Superiority of photon subtraction to addition for entanglement in a multimode squeezed vacuum

Tamoghna Das,[1] R. Prabhu,[1,2] Aditi Sen(De),[1] and Ujjwal Sen[1]
[1]*Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India*
[2]*Department of Physics, Indian Institute of Technology Patna, Bihta 801103, Bihar, India*

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We investigate the entanglement patterns of photon-added and photon-subtracted four-mode squeezed vacuum states. Entanglements in different scenarios are analyzed by varying the number of photons added or subtracted in certain modes, which are referred to as the "player" modes, the others being "spectators." We find that the photon-subtracted state can give us higher entanglement than the photon-added state which is in contrast to the two-mode situation. We also study the logarithmic negativity of the two-mode reduced density matrix obtained from the four-mode state which again shows that the state after photon subtraction can possess higher entanglement than that of the photon-added state, and we then compare it to that of the two-mode squeezed vacuum state. Moreover, we examine the non-Gaussianity of the photon-added and photon-subtracted states to find that the rich features provided by entanglement cannot be captured by the measure of nonclassicality.

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I. INTRODUCTION

Distribution of entanglement in a multipartite quantum system is known to be a useful resource in several quantum communication and quantum computational tasksx [1]. Notable ones include quantum secret sharing [2], distributed quantum dense coding [3], distribution and concentration of quantum state [4], and cluster state quantum computing [5]. Such protocols have successfully been realized in physical systems such as photons [6], ions [7], nuclear magnetic resonance [8], nitrogen vacancy centers [9], etc.

One of the physical systems in which quantum information tasks have been realized in the laboratory is the class of continuous variable (CV) systems. Historically, the notion of the quantum correlated state of two particles in CV systems first arrived in the seminal paper of Einstein, Podolosky, and Rosen in 1935 [10]. In recent years, several communication schemes such as teleportation [11] and classical information transfer by quantum channels [12], have extensively been investigated both theoretically and experimentally, in CV systems, especially in Gaussian states [13–16]. However, it has been discovered that there are several protocols which cannot be implemented using Gaussian states with Gaussian operations. Examples include entanglement distillation [17], measurement-based universal quantum computation [18], teleportation [19], and quantum error correction [20].

Non-Gaussian states are increasingly being found to be important in several applications. They have also been realized in the laboratory [21].

An important method to make such states is by adding and subtracting photons, when the initial state is the squeezed vacuum state. Starting with the single-mode squeezed vacuum state, whose Wigner function [22] is always positive, it was shown that photon addition can generate a negative dip of the Wigner function in the phase space [23] and hence can deviate from being a Gaussian state. In the case of the two-mode squeezed vacuum (TMSV) state as input state, both entanglement and fidelity of teleportation can be increased by adding and subtracting photons to (from) one or two modes [24]. For such experiments, see [25,26]. Moreover, the entanglement content of the photon-added state obtained from the TMSV state was shown to be always higher than that of the photon-subtracted state [24].

Investigations of the squeezed vacuum state with respect to photon addition and subtraction are usually restricted to the two-mode case, even though the importance of the multimode CV system is unquestionable [27–30]. For example, several preparation schemes have been proposed for preparing of multimode cluster states which form one of the main ingredients in measurement-based quantum computation [29]. Moreover, it is believed that multimode entangled states can be a resource to build quantum communication networks, for transferring both classical as well as quantum information between several senders and several receivers [30].

In this paper, we consider the four-mode squeezed vacuum (FMSV) state as input to a de-Gaussification scheme. The latter is being carried out by adding and subtracting photons in different modes. We evaluate entanglement between different modes in all possible bipartitions and compare the results of the photon-added state with the subtracted ones. We call a mode as "player" mode when we analyze the effect on entanglement, by varying the number of photons added or subtracted in that mode. There could be several such player modes. The remaining modes, in which either no photons or a fixed number of photons are added or subtracted, are referred to as the spectator modes. We here investigate two scenarios: (1) one player mode and (2) two player modes. We analytically show that in the single player case, i.e., when photons have been added (subtracted) to (from) a single mode, in the player:spectator bipartition, entanglements of the photon-added and photon-subtracted states coincide. In this situation, we prove that entanglements in both photon-added and photon-subtracted states monotonically increase with the number of photons added or subtracted. Unlike the TMSV case, we observe that there exists scenarios in which photon-subtracted output states obtained by subtracting photons from one or two modes contain higher entanglement, compared to the photon-added state. Specifically, we find that the photon-subtracted state contains more entanglement in the spectator:rest bipartition than that of the photon-added state, when a single mode acts as a player. Similar hierarchy can also be obtained when any two modes act as players. Interestingly, the advantageous situation for photon subtraction can be reversed by adding a fixed number of photons in the spectator modes.

Such behavior can also be viewed by analyzing the logarithmic negativity of the output two party state which can be obtained by discarding either two player modes or two spectator modes.

Finally, we study a distance-based measure of non-Gaussianity in these scenarios and find that the non-Gaussianity in general is higher for the photon-added state than that of the photon-subtracted state. In the case of two modes, photon-added states are known to be more non-Gaussian than the photon-subtracted states. However, as already noticed in [24], the relation between entanglement and non-Gaussianity is not straightforward. In the four-mode case, we again find that the photon-added state has always higher non-Gaussianity than that of the photon-subtracted states, and hence reflects that entanglement and non-Gaussianity are possibly not directly connected.

One should note here that there are several experimentally viable methods available, by which a single photon can be added (subtracted) to (from) a given continuous variable state [25,26]. In recent years, it has been proposed that such schemes can be extended to the case of addition (subtraction) of a large number of photons to (from) a given state [31] by using currently available experimental techniques. The existence of possible experimental strategies to prepare and manipulate such states is also motivates us to investigate nonclassical properties of the multimode states after photon addition (subtraction).

The paper is organized in the following way. In Sec. II, we discuss briefly the N-mode squeezed vacuum state, and two special cases, the two-mode squeezed vacuum state, in Sec. II A, and the four-mode squeezed vacuum state, in Sec. II B. In Sec. II B 1, we consider the FMSV state, when m_i number of photons are added to or subtracted from the mode i. In Secs. III and IV, we briefly introduce a nonclassicality measure of a quantum state in a CV system, and the quantum correlation measures which are relevant in the paper, respectively. In Sec. V, we present the main results in which we systematically compare the entanglement of the four-mode photon-added state with that of the photon-subtracted state by considering the von Neumann entropy in different bipartitions. Another entanglement measure, the logarithmic negativity, for the photon-added and photon-subtracted states are evaluated in Sec. VI, while the behavior of non-Gaussianity for the output state is studied in Sec. VII. We summarize in Sec. VIII.

II. N-MODE SQUEEZED VACUUM STATE

In this section, we discuss the N-mode squeezed vacuum state (NMSV), specifically the two-mode and four-mode squeezed vacuum states, and a state obtained after adding (subtracting) an arbitrary number of photons at the mode i. These states are examples of entangled states in continuous variables which can be used in various quantum information tasks. To define such states, let us first denote the bosonic creation and annihilation operators at the mode i, as $\hat{a}_i^\dagger$ and $\hat{a}_i$, respectively, which satisfy the bosonic commutation relations, $[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij}$, and $[\hat{a}_i,\hat{a}_j] = 0$, $[\hat{a}_i^\dagger,\hat{a}_j^\dagger] = 0$. By using bosonic operators, an N-mode squeezing operator can be defined as

$$\mathcal{S}(\epsilon) = \exp\left[\frac{1}{2}\sum_{i=1}^{N}(\epsilon^*\hat{a}_i\hat{a}_{i+1} - \epsilon\hat{a}_i^\dagger\hat{a}_{i+1}^\dagger)\right], \tag{1}$$

where $\hat{a}_{N+1} = \hat{a}_1$. The corresponding NMSV state is given by

$$\begin{aligned}|\psi_N\rangle &= \mathcal{S}(\epsilon)|0_1 0_2\cdots 0_N\rangle \\ &= \frac{1}{N_S}\exp\left(-\frac{1}{2}\sum_{j,k=1}^{N}\hat{a}_j^\dagger \tanh(rQ)_{jk}\hat{a}_k^\dagger e^{i\theta}\right) \\ &\quad\times|0_1 0_2\cdots 0_N\rangle,\end{aligned} \tag{2}$$

where $|0_1 0_2\cdots 0_N\rangle$ is the N-mode vacuum state, N_S is a normalization constant, and $\epsilon = re^{i\theta}$, with r being the squeezing parameter. Here the matrix Q is obtained from the following relation:

$$\begin{aligned}\mathcal{S}(\epsilon)^\dagger\hat{a}_j\mathcal{S}(\epsilon) &= \sum_{k}^{N}[\cosh(rQ)_{jk}\hat{a}_k - \sinh(rQ)_{jk}e^{i\theta}\hat{a}_k^\dagger] \\ &\forall\ j = 1,\ldots,N.\end{aligned} \tag{3}$$

Let us now define the position and momentum operators for each mode, given by

$$q_i = (\hat{a}_i + \hat{a}_i^\dagger), \tag{4}$$

$$p_i = \frac{1}{i}(\hat{a}_i - \hat{a}_i^\dagger), \tag{5}$$

to show that Eq. (2) indeed represents a squeezed state. The variances of the N-mode quadrature operators, $X_1 = \frac{1}{2\sqrt{N}}\sum_j(\hat{a}_j + \hat{a}_j^\dagger)$ and $X_2 = \frac{1}{2i\sqrt{N}}\sum_j(\hat{a}_j - \hat{a}_j^\dagger)$, are given by

$$\Delta X_1^2 = \tfrac{1}{4}[e^{2r}\sin^2(\theta/2) + e^{-2r}\cos^2(\theta/2)] \tag{6}$$

and

$$\Delta X_2^2 = \tfrac{1}{4}[e^{2r}\cos^2(\theta/2) + e^{-2r}\sin^2(\theta/2)]. \tag{7}$$

Thus for $\theta = 0$ or π, we have $\Delta X_1\Delta X_2 = \frac{1}{4}$. However, for any one of the $i = 1,2$, $\Delta X_i \leqslant \frac{1}{2}$, while $\Delta X_i \geqslant \frac{1}{2}$ for the other i. This guarantees that the state given in Eq. (2) is a squeezed state. We assume $\theta = 0$ throughout the paper.

A. Two-mode squeezed vacuum state

The two-mode squeezed vacuum state can be obtained by putting $N = 2$ in Eq. (2), where $Q = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, and

$$\tanh rQ = \begin{pmatrix}0 & \tanh r\\ \tanh r & 0\end{pmatrix}. \tag{8}$$

Thus, the TMSV state with $\theta = 0$ is given by

$$\begin{aligned}|\psi_2\rangle &= \operatorname{sech} r\, e^{-\tanh r\hat{a}_1^\dagger\hat{a}_2^\dagger}|00\rangle \\ &= \operatorname{sech} r\sum_{n=0}^{\infty}(-\tanh r)^n|n\rangle|n\rangle,\end{aligned} \tag{9}$$

where $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n}}|0\rangle$ is the occupation number state.

Taking $|\psi_2\rangle$ as the initial state, the behavior of entanglement and non-Gaussianity after adding or subtracting photons have extensively been investigated [24].

B. Four-mode squeezed vacuum state

Let us now consider the FMSV state obtained by setting $N = 4$ in Eq. (2). The 4×4 matrix, Q, in this case, takes the form

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

The FMSV state with $\theta = 0$ is then given by [32]

$$|\psi_4\rangle = \frac{1}{\cosh r} e^{-(\tanh r/2)(\hat{a}_1^\dagger + \hat{a}_3^\dagger)(\hat{a}_2^\dagger + \hat{a}_4^\dagger)}|0000\rangle. \tag{11}$$

Expanding the exponential in Eq. (11), we have

$$|\psi_4\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\left(-\frac{\tanh r}{2}\right)^n \sum_{r_1,r_2=0}^{n}\sqrt{\binom{n}{r_1}\binom{n}{r_2}} \times |n-r_1\rangle|n-r_2\rangle|r_1\rangle|r_2\rangle. \tag{12}$$

Such an FMSV state can be prepared in the laboratory by using currently available technology [28]. In particular, to create an FMSV state, one has to first create a TMSV state, given in Eq. (9), which can be obtained by passing two single-mode squeezed vacuum states through a 50:50 beam splitter. After that, each part of the TMSV state along with two vacuum states are sent through two 50:50 beam splitters, resulting in the FMSV state.

1. Photon-added and photon-subtracted four-mode state

In this paper, we consider the FMSV state, $|\psi_4\rangle$, as an initial state and our aim is to find the characteristics of its entanglement and the measure of non-Gaussianity after adding and subtracting a finite number of photons. Suppose m_i number of photons are added at each mode i, with $i = 1,2,3,4$. Then the output four-mode (FM) state reads as

$$\begin{aligned}\left|\psi^{\rm add}_{\{m_i\}}\right\rangle &= \frac{1}{N^{\rm add}}\sum_{n=0}^{\infty}\left(-\frac{\tanh r}{2}\right)^n\sum_{r_1,r_2=0}^{n}\sqrt{\binom{n}{r_1}\binom{n}{r_2}}\sqrt{\frac{(n-r_1+m_1)!}{(n-r_1)!}}\sqrt{\frac{(n-r_2+m_2)!}{(n-r_2)!}}\sqrt{\frac{(r_1+m_3)!}{r_1!}}\sqrt{\frac{(r_2+m_4)!}{r_2!}} \\ &\quad\times|n-r_1+m_1\rangle|n-r_2+m_2\rangle|r_1+m_3\rangle|r_2+m_4\rangle \\ &\equiv \sum_{n=0}^{\infty}\sum_{r_1,r_2=0}^{n} p^{\{m_i\}}_{n,r_1,r_2}|n-r_1+m_1\rangle|n-r_2+m_2\rangle|r_1+m_3\rangle|r_2+m_4\rangle,\end{aligned} \tag{13}$$

where $N^{\rm add}$ is the normalization constant. Similarly, after subtracting $\{m_i\}$ $(i = 1,2,3,4)$ number of photons from each mode of the FMSV state, the resulting state is given by

$$\begin{aligned}\left|\psi^{\rm sub}_{\{m_i\}}\right\rangle &= \frac{1}{N^{\rm sub}}\sum_{n=M}^{\infty}\left(-\frac{\tanh r}{2}\right)^n\sum_{r_1=m_3}^{n-m_1}\sum_{r_2=m_4}^{n-m_2}\sqrt{\binom{n}{r_1}\binom{n}{r_2}}\sqrt{\frac{(n-r_1)!}{(n-r_1-m_1)!}}\sqrt{\frac{(n-r_2)!}{(n-r_2-m_2)!}}\sqrt{\frac{r_1!}{(r_1-m_3)!}}\sqrt{\frac{r_2!}{(r_2-m_4)!}} \\ &\quad\times|n-r_1-m_1\rangle|n-r_2-m_2\rangle|r_1-m_3\rangle|r_2-m_4\rangle \\ &\equiv \sum_{n=M}^{\infty}\sum_{r_1=m_3}^{n-m_1}\sum_{r_2=m_4}^{n-m_2} q^{\{m_i\}}_{n,r_1,r_2}|n-r_1-m_1\rangle|n-r_2-m_2\rangle|r_1-m_3\rangle|r_2-m_4\rangle,\end{aligned} \tag{14}$$

where $N^{\rm sub}$ is the normalization constant and $M = \max\{m_1+m_3, m_2+m_4\}$. The above two equations will help us to obtain several single- and two-mode reduced density matrices which are required to study the behavior of entanglement for a four-mode state in different bipartitions.

III. MEASURE OF NONCLASSICALITY IN CONTINUOUS VARIABLE SYSTEMS

The negative Wigner function of a given state indicates the nonclassical nature of the corresponding state while the positivity implies the opposite. On the other hand, it is known that the Wigner function of a Gaussian state is always positive [33]. Therefore, one can define a measure of non-Gaussianity or nonclassicality by measuring the departure of a given state, ρ, in a CV system from a Gaussian state. In terms of relative entropy distance, it is given by [34–36]

$$\delta_{NG}(\varrho) = S(\varrho||\varrho_G) = S(\varrho_G) - S(\varrho), \tag{15}$$

where $S(\eta||\sigma) = -\mathrm{tr}(\eta\ \log_2\sigma) - S(\eta)$, and ρ_G is a Gaussian state which has the same covariance matrix and first moment as ρ. Here, $S(\sigma) = -\mathrm{tr}(\sigma\ \log_2\sigma)$ is the von Neumann entropy of σ.

The von Neumann entropy, $S(\varrho_G)$, of any Gaussian state can be calculated by using its covariance matrix, σ. For an N-mode Gaussian state, ϱ_G, the von Neumann entropy is defined [14] as

$$S(\varrho_G) = \sum_{k=1}^{N} g(\nu_k), \tag{16}$$

where ν_k is the Williamson normal form of the covariance matrix of the N-mode Gaussian state ϱ_G, and the function $g(x)$ is given by

$$g(x) = -\frac{x+1}{2}\log_2\left(\frac{x+1}{2}\right) - \frac{x-1}{2}\log_2\left(\frac{x-1}{2}\right). \tag{17}$$

In this paper, we will take a Gaussian state as the input state, and after photon addition (subtraction), the diversion of the output state from the input Gaussian state will be quantified by δ_{NG}.

IV. QUANTUM CORRELATION MEASURES

Quantum correlation measures in bipartite systems, especially for two qubit systems, are well understood. Such quantifications include the von Neumann entropy of local density matrices for pure states [37], entanglement of formation [38], concurrence [39], and logarithmic negativity [40]. However, measures of quantum correlation in a multipartite scenario, both in discrete and CV systems, are limited [1,15]. To characterize entanglement in a CV system with multiple modes, one possibility is to compute von Neumann entropy in different bipartitions of modes. Another possibility is to study the logarithmic negativity of two modes which can be obtained after discarding all the modes except two. In this section, we briefly discuss the local von Neumann entropy and the logarithmic negativity in CV systems.

A. Entanglement of a pure state

Entanglement of a bipartite pure state can be defined by the von Neumann entropy of the reduced density matrix of a given state [37], i.e.,

$$E(|\psi\rangle_{AB}) = S(\rho_A), \tag{18}$$

where $\rho_A = \text{tr}_B(|\psi\rangle_{AB}\langle\psi|)$. In CV systems, entanglement of a two-mode pure state can be quantified by the von Neumann entropy of a single mode. The single-mode density matrix can be a matrix of infinite dimension which has to be diagonalized to evaluate its von Neumann entropy. The calculation of the entropy can be carried out after truncating the matrix to a large block. The block size is determined by checking for convergence of trace, with increasing block size, to unity up to a certain significant digit. We will discuss this issue in detail for a specific scenario.

In the multipartite domain, entanglement is difficult to characterize even for pure states [1]. However, if one divides a multipartite system into two blocks, then the entanglement between the two subsystems is the von Neumann entropy of one of the blocks, provided the system is in a pure state. The entanglement between two blocks of a multiparty state can capture entanglement distribution in the multipartite domain. Such a quantification has been extensively used in many-body systems [41]. Here, we divide the multimode system into two parts and investigate the behavior of the entanglement content in the bipartition by adding (subtracting) photons in various modes.

B. Logarithmic negativity

In CV systems, logarithmic negativity (LN) is an important entanglement measure [40]. For a state ρ_N, with $N = N_1 + N_2$ modes, it is given by

$$\text{LN}(\rho_N) = \log_2 \mathcal{N}(\rho_N), \tag{19}$$

where the negativity of the given state is given by

$$\mathcal{N}(\rho_N) = 1 + 2\left|\sum_i \mu_i\right|. \tag{20}$$

Here μ_i's are the negative eigenvalues of the partially transposed density matrix, $\rho_N^{T_{N_1}}$, where partial transposition is taken with respect to the N_1 modes [42]. As mentioned for the evaluation of the von Neumann entropy, LN is also calculated by truncating to a large block of the infinite dimensional matrix.

V. COMPARISON OF ENTANGLEMENT ENHANCEMENT BETWEEN PHOTON ADDITION AND SUBTRACTION

In this section, our aim is to investigate the effects on entanglement in different bipartitions, when photons are added (subtracted) in (from) different modes of a four-mode squeezed vacuum state. To study such behavior, we divide the modes into two different categories, viz., (1) player modes—the modes in which the number of photons that we add (subtract) varies, and (2) spectator modes—the modes in which either no photon or a fixed number of photons are added (subtracted) and hence plays a spectator role in the de-Gaussification process. A comparison has been made between the situations, when the m_i, $i = 1,2,\ldots$ photons are added in the player modes, and the scenario when the same number of photons are subtracted from the player modes. To execute such comparison, we introduce a quantity

$$\delta_{\mathcal{A}}^{E}(\{m_i\}) = E\left(\rho_{\mathcal{A}:\mathcal{B}}^{\text{add}\{m_i\}}\right) - E\left(\rho_{\mathcal{A}:\mathcal{B}}^{\text{sub}\{m_i\}}\right), \tag{21}$$

where $\mathcal{A}:\mathcal{B}$ is a bipartition with $\mathcal{A}\cap\mathcal{B} = \emptyset$. The positivity of $\delta^E(\{m_i\})$ implies that addition is better than subtraction from an entanglement perspective. It is clear that the behavior of $\delta_{\mathcal{A}}^E(\{m_i\})$ with $\{m_i\}$ depends on the number of player and spectator modes as well as the bipartite splits.

A. Photon added and subtracted with one player mode

Let us first consider a situation in which one mode acts as a player while the rest are the spectator modes. We first restrict ourselves in the 1 : 234 cut irrespective of the choice of the player mode. In this case, there exist three different possibilities of choosing a player mode: (a) first mode as player and the rest as spectators, (b) second mode as player, and (c) third mode as player (see Fig. 1). From Eqs. (13) and (14), it is clear that fourth mode as a player is equivalent with case (b), and hence we exclude this case.

1. Single player mode in the smallest bipartition

Suppose that we add or subtract m_1 photons in the first mode without putting any number of photons in the rest of the modes, as shown in Fig. 1(a). Here the first mode acts as a player. The reduced density matrices can be calculated from Eqs. (13) and (14), which read as

$$\rho_{1,m_1}^{\text{add}} = \frac{1}{N_1^{\text{add}}}\sum_{n=0}^{\infty}\frac{\tanh^{2n} r}{2^n}\sum_{r_1=0}^{n}\binom{n}{r_1}\frac{(n+m_1-r_1)!}{(n-r_1)!} \times |n+m_1-r_1\rangle\langle n+m_1-r_1| \tag{22}$$

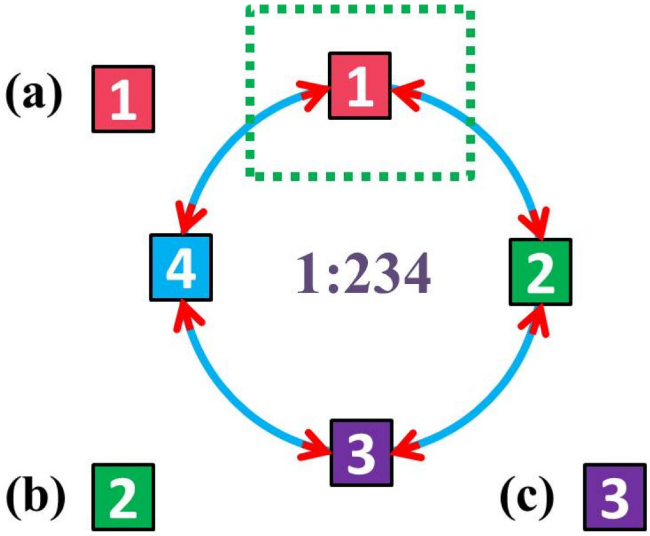

FIG. 1. Schematic diagram of choices of player and spectator modes as well as partitions. If we fix the bipartition to be 1:234 there are three nontrivial possibilities of choosing a single player in the photon-added and the photon-subtracted FM state. There are the cases (a)–(c), and the number in the square mentioned for each case is the mode at which the photon is added or subtracted.

for photon addition, and

$$\rho^{\text{sub}}_{1,m_1} = \frac{1}{N_1^{\text{sub}}} \sum_{n=0}^{\infty} \frac{\tanh^{2n} r}{2^n} \sum_{r_1=0}^{n} \frac{(m_1+r_1)!}{r_1!} \binom{n+m_1}{r_1+m_1} \times |r_1\rangle\langle r_1| \tag{23}$$

for photon subtraction. We now analytically establish that entanglement in the bipartition of the player and the spectator modes increases with the number of photons added.

Proposition. Entanglement increases with the addition of a single photon in a four-mode photon-added state, i.e.,

$$E\big(\big|\psi^{\text{add}}_{m_1+1}\big\rangle\big)_{1:234} \geqslant E\big(\big|\psi^{\text{add}}_{m_1}\big\rangle\big)_{1:234}, \tag{24}$$

where $|\psi^{\text{add}}_{m_1+i}\rangle$, $i = 0,1$ denotes the state in which $m_1 + i$ number of photons are added at the mode 1.

Proof. To evaluate entanglement in the 1:234 bipartition, we have to study the single-mode reduced density matrix $\rho^{\text{add}}_{1,m_1}$ of the four-mode state $|\psi^{\text{add}}_{m_1}\rangle$. To prove $E(|\psi^{\text{add}}_{m_1+1}\rangle)_{1:234} \geqslant E(|\psi^{\text{add}}_{m_1}\rangle)_{1:234}$, it is equivalent to show $S(\rho^{\text{add}}_{1,m_1+1}) \geqslant S(\rho^{\text{add}}_{1,m_1})$. After inserting the normalization constant in Eq. (22), we get

$$\rho^{\text{add}}_{1,m_1} = 2^{m_1} \frac{(1-x)^{m_1+1}}{(2-x)^{m_1}} \sum_{r_1=0}^{\infty} f(r_1,x) \binom{m_1+r_1}{m_1} \times |m_1+r_1\rangle\langle m_1+r_1| = \sum_{r_1=0}^{\infty} g(x,m_1,r_1) |m_1+r_1\rangle\langle m_1+r_1|, \tag{25}$$

where $x = \tanh^2 r$,

$$f(r,x) = \sum_{n=r}^{\infty} \frac{x^n}{2^n} \binom{n}{r} \tag{26}$$

and

$$g(x,m,r) = 2^m \frac{(1-x)^{m+1}}{(2-x)^m} f(r,x) \binom{m+r}{m}. \tag{27}$$

Therefore, entanglement in the player:spectator bipartition is given by $E(|\psi^{\text{add}}_{m_1}\rangle_{1:\text{rest}}) = S(\rho^{\text{add}}_{1,m_1}) = -\sum_{r_1=0}^{\infty} g(x,m_1,r_1) \log_2 g(x,m_1,r_1)$.

Now if we add one more photon to the state in Eq. (22), the entanglement is going to be $E(|\psi^{\text{add}}_{m_1+1}\rangle_{1:\text{rest}}) = -\sum_{r_1=0}^{\infty} g(x,m_1+1,r_1) \log_2 g(x,m_1+1,r_1)$. Let us now evaluate $g(x,m_1+1,r_1)$. It simplifies as

$$g(x,m_1+1,r_1) = \frac{2(1-x)}{(2-x)} g(x,m_1,r_1) + \frac{x}{2-x} g(x,m_1+1,r_1-1), \tag{28}$$

by using Pascal's identity, and the recursion relation of $f(r,x)$, which is given by

$$f(r,x) = \frac{x}{2-x} f(r-1,x). \tag{29}$$

Using the concavity of the function $h(x) = -x \log_2 x$, we get

$$h[g(x,m_1+1,r_1)] \geqslant \frac{2(1-x)}{(2-x)} h[g(x,m_1,r_1)] + \frac{x}{2-x} h[g(x,m_1+1,r_1-1)]. \tag{30}$$

Taking the sum over r_1 in both sides, we have

$$S\big(\rho^{\text{add}}_{1,m_1+1}\big) \geqslant \frac{2(1-x)}{(2-x)} S\big(\rho^{\text{add}}_{1,m_1}\big) + \frac{x}{2-x} S\big(\rho^{\text{add}}_{1,m_1+1}\big) \tag{31}$$

which immediately implies

$$S\big(\rho^{\text{add}}_{1,m_1+1}\big) \geqslant S\big(\rho^{\text{add}}_{1,m_1}\big). \tag{32}$$

Hence the proof. ■

Similarly one can also show that entanglement of the photon-subtracted state in the player:spectator split increases with the number of photons subtracted from the state.

We are now going to analyze the effects on entanglement under addition and subtraction of same number of photons.

Proposition. When a single mode acts as a player, entanglement between the player and the spectator modes of the photon-added state coincide with that of the photon-subtracted state.

Proof. To prove that the increase of entanglement in the multimode state is the same for addition and subtraction, we consider the single-mode reduced density matrix. The single site reduced density matrix of photon-subtracted state after inserting N_1^{sub} is given by

$$\rho^{\text{sub}}_{1,m_1} = (1-x)^{m_1+1} \sum_{r_1=0}^{\infty} \binom{m_1+r_1}{m_1} \times \underbrace{\sum_{n=r_1}^{\infty} \frac{x^n}{2^n} \binom{n+m_1}{r_1+m_1}}_{f_{\text{sub}}(r_1,m_1,x)} |r_1\rangle\langle r_1|, \tag{33}$$

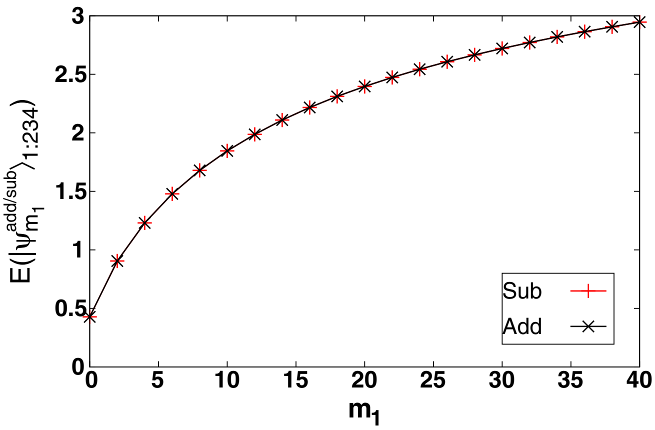

FIG. 2. Behavior of $E(|\psi^{\text{add}}_{m_1}\rangle_{1:234})$ and $E(|\psi^{\text{sub}}_{m_1}\rangle_{1:234})$ vs m_1. We add ($\times$) and subtract ($+$) up to 40 photons in (from) the first mode, and calculate entanglement in the $1:234$ bipartition, when no photons are added (subtracted) in (from) the spectator modes. As shown in the propositions, entanglement in both the cases increases monotonically with m_1 and they coincide. Here the entanglement is plotted in the unit of ebits while the abscissa is dimensionless.

where

$$f_{\text{sub}}(r_1,m_1,x) = \frac{x^{r_1}2^{m_1+1}}{(2-x)^{r_1+m_1+1}}, \tag{34}$$

which can be obtained by a recursion relation similar to that given in Eq. (29). On the other hand, the reduced density matrix after adding the same number of photons reads as

$$\rho^{\text{add}}_{1,m_1} = \sum_{r_1=0}^{\infty} 2^{m_1}\frac{(1-x)^{m_1+1}}{(2-x)^{m_1}}\frac{2}{2-x}\left(\frac{x}{2-x}\right)^{r_1} \times \binom{m_1+r_1}{m_1}|r_1+m_1\rangle\langle r_1+m_1|. \tag{35}$$

Comparing Eqs. (33) and (35), we have $S(\rho^{\text{add}}_{1,m_1}) = S(\rho^{\text{sub}}_{1,m_1})$. ■

To visualize the above propositions, we plot $S(\rho^{\text{add/sub}}_{1,m_1})$, with respect to m_1 by fixing the squeezing parameter $r = 0.4$ in Fig. 2. It clearly shows that the curve for photon addition merges with the curve of photon subtraction. Moreover, it shows that entanglement in that bipartition monotonically increases with the addition or subtraction of photons as shown in Proposition 1. Note here that although the results presented here are when the photons are added at mode 1 and the bipartition is considered as player:spectator mode, the propositions remain unaltered if another mode also acts as a player by keeping the similar bipartition.

2. Effects on entanglement due to change of partition

We now consider the entanglement in the same bipartition as in the previous case, i.e., $1:234$. However, the second or third mode now acts as player and no photons are added in the rest of the modes. In the previous case, one block contained only the player mode, while the other contains all the spectator modes. In this case, one part of the partition contains one spectator mode while the other one consists of both the player and the rest of the spectator modes. In the previous case, we have already shown that the effects on entanglement due to addition and subtraction of photons are similar. We will now show whether such observation remains invariant even in this scenario.

Let us now take the four-mode squeezed vacuum state as input, and add (subtract) m_2 photons in (from) the second mode. As depicted in Fig. 3(a), we find that unlike the previous case, the photon-subtracted state possesses more entanglement in the $1:234$ bipartition than that of the photon-added state. The ordering remains unchanged if one takes the third mode as player and considers entanglement in the $1:234$ split [see Fig. 3(b)]. Moreover, we observe that the amount of entanglement decreases in this scenario, compared to the case when the second mode acts as a player. Note here that if one takes the two-mode squeezed vacuum state as input, it was observed that the bipartite entanglement content of the photon-subtracted state is always lower than that of the photon-added state.

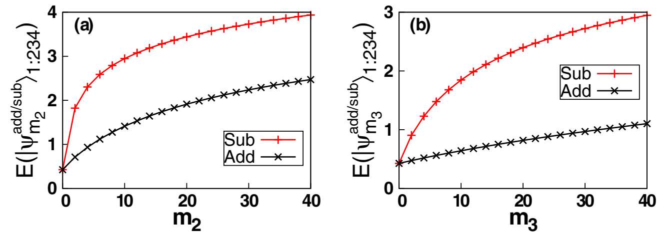

FIG. 3. (a) Trends of $E(|\psi^{\text{add}}_{m_2}\rangle_{1:234})$ and $E(|\psi^{\text{sub}}_{m_2}\rangle_{1:234})$ with the number of photons added (subtracted) in (from) the second mode. (b) A similar study has been carried out when the third mode acts as a player. Both the cases reveal that subtraction is better than addition. Ordinates are plotted in the unit of ebits, while the abscissas are dimensionless.

3. Bipartition with both player and spectator modes

We still restrict ourselves to the case of a single player. But we now move to the situations in which entanglement of a

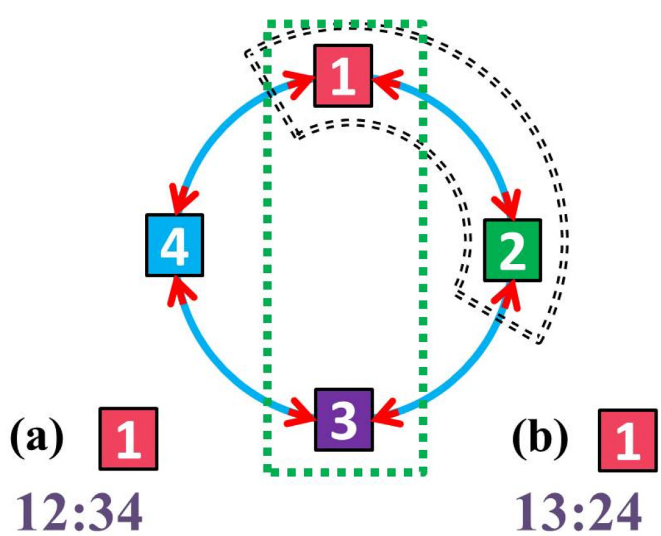

FIG. 4. Schematic diagram of two different blocks, when a single mode, specifically the first mode, acts as a player.

four-mode state is studied by considering a bipartition in which both sides of the split contain two modes, namely, 12:34 and 13:24. The other split between modes, i.e., 14:23, reflects a similar behavior, due to the symmetry of the four-mode state. In these two scenarios, photons are added or subtracted in the first mode, as shown in Fig. 4, and no photons are added or subtracted in the other spectator modes.

To study entanglement of $|\psi_{m_1}^{\text{add}}\rangle$ ($|\psi_{m_1}^{\text{sub}}\rangle$) in the 12:34 or 13:24 bipartition, we require the two party reduced density matrices $\rho_{12,m_1}^{\text{add}}$, $\rho_{13,m_1}^{\text{add}}$, $\rho_{12,m_1}^{\text{sub}}$, and $\rho_{13,m_1}^{\text{sub}}$.

We have

$$\rho_{12,m_1}^{\text{add}} = \frac{1}{N_{12}^{\text{add}}} \sum_{n,n'=0}^{\infty} \frac{x^{(n+n')/2}}{2^{n+n'}} \sum_{r_1,r_2=0}^{\min\{n,n'\}} \sqrt{\binom{n}{r_1}\binom{n}{r_2}} \sqrt{\binom{n'}{r_1}\binom{n'}{r_2}} \sqrt{\frac{(n+m_1-r_1)!}{(n-r_1)!}} \sqrt{\frac{(n'+m_1-r_1)!}{(n'-r_1)!}}$$
$$\times |n+m_1-r_1\rangle_1 |n-r_2\rangle_2 \langle n'+m_1-r_1|_1 \langle n'-r_2|_2 \quad (36)$$

and

$$\rho_{12,m_1}^{\text{sub}} = \frac{1}{N_{12}^{\text{sub}}} \sum_{n,n'=m_1}^{\infty} \frac{x^{(n+n')/2}}{2^{n+n'}} \sum_{r_1=0}^{\min\{n,n'\}-m_1} \sum_{r_2=0}^{\min\{n,n'\}} \sqrt{\binom{n}{r_1}\binom{n}{r_2}} \sqrt{\binom{n'}{r_1}\binom{n'}{r_2}} \sqrt{\frac{(n-r_1)!}{(n-m_1-r_1)!}} \sqrt{\frac{(n'-r_1)!}{(n'-m_1-r_1)!}}$$
$$\times |n-m_1-r_1\rangle_1 |n-r_2\rangle_2 \langle n'-m_1-r_1|_1 \langle n'-r_2|_2. \quad (37)$$

Note that in the previous cases, where one partition contains only a single mode, we required single-site density matrices to calculate the entanglement, and they are always diagonal in the number basis. The same is not the case for two-site density matrices. Similarly, one can find out the reduced density matrices of $\rho_{13,m_1}^{\text{add}}$ and $\rho_{13,m_1}^{\text{sub}}$. In both scenarios, we observe that entanglement increases against the number of photons added, m_1, and the same is true for subtraction of photons (see Fig. 5). Moreover, as observed in the previous case with the smallest partition consisting of the spectator mode, the photon-subtracted state contains higher entanglement in the 12:34 as well as 13:24 partitions than that of the corresponding photon-added state [see Figs. 5(a) and 5(b)].

We briefly mention here the method used to calculate $S(\rho_{12,m_1}^{\text{add}})$, and the other local entropies. The von Neumann entropy of $\rho_{12,m_1}^{\text{add}}$ can be obtained if one can diagonalize the infinite dimensional matrix, given in Eq. (36). To calculate it, for fixed m_1, we have to truncate the summation up to a large value of n and n', say N for both, and calculate its trace, i.e., $\text{tr}_N(\rho_{12,m_1}^{\text{add}})$, as well as von Neumann entropy, $S^N(\rho_{12,m_1}^{\text{add}})$. We then choose $2N$ as maximum of n and n' and obtain the quantities. When the difference between $S^N(\rho_{12,m_1}^{\text{add}})$ and $S^{2N}(\rho_{12,m_1}^{\text{add}})$ is of the order of 10^{-6}, we take $S^N(\rho_{12,m_1}^{\text{add}})$ as the actual entropy. In Fig. 6, for a fixed value of m_1, we plot $S^N(\rho_{12}^{\text{add}})$ and $\text{tr}_N(\rho_{12}^{\text{add}})$ with the variation of N. With the increase of m_1, we observe that we require higher values of N. However, the figure shows both the quantities converge when $N \geqslant 10$, irrespective of the value of m_1. When we compute entropy or LN, we always carry out a similar scaling analysis for choosing N.

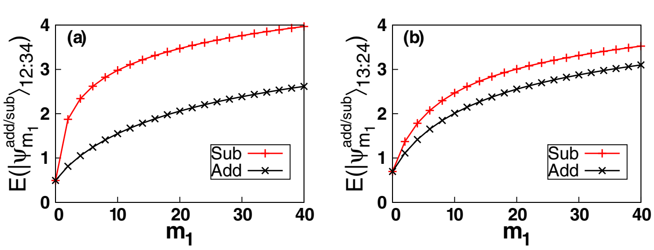

FIG. 5. Plots of entanglements of photon-added and photon-subtracted states in the 12:34 (a) and 13:24 (b) bipartitions with m_1. The ordinates are plotted in the unit of ebits, while the abscissas are dimensionless.

B. Behavior of entanglement of photon-added and photon-subtracted states with two player modes

In this section, keeping the four-mode squeezed vacuum state as the input state, we increase the number of players from one to two modes, and hence the possibilities of choosing the player modes with nontrivial bipartition grows substantially. For a fixed bipartition, we investigate the nature of entanglement by changing the modes in which photons are added or subtracted. Up to now, we have shown that the entanglement content of the resulting state after subtracting photons is either equal or higher than that of the photon-added states. Let us now investigate whether such situation persists when two modes are players.

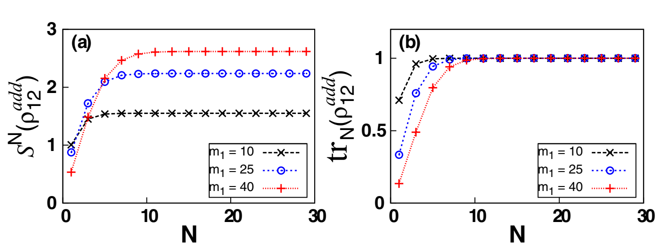

FIG. 6. Plot of convergence of von Neumann entropy $S^N(\rho_{12,m_1}^{\text{add}})$ in (a) and $\text{tr}_N(\rho_{12,m_1}^{\text{add}})$ in (b) against N which is the maximum value of n and n'. We choose three different values of m_1, viz., $m_1 = 10, 25, 40$. We find that, for example, for $m_1 = 40$, trace goes to unity and the entropy (entanglement) converges for $N \geqslant 10$. Here the von Neumann entropy is measured in the unit of bits and the trace and the abscissas are dimensionless.

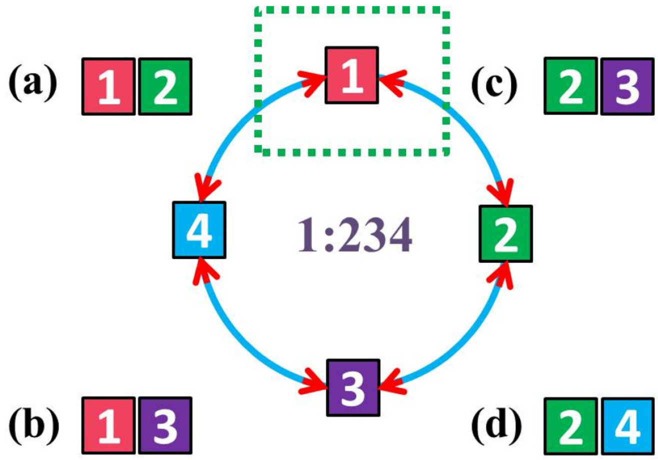

FIG. 7. Schematic diagram of four nontrivial possibilities of choosing two modes as players in the 1 : 234 bipartition. Other choices can be shown as repetitions due to the symmetry of the FM state.

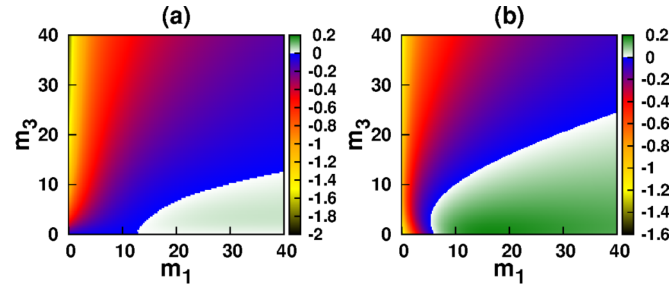

FIG. 8. Behavior of $\delta_1^E(m_1,m_3)$ against m_1 (horizontal axes) and m_3 (vertical axes). Panels (a) and (b) correspond to inactive and active spectator modes, respectively. In (b) we add (subtract) $m_2 = 5$ photons in (from) the second mode. If both second and fourth modes are active spectators, the region of $\delta_1(m_1,m_3)$ increases in the (m_1,m_3) plane. For example, if we choose $m_2 =$, and $m_4 =$, the region for which $\delta_1^E(m_1,m_3) > 0$ increases. All the axes are dimensionless, while entanglements are plotted in the unit of ebits.

1. One part of the bipartite split contains a single mode

We begin by concentrating on the entanglement of the FM state after the addition (subtraction) of photons in the 1 : 234 bipartition. In this scenario, there are four possibilities for adding and subtracting photons. As shown in Fig. 7, the modes that act as players are as follows: (a) the first and the second mode, (b) the first and the third mode, (c) the second and the third mode, and (d) the second and the fourth mode. Other possibilities can be reduced to any one of the above four cases due to the symmetry in the four-mode squeezed state. Moreover, it can be shown that the entanglement pattern of cases (a) and (b) are qualitatively similar, while cases (c) and (d) are analogous and hence the entanglement features will be studied in pairs.

Cases (a) and (b). We now consider the situation where either the first and the second modes act as players or the first and the third modes are players. We calculate the $\delta_1^E(m_1,m_i)$ $(i \neq 1)$, when no photons are added and subtracted from the spectator modes.

We observe that there exists a region for which $\delta_1^E(m_1,m_i) > 0$, which is in contrast with the case when one mode was player in the preceding subsection [see Fig. 8(a)]. As seen from the figure, for moderate values of m_1, the boundary between the positive and negative regions is almost a straight line and hence we can find the slope of the straight line which can help to study these cases quantitatively. We find that for high values of m_1, the slope of $\delta_1^E(m_1,m_3) = 0$ is approximately 0.28, which is small compared to the slope of $\delta_1^E(m_1,m_2) = 0$, which is 0.64. Moreover, we notice that $\max[\delta_1^E(m_1,m_3)] = 0.2 < \max[\delta_1^E(m_1,m_2)] = 0.4$, while the minimum value of $\delta_1^E(m_1,m_3)(=-2.0)$ is smaller than that of $\delta_1^E(m_1,m_2)(=-1.6)$, in the regions surveyed. Therefore, we can conclude that to create maximal entanglement in this scenario, photon addition is advantageous when one adds photons in the first and the second modes compared to the case of m_1 and m_3 being players (with $m_1 \gg m_i$, $i = 2,3$).

In both cases, spectator modes play an important role in the behavior of entanglement in the 1 : 234 bipartition. As depicted in Fig. 8(b), entanglement in the photon-added state can be increased by adding photons in the spectator modes. For example, when $m_{2(4)} = 5, \delta_1^E(m_1,m_3)$ against m_1 and m_3 is depicted in Fig. 8(b). A quantitative comparison can be made between Figs. 8(a) and 8(b). In particular, for $m_1 \gg m_3$, the region with $\delta_1^E(m_1,m_3) > 0$ when no photons are added (subtracted) in the spectator modes can be calculated. In this limit, we assume that the boundary is a straight line and hence the area is the area of a quadrilateral. Let us call the area as Δ_0. In this case, we calculate the area of the quadrilateral when $m_1 \geqslant 25$ and $m_1 \leqslant 40$, and we find $\Delta_0 \approx 160$. After adding (subtracting) five photons in the second or fourth modes, we find that the area Δ_5 of the corresponding quadrilateral increases and $\Delta_5 \approx 253$.

Behavior of entanglement in the 1 : 234 split for cases (c) and (d) are almost identical with the previous cases. The only difference is that entanglement of the subtracted state is always better than that of the added state when spectator modes are inactive. The picture changes, i.e., entanglement of the photon-added states starts increasing faster than the photon-subtracted states, like in the preceding cases, when fixed numbers of photons are added (subtracted) in the spectator mode(s).

2. Bipartition containing equal number of modes

We will now consider the case where we still keep two modes as players but we now divide four modes into two blocks consisting of two modes instead of one mode in the preceding discussion. In this case, the two nontrivial bipartitions are 12 : 34 and 13 : 24. Let us first concentrate on the bipartition 13 : 24. In this case, the symmetry of the FMSV state after the addition or subtraction of an arbitrary number of photons in all the modes, given in Eqs. (13) and (14), ensures that there are only two nontrivial situations in the case of two player modes (see Fig. 9). They are (a) when the players are the first and the second modes, and (b) when the first and third modes act as players. Cases (a) and (b) show similar entanglement behavior like previous cases, when one part of the bipartition contains a single mode, and hence we only discuss the situation when two spectator modes are active, which has not been analyzed before.

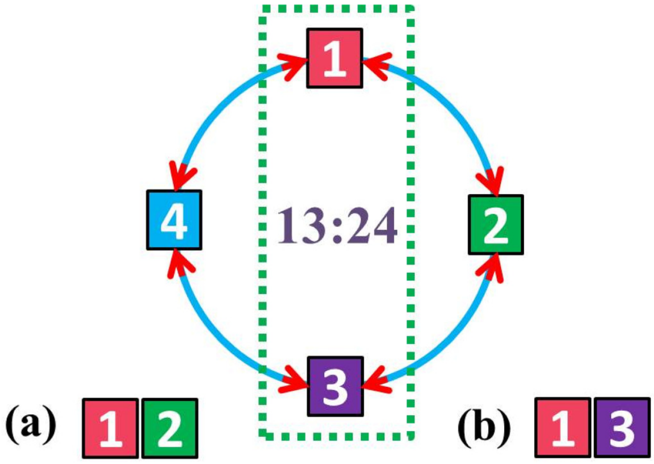

FIG. 9. Distinct scenarios of two player modes in the 13 : 24 split. There are two possibilities: (a) first and second as players and (b) first and third as players.

Case (a). The reduced density matrix of the first and the third mode, for the photon-added state, is given by

$$\rho^{\text{add}}_{13,\{m_i\}} = \text{tr}_{24}\big(\big|\psi^{\text{add}}_{\{m_i\}}\big\rangle\big\langle\psi^{\text{add}}_{\{m_i\}}\big|\big) = \sum_{n=0}^{\infty}\sum_{r_1=0}^{n} a_{n,r_1,q}|n+m_1-r_1\rangle_1|m_3+r_1\rangle_3 \times\langle n+m_1-q|_1\langle m_3+q|_3, \tag{38}$$

where we write

$$a_{n,r_1,q} = \frac{1}{N^{\text{add}}_{13}}\frac{x^n}{2^n}\sum_{r_2=0}^{n}\left[\binom{n}{r_1}\binom{n}{q}\right]^{1/2}\binom{n}{r_2} \times\left(\frac{(n+m_1-r_1)!}{(n-r_1)!}\frac{(n+m_1-q)!}{(n-q)!}\frac{(m_3+r_1)!}{r_1!}\frac{(m_3+q)!}{(q)!}\right)^{1/2} \times\frac{(n+m_2-r_2)!}{(n-r_2)!}\frac{(m_4+r_2)!}{r_2!}. \tag{39}$$

Similarly, one can also find the two party reduced density matrix, $\rho^{\text{sub}}_{13,\{m_i\}}$, for photon subtraction by tracing out the second and fourth modes in Eq. (14).

If the first and second modes act as players, we find that subtraction is always better than addition for arbitrary values of m_1 and m_2. This case is similar to the case with a single mode being player and cases with second and third modes or second and fourth modes being players. To show once more that spectators play a fundamental role in interchanging the entanglement property for photon addition and subtraction, we elaborate the analysis in two scenarios: (1) when a fixed number of photons are added (subtracted) in a single spectator mode, a positive region emerges, which indicates that the quantum correlation in the 13 : 24 bipartition is greater for photon addition than that for subtraction, as already seen before. An interesting point to note here is that a positive region appears for small values of m_2 and almost for all values of m_1. This is probably due to the fact that we add photons in the third mode which belongs to the same block as the first mode. (2) When both the spectator modes are active, the

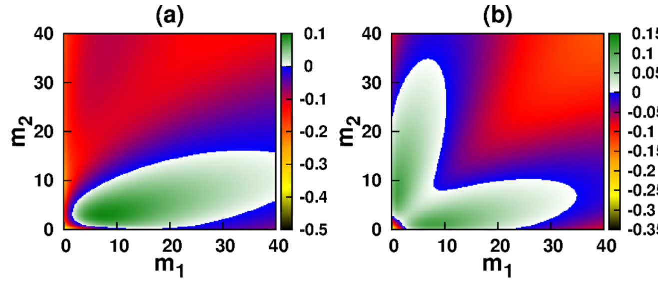

FIG. 10. Role of spectator modes in $\delta^E_{13}(m_1,m_2)$. In (a), $m_3 = 10$ and $m_4 = 0$, while in (b) $m_3 = m_4 = 5$. We see that spectator modes help to enhance entanglement in the photon-added state. All the axes are dimensionless, while entanglements are plotted in the unit of ebits.

positive region can be seen in both the axes due to symmetry present in the FM state, as depicted in Fig. 10(b).

Finally, we concentrate on a nontrivial partition, the 12 : 34 cut (see Fig. 11). From the perspective of entanglement, this partition is unique. In this scenario, there are three ways to choose the players. We find that with and without the participation of spectator modes, the entanglement of photon subtraction is always higher or equal to that of the photon addition which makes this situation exclusive from others.

VI. COMPARISON OF LOGARITHMIC NEGATIVITY BETWEEN TWO-MODE AND FOUR-MODE STATES

Up to now, we have considered an FMSV state as input and have compared the behavior of entanglement between photon-added and photon-subtracted states as well as entanglement of an output state in different bipartitions having different player and spectator modes. In this section, our aim is to make a comparison between the output state obtained from the TMSV state after adding or subtracting photons and the two-mode state obtained from the FMSV state. To perform such comparison, we discard two modes from the four-mode state and calculate the LN of the two-mode reduced state, which we then compare with the LN of the photon-added (subtracted) state that is obtained from the TMSV state as the input [24].

When the input state is a TMSV state, the output state after photon addition or subtraction remains a pure state. When the

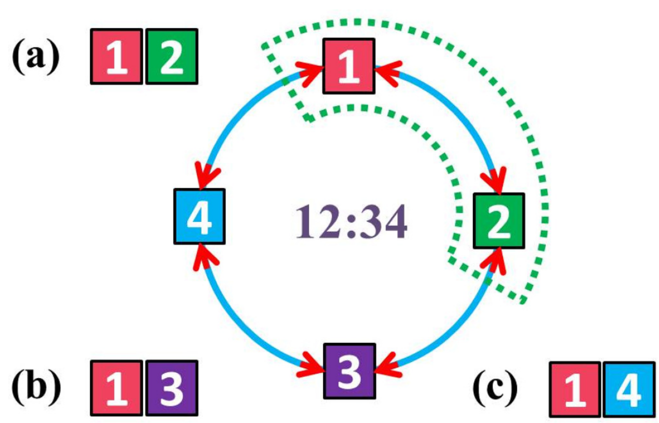

FIG. 11. Schematic diagram of choices of two player modes in the 12 : 34 split.

input state is a FMSV state, the state after photon addition and subtraction is again a pure state, but of four modes. The entanglement of this four-mode state can be analyzed in several ways. If one analyzes the entanglement properties of this state in bipartite partitions, then entanglement entropy serves as a good measure. However, if one wishes to analyze its entanglement properties between two modes, when the other two modes have been traced out, then one needs to use a measure that is well defined and computable for two-mode mixed quantum states. One is then naturally led to use the logarithmic negativity. To perform a comparison between the entanglements of the two-mode output states obtained from the TMSV state as input, and of the two-mode state obtained from the corresponding FMSV state as input, we use the logarithmic negativities for both. This is because the former state is a pure state, while the latter is a mixed one, and logarithmic negativity is well defined and computable for both the states.

In the case of the TM state, the output state, after adding (subtracting) photons, still remains pure and hence LN can be calculated analytically [28]. However, for the FM case, the output state is mixed which is obtained by discarding two modes and we adopt the same mechanism as we have done to calculate the von Neumann entropy of reduced density matrices, described in Sec. V A 3. In particular, we evaluate LN as well as trace for large $n = N$, and then by increasing N, we check whether trace goes to unity up to six decimal points. We truncate the system when trace has already converged to unity, up to six decimal points.

In the TMSV case, photons can be added to either of the modes or to both the modes. On the other hand, there are several scenarios for the four-mode states. If there is a single player, either one of the modes of the output state can act as player or none of the modes of the output state is the player. In the case of two players, (i) two players can be the two modes of the output state, (ii) one mode of the output state can be a player, or (iii) the discarded modes can be the player modes.

Before considering the FMSV state, let us first consider the TMSV state as input. Note that the nature of LN qualitatively matches with the von Neumann entropy of the reduced density matrix. As shown in [24], when the single mode acts as player, the LN for photon addition coincides with the subtraction, which is also the case for the von Neumann entropy. If both the modes act as players, photon addition is always more beneficial for entanglement than photon subtraction [24], which can be seen from the behavior of LN of the output state.

In the case of a single player or two players in the FM state, if the output state contains the player mode(s), then the reduced two-mode state obtained from the photon-added state has higher LN than that of the photon-subtracted state. Hence, the behavior of the LN of the output state from the TM and FM states is identical. As we have shown, this is not the case if we consider the behavior of entanglement of the pure four-mode output state in bipartitions. Figure 12 depicts the behavior of the LN of the two-mode reduced state from the four-mode output state when the first mode acts as player as well as both the modes of the two-mode state are players. In all these situations, no photons are added (subtracted) in the spectator modes. We observe that when there is a single player, e.g., the first mode of the reduced state, entanglement increases (decreases) monotonically, if

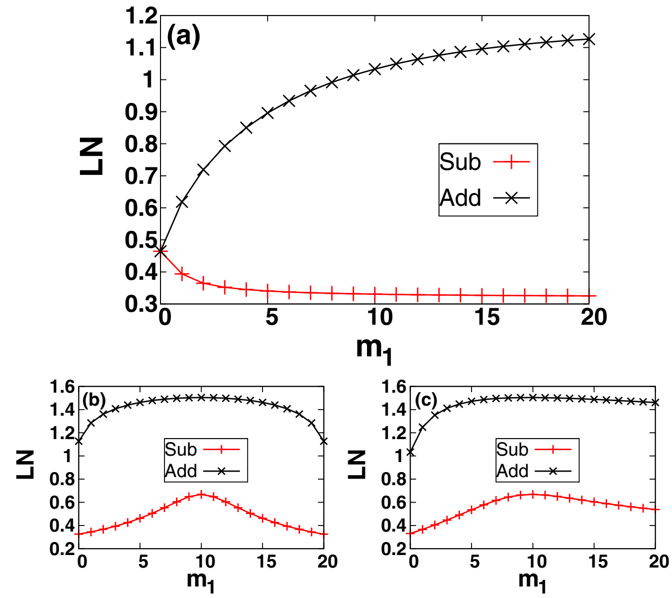

FIG. 12. LN between the first and second modes obtained from the FM output state. (a) First mode is player. (b) First and second modes are players with $m_1 + m_2 = 20$. (c) First mode is player while the second one is spectator with $m_2 = 10$. The ordinates are plotted in the unit of ebits, while the abscissas are dimensionless.

photons are added (subtracted). However, such monotonicity with respect to the number of photons added (subtracted) is lost if photons are added (subtracted) in both the modes with total number of photons being fixed as shown in Fig. 12(b). A similar qualitative feature in entanglement is seen when the first mode acts as player while second mode is a spectator having a fixed finite number of photons [see Fig. 12(c)]. We find that the bipartite entanglement reaches its maximum with respect to m_1, when an equal number of photons are added (subtracted) in both the modes, i.e., $m_1 = m_2$, in Fig. 12(b) and $m_1 \approx m_2$ in Fig. 12(c).

Lastly, we consider the scenario when we add and subtract photons in the discarded modes, i.e., in the third and fourth modes, and we find the LN between the first and the second modes, which are spectators. The LN of the output state decreases if one of the discarded modes acts as a player. For example, by taking the third mode as player, we plot the LN of the first and the second mode with m_3 in Fig. 13(a). Unlike previous cases, the LN of the photon-subtracted state is higher than that of the added state when $m_3 \geqslant 9$, which can never be observed for the TM case. The LN of the photon-subtracted state is more pronounced than that of the added one if both the discarded modes act as players. The same number of photons are added (subtracted) in (from) both the spectator modes, i.e., $m_3 = m_4$, as shown in Fig. 13(b) in which $\text{LN}(\rho_{34}^{\text{sub}}) \geqslant \text{LN}(\rho_{34}^{\text{add}})$.

VII. NONCLASSICALITY MEASURE OF THE PHOTON-ADDED (-SUBTRACTED) FM STATE

As mentioned in the Introduction, the photon addition and subtraction is one of the ways to create a non-Gaussian state. In this section, we quantify the departure of the photon-added

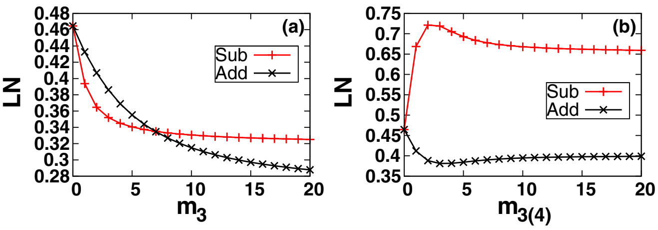

FIG. 13. (a) The nature of LN of the first and second modes against the number of photons added (subtracted) in (from) the third mode which has been traced out. (b) LN of the same state in which third as well as fourth modes act as players. The equal number of photons are added (subtracted) in (from) both the modes. The ordinates are plotted in the unit of ebits, while the abscissas are dimensionless.

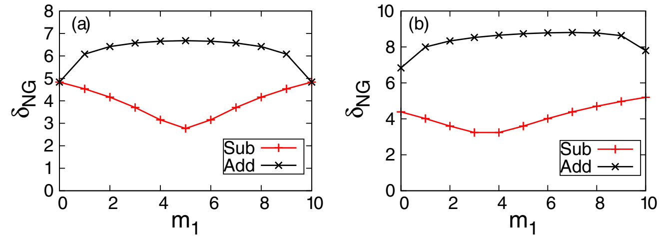

FIG. 14. Behavior of non-Gaussianity measure δ_{NG} against m_1. (a) First and second modes are players with $m_1 + m_2 = 10$. Spectator modes are ineffective. (b) Spectator modes are active with $m_3 = 4$ and in player modes, $m_1 + m_2 = 10$. The ordinates are plotted in the unit of bits, while the abscissas are dimensionless.

(-subtracted) FMSV state from Gaussianity, as a function of added (subtracted) photons from the player modes, which was introduced in Sec. III.

Since the photon-added (-subtracted) FM state is in a pure state, the second term of $\delta_{NG}(\rho)$, given in Eq. (15), vanishes. To calculate $\delta_{NG}(\rho)$, we have to find the covariance matrix of ρ_G, which is same as $\rho^{\text{add/sub}}_{\{m_i\}} = |\psi\rangle\langle\psi|^{\text{add/sub}}_{\{m_i\}}$. It is given by

$$\sigma_\varrho = \begin{pmatrix} \langle q_1^2\rangle I & \langle q_1q_2\rangle\sigma_z & \langle q_1q_3\rangle I & \langle q_1q_4\rangle\sigma_z \\ \langle q_1q_2\rangle\sigma_z & \langle q_2^2\rangle I & \langle q_2q_3\rangle\sigma_z & \langle q_2q_4\rangle I \\ \langle q_1q_3\rangle I & \langle q_2q_3\rangle\sigma_z & \langle q_3^2\rangle I & \langle q_3q_4\rangle\sigma_z \\ \langle q_1q_4\rangle\sigma_z & \langle q_2q_4\rangle I & \langle q_3q_4\rangle\sigma_z & \langle q_4^2\rangle I \end{pmatrix}, \quad (40)$$

where $q_i = \hat{a}_i + \hat{a}_i^\dagger$, and the expectations are taken over the photon-added and photon-subtracted FM states, given in Eqs. (13) and (14) (for details, see the Appendix).

The Williamson normal form of Eq. (40) can be evaluated by using the prescription given in [43]. We numerically calculate the Williamson normal form of the matrix in Eq. (40) for both photon addition and subtraction and calculate the non-Gaussianity, which in this case reduces to $S(\rho^{\text{add/sub}}_{G,\{m_i\}})$.

In all the cases, photon addition leads to a more rapid departure of Gaussianity than that of the photon subtraction. We also notice that if among four modes, photons are added only in two modes, then the behavior of δ_{NG} obtained in the FM state and the TM state are qualitatively similar. It is clear from the behavior of the non-Gaussianity measure that the photon-subtracted state slowly becomes non-Gaussian as compared to the photon-added state and the behavior remains unchanged irrespective of the choices of the player and the spectator modes (see Fig. 14). The rich picture of the role of different modes, captured by entanglement, is not seen by the non-Gaussianity measure and hence indicates that there is possibly no direct connection between non-Gaussianity and entanglement content of the output state obtained after photon addition (subtraction) [24].

VIII. CONCLUSION

Photon addition and subtraction constitute useful methods to prepare non-Gaussian states. It has already been established that non-Gaussian states are useful in various quantum-mechanical tasks ranging from entanglement distillation to quantum error correction. We have investigated the entanglement properties of the non-Gaussian states generated by adding or subtracting photons in Gaussian states. In the case of two-mode states, entanglement of photon-added states are known to be equal or higher than that of the photon-subtracted ones.

We have shown that this is not the case when one increases the number of modes. We found that for four-mode states, the trend of entanglement distribution in different bipartitions of the photon-added (-subtracted) states is much richer than that in the two-mode states. Specifically, we showed that there exists a scenario, in which multimode entanglement content of the photon-subtracted state is always higher than that of the corresponding photon-added one. The results remained unchanged even if one discarded two modes from the four-mode output state. Moreover, we showed that the picture that emerges from entanglement of the output state does not match with the behavior in the same states of distance-based non-Gaussianity measures. Up to now, it was known that among addition and subtraction, addition is more beneficial. But our work shows that photon subtraction can also be advantageous if we consider a state of a higher number of modes.

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APPENDIX

The expectations used in Eq. (40), for the calculation of non-Gaussianity, taken over the photon-added and photon-subtracted states are given below.

$$\langle q_1^2\rangle^{\text{add}} = 1 + 2m_1 + 2\sum_{n,r_1,r_2}\left(p^{\{m_i\}}_{n,r_1,r_2}\right)^2(n - r_1), \quad (A1)$$

$$\langle q_2^2\rangle^{\text{add}} = 1 + 2m_2 + 2\sum_{n,r_1,r_2}\left(p^{\{m_i\}}_{n,r_1,r_2}\right)^2(n - r_2), \quad (A2)$$

$$\langle q_3^2\rangle^{\text{add}} = 1 + 2m_3 + 2\sum_{n,r_1,r_2}\left(p^{\{m_i\}}_{n,r_1,r_2}\right)^2 r_1, \quad (A3)$$

$$\langle q_4^2\rangle^{\text{add}} = 1 + 2m_4 + 2\sum_{n,r_1,r_2}\left(p^{\{m_i\}}_{n,r_1,r_2}\right)^2 r_2, \quad (A4)$$

$$\langle q_1 q_2\rangle^{\text{add}} = 2\sum_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n+1,r_1,r_2}\sqrt{(n+m_1-r_1+1)(n+m_2-r_2+1)}, \tag{A5}$$

$$\langle q_1 q_3\rangle^{\text{add}} = 2\sum_{n,r_2}\sum_{r_1=0}^{n-1} p^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1+1,r_2}\sqrt{(n+m_1-r_1)(m_3+r_1+1)}, \tag{A6}$$

$$\langle q_1 q_4\rangle^{\text{add}} = 2\sum_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n+1,r_1,r_2+1}\sqrt{(n+m_1-r_1+1)(m_4+r_2+1)}, \tag{A7}$$

$$\langle q_2 q_3\rangle^{\text{add}} = 2\sum_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n+1,r_1+1,r_2}\sqrt{(n+m_2-r_2+1)(m_3+r_1+1)}, \tag{A8}$$

$$\langle q_2 q_4\rangle^{\text{add}} = 2\sum_{n,r_1}\sum_{r_2=0}^{n-1} p^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1,r_2+1}\sqrt{(n+m_2-r_2)(m_4+r_2+1)}, \tag{A9}$$

$$\langle q_3 q_4\rangle^{\text{add}} = 2\sum_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n+1,r_1+1,r_2+1}\sqrt{(m_3+r_1+1)(m_4+r_2+1)}, \tag{A10}$$

where $\sum_{n,r_1,r_2}$ in the photon-added states, is the short form of $\sum_{n=0}\sum_{r_1=}^{n}\sum_{r_2=0}^{n}$. Further, for the photon-subtracted states,

$$\langle q_1^2\rangle^{\text{sub}} = 1-2m_1+2\sum_{n,r_1,r_2}\left(q^{\{m_i\}}_{n,r_1,r_2}\right)^2(n-r_1), \tag{A11}$$

$$\langle q_2^2\rangle^{\text{sub}} = 1-2m_2+2\sum_{n,r_1,r_2}\left(q^{\{m_i\}}_{n,r_1,r_2}\right)^2(n-r_2), \tag{A12}$$

$$\langle q_3^2\rangle^{\text{sub}} = 1-2m_3+2\sum_{n,r_1,r_2}\left(q^{\{m_i\}}_{n,r_1,r_2}\right)^2 r_1, \tag{A13}$$

$$\langle q_4^2\rangle^{\text{sub}} = 1-2m_4+2\sum_{n,r_1,r_2}\left(q^{\{m_i\}}_{n,r_1,r_2}\right)^2 r_2, \tag{A14}$$

$$\langle q_1 q_2\rangle^{\text{sub}} = 2\sum_{n,r_1,r_2} q^{\{m_i\}}_{n,r_1,r_2} q^{\{m_i\}}_{n+1,r_1,r_2}\sqrt{(n-m_1-r_1+1)(n-m_2-r_2+1)}, \tag{A15}$$

$$\langle q_1 q_3\rangle^{\text{sub}} = 2\sum_{n,r_2}\sum_{r_1=m_3}^{n-m_1-1} q^{\{m_i\}}_{n,r_1,r_2} p^{\{m_i\}}_{n,r_1+1,r_2}\sqrt{(n-m_1-r_1)(r_1-m_3+1)}, \tag{A16}$$

$$\langle q_1 q_4\rangle^{\text{sub}} = 2\sum_{n,r_1,r_2} q^{\{m_i\}}_{n,r_1,r_2} q^{\{m_i\}}_{n+1,r_1,r_2+1}\sqrt{(n-m_1-r_1+1)(r_2-m_4+1)}, \tag{A17}$$

$$\langle q_2 q_3\rangle^{\text{sub}} = 2\sum_{n,r_1,r_2} q^{\{m_i\}}_{n,r_1,r_2} q^{\{m_i\}}_{n+1,r_1+1,r_2}\sqrt{(n-m_2-r_2+1)(r_1-m_3+1)}, \tag{A18}$$

$$\langle q_2 q_4\rangle^{\text{sub}} = 2\sum_{n,r_1}\sum_{r_2=m_4}^{n-m_2-1} q^{\{m_i\}}_{n,r_1,r_2} q^{\{m_i\}}_{n,r_1,r_2+1}\sqrt{(n-m_2-r_2)(r_2-m_4+1)}, \tag{A19}$$

$$\langle q_3 q_4\rangle^{\text{sub}} = 2\sum_{n,r_1,r_2} q^{\{m_i\}}_{n,r_1,r_2} q^{\{m_i\}}_{n+1,r_1+1,r_2+1}\sqrt{(r_1-m_3+1)(r_2-m_4+1)}, \tag{A20}$$

where $\sum_{n,r_1,r_2}$ is the short form of $\sum_{n=M}\sum_{r_1=m_3}^{n-m_1}\sum_{r_2=m_4}^{n-m_2}$, and $M=\max\{m_1+m_3,m_2+m_4\}$.

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