One-dimensional Bose-Einstein condensation of photons in a microtube

Alex J. Kruchkov*

Laboratory for Quantum Magnetism (LQM), École Polytechnique Fédérale de Lausanne (EPFL), Station 3, CH-1015 Lausanne, Switzerland

(Received 6 January 2016; revised manuscript received 17 February 2016; published 11 April 2016)

This paper introduces a quasiequilibrium one-dimensional Bose-Einstein condensation of photons trapped in a microtube. Light modes with a cutoff frequency (a photon's "mass") interact through different processes of absorption, emission, and scattering on molecules and atoms. In this paper we study the conditions for the one-dimensional condensation of light and the role of photon-photon interactions in the system. The technique in use is the Matsubara Green's functions formalism modified for the quasiequilibrium system under study.

DOI: 10.1103/PhysRevA.93.043817

I. INTRODUCTION

Light in vacuum is quantized as massless photons, which in equilibrium obey Bose-Einstein statistics. If the photons were massive, they could break the gauge symmetry under certain conditions and condense to the lowest energy state sharing a single wave function. However, the light nowadays is considered to be massless, so can one expect photons to form the Bose-Einstein condensate (BEC)? A short answer is yes.

To observe it, the group of Weitz [1,2] used an optical microcavity, where the spectra of light modes have a cutoff due to a geometrical constraint. This cutoff acts as an effective mass for a two-dimensional (2D) photon. Dimensionality here refers to the motional degree of freedom of photons. Although the 2D photons now possess a mass, it is not enough: There is no BEC transition in a uniform two-dimensional system. The condition of uniformity, however, is broken by the slight curvature of the cavity walls, so the trapped light can be mapped on a 2D field of massive nonrelativistic quasiparticles experiencing harmonic potential [2], the system that is known to undergo the BEC transition. In the experiments [2], the controllable thermalization process [1,3] picks up a single light mode, and small photon losses are compensated by a weak external laser pumping. Therefore, it was shown that the number of photons is conserved on average, and the researchers can keep the system close to its thermodynamical equilibrium. The quasiequilibrium BEC of photons is observed at room temperatures [2,4].

The system being argued to be different from conventional lasers[1] becomes of interest for various theoretical [5–20] and experimental studies [4,21,22]. The growing experimental and theoretical interest in the topic requires broadening the variety of systems for which the condensation of photons could be observed. In particular, it was explicitly discussed for dimensionalities $D = 2$ (see, e.g., [3,6,8,23]) and in different contexts for $D = 3$ (see, e.g., [7,24,25]), but never in one dimension. Therefore, there is a need to complete the study for photons with the one-dimensional degree of freedom.

The theoretical methods applied to the system strongly vary. Some authors are using a phenomenological nonlinear Schrödinger equation (Gross-Pitaevskii equation) in different forms [2,11,15]. The noninteracting $T \neq 0$ theory is also in use in different forms [2,6–8]. The fully off-equilibrium condensate is studied with either an effective kinetic equation with Jaynes-Cummings interaction [9,19] in the approximation with real-time propagators or the off-equilibrium $T \neq 0$ Green's function formalism (Schwinger-Keldysh formalism, complex-time propagators, Matsubara frequencies, etc.) [10,12–14]. In my opinion, the Schwinger-Keldysh formalism is the most general and the most powerful approach here.

This paper introduces the one-dimensional quasiequilibrium condensation of photons in a microtube. In my opinion, the Matsubara Green's function formalism is appropriate here for the near-equilibrium system. Of course, the Schwinger-Keldysh formalism may be used here but once the near-equilibrium properties of the system are well understood. Matsubara's formalism describes the finite-temperature close-to-equilibrium systems, and it is valid in the symmetrical phase. The main advantage of the approach for this study is that one can calculate the critical parameters of the interacting system. In this paper, I write the Hamiltonian, which takes into account one-photon and two-photon processes of interaction with atoms, and treat it perturbatively. As a result, I can describe the influence of indirect photon-photon interactions on the critical parameters.

There are some limitations of the model that I use. First, I do not study the thermalization process, and the time evolution of the system in general, bounding myself to the steady state only. Second, I restrict myself to the (first two) leading corrections to self-energy, which, in terms of direct photon-photon interactions, if they were present, would correspond to the Hartree-Fock mean-field theory. The system under study is a bit more subtle and to obtain these effectively mean-field contributions one needs to go to the fourth order in perturbation theory. For the same reasons, optical collisions, i.e., two-atom mechanical collisions leading to creation of photons, are not taken into account, even though the model in use can do it. Other conditions of validity, which shape up the

*alex.kruchkov@epfl.ch

[1]Photon BECs and lasers are different in several ways in their underlying physical mechanisms, coherence, and statistical and thermodynamical properties. In particular, in contrast to lasers, the photon BECs do not require the inversion of the atomic population. The photon BEC is a thermalized system; i.e. the temperature of the cavity is assigned to photon statistics. The macroscopic population of the mode in case of photon BECs is a consequence of equilibrium Bose statistics. The observed coherence of photon BECs is reported to be competitive with some lasers. For further details, please consider Refs. [2–6,9,21].

model, are discussed in the main text and the footnotes as they appear.

This paper is organized into three sections followed by appendices. In the first section of the main body of the paper (Sec. II), the effective mass of light modes is introduced in tubes with varying geometry; then, I discuss the conditions for condensation in 1D and estimate critical parameters. The next section (Sec. III) deals with the interacting theory, where the effective Hamiltonian of photon-photon interaction is derived; this section is the heart of the paper and is organized into subsections for more comfortable reading: It describes the perturbation theory for both the uniform case and the trapped case. The summarizing section is organized more like an outline and discussion, yet the deeper study of the problem is still needed.

This paper should be considered as an introduction of the concept of light condensation in one dimension, complemented by a collection of relevant estimates, not as a strict and general theory from the first principles.

II. GENERAL IDEA: LIGHT TRAPPING AND CONDENSATION

To condense photons in a cavity means to transform their states into the lowest-energy thermodynamical state in the system [2,3,6,8]. In this section I skip the details of the thermalization process because they are studied sufficiently well [1,3,9,10], thus restricting myself to mentioning the three important ideas. First, the losses of photons are compensated by a weak external pumping, so the number of photons conserves on average. Second, the cavity gives a discrete set of light modes with different cutoffs. Third, it is possible to thermalize one of the modes, hence ensuring the single cutoff frequency for all the thermalized photons. As a result, supporting only one of the modes, we ascribe the effective mass to a photon as it is described in the first subsection of this section.

The second challenge for condensing photons in 1D is to choose the shape of the waveguide (microtube) where the condensation is possible. This choice is done in the context of the noninteracting model in the second subsection. At the end of this section, we discuss the conditions for condensation and estimate the critical number of photons for the set of parameters that are similar to those from Refs. [1–4].

A. Light trapping and effective mass of photons

For simplicity, we consider here the waveguides made of a microtube with axial symmetry, as it is sketched in Fig. 1. The shape of the tube in the general case is given by a rather smooth function $\rho(z)$ (see Fig. 1). Due to the cylindrical symmetry, a photon's energy $\hbar\omega$ is described by two quantum numbers k_z and k_ρ,

$$\hbar\omega(\mathbf{k}) = \hbar\tilde{c}|\mathbf{k}| = \hbar\tilde{c}\left(k_z^2 + k_\rho^2\right)^{1/2}, \tag{1}$$

where ω is the frequency of a photon with the momentum $\mathbf{k}$ decomposed for convenience into longitudinal k_z and polar k_ρ components. In the microtube the polar wave number k_ρ is strongly discrete while the longitudinal component k_z can be taken to be continuous because of the strong inequality $R_0 \ll l$.

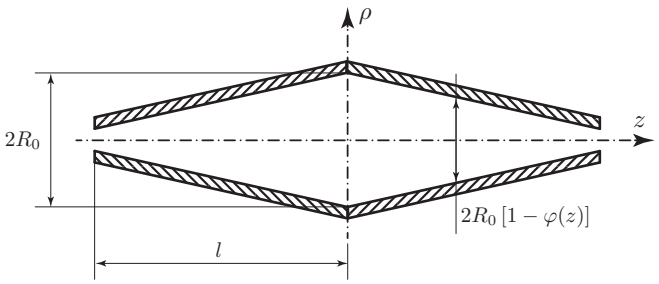

FIG. 1. A scheme of a microtube waveguide for trapping photons. The shape of the tube is determined by the relative deviation $\varphi(z)$ of the inner radius. To ease visual presentation, the function $\varphi(z)$ is taken to be linear.

In the general case, the set of k_ρ's follows from Maxwell equations in the microtube shaped as $\rho(z)$ with the boundary conditions on its walls. We consider a tube with the closed ends.[2] For the mirror walls one gets

$$k_\rho(z) = \frac{q_{mn}}{\rho(z)}, \tag{2}$$

where q_{mn} is the nth root of Bessel function of the mth order, $J_m(q_{mn}) = 0$ (see, e.g., [26]). The formula (2) was obtained in the approximation of a tube with a slightly changing cross-section radius,

$$\rho(z) = R_0[1 - \varphi(z)], \quad \varphi(z) \ll 1, \tag{3}$$

where R_0 is the radius of the tube at $z = 0$; see Fig. 1. The dimensionless quantity $\varphi(z)$ shows the small relative deviation of the tube's radius. Due to the strong requirement $R_0 \ll l$, where l is the half length of the microtube, one expects $\overline{k_z} \ll k_0$, where $k_0 = q_{mn}/R_0$ is the minimal polar wave number. As a consequence, expression (1) can be asymptotically expanded,

$$\hbar\omega \simeq \hbar\tilde{c}k_0\left[1 + \frac{k_z^2}{2k_0^2} + \varphi(z)\right]. \tag{4}$$

This expression can be rewritten in a more intuitive way,

$$\hbar\omega \simeq \hbar\omega_0 + \frac{\hbar^2 k_z^2}{2m^*} + V(z), \tag{5}$$

which reminds one of a particle with a mass m^* and one-dimensional degree of freedom k_z in the field of external potential $V(z)$. In our case, the effective mass of a photon, as follows from comparison between Eqs. (5) and (4), is defined as

$$m^* = \frac{\hbar\, q_{mn}}{\tilde{c}\, R_0} \tag{6}$$

and is related to the cutoff frequency ω_0 as $\hbar\omega_0 = m^*\tilde{c}^2$, which is a measure of the minimum energy of photons. The trapping

[2]The approach of this subsection uses the slowly varying envelope model (parabolic approximation), which is not valid for description of EM field near the reflection cross section and open ends of the waveguide. The presented model will be therefore valid only for the sufficiently long microtube with closed ends.

(pseudo)potential, caused by the geometry of the reflective inner surface, is

$$V(z) = \frac{q_{mn}\hbar\tilde{c}}{R_0}\varphi(z). \tag{7}$$

Thus, both the effective mass of a photon inside the tube and the effective potential take their origin from the specific geometry under consideration or, strictly speaking, from the $k_\rho(z)$ component of a photon's wave vector.

Summarizing the main idea of the subsection, one can say that the system of photons trapped inside the microtube can be considered as an ensemble of quasiparticles with the mass m^* and the one-dimensional degree of freedom $\kappa = k_z$ placed in the potential $V(z)$. The form of the potential $V(z)$ is determined by the shape of the microtube waveguide $\varphi(z)$. Therefore, changing the shape of this waveguide, one can change the trapping potential.

B. Noninteracting theory and critical number of photons

The noninteracting model is good for a primal estimate. In this model, the photons in the microtube are considered to be noninteracting particles with the one-dimensional degree of freedom. The total number of photons is given by integrating the Bose-Einstein distribution over the configurational space. The condensation condition can be expressed as follows: The chemical potential of photons μ_0 at the critical point reaches the minimum energy of photons in the system, i.e., $\mu_0 = \hbar\omega_0$ (see [2,6–8]). For the ideal photon gas with the one-dimensional degree of freedom, the critical number of particles for Bose-Einstein condensation can be estimated in Wigner approximation,

$$N_0 = \int \frac{dk_z dz}{2\pi} g^* \left\{ \exp\left[\frac{\hbar^2 k_z^2/2m^* + V(z)}{T} \right] - 1 \right\}^{-1}, \tag{8}$$

where g^* takes into account the possible degeneracy in photon energy (see, e.g., [7,8]). The integral in (8) may or may not converge, which is a consequence of the Bogoliubov theorem stating, in particular, that there is no BEC in dimensions below three if the system is uniform. However, the presence of the external potential can be considered as nonhomogeneity and the integral in (8) is convergent for certain types of potentials; for example, in the case of the one-minimum symmetrical potentials, the singularity is integrable if only the dimensionless potential $\varphi(z) = V(z)/\hbar\omega_0$ grows slower than a parabolic function,

$$\varphi(z) = |z/L_0|^\alpha, \quad \alpha \in (0,2), \tag{9}$$

where L_0 is a parameter in units of length. Even though one can imagine more sophisticated potentials (for example, multiple-minimum potentials), for simplicity, we restrict ourselves to the case of the one-minimum potential of the form (9). To calculate the integral (8), we introduce new variables ξ and η, such as

$$\xi^2 = \frac{\hbar^2 k_z^2}{2m^* T}, \quad \eta^\alpha = \frac{\hbar\omega_0}{T}\left|\frac{z}{L_0}\right|^\alpha, \tag{10}$$

and after some hackneyed algebra we obtain the expression for the critical number of photons in the system to observe the phase transition at temperature T,

$$N_0(T;\alpha) = g^* \frac{2\sqrt{2}}{\pi}\frac{L_0\omega_0}{\tilde{c}}\left(\frac{T}{\hbar\omega_0}\right)^{\frac{1}{2}+\frac{1}{\alpha}} I(\alpha), \tag{11}$$

where we have introduced the dimensionless normalization integral,

$$I(\alpha) = \int_0^\infty \int_0^\infty \frac{d\xi\, d\eta}{\exp(\xi^2 + \eta^\alpha) - 1}. \tag{12}$$

The normalization integral remains finite while the trapping parameter is $\alpha \in (0,2)$. The dependence $I(\alpha)$ is shown in Fig. 2. It is noteworthy that there is a minimum at the trapping parameter $\alpha_{\min} = 0.71, I(\alpha_{\min}) = 1.9$. However, a more fruitful trapping parameter is $\alpha = 1$, i.e., $\varphi(z) \propto |z|$, where the value of the normalization integral (as also other quantities of noninteracting and interacting theory) can be calculated analytically, $I(1) = \Gamma(3/2)\zeta(3/2)$. In this case, the expression (11) for the critical number of photons simplifies,

$$N_0 = \sqrt{\frac{2}{\pi}}\, g^* \zeta(3/2)\frac{L_0\omega_0}{\tilde{c}}\left(\frac{T}{\hbar\omega_0}\right)^{3/2}. \tag{13}$$

Taking into account the explicit expression for ω_0 and taking rough values $\zeta(3/2) \approx 2.6$, $g^* \approx 3$, formula (13) can be simplified and given in terms of direct experimental parameters,

$$N_0 \approx \left(\frac{T}{\hbar\tilde{c}k_\Lambda}\right)^{3/2}, \quad k_\Lambda = \left(\frac{q_{mn}}{4\pi^2 R_0 L_0^2}\right)^{1/3}. \tag{14}$$

Formula (14) defines the critical number of photons in the tube with V-like trapping profile, $\varphi(z) \propto |z|$. Such a biconical waveguide can be indeed manufactured[3] [27]. It is remarkable that the $T^{3/2}$ dependence, as in formula (14) for a 1D gas of

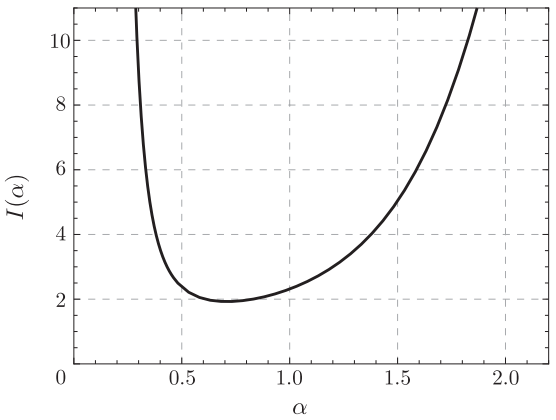

FIG. 2. Normalization integral $I(\alpha)$ as a function of the trapping parameter $\alpha, V(z) \propto |z|^\alpha$. The local minimum is situated at $\alpha_{\min} = 0.71$. The two asymptotes (not shown) are $I(\alpha) \to +\infty$ as $\alpha \to 0$ and $\alpha \to 2$, limiting the region of desirable trapping parameters to $\alpha \in (0,2)$, where the condensation of an ideal gas of photons is allowed in one dimension.

[3]In this paper, I also have not considered the finite-size effects, which could favor the condensation in a microtube with the closed ends.

particles under V-like potential, also holds for a 3D uniform gas of bosons. This similarity arises from the composition of the Wigner integral (8).

An estimate for the biconical waveguide gives the following. For definiteness, we take the lowest Bessel root $q_{01} \approx 2.4$. The radius of the waveguide should be such to ensure closeness of cutoff and the atomic transition frequencies, giving $R_0 \approx q_{01}\lambda\!\!\!^{-}_{\rm at}$, where $\lambda\!\!\!^{-}_{\rm at}$ is the atomic transition wavelength reduced by 2π. Taking now for estimate $\lambda\!\!\!^{-}_{\rm at} \sim 10^{-6}$ m, $\tilde{c} \approx 2.2 \times 10^8$ m/s, $L_0 \sim 10^{-2}$ m, and the room temperature $T = 300$ K, the threshold number of photons to trigger condensation is $N_0 \sim 10^4$, which is even smaller than the one reported for the 2D condensation, $N_0 \sim 10^5$. Thus, one may conclude that even at room temperatures the one-dimensional condensation of photons is possible.

Here, however, I should mention a very important feature of the finite-number BECs: Strictly speaking, there are no "phases" and "phase transitions" and, therefore, the finite-number consideration can smoothen different singularities in a significant way. Indeed, in my calculations above, in the experiments on photon condensation, and in the many experiments on atomic BECs the number of particles under consideration is mesoscopic, 10^3–10^6. The issue on the Bose-Einstein condensation with a finite number of particles was studied by Ketterle and Van Druten [28] in the early era of alkali-metal atoms BEC experiments. The inference of their paper [28] can be summarized as follows: There is no fundamental difference between the thermodynamical and finite-number BEC if the density of states of the particles is correctly approximated. In particular, they show that if the treatment with thermodynamical-limit integrals is convergent, the finite-number effects modify physical quantities by just a few percent even for the extremely mesoscopic case $N \sim 10^2$. To avoid the confusion, throughout the paper we consider the case with $\alpha \approx 1$ when the integrals are converging fast; however, we do not redefine the quantity $g^*(k)$ for the sake of simplicity. Therefore, the present model should be considered only as a main-order estimate.

C. Fluctuation-free photon gas below T_C

In this subsection, we consider several estimates concerning some of the properties of the photon gas below the critical temperature. For simplicity, we consider only $\alpha = 1$, and we will work with the time-averaged physical quantities. The number of atoms in the condensate in an ideal-gas model is given by

$$N_{k_z=0}(T < T_C) \approx N - \sqrt{\frac{2}{\pi}}\, \mathrm{g}^* \zeta(3/2) \frac{L_0 \omega_0}{\tilde{c}} \left(\frac{T}{\hbar\omega_0}\right)^{3/2}, \quad (15)$$

where the quantities $N_{k_z=0}$ correspond to the time-averaged number of photons in the condensate, i.e., neglecting the condensate fluctuations.[4] In this model, the spectral density of the ideal gas of photons is

$$\nu_{k_z} = N_{k_z=0}\, 2\pi\delta(k_z) + \nu_{k_z\neq 0}, \qquad \int \frac{dk_z}{2\pi}\nu_{k_z} \equiv N, \quad (16)$$

where $\delta(x)$ is the Dirac δ function of the argument x. The factor of 2π was introduced with the δ function to fulfill the convention in use [see the normalization condition in (16)]. The spectral density of noncondensed photons $\nu_{k_z\neq 0}$ can be calculated similarly to the case with 2D system (see [8]). In the case of V-like potential, $V(z) = \hbar\omega|z|/L_0 = u|z|$, one has

$$\nu_{k_z\neq 0} = 2\mathrm{g}^* \frac{T}{u} \ln\left[1 - \exp\left(-\frac{\hbar^2 k_z^2}{2m^* T}\right)\right]^{-1}. \quad (17)$$

We emphasize that the dependence of the spectral density of photons (17) on the "kinetic energy" in a 1D system with $\alpha = 1$ potential is the same as in the case of a 2D system with $\alpha = 2$ (harmonic) potential, the difference is only in the normalization prefactors; see Ref. [8].

We calculate quasiclassically the mean value of the longitudinal wave vector squared,

$$\langle k_z^2\rangle = \frac{1}{N}\int_{-\infty}^{+\infty} k_z^2\, \nu_{k_z} \frac{dk_z}{2\pi} = 1.07 \frac{g^*}{N}\left(\frac{m^*}{\hbar^2}\right)^{3/2} \frac{T^{5/2}}{u}. \quad (18)$$

The deviation in the frequency of photons is therefore given by

$$\hbar\langle\omega - \omega_0\rangle = 0.54 \frac{g^*}{N}\frac{\sqrt{m^*}}{\hbar u} T^{5/2}. \quad (19)$$

At first glance, this formula looks quite different from the 2D case (see Ref. [8]); however, one can verify that it also reduces to the form $\hbar\langle\omega - \omega_0\rangle \sim \frac{N_{k_z\neq 0}}{N} T$, with the amount of noncondensed photons given by $N_{k_z\neq 0} = \lim_{\delta\to 0^+} \int_\delta^\infty \nu_{k_z}\, dk_z/\pi$, thus depending crucially on the fraction of the condensate. The formula (19), however, does not take into account the quantum effects, which will be estimated further, and therefore it is valid only approximately at $T \neq 0$.

Let us also estimate some of the thermodynamical properties of the photons in condensate according to the ideal-gas model from Ref. [8]. The total energy of photons in a partially condensed state is given quasiclassically by

$$E = N_{k_z=0}\hbar\omega_0 + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dz dk_z}{2\pi} \frac{g^*\,\hbar\omega(k_z,z)}{\exp\left[\frac{\hbar^2 k_z^2}{2m^*T} + \frac{u|z|}{T}\right] - 1}$$
$$= N\,\hbar\omega_0 + 1.13 g^* \left(\frac{2m^*}{\hbar^2}\right)^{1/2} \frac{T^{5/2}}{u}. \quad (20)$$

The heat capacity of the photon gas with condensate can be found as a temperature derivative,

$$C(T < T_C) = 2.83 g^* \left(\frac{2m^*}{\hbar^2}\right)^{1/2} \frac{T^{3/2}}{u} \propto T^{3/2}. \quad (21)$$

[4] As it was explained from the first principles of statistical physics in Refs. [5,6], the conservation of photons requires an accurate account of free photons, photons absorbed by atoms, and the "hidden" photons, with a subsequent calculation of the chemical potential. In the present study, I consider a simplified model similar to Refs. [3,7,8]. The effect of the "hidden" photons can be quite significant: During the nonequilibrium condensation, these photons can lead to the strong fluctuations in photon number, observed by Schmitt *et al.* [21] and predicted previously by Sob'yanin [6].

Now we consider the condensate of photons at sufficiently "low" temperatures, $T \ll T_0$, or at the fixed temperature T but in the limit $N/N_0 \gg 1$, so with the good accuracy the photons can be considered to be in the coherent state. In this limit their behavior can be described by a collective wave function. In ideal-gas approximation, the evolution of the wave function is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi(z,t)}{\partial t} = \left[-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial z^2} + V(z)\right]\psi(z) = 0. \tag{22}$$

To solve this equation, we first make the phase rotation $\psi(z,t) = e^{-i\mu t}\psi(z)$, where μ is the chemical potential of the noninteracting Bose-Einstein condensate in the thermodynamical limit $N/N_0 \to \infty$, to obtain the stationary Schrodinger equation,

$$\left[-\frac{\hbar^2}{2m^*}\frac{d^2}{dz^2} + V(z) - \mu\right]\psi(z) = 0, \tag{23}$$

with the two boundary conditions $\psi'(0) = 0$ and $\psi(\infty) = 0$. The solution of this equation for $V(z) = u|z|$ is straightforward,

$$\psi(z) = \sqrt{\rho}_0\,\mathrm{Ai}\left(\frac{|z|}{a} + \varphi_1\right), \quad a = \left(\frac{\hbar^2}{2m^*u}\right)^{1/3}, \tag{24}$$

where $\mathrm{Ai}(x)$ is Airy function of the first kind, $\sqrt{\rho}_0$ is a normalization factor, and $\varphi_1 = -1.0188$ is the first zero of the Airy prime, $\mathrm{Ai}'(\varphi_1) = 0$. Choosing the normalization condition $\int_{-\infty}^{+\infty}|\psi(z,t)|^2 = N$, one obtains

$$\rho_0 = 1.71\frac{N}{a} = 1.71\,N(2m^*u/\hbar^2)^{1/3}. \tag{25}$$

Finally, the chemical potential of the condensate μ can be found from Eqs. (23) and (24),

$$\mu = -ua\varphi_1 = |\varphi_1|(\hbar^2u^2/2m^*)^{1/3}. \tag{26}$$

Therefore, the wave function of light BEC in this regime behaves like (see also Fig. 3)

$$\begin{aligned}\psi(z,t) \simeq\ & 1.47\,N^{1/2}(m^*u/\hbar^2)^{1/6}\\ &\times\mathrm{Ai}(\sqrt[3]{2m^*u/\hbar^2}\,|z| + \varphi_1)\\ &\times\exp[-i|\varphi_1|(\hbar^2u^2/2m^*)^{1/3}t + i\Phi_0].\end{aligned} \tag{27}$$

There are deep physical consequences of the above-derived formulas. The physical parameters of the light condensate in the coherent state for $\alpha = 1$ are the following. First, the condensate is mainly localized in the region of $2a$, with $a \sim (\hbar^2/2m^*u)^{1/3}$, the chemical potential of the light condensate is of order $\mu \sim ua \sim \hbar\omega_0$, and the average density at the center of the condensate (number of photons per unit length) is given by $\rho_0 \sim N/a$. These results for arbitrary α can be generalized by dimensional analysis of the differential equation (23) with $V(z) = u(\alpha)|z|^\alpha$, giving in the general case $a \sim (\hbar^2/2m^*u)^{1/(2+\alpha)}$ and $\mu \sim u(\alpha)a^\alpha$; the wave function, however, can be found only numerically.

Finally, let us discuss some more statistical properties of the photon condensate in the coherent phase. First, let us consider the variance

$$\int_{-\infty}^{+\infty} dz z^2|\psi(z)|^2 = 0.438\rho_0a^3 \tag{28}$$

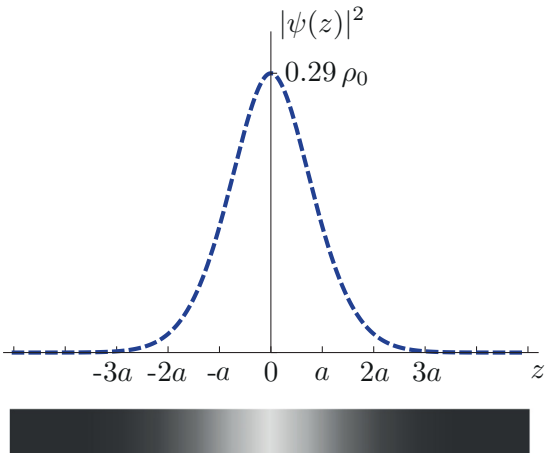

FIG. 3. Density of photons $|\psi(z)|^2$ in the condensate as a function of distance z in the case of V-like potential $V(z) = u|z|$ and (below) the corresponding density plot of light intensity inside the microtube. The wave function $\psi(z)$ is normalized on the total number of free photons in the system N. The density ρ_0 and space units a are explicitly given in the main text. The condensate is mostly localized in the area $2a$ around the center of the microtube ($z = 0$).

(the expectation value of z is zero due to the symmetry of the wave function). The standard deviation is therefore given by

$$\langle\Delta z\rangle \equiv \sigma_z = 0.971a\sqrt{N}. \tag{29}$$

Using the Heisenberg uncertainty principle, we find $\langle\Delta k_z\rangle \geqslant 1/(2\langle\Delta z\rangle)$. Therefore, the uncertainty in the longitudinal wave number is

$$\langle\Delta k_z\rangle \sim \frac{1}{a\sqrt{N}}. \tag{30}$$

This allows us to estimate roughly the principal uncertainty in the energy of the emitted Bose-Einstein-condensed light from a microtube as $\Delta\omega \sim \frac{1}{N}(\hbar^2u^2/m^*)^{1/3}$.

III. INTERACTING THEORY

As we have seen in the previous section, the noninteracting model is good. It works for a range of potentials and predicts the BEC transition of photons in one dimension. The noninteracting model would be exact for photons in a vacuum where their scattering cross section is negligible. However, the photons in the system under study do interact with each other, albeit indirectly. These interactions arise from the multiple acts of scattering, absorption, and emission of photons. One can classify all these processes by the number of photons involved in a single act and then construct a hierarchy of irreducible acts (the events, which cannot be represented as a product of two different acts). This hierarchy defines the form of an effective interacting Hamiltonian, which then is treated perturbatively.

To build up the perturbation theory, first I write down the Hamiltonian of the irreducible interactions, which is done in the first subsection. In the second subsection, the renormalized Green's functions of photons are derived for the uniform (nontrapped) case. The effect of the trapping potential is considered in the third subsection. Finally, the last subsection gives contributions for all the one-photon and two-photon

processes. This section is rich in physics and intended to be longer.

A. Interacting Hamiltonian and observables

The interacting Hamiltonian $\mathcal{H}$ should include processes of absorption, emission, and scattering of photons on atoms. It can be naturally written in a second-quantized form. For this, we introduce the operators of creation $\phi^\dagger_{\kappa_\mathbf{k}}$ and annihilation $\phi_{\kappa_\mathbf{k}}$ of a photon as a massive quasiparticle with the one-dimensional degree of freedom $\kappa_\mathbf{k} = (\mathbf{k}^2 - k_0^2)^{1/2} \equiv \kappa$. In the cylindrical microtube, as we have already shown, these quasiparticles have the quadratic dispersion law $\hbar\omega_\kappa = \hbar\omega_0 + \hbar^2\kappa^2/2m^*$ and are placed in the field of trapping potential. Consequently, the second-quantized Hamiltonian is given in a general form as

$$\mathcal{H} = \sum_\mathbf{k} \hbar\omega_{\kappa_\mathbf{k}} \phi^\dagger_{\kappa_\mathbf{k}} \phi_{\kappa_\mathbf{k}} + \sum_{\mathbf{k},\mathbf{q}} V_\mathbf{q}\, \phi^\dagger_{\kappa_{\mathbf{k}+\mathbf{q}}} \phi_{\kappa_\mathbf{k}} + \mathcal{H}_I + \mathcal{H}_{\rm at}, \quad (31)$$

where $V_\mathbf{q}$ is the Fourier transform of the trapping potential $V(z)$ and $\mathcal{H}_I$ reflects the photon-atom interactions and $\mathcal{H}_{\rm at}$ is the two-level-atoms Hamiltonian.[5] Here are some examples of the elementary interaction acts: A photon can be absorbed by an atom in the ground state; a photon can be emitted by an atom in an excited state; a photon can be scattered by an atom (or in second-quantized language destroyed and then created again). To describe these events quantomechanically, an adequate atomic model in use is the so-called two-level model, where an atom can be in two states: the ground state $|E_{\sigma_1}(\mathbf{p})\rangle$ and the excited state $|E_{\sigma_2}(\mathbf{p})\rangle$. The validity of this model for this study is satisfied by the two reasons, mainly: First, the cutoff of photons is set closer to the chosen atomic transition, $\hbar\omega_0 \approx \hbar\omega_{\rm at}$, and the other transitions are energetically separated; second, the number of photons is small in comparison to the number of atoms, $N_\phi \ll N_{\rm at}$. As a consequence, the probability of exciting the higher states is strongly suppressed and can be neglected in the main approximation.

We introduce the operators of creation and annihilation of atoms in the ground state $a^\dagger_\mathbf{p}$,$a_\mathbf{p}$ and in the excited state $\tilde{a}^\dagger_\mathbf{p}$,$\tilde{a}_\mathbf{p}$, where indices "$\mathbf{p}$" label the atomic momenta. The photon-atom interactions are described now as all the possible combinations of ϕ's, a's, and $\tilde{a}$'s (times the complex-valued coupling vertices), and the number of these combinations is, in principle, infinite. The good news here is that one can, for example, build up a hierarchy of irreducible processes based on the number of photons involved in a process. "Irreducible" stands here for a process which cannot be decomposed into two (or more) simpler processes.

The important one-photon and two-photon processes are sketched in Fig. 4. For example, the left diagram in panel (a) of Fig. 4 shows a simplest one-photon process: A ground-state atom absorbs a photon and goes to the excited state. In the conjugated process, shown in the right diagram of panel (a), the excited atom emits a photon and goes to the ground state. Hence, the interacting Hamiltonian can be expressed as

$$\mathcal{H}_I = \mathcal{H}_I^{11} + \mathcal{H}_I^{12} + \mathcal{H}_I^{13} + \mathcal{H}_I^{21} + \mathcal{H}_I^{22} + \mathcal{H}_I^{23} + \cdots, \quad (32)$$

where the one-photon processes are described by

$$\begin{aligned}
\mathcal{H}_I^{11} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k}} \Gamma^{11}_\mathbf{k}\, \tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}}\, a_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{12} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k}} \Gamma^{12}_\mathbf{k}\, a^\dagger_{\mathbf{p}+\mathbf{k}}\, a_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{13} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k}} \Gamma^{13}_\mathbf{k}\, \tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}}\, \tilde{a}_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},
\end{aligned} \quad (33)$$

the two-photon processes are described by

$$\begin{aligned}
\mathcal{H}_I^{21} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \Gamma^{21}_{\mathbf{k}\mathbf{q}}\, \tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}-\mathbf{q}}\, \phi^\dagger_{\kappa_\mathbf{q}}\, a_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{22} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \Gamma^{22}_{\mathbf{k}\mathbf{q}}\, a^\dagger_{\mathbf{p}+\mathbf{k}-\mathbf{q}}\, \phi^\dagger_{\kappa_\mathbf{q}}\, a_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{23} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \Gamma^{23}_{\mathbf{k}\mathbf{q}}\, \tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}-\mathbf{q}}\, \phi^\dagger_{\kappa_\mathbf{q}}\, \tilde{a}_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{24} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \Gamma^{24}_{\mathbf{k}\mathbf{q}}\, \tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}+\mathbf{q}}\, \phi_{\kappa_\mathbf{q}}\, a_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{25} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \Gamma^{25}_{\mathbf{k}\mathbf{q}}\, a^\dagger_{\mathbf{p}+\mathbf{k}+\mathbf{q}}\, \phi_{\kappa_\mathbf{q}}\, a_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},\\
\mathcal{H}_I^{26} &= \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \Gamma^{26}_{\mathbf{k}\mathbf{q}}\, \tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}+\mathbf{q}}\, \phi_{\kappa_\mathbf{q}}\, \tilde{a}_\mathbf{p}\, \phi_{\kappa_\mathbf{k}} + \text{H.c.},
\end{aligned} \quad (34)$$

and so on. For aesthetics, I adopt $\hbar = 1$ in indices labeling; for example, $\tilde{a}^\dagger_{\mathbf{p}+\mathbf{k}} \equiv \tilde{a}^\dagger_{\mathbf{p}+\hbar\mathbf{k}}$ reads as an excitation of an atom with the momentum $\mathbf{p} + \hbar\mathbf{k}$. In general, I tend to keep $\mathbf{p}$ and $\mathbf{p}'$ for atomic momenta and $\mathbf{k}$ and $\mathbf{q}$ for photon wave vectors, so it is easy to distinguish. The coupling parameters $\Gamma_\mathbf{k}$ should be also read as $\Gamma_\mathbf{k} = \Gamma(\omega_\mathbf{k})$ due to their scalar nature. In the present paper, we neglect the contributions from optical collisions, i.e., the processes of form $a^\dagger_{\mathbf{p}_1}\, a^\dagger_{\mathbf{p}_1}\, \phi^\dagger_{\kappa_\mathbf{q}}\, a_{\mathbf{p}'_1}\, a_{\mathbf{p}'_2}$ (with $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2$) and others, even though these processes can be taken into account in this model by writing their one-photon and two-photon Hamiltonians in the second-quantized form. For simplicity, here we consider $\hbar = 1$ until the end of the section, where it is restored.

We define the Matsubara Green's function of a photon as

$$\mathcal{G}(\kappa,\tau,\tau_0) = -\langle\, \mathrm{T}_\tau\, \phi_\kappa(\tau)\, \phi^\dagger_\kappa(\tau_0)\rangle_{\rm th}, \quad (35)$$

[5]The two-level-atom Hamiltonian $\mathcal{H}_{\rm at} = \frac{1}{2}\hbar\omega_{\rm at} \sum_\mathbf{p} \tilde{a}^\dagger_\mathbf{p}\tilde{a}_\mathbf{p} - a^\dagger_\mathbf{p} a_\mathbf{p}$ describes the atomic contribution, and, in general, it should be treated self-consistently with the photonic and the mixed parts of the Hamiltonian (31). This procedure usually leads to redefinition of the quasiparticles in the system by introducing the so-called atomic polaritons (dressed atom-light states) because the photon description is no more applicable in the general case of arbitrary coupling (see, e.g., Refs. [27,29–31]). However, as discussed in Refs. [1,2,29], for some range of parameters the conditions for the strong atom-photon coupling are not met, and the relevant quasiparticles in the system are photons (the two branches of polaritons degenerate into the single branch of photons). Therefore, in this case the atomic Green's functions can be considered uncoupled from the influence of photonic Green's functions, which allows us to drop $\mathcal{H}_{\rm at}$ out of consideration in photonic perturbation theory.

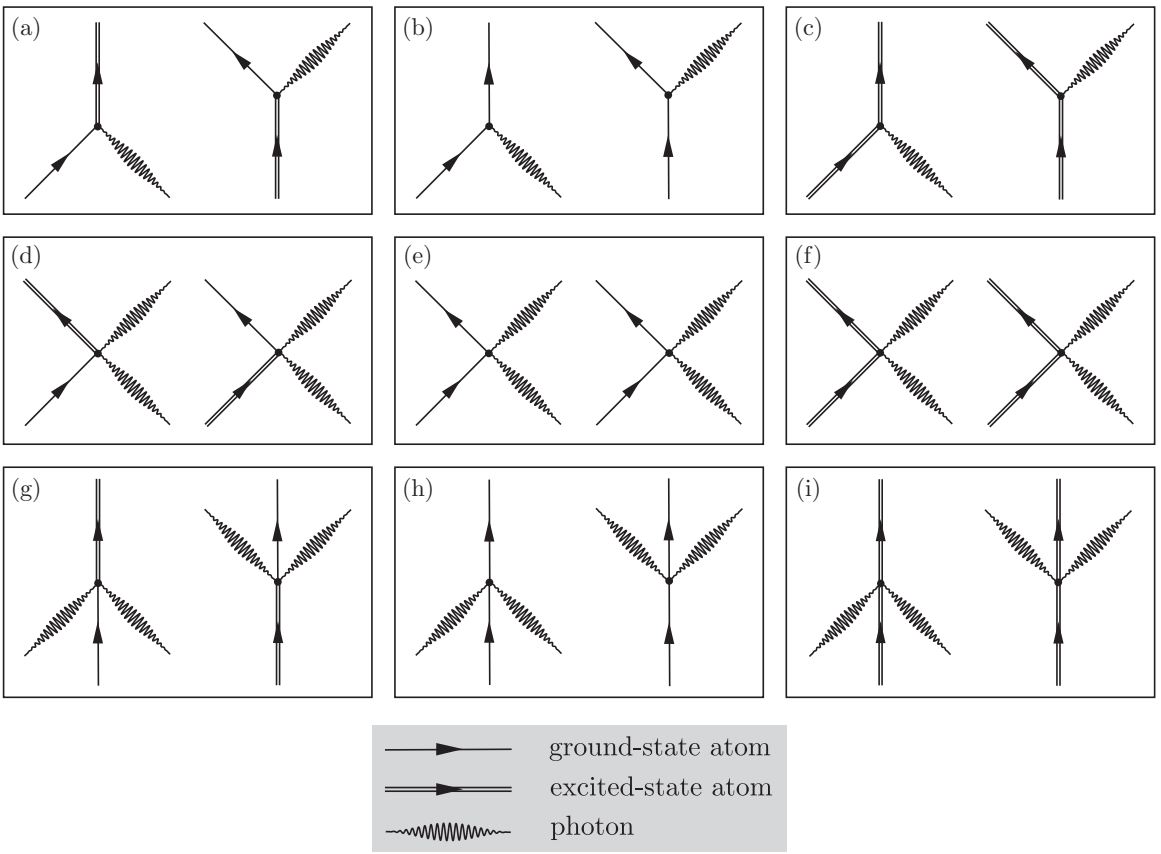

FIG. 4. Hierarchy of the interaction processes: (a)–(c) one-photon processes; (d)–(i) two-photon processes. The notations are as follows: A single line designates a ground-state atom; a double line designates an excited-state atom; a curly line designates a photon. Time evolution is vertical by convention.

where all the operators are in the (imaginary-time) Heisenberg representation, and T_τ stands for Matsubara time ordering. We introduce now Fourier-transformed Green's functions,

$$\mathcal{G}(\kappa,i\omega_n)=\int_0^\beta d\tau e^{i\omega_n\tau}\mathcal{G}(\kappa,\tau), \tag{36}$$

where $i\omega_n$ are Matsubara frequencies, which are of discrete nature, $\omega_n=2\pi n/\beta$ for bosons, and $\beta=1/T$ is the inverse temperature, as usual. We also introduce the atomic Green's functions as

$$\begin{aligned}\mathcal{G}_{\mathrm{at}}(\mathbf{p},\tau,\tau_0)&=-\langle\,\mathrm{T}_\tau\,a_{\mathbf{p}}(\tau)\,a^\dagger_{\mathbf{p}}(\tau_0)\rangle_{\mathrm{th}},\\ \tilde{\mathcal{G}}_{\mathrm{at}}(\mathbf{p},\tau,\tau_0)&=-\langle\,\mathrm{T}_\tau\,\tilde{a}_{\mathbf{p}}(\tau)\,\tilde{a}^\dagger_{\mathbf{p}}(\tau_0)\rangle_{\mathrm{th}}.\end{aligned} \tag{37}$$

Some properties of the atomic Green's functions are given in Appendix A.

Matsubara Green's functions link observables at finite temperatures with the equal-time response of quantum operators. For instance, the occupation number of photons with one-dimensional degree of freedom κ is given by

$$f_\kappa=\langle\phi^\dagger_\kappa\phi_\kappa\rangle=-\lim_{\tau\to0^-}\mathcal{G}(\kappa,\tau,0), \tag{38}$$

and similar expressions for the occupation number of atoms in the three-dimensional reciprocal space $\mathbf{p}$ are

$$\begin{aligned}n_{\mathbf{p}}&=\langle a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\rangle=-\lim_{\tau\to0^-}\mathcal{G}_{\mathrm{at}}(\mathbf{p},\tau,0),\\ \tilde{n}_{\mathbf{p}}&=\langle\tilde{a}^\dagger_{\mathbf{p}}\tilde{a}_{\mathbf{p}}\rangle=-\lim_{\tau\to0^-}\tilde{\mathcal{G}}_{\mathrm{at}}(\mathbf{p},\tau,0).\end{aligned} \tag{39}$$

One may notice that the photon occupation number f_κ and atomic occupation numbers $n_{\mathbf{p}},\tilde{n}_{\mathbf{p}}$ are introduced by different literals. This is as intended. First, I wish to distinguish between them two without adding additional indices. Second, the two quantities are of different units, so they represent different physical notions.

B. Perturbation theory in a uniform medium

In this subsection, we derive the renormalized photon propagator and the corresponding self-energies in the absence of the trapping potential. In the Matsubara formalism, the perturbed Green's function is given by a series expansion,

$$\begin{aligned}\mathcal{G}(\kappa,\tau,\tau_0)=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n\\ \times\langle\mathrm{T}_\tau\,\phi_\kappa(\tau)\,\phi^\dagger_\kappa(\tau_0)\,\mathcal{H}_I(\tau_1)\cdots\mathcal{H}_I(\tau_n)\,\rangle_0,\end{aligned} \tag{40}$$

where the thermal averaging includes only connected diagrams and 0 stands for the noninteracting eigenstates. To simplify the discussion, in this subsection I consider a single process only,

namely the first one,

$$\mathcal{H}_I^{11} = \frac{1}{\sqrt{N_{\rm at}}} \sum_{\mathbf{p},\mathbf{k}} \Gamma_{\mathbf{k}}^{11}\, \tilde{a}_{\mathbf{p}+\mathbf{k}}^{\dagger}\, a_{\mathbf{p}}\, \phi_{\kappa_{\mathbf{k}}} + \Gamma_{\mathbf{k}}^{11*}\, \phi_{\kappa_{\mathbf{k}}}^{\dagger}\, a_{\mathbf{p}}^{\dagger}\, \tilde{a}_{\mathbf{p}+\mathbf{k}}. \quad (41)$$

This choice is not arbitrary. Indeed, it is $\mathcal{H}_I^{11}$ that gives one of the most significant contributions to the self-energy. At the end of this section, the contributions from all the other one-photon and two-photon processes are taken into account. To simplify the relevant formulas further, I drop in this subsection all the "11" superscripts of expression (41).

According to the formula (40), the first nonvanishing correction that is given due to acts of absorption and reemission is given by the first mean-field correction,

$$\delta\mathcal{G}^{(1)}(\kappa,\tau) = -\frac{1}{2!}\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \times \langle \mathrm{T}_{\tau}\, \phi_{\kappa}(\tau)\, \phi_{\kappa}^{\dagger}(0)\, \mathcal{H}_I(\tau_1)\, \mathcal{H}_I(\tau_2)\rangle_0. \quad (42)$$

Using the explicit expression (41), and decoupling the time-ordered average by using Wick's theorem, the contributions of the connected diagrams are determined as

$$\delta\mathcal{G}^{(1)}(\kappa_{\mathbf{k}},\tau) = \frac{|\Gamma_{\mathbf{k}}|^2}{N_{\rm at}} \int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \mathcal{G}(\kappa_{\mathbf{k}},\tau-\tau_2)\mathcal{G}(\kappa_{\mathbf{k}},\tau_1) \times \sum_{\mathbf{p}} \mathcal{G}_{\rm at}(\mathbf{p},\tau_1-\tau_2)\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},\tau_2-\tau_1). \quad (43)$$

Going to the $i\omega_n$ representation of formula (43), one shows that the nonvanishing contribution corresponds to $\tau_1 = \tau_2$. For the nondegenerate ensemble of atoms, the product of the atomic propagators is given as $\mathcal{G}_{at}(\mathbf{p},\tau_1-\tau_2)\tilde{\mathcal{G}}_{at}(\mathbf{p}+\mathbf{k},\tau_2-\tau_1) = n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}$. Thus, after Fourier-transforming Eq. (43), one obtains the first renormalization to the Green's function,

$$\delta\mathcal{G}^{(1)}(\kappa,i\omega_n) = \Sigma^{(1)}(\kappa,i\omega_n)\,\mathcal{G}_0^2(\kappa,i\omega_n), \quad (44)$$

where the first (on-shell) self-energy $\Sigma^{(1)}(\kappa,i\omega_n) = \Sigma_{\kappa}^{(1)}$ is contributed by one-photon emission/absorption,

$$\Sigma_{\kappa_{\mathbf{k}}}^{(1)} = \frac{|\Gamma_{\mathbf{k}}|^2}{N_{\rm at}} \sum_{\mathbf{p}} n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}. \quad (45)$$

To calculate the density of particles, we go back to the τ representation, namely considering the perturbed Green's function as

$$\mathcal{G}(\kappa,\tau) \approx \frac{1}{\beta}\sum_{i\omega_n} e^{-i\omega_n\tau}\mathcal{G}_0(\kappa,i\omega_n) + \frac{1}{\beta}\sum_{i\omega_n} e^{-i\omega_n\tau}\mathcal{G}_0^2(\kappa,i\omega_n)\Sigma_{\kappa}^{(1)}. \quad (46)$$

As a result, the occupation number of photons, renormalized due to interactions, is obtained through the $\tau \to 0^-$ limit. In the main order one obtains

$$f_{\kappa} = -\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}_0(\kappa,i\omega_n) - \frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}_0^2(\kappa,i\omega_n)\,\Sigma_{\kappa}^{(1)} = -\frac{1}{\beta}\sum_{i\omega_n}\frac{1}{i\omega_n-\omega_{\kappa}} - \frac{1}{\beta}\sum_{i\omega_n}\frac{\Sigma_{\mathbf{k}}^{(1)}}{(i\omega_n-\omega_{\kappa})^2}. \quad (47)$$

Using now the Matsubara frequencies summation rules (see Ref. [32]), one derives

$$f_{\kappa} = \frac{1}{e^{\beta\omega_{\kappa}}-1} - \frac{\beta\Sigma_{\kappa}^{(1)}e^{\beta\omega_{\kappa}}}{(e^{\beta\omega_{\kappa}}-1)^2}. \quad (48)$$

So far the existence of the nonzero chemical potential μ was for simplicity avoided in (48), but it is easily included switching $\omega_{\kappa} \to \omega_{\kappa} - \mu$ in the Green's functions of photons. For the nonzero chemical potential, the renormalized occupation number in the first order is

$$f(\omega_{\kappa} + \Sigma_{\kappa}^{(1)} - \mu) \approx \frac{1}{e^{\beta(\omega_{\kappa}-\mu)}-1} - \frac{\beta\Sigma_{\kappa}^{(1)}e^{\beta(\omega_{\kappa}-\mu)}}{(e^{\beta(\omega_{\kappa}-\mu)}-1)^2}. \quad (49)$$

The first term in (49) is intuitively understandable as it describes the occupation number for the noninteracting gas of photons while the second term gives the first interacting correction. We still keep here $\hbar = 1$.

The next nonvanishing correction is given by the fourth order of the perturbation theory,

$$\delta\mathcal{G}^{(2)}(\kappa,\tau) = -\frac{1}{4!}\int_0^{\beta}\int_0^{\beta}\int_0^{\beta}\int_0^{\beta} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \times \langle \mathrm{T}_{\tau}\, \phi_{\kappa}(\tau)\phi_{\kappa}^{\dagger}(0)\mathcal{H}_I(\tau_1)\mathcal{H}_I(\tau_2)\mathcal{H}_I(\tau_3)\mathcal{H}_I(\tau_4)\rangle. \quad (50)$$

The unique connected diagrams are given, for example, by $\tau_4 = \tau_1, \tau_3 = \tau_2$. Decoupling operators with the use of Wick's theorem, one finds the quantity under averaging $\langle \mathrm{T}_{\tau}\cdots\rangle$ in Eq. (50) containing a combination of the atomic Green's functions,

$$\begin{aligned}&\mathcal{G}_{\rm at}(\mathbf{p},\tau_{12})\mathcal{G}_{\rm at}(\mathbf{p}',\tau_{21})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},\tau_{21})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}'+\mathbf{q},\tau_{12})\\ &+\mathcal{G}_{\rm at}(\mathbf{p},0)\mathcal{G}_{\rm at}(\mathbf{p}',0)\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},0)\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}'+\mathbf{k},0)\\ &+\mathcal{G}_{\rm at}(\mathbf{p},\tau_{12})\mathcal{G}_{\rm at}(\mathbf{p}+\mathbf{k}-\mathbf{q},\tau_{21})\tilde{\mathcal{G}}_{\rm at}^2(\mathbf{p}+\mathbf{k},0)\\ &+\mathcal{G}_{\rm at}^2(\mathbf{p},0)\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},\tau_{21})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{q},\tau_{12}), \end{aligned} \quad (51)$$

where we have introduced $\tau_{12} = \tau_1 - \tau_2$ and $\tau_{21} = \tau_2 - \tau_1$ for brevity. Using the properties of the Matsubara Green's functions, this expression can be transformed to

$$\begin{aligned}&\mathcal{G}_{\rm at}(\mathbf{p},\tau_{12})\mathcal{G}_{\rm at}(\mathbf{p}',\tau_{21})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},\tau_{21})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}'+\mathbf{q},\tau_{12})\\ &\times(1+\delta_{\mathbf{p},\mathbf{p}'}+\delta_{\mathbf{k},\mathbf{q}}+\delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{q}}). \end{aligned} \quad (52)$$

This, in turn, gives the second perturbative correction to the photon Green's function as

$$\begin{aligned}\delta\mathcal{G}^{(2)}(\kappa_{\mathbf{k}},\tau) = \frac{|\Gamma_{\mathbf{k}}|^2}{N_{\rm at}}\sum_{\mathbf{q}}\frac{|\Gamma_{\mathbf{q}}|^2}{N_{\rm at}}\sum_{\mathbf{p},\mathbf{p}'}\int_0^{\beta}\int_0^{\beta} d\tau_1 d\tau_2\\ \times\mathcal{G}(\kappa_{\mathbf{k}},\tau-\tau_2)\,\mathcal{G}(\kappa_{\mathbf{q}},\tau_2-\tau_1)\,\mathcal{G}(\kappa_{\mathbf{k}},\tau_1)\\ \times\mathcal{G}_{\rm at}(\mathbf{p},\tau_1-\tau_2)\mathcal{G}_{\rm at}(\mathbf{p}',\tau_2-\tau_1)\\ \times\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},\tau_2-\tau_1)\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}'+\mathbf{q},\tau_1-\tau_2)\\ \times(1+\delta_{\mathbf{p},\mathbf{p}'}+\delta_{\mathbf{k},\mathbf{q}}+\delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{q}}). \end{aligned} \quad (53)$$

Now we use again the condition that the atomic ensemble is nondegenerate, yielding a quasiclassical propagator $\mathcal{G}_{\rm at}(\mathbf{p},\tau) = -e^{-E_{\mathbf{p}}\tau}n_{\mathbf{p}}$ and a similar expression for $\tilde{\mathcal{G}}_{\rm at}(\mathbf{p},\tau)$.

Therefore, one obtains

$$\mathcal{G}_{\rm at}(\mathbf{p},\tau_{12})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}+\mathbf{k},\tau_{21})\mathcal{G}_{\rm at}(\mathbf{p}',\tau_{21})\tilde{\mathcal{G}}_{\rm at}(\mathbf{p}'+\mathbf{q},\tau_{12}) = n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}\, n_{\mathbf{p}'}\tilde{n}_{\mathbf{p}'+\mathbf{q}}\, e^{(\omega_{\mathbf{q}}-\omega_{\mathbf{k}})\tau_{21}}, \quad (54)$$

where we have used the energy conservation law, $E_{\mathbf{p}}+\omega_{\mathbf{k}} = E_{\mathbf{p}+\mathbf{k}}+\Delta$, with $\Delta=\omega_{\rm at}$ as the energy distance between the ground state and the excited state, and a similar expression for $\mathbf{q}$. The vector labeling of photon energies is used here to emphasize that this expression (54) holds in general. For definiteness, we consider $\tau_2>\tau_1$ in the present calculation. The next step is proceeding to the Matsubara frequencies, and we are doing the Fourier transform to Matsubara frequencies,

$$\delta\mathcal{G}^{(2)}(\kappa_{\mathbf{k}},i\omega_n) = \frac{|\Gamma_{\mathbf{k}}|^2}{N_{\rm at}^2}\sum_{\mathbf{q}}|\Gamma_{\mathbf{q}}|^2\sum_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}\, n_{\mathbf{p}'}\tilde{n}_{\mathbf{p}'+\mathbf{q}} \times\mathcal{G}(\kappa_{\mathbf{k}},i\omega_n)\mathcal{G}(\kappa_{\mathbf{k}},i\omega_n),\mathcal{G}(\kappa_{\mathbf{q}},i\omega_n+\omega_{\mathbf{q}}-\omega_{\mathbf{k}}) \times(1+\delta_{\mathbf{p},\mathbf{p}'}+\delta_{\mathbf{k},\mathbf{q}}+\delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{q}}). \quad (55)$$

We introduce the effective interaction coupling parameter

$$F(\mathbf{k};\mathbf{q}) = \frac{1}{N_{\rm at}^2}\sum_{\mathbf{p},\mathbf{p}'}(1+\delta_{\mathbf{p},\mathbf{p}'}+\delta_{\mathbf{k},\mathbf{q}}+\delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{q}}) \times n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}\, n_{\mathbf{p}'}\tilde{n}_{\mathbf{p}'+\mathbf{q}}, \quad (56)$$

which simplifies the formula for the perturbed Green's function,

$$\delta\mathcal{G}^{(2)}(\kappa,i\omega_n) = |\Gamma_{\mathbf{k}}|^2\,\mathcal{G}^3(\kappa,i\omega_n)\sum_{\mathbf{q}}|\Gamma_{\mathbf{q}}|^2F(\mathbf{k};\mathbf{q}). \quad (57)$$

The function $F(\mathbf{k};\mathbf{q})$ can be calculated explicitly (see Appendix C). For the next step, we sum up over Matsubara frequencies to obtain the equal-time response,

$$\mathcal{G}^{(2)}(\kappa,\tau=0) = |\Gamma_{\mathbf{k}}|^2\sum_{\mathbf{q}}|\Gamma_{\mathbf{q}}|^2F(\mathbf{k};\mathbf{q})\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}^3(\kappa,i\omega_n). \quad (58)$$

To proceed in the on-shell approximation, we use noninteracting propagators. The sum can be calculated using the standard Matsubara machinery [32], yielding the result

$$\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}_0^3(\kappa,i\omega_n) = \frac{\beta^2e^{\beta\omega_\kappa}}{2}f_0^2(\omega_\kappa)-\beta^2e^{2\beta\omega_\kappa}f_0^3(\omega_\kappa), \quad (59)$$

where again $f_0(\omega)=\{e^{\beta\omega}-1\}^{-1}$ is the Bose-Einstein distribution for photons. In the leading order, the perturbed Green's function is given by ($\mu=0$)

$$\mathcal{G}(\kappa,\tau=0) = \mathcal{G}_0(\kappa,\tau=0)+\beta e^{\beta\omega_\kappa}\left(\Sigma_\kappa^{(1)}+\Sigma_\kappa^{(2)}\right)f_0^2(\omega_\kappa) + \mathcal{O}[f_0^3(\omega_\kappa)], \quad (60)$$

where the second self-energy contribution is determined as

$$\Sigma_{\kappa_{\mathbf{k}}}^{(2)} = \frac{\beta}{2}|\Gamma_{\mathbf{k}}|^2\sum_{\mathbf{q}}|\Gamma_{\mathbf{q}}|^2F(\mathbf{k};\mathbf{q}). \quad (61)$$

The formulas (60) and (61) completely describe the renormalized equal-time response for the one-photon process $\mathcal{H}^{11}$ in the leading orders for a uniform system at $T\neq0$. The interactions thus modify the photon's spectrum,

$$\tilde{\omega}_\kappa = \omega_\kappa+\Sigma_\kappa^{(1)}+\Sigma_\kappa^{(2)}, \quad \omega_\kappa=\omega_0+\frac{\kappa^2}{2m^*}. \quad (62)$$

Including now the nonvanishing chemical potentials μ, and keeping only linear terms in $\delta\mu=\mu-\mu_0=\lim_{\kappa\to0}[\Sigma_\kappa^{(1)}+\Sigma_\kappa^{(2)}]$, one obtains the distribution function of photons, modified by interactions, as

$$f(\tilde{\omega}_\kappa-\mu)\approx f_0(\omega_\kappa-\mu_0)-\beta e^{\beta(\omega_\kappa-\mu_0)}f_0^2(\omega_\kappa-\mu_0) \times\left[\Sigma_\kappa^{(1)}+\Sigma_\kappa^{(2)}-\delta\mu\right]. \quad (63)$$

If integrated over κ, Eq. (63) gives the critical number of photons at temperature T. Unfortunately, in one dimension the Bose-Einstein singularity is not integrable, so we need to modify the formalism by considering the trapping potential.

C. Perturbation theory in the trapping potential

In this subsection, we calculate the renormalization of photon's propagators $\mathcal{G}$ as they are trapped in the microtube waveguide. This is done again by means of the perturbation theory. First, we look for the first corrections to the nontrapped propagator and then sum up corrections in all orders. This procedure gives a new propagator G, describing noninteracting but trapped photons. At the end of this subsection, this result is merged with the outcome of the previous subsection, giving the interacting trapped propagator in the leading order.

Consider noninteracting photons trapped in the potential $V(\mathbf{r})$ with its Fourier image $V_{\mathbf{k}}$. The Hamiltonian responsible for this reads

$$\mathcal{H} = \sum_{\mathbf{k}}\omega_{\kappa_{\mathbf{k}}}\phi^\dagger_{\kappa_{\mathbf{k}}}\phi_{\kappa_{\mathbf{k}}}+\sum_{\mathbf{k},\mathbf{q}}V_{\mathbf{q}}\,\phi^\dagger_{\kappa_{\mathbf{k}+\mathbf{q}}}\phi_{\kappa_{\mathbf{k}}}. \quad (64)$$

The perturbation theory is given by the same formalism [see (40)]. The renormalized photon propagator G originates from the nontrapped propagator $G_0\equiv\mathcal{G}$ and is augmented by the series of perturbative corrections,

$$G(\kappa,\tau) = \mathcal{G}(\kappa,\tau)+\delta G^{(1)}(\kappa,\tau)+\delta G^{(2)}(\kappa,\tau)+\cdots, \quad (65)$$

as sketched in Fig. 5. The first correction to the propagator due to interaction with the external potential (second diagram on Fig. 5) is expressed as

$$\delta G^{(1)}(\kappa_{\mathbf{k}},\tau) = \sum_{\mathbf{q},\mathbf{k}'}V_{\mathbf{q}}\int_0^\beta d\tau_1 \times\langle{\rm T}_\tau\,\phi_{\kappa_{\mathbf{k}}}(\tau)\,\phi^\dagger_{\kappa_{\mathbf{k}}}(0)\,\phi^\dagger_{\kappa_{\mathbf{k}'+\mathbf{q}}}(\tau_1)\,\phi_{\kappa_{\mathbf{k}'}}(\tau_1)\rangle. \quad (66)$$

Using Wick's theorem, one obtains

$$\delta G^{(1)}(\kappa_{\mathbf{k}},\tau) = V_0\int_0^\beta d\tau_1\mathcal{G}(\kappa_{\mathbf{k}},\tau-\tau_1)\mathcal{G}(\kappa_{\mathbf{k}},\tau_1), \quad (67)$$

where $V_0\equiv V_{\kappa_{\mathbf{k}=0}}$. The convolution vanishes as we go to the Fourier transform, which gives

$$\delta G^{(1)}(\kappa,i\omega_n) = V_0\,\mathcal{G}^2(\kappa,i\omega_n). \quad (68)$$

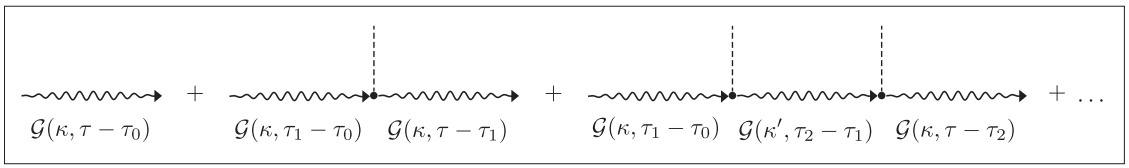

FIG. 5. Renormalization of a photon's propagator in the external field. Wavy propagators stand for one-dimensional photon Green's functions $\mathcal{G}(\kappa,\tau)$, whereas dashed lines denote interaction with external potential. The series converge up to the trapped photon propagator $G(\kappa,\tau)$.

As in the previous calculations, the equal-time response is extracted from summing up over Matsubara frequencies,

$$\delta G^{(1)}(\kappa,\tau=0)=V_0\frac{1}{\beta}\sum_{i\omega_n}\mathcal{G}^2(\kappa,i\omega_n)=V_0\frac{\beta e^{\beta\omega_\kappa}}{(e^{\beta\omega_\kappa}-1)^2}. \tag{69}$$

The long-wavelength asymptote $\kappa\to 0$ of the Fourier image V_κ of potential $V(z)$ is finite and given by $V_0=V(l)/(1+\alpha)$ (see Appendix B). The theoretical model presented in this section is not limited to $V(z)\propto|z|^\alpha$ with $\alpha=1$; however, for the sake of simplicity and aesthetics of the expressions we choose the linearly growing trapping potential, $\alpha=1$, which gives

$$\delta G^{(1)}(\kappa,\tau=0)=\frac{e^{\beta\omega_k}f_0^2(\omega_\kappa)}{2}\beta V(l). \tag{70}$$

The next correction is given by the third diagram in Fig. 5, which is the only connected diagram one can build up with the perturbed Hamiltonian (64). This diagram is expressed mathematically as

$$\delta G^{(2)}(\kappa_{\mathbf{k}},\tau)=\sum_{\mathbf{q}}V_{\mathbf{q}}V_{-\mathbf{q}}\int_0^\beta\int_0^\beta d\tau_1 d\tau_2 \times\mathcal{G}(\kappa_{\mathbf{k}},\tau_1)\mathcal{G}(\kappa_{\mathbf{k}+\mathbf{q}},\tau_2-\tau_1)\mathcal{G}(\kappa_{\mathbf{k}},\tau-\tau_2), \tag{71}$$

which upon Fourier transform yields to

$$\delta G^{(2)}(\kappa_{\mathbf{k}},i\omega_n)=\mathcal{G}^2(\kappa_{\mathbf{k}},i\omega_n)\sum_{\mathbf{q}}|V_{\mathbf{q}}|^2\mathcal{G}(\kappa_{\mathbf{k}+\mathbf{q}},i\omega_n). \tag{72}$$

Therefore, the equal-time response is expressed in terms of the Matsubara frequency sum,

$$\delta G^{(2)}(\kappa_{\mathbf{k}},\tau=0)=\frac{1}{\beta}\sum_{\mathbf{q}}|V_{\mathbf{q}}|^2\sum_{i\omega_n}\mathcal{G}^2(\kappa_{\mathbf{k}},i\omega_n)\mathcal{G}(\kappa_{\mathbf{k}+\mathbf{q}},i\omega_n)=\sum_{\mathbf{q}}|V_{\mathbf{q}}|^2\frac{1}{\beta}\sum_{i\omega_n}\frac{1}{(i\omega_n-\omega_{\kappa_{\mathbf{k}}})^2(i\omega_n-\omega_{\kappa_{\mathbf{k}+\mathbf{q}}})}. \tag{73}$$

Using the rules of summation, one obtains the correction to the density of photons due to the interaction with external potential as

$$\delta G^{(2)}(\kappa_{\mathbf{k}},\tau=0)=-\sum_{\mathbf{q}}|V_{\mathbf{q}}|^2\left\{\frac{f_0(\omega_{\kappa_{\mathbf{k}+\mathbf{q}}})-f_0(\omega_{\kappa_{\mathbf{k}}})}{(\omega_{\kappa_{\mathbf{k}+\mathbf{q}}}-\omega_{\kappa_{\mathbf{k}}})^2}+\frac{\beta e^{\beta\omega_{\kappa_{\mathbf{k}}}}f_0^2(\omega_{\kappa_{\mathbf{k}}})}{\omega_{\kappa_{\mathbf{k}+\mathbf{q}}}-\omega_{\kappa_{\mathbf{k}}}}\right\}. \tag{74}$$

It is clear that the main contribution to the sum in Eq. (74) is given by the region where $\omega_{\kappa_{\mathbf{k}+\mathbf{q}}}\approx\omega_{\kappa_{\mathbf{k}}}$. Therefore, we can expand the numerators in the Taylor series around $\omega_{\kappa_{\mathbf{k}}}$. It is important to go up to the third order because some terms get canceled. This, in turn, leads to

$$\delta G^{(2)}(\kappa,\tau=0)=-\frac{e^{\beta\omega_\kappa}(1+e^{\beta\omega_\kappa})}{2}\beta^2 f_0^3(\omega_\kappa)\sum_{\mathbf{q}}|V_{\mathbf{q}}|^2. \tag{75}$$

Now one can calculate the sum here explicitly, which gives $\sum_{\mathbf{q}}|V_{\mathbf{q}}|^2=V^2(l)/(1+2\alpha)$ (see Appendix B). Taking again the linearly growing potential, $\alpha=1$, one therefore obtains the expression

$$\delta G^{(2)}(\kappa,\tau=0)=-\frac{e^{\beta\omega_\kappa}(1+e^{\beta\omega_\kappa})f_0^3(\omega_\kappa)}{6}\beta^2V^2(l), \tag{76}$$

which is the second-order correction to the free propagator due to the light trapping. Therefore, the first two corrections are given by Eqs. (70) and (76). For this study, it is important to go to the further orders. One can verify that in all orders the series (65) converges to

$$G(\kappa,\tau=0)=-\frac{1}{\beta V(l)}\ln\frac{1-e^{\beta\omega_\kappa}e^{\beta V(l)}}{(1-e^{\beta\omega_\kappa})e^{\beta V(l)}}, \tag{77}$$

which describes the propagator in the external potential $V(z)=u|z|$. Again, the results can be obtained in the $l/R_0\to\infty$ approximation and proceeding to the continuous spectrum by $\sum_{\mathbf{k}}\to 2l\int\frac{d\kappa}{2\pi}$. Therefore, throughout the machinery of the previous derivations, one needs to replace the nontrapped propagators $\mathcal{G}(\kappa,\tau)$ by the propagators $G(\kappa,\tau)$ of trapped photons. Up to the first order in photon-atom interactions (i.e., neglecting all the higher-order terms), one

obtains

$$\tilde{f}(\tilde{\omega}_\kappa - \mu) \approx -G(\kappa,\tau=0) - \beta e^{\beta(\omega_\kappa-\mu)} G^2(\kappa,\tau=0) \times(\Sigma_\kappa - \delta\mu). \quad (78)$$

To calculate the critical photon number, one needs to sum up over momenta. However, a photon in the system under study is described by the motional degree of freedom $\kappa = k_z$, but there also could be other degrees of freedom, for example, a polarizational degree of freedom. Recall that it was taken into account in formula (8) by introducing $g = g(k)$, which describes degeneracy in photon energy levels. For the estimate in (8), we took an effective $g^* \approx 3$, because a massive boson can exist in three polarization states even in on-shell approximation. Therefore, in this approximation the continuous limit is introduced as $\sum_{\mathbf{k}} \to \frac{gl}{\pi}\int_{-\infty}^{\infty} d\kappa$. This essentially leads to multiplying by g^* each time we have wave-vector summation. Therefore, taking into account Eq. (78), one obtains

$$N_\phi \approx \frac{2g^* l}{\pi\beta V(l)} \int_0^\infty d\kappa \ln\left[\frac{1-e^{\beta(\omega_\kappa-\mu_0)}e^{\beta V(l)}}{(1-e^{\beta(\omega_\kappa-\mu_0)})\,e^{\beta V(l)}}\right] - \frac{2g^* l}{\pi\beta V^2(l)} \int_0^\infty d\kappa e^{\beta(\omega_\kappa-\mu_0)}(\Sigma_\kappa - \delta\mu) \times \ln^2\left[\frac{1-e^{\beta(\omega_\kappa-\mu_0)}\,e^{\beta V(l)}}{(1-e^{\beta(\omega_\kappa-\mu_0)})\,e^{\beta V(l)}}\right]. \quad (79)$$

One can verify — either analytically or numerically — that the first term is exactly the noninteracting result we discussed before [see the expression (13)]. Indeed, taking $l \to \infty$ and $\mu_0 = \hbar\omega_0$, the first term turns into

$$N_0 = \frac{2g^* T}{\pi u}\int_0^\infty d\kappa \ln\left[1 + \frac{1}{\exp(\hbar^2\kappa^2/2m^*T) - 1}\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{d\kappa dz}{2\pi} \frac{g^*}{\exp\left[\frac{\hbar^2}{2m^*T}\kappa^2 + \frac{u}{T}|z|\right] - 1}, \quad (80)$$

where $V(z) = u|z|$, i.e., $u = \hbar\omega_0/L_0$. One notices that it is exactly formula (8) after relabelling motional degree of freedom κ as k_z.

D. Contributions from one-photon and two-photon processes

Finally, in this subsection we list the contributions to self-energies from one-photon and two-photon processes (32)–(34) without a detailed derivation. The derivation procedure is the same as for the process $\mathcal{H}^{11}$ in the two previous subsections. The critical number of photons in the interacting case involves the self-energies from the processes $\mathcal{H}^{11}, \mathcal{H}^{12}, \mathcal{H}^{13}, \mathcal{H}^{21}, \mathcal{H}^{22}, \mathcal{H}^{23}, \mathcal{H}^{24}, \mathcal{H}^{25}, \mathcal{H}^{26}$, which are given by

$$\Sigma_\kappa \approx \Sigma_\phi(\kappa) + \Sigma_{\phi\phi}(\kappa), \quad (81)$$

where the number of "ϕ" in the subscripts stands for the number of photons in an irreducible process; in this study it is either one or two. The self-energies like $\Sigma_{\phi\phi\phi}(\kappa)$ and of higher orders are neglected. The contributions from one-photon processes are given by

$$\Sigma_\phi^{(1)}(\kappa_{\mathbf{k}}) = \sum_{\mathbf{p}} \gamma_{\mathbf{k}}^{11}\, n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}} + \gamma_{\mathbf{k}}^{12}\, n_{\mathbf{p}} n_{\mathbf{p}+\mathbf{k}} + \gamma_{\mathbf{k}}^{13}\, \tilde{n}_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}},$$

$$\Sigma_\phi^{(2)}(\kappa_{\mathbf{k}}) = \frac{\beta}{2}\sum_{\mathbf{q}} \gamma_{\mathbf{k}}^{11}\gamma_{\mathbf{q}}^{11} F_1(\mathbf{k};\mathbf{q}) + \gamma_{\mathbf{k}}^{12}\gamma_{\mathbf{q}}^{12} F_2(\mathbf{k};\mathbf{q}) + \gamma_{\mathbf{k}}^{13}\gamma_{\mathbf{q}}^{13} F_3(\mathbf{k};\mathbf{q}), \quad (82)$$

where $\gamma_{\mathbf{k}}^{11} \approx |\Gamma_{\mathbf{k}}^{11}|^2$, $\gamma_{\mathbf{k}}^{12} = |\Gamma_{\mathbf{k}}^{12}|^2$, $\gamma_{\mathbf{k}}^{13} = |\Gamma_{\mathbf{k}}^{13}|^2$ are positive factors. In the main approximation the effective interaction parameters (see Appendix C) are given by $F_\alpha(\mathbf{k};\mathbf{q}) \approx \sigma_\alpha(\mathbf{k})\,\sigma_\alpha(\mathbf{q})$, where the quantities

$$\sigma_1(\mathbf{k}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}, \quad \sigma_2(\mathbf{k}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} n_{\mathbf{p}} n_{\mathbf{p}+\mathbf{k}},$$

$$\sigma_3(\mathbf{k}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} \tilde{n}_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}, \quad (83)$$

can be calculated analytically (see Appendix C). The contribution of the two-photon processes are given by

$$\Sigma_{\phi\phi}^{(1)}(\kappa_{\mathbf{k}}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} r_{\mathbf{k}}^{22}\, n_{\mathbf{p}} + r_{\mathbf{k}}^{23}\,\tilde{n}_{\mathbf{p}} = r_{\mathbf{k}}^{22}\frac{N_a}{N_{\rm at}} + r_{\mathbf{k}}^{23}\frac{\tilde{N}_a}{N_{\rm at}},$$

$$\Sigma_{\phi\phi}^{(2)}(\kappa) \approx \frac{\beta}{2}\sum_{\mathbf{q}} \gamma_{\mathbf{kq}}^{21}\,\sigma_1(\mathbf{k}-\mathbf{q}) + \gamma_{\mathbf{kq}}^{22}\,\sigma_2(\mathbf{k}-\mathbf{q}) + \gamma_{\mathbf{kq}}^{23}\,\sigma_3(\mathbf{k}-\mathbf{q}), \quad (84)$$

where $r_{\mathbf{k}}^{ab} = \mathrm{Re}\;\Gamma_{\mathbf{k}}^{ab}$ and $\gamma_{\mathbf{kq}}^{2a} = |\Gamma_{\mathbf{k}}^{2a}|^2$. The expressions (82)–(84) describe the one-photon and two-photon processes without taking into account optical collisions of atoms (or molecules). Therefore, the critical number of photons to observe the Bose-Einstein condensation at the temperature T is defined by Eq. (79), with the self-energies given by expressions (82)–(84) and the renormalization of the noninteracting chemical potential, $\delta\mu = \lim_{\kappa\to 0}\Sigma_\kappa$.

An important simplification comes in the limit $l \to \infty$. In this case, the critical number of photons is given by

$$N_C \approx N_0 - \frac{2g^* L_0^2}{l}\frac{T}{\hbar\omega_0}\int_0^\infty d\kappa \exp\left(\frac{\hbar^2\kappa^2}{2m^*T}\right)\frac{\Sigma_\kappa - \delta\mu}{\hbar\omega_0} \times \ln^2\left[1 + f_0\left(\frac{\hbar^2\kappa^2}{2m^*T}\right)\right], \quad (85)$$

where still $f_0(x) = (e^x - 1)^{-1}$; the noninteracting critical number N_0 is given by formulas (13) and (14). Note that in the case $l \to \infty$ contributions from the lowest mean-field self-energies $\Sigma_\phi^{(1)}$ and $\Sigma_{\phi\phi}^{(1)}$ vanish, but contributions from $\Sigma_\phi^{(2)}$ and $\Sigma_{\phi\phi}^{(2)}$ remain finite as they involve continuous-limit summation over photon momenta, returning the factor l.

We note that the self-energies (82) and (84) are responsible for the interaction effects for the degenerate photon gas,[6] and

[6]It may seem that the quantities Σ_κ, which behave essentially like renormalization of the photon's energy in the cavity with media, could be derived from the well-known methods of electrodynamics of continuous media. This is, however, not true. The quantities Σ_κ are the expectation values of the quantum operators of self-energies, which means that they inherently contain the information on the possible macroscopical occupation of one of the modes through the finite-T Green's functions. The renormalization of the EM modes

in particular they shift down the critical number of particles in the way it is presented in formula (85).

Formula (85) should be considered only as an estimate of the contribution of photon-photon interactions to the critical number of particles. The perturbation theory in the case of bosons has subtle issues at the critical point, and, generally speaking, fails at $T = T_C$. The reason is that correlation length ξ diverges as $T \to T_C$. As a consequence, at $T = T_C$ one cannot take the intermediate propagators the same way as they are considered in the symmetric phase. Therefore, there is a question as to what extent formula (85) could evaluate the critical number of particles for a real system with weakly interacting photons. In general, the range of validity of formula (85) can be addressed only numerically, for example, by using the path-integral Monte Carlo techniques (see Refs. [33–36]). This, however, goes far beyond the scope of the present paper. The following estimate, however, can be done. The study of Pilati *et al.* [33] reports the accuracy around 2% of different analytical and semianalytical perturbation approaches for predicting the critical parameters for the gaseous parameter $na_s^3 \sim 10^{-4}$, where a_s is the s-wave scattering length. In this regard, it is possible to estimate in order of magnitudes the range of validity of the expression (85): It was reported in Refs. [2,3] that the dimensionless interaction parameter for photon-photon interactions in rhodamine 6G dye is 7×10^{-4}, giving an estimate of the s-wave scattering length $a_s \sim 10^{-4}$ m. Considering a similar demand for convergence of the present theory, $n_{\rm ph}a_s^3 \ll 10^{-4}$, one could expect that the region of validity is roughly given as $n_{\rm ph} \ll 10^{16}$. In other words, the density of the photons in the center of the cavity should be sufficiently small that (1) the two-level atoms approximation is valid and (2) the perturbation theory gives an output with a controllable accuracy.

IV. DISCUSSION AND OUTLINE

The main goal of this paper was to introduce the condensation of photons in one dimension: Find the necessary conditions, estimate the critical parameters, and look for the role of light-matter interactions. However, it was not my goal to plan a particular experiment, and neither was it to plot the observables, since such calculations make sense only after (and only if) the experiment succeeds.

The analysis, presented in Sec. II, shows that in the weakly interacting case, the condensation is possible if the light is trapped in a prolongated microtube, $l \gg R_0$, which is slowly narrowing towards the ends as a power-law function weaker than parabolic. The analysis has not been done for the strongly varying shape, $l \sim R_0$, as the quantization procedure in that case is not straightforward. However, I would not be surprised if a similar phenomenon, yet less distinct, could be observed for $l \sim R_0$.

The experiments on Bose-Einstein condensation of photons [1–4] have an interesting distinguishing feature: the temperature of the setup is fixed, and the number of particles is tuned by external pumping. Therefore, one of the natural observables is $N_C(T)$ [compare to $T_C(N)$ in atomic BECs]. In this study, the noninteracting model contributes to the critical number of photons as $N_0 \propto T^{3/2}$, whereas the first perturbative corrections contribute in a more complicated manner [see formulas (79) and (85) and the linked expressions in Appendix C]. Again, as in the case of the two-dimensional BEC of photons [2,8], the geometry of the system is important for tuning the system, since the parameters R_0, l, L_0 appear both in the noninteracting and interacting context.

The other interesting feature here is the sharp response to the atomic frequency resonance. As already mentioned, for the thermalization process based on the repeated processes of absorption and emission, it is important to ensure the closeness of the cut-off frequency ω_0 and the main atomic transition frequency $\omega_{\rm at}$, so the absorption processes are favorable enough compared to scattering processes. Even though the thermalization processes were not considered implicitly in this study, just referring to the earlier studies, the importance of the relation $\omega_0 \approx \omega_{\rm at}$ is apparent, as it appears throughout the paper, in both the noninteracting and the interacting cases: The quantity $\Theta = e^{(\hbar\omega_0 - \hbar\omega_{\rm at})/T}$ reflects the strength of this resonance for this system (see, for example, Appendix C); also, the coupling parameters Γ introduced in the Hamiltonian (32)–(34) will have local extrema for momenta of photons, satisfying the relation $\hbar\omega(\mathbf{k}) \simeq \hbar\omega_{\rm at}$. Finally, for the completeness of this consideration, one should also add into account the average number of photons that are coupled with atoms, which also depends on the closeness to the atomic resonance. In equilibrium this quantity is linearly proportional to the number of atoms and is given by $N_{\rm at}[1 + g_{12}\exp(\frac{\hbar\omega_{\rm at} - \hbar\omega_0}{T})]^{-1}$, where g_{12} is the ratio between degeneracy of the atomic ground state and the first excited state; for details, see Refs. [7,8].

The influence of indirect photon-photon interactions, mediated through the different processes of absorption, emission, and scattering, was studied in terms of an effective Hamiltonian, taking into account the hierarchy of multiphoton processes. Because the photon number in the system under study is significantly smaller than the number of atoms, the hierarchy graph can be truncated on the one-photon and two-photons processes, which give the leading contributions to self-energy, corresponding to the effectively Hartree-Fock terms if the direct photon-photon scattering was present. The temperature-dependent perturbation theory, represented here by Matsubara formalism, is valid in the symmetrical phase, thus allowing us to estimate the critical parameters of the system. I should make here two important remarks. First, the different combinations of one-photon and two-photon processes can give interfering terms which, of course, will contribute to the self-energy; however, this contribution appears to be significantly smaller; the lowest contributions of the three-photon processes are of the higher order, at least with eight photon operators, which is beyond the present study. Second, there could also be present different one-photon and two-photon processes, involving more than a pair of atoms, for example, optical collisions of a form $a^\dagger a^\dagger \phi^\dagger a\, a$, $a^\dagger a^\dagger \phi^\dagger \tilde{a}\, \tilde{a}$,

based on the continuous-media electrodynamics may or may not be linked to the expectation values of the self-energy operators for the thermalized photons. Maxwell equations do not contain information on Bose-Einstein statistics, and therefore replacing the true self-energies by their out-of-equilibrium classical analogs will break the self-consistency of calculations.

etc. Even though formally these processes contribute into effectively Hartree-Fock decouplings, at least for the values of parameters used in the present paper the corresponding self-energies are negligible compared to the self-energy contributions given by formulas (82)–(84).

The problem, however, requires further study. For example, for the photons in Bose-Einstein condensate, a more general formalism, allowing broken symmetry, is required. A suitable machinery is given by the Popov approximation; I am currently working on it. It will be published elsewhere.

ACKNOWLEDGMENTS

This work was supported by the Swiss National Science Foundation, its Sinergia network MPBH (Mott Physics Beyond the Heisenberg model). The author would like to thank Henrik M. Rønnow, Ihor Vakulchyk, and Hélène Piet for their valuable help and support.

APPENDIX A: ATOMIC GREEN'S FUNCTIONS

The ground-state atoms and excited-state atoms are physically the same objects (although with different quantum numbers of electronic orbitals), so their propagators should be physically linked. On the other hand, in the model of two-level atoms, the dynamics of the system is described by two independent quantum operators $a_{\mathbf{p}}$ (ground state) and $\tilde{a}_{\mathbf{p}}$ (excited state). The generalized atomic operator is introduced as

$$A_{\mathbf{p}} = \begin{pmatrix} a_{\mathbf{p}} \\ \tilde{a}_{\mathbf{p}} \end{pmatrix}. \tag{A1}$$

One can verify that the number operator $A^{\dagger}_{\mathbf{p}} A_{\mathbf{p}}$ returns the total number of atoms in the system, that is, the number of atoms in the ground state N_a together with the number of atoms in the excited state $N_{\tilde{a}}$,

$$N_{\rm at} = \sum_{\mathbf{p}} A^{\dagger}_{\mathbf{p}} A_{\mathbf{p}} = \sum_{\mathbf{p}} a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} + \tilde{a}^{\dagger}_{\mathbf{p}} \tilde{a}_{\mathbf{p}} = N_a + N_{\tilde{a}}. \tag{A2}$$

The generalized atomic Green's function is defined as follows:

$$\mathfrak{G}(\mathbf{p},\tau,\tau_0) = -\langle \mathrm{T}_{\tau}\ A_{\mathbf{p}}(\tau) A^{\dagger}_{\mathbf{p}}(\tau_0)\rangle_{\rm th}. \tag{A3}$$

Alternatively, it can also be presented in a matrix form,

$$\mathfrak{G}(\mathbf{p},\tau,\tau_0) = \begin{pmatrix} \mathcal{G}^A_{11}(\mathbf{p},\tau,\tau_0)\ \mathcal{G}^A_{12}(\mathbf{p},\tau,\tau_0) \\ \mathcal{G}^A_{21}(\mathbf{p},\tau,\tau_0)\ \mathcal{G}^A_{22}(\mathbf{p},\tau,\tau_0) \end{pmatrix}, \tag{A4}$$

with the matrix elements given by

$$\begin{aligned} \mathcal{G}^A_{11}(\mathbf{p},\tau,\tau_0) &= -\langle \mathrm{T}_{\tau} a_{\mathbf{p}}(\tau) a^{\dagger}_{\mathbf{p}}(\tau_0)\rangle, \\ \mathcal{G}^A_{12}(\mathbf{p},\tau,\tau_0) &= -\langle \mathrm{T}_{\tau} a_{\mathbf{p}}(\tau) \tilde{a}^{\dagger}_{\mathbf{p}}(\tau_0)\rangle, \\ \mathcal{G}^A_{21}(\mathbf{p},\tau,\tau_0) &= -\langle \mathrm{T}_{\tau} a_{\mathbf{p}}(\tau) a^{\dagger}_{\mathbf{p}}(\tau_0)\rangle, \\ \mathcal{G}^A_{22}(\mathbf{p},\tau,\tau_0) &= -\langle \mathrm{T}_{\tau} a_{\mathbf{p}}(\tau) \tilde{a}^{\dagger}_{\mathbf{p}}(\tau_0)\rangle. \end{aligned} \tag{A5}$$

The off-diagonal propagators compensate each other near equilibrium. Throughout the main text of the paper I use the notation $\mathcal{G}_{\rm at}(\mathbf{p},\tau) \equiv \mathcal{G}^A_{11}(\mathbf{p},\tau,0)$ and $\tilde{\mathcal{G}}_{\rm at}(\mathbf{p},\tau) \equiv \mathcal{G}^A_{22}(\mathbf{p},\tau,0)$.

APPENDIX B: SUMMATION RULES FOR POTENTIAL HARMONICS

The Fourier transforms $V_{\mathbf{k}} = V_{\kappa_{\mathbf{k}}}$ of the potential $V(z)$ are defined as

$$V(z) = \sum_{\kappa} V_{\kappa} e^{i\kappa z}, \quad V_{\kappa} = \frac{1}{2l}\int_{-l}^{+l} dz e^{-i\kappa z} V(z). \tag{B1}$$

One can verify by direct calculation that the following property holds:

$$\sum_{\kappa} V^2_{\kappa} = \int_{-l}^{l} \frac{dz}{2l} V(z) V(-z). \tag{B2}$$

For symmetric potentials $V(z) = V(-z)$, thus one obtains

$$\sum_{\kappa} V^2_{\kappa} = \frac{1}{l}\int_{0}^{l} dz V^2(z). \tag{B3}$$

For the power-law models, $V(z) \propto |z|^{\alpha}$, one can carry out the integral explicitly as

$$\sum_{\kappa} V^2_{\kappa} = \frac{V^2(z=l)}{1+2\alpha}. \tag{B4}$$

This sum remains finite for the important region $\alpha \in (0,2)$ and decreases monotonically, approaching zero as $\alpha \to \infty$.

We also need to calculate the $\kappa = 0$ term of this sum separately, which can be done as

$$V^2_0 = \left[\frac{1}{2l}\int_{-l}^{l} dz V(z)\right]^2 = \frac{V^2(z=l)}{(1+\alpha)^2}, \tag{B5}$$

where the last equality sign stands for $V(z) \propto |z|^{\alpha}$. Thus, the quantity $V_0 = V(l)/(1+\alpha)$ describes the long-scale physics of the problem. Note that the main contribution to the sum (B2) is made by the low-energy photons, $\omega \sim \omega_0$.

APPENDIX C: EFFECTIVE INTERACTION PARAMETERS

The effective interaction parameters $F(\mathbf{k},\mathbf{q})$ determine the self-energy corrections to photon's spectrum, thus renormalizing the interacting vertices, reducing the photon-atoms-photon interaction to effective photon-photon interaction. Consider the first interaction parameter, given by

$$\begin{aligned} F_1(\mathbf{k};\mathbf{q}) = \frac{1}{N^2_{\rm at}} \sum_{\mathbf{p},\mathbf{p}'} &(1 + \delta_{\mathbf{p},\mathbf{p}'} + \delta_{\mathbf{k},\mathbf{q}} + \delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{q}}) \\ &\times n_{\mathbf{p}} \tilde{n}_{\mathbf{p}+\mathbf{k}}\, n_{\mathbf{p}'} \tilde{n}_{\mathbf{p}'+\mathbf{q}}. \end{aligned} \tag{C1}$$

Even though the entire sum can be calculated analytically, for the consistency of the main-order approximation the δ's should be dropped, thus leading to

$$F_1(\mathbf{k};\mathbf{q}) \approx \frac{1}{N^2_{\rm at}} \sum_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}} \tilde{n}_{\mathbf{p}+\mathbf{k}}\, n_{\mathbf{p}'} \tilde{n}_{\mathbf{p}'+\mathbf{q}}, \tag{C2}$$

and the sum factorizes to

$$F_1(\mathbf{k};\mathbf{q}) \approx \sigma_1(\mathbf{k})\,\sigma_1(\mathbf{q}), \quad \sigma_1(\mathbf{k}) = \frac{1}{N_{\rm at}} \sum_{\mathbf{p}} n_{\mathbf{p}} \tilde{n}_{\mathbf{p}+\mathbf{k}}. \tag{C3}$$

Therefore, we basically need to calculate $\sigma_{\mathbf{k}}$. We do it in the following steps. First, let us calculate the total number

of atoms, which in nondegenerate system is given by

$$N_{\rm at} = \sum_{\mathbf{p}} n_{\mathbf{p}} + \tilde{n}_{\mathbf{p}} = \sum_{\mathbf{p}} e^{\beta(\mu_1 - E_{\mathbf{p}})} + e^{\beta(\mu_2 - E_{\mathbf{p}} - \Delta)}, \quad \text{(C4)}$$

where $\Delta = \hbar\omega_{\rm at}$ is the energy difference between the two atomic eigenstates. The normalization is included as an additive constant to the chemical potentials. The chemical potentials $\mu_{1,2}$ of the ground-state and excited-state atoms are linked by the condition of chemical equilibrium, $\mu_1 + \mu = \mu_2$, with μ to be the chemical potential of photons. Because $N_{\rm at}$ is an observable, one can first derive the condition for the chemical potential for one of the atomic subsystem,

$$e^{\beta\mu_1} = \frac{N_{\rm at}}{1+\Theta}\left(\sum_{\mathbf{p}} e^{-\beta E_{\mathbf{p}}}\right)^{-1}, \quad \Theta(\mu) = e^{\beta(\mu-\Delta)}, \quad \text{(C5)}$$

and then calculate the effective interaction factor

$$\sigma_1(\mathbf{k}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} n_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}} = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} e^{\beta(\mu_1 - E_{\mathbf{p}})} e^{\beta(\mu_2 - E_{\mathbf{p}+\mathbf{k}} - \Delta)}. \quad \text{(C6)}$$

After this, upon the substitution of the previous formula (C5), and use of the energy-conservation relation $E_{\mathbf{p}} + \hbar\omega_{\mathbf{k}} = E_{\mathbf{p}+\mathbf{k}} + \Delta$, one obtains

$$\sigma_1(\mathbf{k}) = \frac{1}{2\sqrt{2}}\, e^{-\beta(\hbar\omega_{\mathbf{k}} - \mu)} \frac{N_{\rm at}}{(1+\Theta)^2}\left(\sum_{\mathbf{p}} e^{-\beta E_{\mathbf{p}}}\right)^{-1}. \quad \text{(C7)}$$

The sum here can be calculated in the continuous approximation,

$$\sum_{\mathbf{p}} e^{-\beta E_{\mathbf{p}}} = \frac{\mathcal{V}}{2\sqrt{2}\pi^{3/2}}\left(\frac{mT}{\hbar^2}\right)^{3/2}, \quad \text{(C8)}$$

where $\mathcal{V}$ is the volume of the cavity with atoms. Therefore, one obtains

$$\sigma_1(\mathbf{k}) = \frac{\pi^{3/2} n_{\rm at}}{(1+\Theta)^2}\left(\frac{\hbar^2}{mT}\right)^{3/2} e^{-(\hbar\omega_{\mathbf{k}} - \mu)/T}, \quad \text{(C9)}$$

where $n_{\rm at} = N_{\rm at}/\mathcal{V}$ is the density of the atoms. One can verify that the quantity $\sigma_1(\mathbf{k})$ is dimensionless.

The similar expression can be obtained for the other effective interaction parameters $F(\mathbf{k},\mathbf{q})$. Again, in the main approximation they are given by

$$F_2(\mathbf{k};\mathbf{q}) \approx \sigma_2(\mathbf{k})\,\sigma_2(\mathbf{q}), \quad \sigma_2(\mathbf{k}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} n_{\mathbf{p}} n_{\mathbf{p}+\mathbf{k}},$$
$$F_3(\mathbf{k};\mathbf{q}) \approx \sigma_3(\mathbf{k})\,\sigma_3(\mathbf{q}), \quad \sigma_3(\mathbf{k}) = \frac{1}{N_{\rm at}}\sum_{\mathbf{p}} \tilde{n}_{\mathbf{p}}\tilde{n}_{\mathbf{p}+\mathbf{k}}. \quad \text{(C10)}$$

The effective interaction factors are calculated within the same approximations and are given by

$$\sigma_2(\mathbf{k}) = \frac{\pi^{3/2}}{(1+\Theta)^2} n_{\rm at}\left(\frac{\hbar^2}{mT}\right)^{3/2} e^{-(\hbar\omega_{\mathbf{k}} - \Delta)/T},$$
$$\sigma_3(\mathbf{k}) = \frac{\pi^{3/2}\Theta}{(1+\Theta)^2} n_{\rm at}\left(\frac{\hbar^2}{mT}\right)^{3/2} e^{-(\hbar\omega_{\mathbf{k}} - \mu)/T}. \quad \text{(C11)}$$

Finally, one can notice that $\sigma_2(\mathbf{k}) = \sigma_1(\mathbf{k})/\Theta$ and $\sigma_3(\mathbf{k}) = \sigma_1(\mathbf{k})\,\Theta$, where Θ is given in formula (C5). As Θ depends on μ, that in the leading order is $\hbar\omega_0$ in the case of condensate, therefore changing the geometry of the cavity one can make certain processes to be more or less important for the system under consideration.

[1] J. Klaers, F. Vewinger, and M. Weitz, Nat. Phys. **6**, 512 (2010).
[2] J. Klaers, J. Schmitt, F. Vewinger, and M. Weitz, Nature (London) **468**, 545 (2010).
[3] J. Klaers, J. Schmitt, T. Damm, F. Vewinger, and M. Weitz, Appl. Phys. B **105**, 17 (2011).
[4] J. Marelic and R. A. Nyman, Phys. Rev. A **91**, 033813 (2015).
[5] D. N. Sob'yanin, Phys. Rev. E **85**, 061120 (2012),
[6] D. N. Sob'yanin, Phys. Rev. E **88**, 022132 (2013).
[7] A. Kruchkov and Yu. Slyusarenko, Phys. Rev. A **88**, 013615 (2013).
[8] A. Kruchkov, Phys. Rev. A **89**, 033862 (2014).
[9] P. Kirton and J. Keeling, Phys. Rev. Lett. **111**, 100404 (2013).
[10] A.-W. de Leeuw, H. T. C. Stoof, and R. A. Duine, Phys. Rev. A **88**, 033829 (2013).
[11] R. A. Nyman and M. H. Szymanska, Phys. Rev. A **89**, 033844 (2014).
[12] A.-W. de Leeuw, H. T. C. Stoof, and R. A. Duine, Phys. Rev. A **89**, 053627 (2014).
[13] E.C.I. van der Wurff, A.-W. de Leeuw, R.A. Duine, and H.T.C. Stoof, Phys. Rev. Lett. **113**, 135301 (2014).
[14] A.-W. de Leeuw, E. C. I. van der Wurff, R. A. Duine, and H. T. C. Stoof, Phys. Rev. A **90**, 043627 (2014).
[15] M. Calvanese Strinati and C. Conti, Phys. Rev. A **90**, 043853 (2014).
[16] E. Sela, A. Rosch, and V. Fleurov, Phys. Rev. A **89**, 043844 (2014).
[17] J. Klaers, J. Schmitt, T. Damm, F. Vewinger, and M. Weitz, Phys. Rev. Lett. **108**, 160403 (2012).
[18] M. Weitz, J. Klaers, and F. Vewinger, Phys. Rev. A **88**, 045601 (2013).
[19] P. Kirton and J. Keeling, Phys. Rev. A **91**, 033826 (2015).
[20] N. Boichenko and Yu. Slyusarenko, Cond. Matt. Phys. **18**, 43002 (2015).
[21] J. Schmitt, T. Damm, D. Dung, F. Vewinger, J. Klaers, and M. Weitz, Phys. Rev. Lett. **112**, 030401 (2014).
[22] J. Schmitt, T. Damm, D. Dung, F. Vewinger, J. Klaers, and M. Weitz, Phys. Rev. A **92**, 011602(R) (2015).
[23] R. Y. Chiao and J. Boyce, Phys. Rev. A **60**, 4114 (1999).
[24] Y. B. Zel'dovich and E. V. Levich, Zh. Eksp. Teor. Fiz. **55**, 2423 (1968) [Sov. Phys. JETP **28**, 1287 (1969)].
[25] V. A. Kuz'min and M. E. Shaposhnikov, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 665 (1978) [JETP Lett. **27**, 628 (1978)].
[26] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part I (McGraw-Hill, New York, 1953).
[27] U. Vogl, A. Sass, F. Vewinger, M. Weitz, A. Solovev, Y. Mei, and O. G. Schmidt, Phys. Rev. A **83**, 053403 (2011).

[28] W. Ketterle and N. J. van Druten, Phys. Rev. A **54**, 656 (1996).
[29] I. Yu. Chestnov, A. P. Alodjants, S. M. Arakelian, J. Nipper, U. Vogl, F. Vewinger, and M. Weitz, Phys. Rev. A **81**, 053843 (2010).
[30] I. Yu. Chestnov, A. P. Alodjants, S. M. Arakelian, J. Klaers, F. Vewinger, and M. Weitz, Phys. Rev. A **85**, 053648 (2012).
[31] A. P. Alodjants, I. Yu. Chestnov, and S. M. Arakelian, Phys. Rev. A **83**, 053802 (2011).
[32] G. Mahan, *Many Particle Phycics* (Plenum Press, New York, 1990).
[33] S. Pilati, S. Giorgini, and N. Prokofev, Phys. Rev. Lett. **100**, 140405 (2008).
[34] G. Baym, J.-P. Blaizot, M. Holzmann, F. Lalo, and D. Vautherin, Eur. Phys. J. B **24** 107 (2001).
[35] P. Gruter, D. Ceperley, and F. Laloe, Phys. Rev. Lett. **79**, 3549 (1997).
[36] K. Nho and D. P. Landau, Phys. Rev. A **70**, 053614 (2004).