter μ, in view of Eqs. (C16) and (C17), one can obtain Eq. (105).

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