# Generic quantum walks with memory on regular graphs

Dan Li,[1,2,*] Michael Mc Gettrick,[2,†] Fei Gao,[1,‡] Jie Xu,[1] and Qiao-Yan Wen[1]

[1] *State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China*

[2] *The De Brún Centre for Computational Algebra, School of Mathematics, The National University of Ireland, Galway, Ireland*

(Received 26 January 2016; published 15 April 2016; corrected 20 April 2016)

Quantum walks with memory (QWM) are a type of modified quantum walks that record the walker's latest path. As we know, only two kinds of QWM have been presented up to now. It is desired to design more QWM for research, so that we can explore the potential of QWM. In this work, by presenting the one-to-one correspondence between QWM on a regular graph and quantum walks without memory (QWoM) on a line digraph of the regular graph, we construct a generic model of QWM on regular graphs. This construction gives a general scheme for building all possible standard QWM on regular graphs and makes it possible to study properties of different kinds of QWM. Here, by taking the simplest example, which is QWM with one memory on the line, we analyze some properties of QWM, such as variance, occupancy rate, and localization.

DOI: 10.1103/PhysRevA.93.042323

## I. INTRODUCTION

Due to constructive quantum interference along the paths in the discrete or the continuous version, quantum walks provide a method to explore all possible paths in a parallel way. Many kinds of models of quantum walks have been proposed, such as single-particle quantum walks [1–4], two-particle quantum walks [5–7], three-state quantum walks [8,9], controlled interacting quantum walks [10,11], indistinguishable-particle quantum walks [12,13], disordered quantum walks [14,15], quantum walks on closed surfaces [16], etc. Each type of quantum walk has its own special features and advantages. Therefore, algorithms based on quantum walks have been established as a dominant technique in quantum computation, ranging from element distinctness [17] to database searching [18–21] and from constructing quantum Hash schemes [10,11] to graph isomorphism testing [22,23].

Most quantum walks that have been studied are quantum walks without memory (QWoM) on regular graphs, such as lines, circles, and lattices. Quantum walks with memory (QWM) have been studied only in Refs. [24–27], while classical walks with memory have been used in research on the behavior of hunting, searching, and building the human-memory search model. Standard QWM are a kind of modified quantum walk that has many extra coins to record the walker's latest path. As we know, only two kinds of QWM have been presented up to now. Rohde *et al.* presented a kind of QWM provided by recycled coins and a memory of the coin-flip history [24]. Mc Gettrick presented another kind of QWM whose coin state decides if the shift is "reflect" or "transmit" [25,26]. Konno and Machida provided limit theorems for Mc Gettrick's QWM [27]. The evolutions of these two QWM agree with our intuition. A general scheme for building all possible standard QWM does not exist. It is desired to design more QWM for research, so that we can explore the potential of QWM. Furthermore, if we want to design a QWM on a complex graph or a position-dependent QWM, it seems impossible to design the unitary evolution for a QWM by intuition.

In this paper, we construct a generic model that includes all possible standard QWM on regular graphs. By analyzing the mathematical formalism of QWoM and two existing QWM, we find that QWM on a regular graph can be transformed into a QWoM on a line digraph of the regular graph. Furthermore, the mapping is one to one. That is, we can study QWoM on a line digraph of a regular graph instead of the corresponding QWM on the regular graph. There is only one coin for QWoM, while there are at least two coins for QWM. Therefore, this replacement decreases the coin space and simplifies the analytic process for QWM. Then we construct a generic model of QWoM on a line digraph of regular graphs. This model is actually the generic model of QWM on regular graphs, and it gives a general scheme for building all possible standard QWM on regular graphs.

With this model, it becomes possible to build any wanted QWM on regular graphs and to study the properties of different kinds of QWM. In this paper, by taking the simplest example, which is QWM with one memory on the line, we analyze some properties of this kind of QWM, such as variance, occupancy rate, and localization. We focus on its relation with the partition and coin shift function, which are introduced for designing QWM. Through analysis and research, we get some interesting and useful results.

This paper is structured as follows. In Sec. II, we present the one-to-one correspondence between QWM on a regular graph and QWoM on a line digraph of the regular graph. In Sec. III, we construct a generic model that includes all possible standard QWM on regular graphs. Then, in Sec. IV, by taking the simplest example, which is QWM with one memory on the line, we analyze some properties of this kind of QWM. Finally, a short summary is given in Sec. V.

## II. RELATION BETWEEN QWM AND QWoM

In this part, we introduce the standard formalization of discrete-time QWoM and two kinds of existing QWM. By analyzing the relation between QWoM and QWM, we show that the evolution of QWM on a regular graph is the same

*lidansusu007@163.com

†michael.mcgettrick@nuigalway.ie

‡gaofei_bupt@hotmail.com

 

as the evolution of QWoM on a line digraph of the regular graph.

For the standard discrete-time QWoM on a linear graph, the walker is a bipartite system $|x,c\rangle$, where $x$ is the position of the walker in the graph and $c$ is the coin which decides the shift of the walker. The evolution is decomposed into two steps, $U = SC$, defined as

$$\begin{aligned} C &: |x,c\rangle \rightarrow \sum_j A_{c,j}|x,j\rangle, \\ S &: |x,c\rangle \rightarrow |x+c,c\rangle, \end{aligned} \tag{1}$$

where $A$ is a unitary coin matrix defining the transition amplitudes. The coin takes values of $\pm 1$ (right and left, respectively). After evolving $t$ steps, the output state is $|\psi_{\text{out}}\rangle = (SC)^t|\psi_{\text{in}}\rangle$.

For QWM, there is not a generic model which includes all possible standard QWM. Until now, there have only been two kinds of QWM.

For QWM in Ref. [24], the evolution is decomposed into two steps, $U = SC$, defined as

$$\begin{aligned} C &: |x,c_1,\ldots,c_{d+1}\rangle \rightarrow \sum_j A_{c_{d+1},j}|x,c_1,\ldots,j\rangle, \\ S &: |x,c_1,\ldots,c_{d+1}\rangle \rightarrow |x+c_{d+1},c_{d+1},c_1,\ldots,c_d\rangle, \end{aligned} \tag{2}$$

where $A$ is still the unitary coin matrix defining the transition amplitudes, $x$ is the current position of the walker, $\{c_i = \pm 1 | i = 1,\ldots,d\}$ records the shift of the walker $i$ steps before, and $c_{d+1}$ is the coin which decides the shift of the walker.

For QWM in Refs. [25,26], the evolution is decomposed into two steps, $U = SC$, defined as

$$\begin{aligned} C &: |x_0,x_1,\ldots,x_d,c\rangle \rightarrow \sum_j A_{c,j}|x_0,x_1,\ldots,x_d,j\rangle, \\ S &: |x_0,x_1,\ldots,x_d,1\rangle \rightarrow |x_1,x_0,x_1,\ldots,x_{d-1},1\rangle, \\ &\quad |x_0,x_1,\ldots,x_d,-1\rangle \rightarrow |2x_0 - x_1,x_0,x_1,\ldots,x_{d-1},-1\rangle, \end{aligned} \tag{3}$$

where $A$ is the unitary coin matrix, $x_0$ is the current position of the walker, and $\{x_i | i = 1,\ldots,d\}$ record the positions of the walker $i$ steps before. The coin $c$ takes values $\pm 1$, and $x_{i+1} = x_i \pm 1$.

To build the bridge between QWM and QWoM, we provide a preface for future needs here. We denote by $G = (V,E)$ a digraph with vertex set $V(G)$ and arc set $E(G)$. With fixed labeling of vertices, the adjacency matrix of a digraph $G$ with $N$ vertices, denoted by $M(G)$, is the $N \times N$ (0,1) matrix with the $ij$th element defined by $M_{i,j}(G) = 1$ if $(x_i,x_j) \in E(G)$ and $M_{i,j}(G) = 0$ otherwise. The line digraph of a digraph $G$, denoted by $\overrightarrow{L}G$, is defined as follows: the vertex set of $\overrightarrow{L}G$ is $E(G)$; for $x_a,x_b,x_c,x_d \in V(G),[(x_a,x_b),(x_c,x_d)] \in E(\overrightarrow{L}G)$ if and only if $(x_a,x_b)$ and $(x_c,x_d)$ are both in $E(G)$ and $x_b = x_c$. The line digraph of $\overrightarrow{L}G$ is denoted by $\overrightarrow{L}^2G$. Similarly, there are $\overrightarrow{L}^dG$s with $d \in N^*$. For simplicity, we call all of them the line digraph of $G$.

Then we show how to transform QWM on a regular graph $G$ to QWoM on a line digraph of $G$. From the definition of a line digraph, we know a vertex of $\overrightarrow{L}^dG$ is a $d$-length path on graph $G$. Therefore, there is a one-to-one correspondence between $|x,c_1,\ldots,c_d\rangle$ and $|(x - c_1 \cdots - c_d,\ldots,x - c_1,x)\rangle$, where $\{(x - c_1 \cdots - c_d,\ldots,x - c_1,x)\}$ is the vertex set of $\overrightarrow{L}^dG$. Then, due to the relation between $|x,c_1,\ldots,c_d,c\rangle$ and $|(x - c_1 \cdots - c_d,\ldots,x - c_1,x),c\rangle$, we build a bridge between the two kinds of quantum walks.

For example, we choose the QWM with two memories on the line, i.e., three qubit coins. For QWM in Ref. [24], the evolution is as follows:

$$\begin{aligned} |x,c_1,c_2,c\rangle &\xrightarrow{C} \sum_j A_{c,j}|x,c_1,c_2,j\rangle \\ &\xrightarrow{S} \sum_j A_{c,j}|x+j,j,c_1,c_2\rangle. \end{aligned} \tag{4}$$

It also can be written as

$$\begin{aligned} &|(x - c_1 - c_2,x - c_1,x),c\rangle \\ &\quad\xrightarrow{C} \sum_j A_{c,j}|(x - c_1 - c_2,x - c_1,x),j\rangle \\ &\quad\xrightarrow{S} \sum_j A_{c,j}|(x - c_1,x,x+j),c_2\rangle. \end{aligned} \tag{5}$$

Equations (4) and (5) correspond to QWM on $G$ and QWoM on $\overrightarrow{L}^2G$, respectively. We can easily know that the evolutions are essentially the same because $|x+j,j,c_1,c_2\rangle$ corresponds to $|(x - c_1,x,x+j),c_2\rangle$, where $(x - c_1,x,x+j)$ is a vertex of $\overrightarrow{L}^2G$. In addition, the evolution space $\overrightarrow{L}^2G \otimes H_2$ for QWoM on the line digraph of $G$ is spanned by $\{|(x - c_1 - c_2,x - c_1,x),j\rangle | c_1,c_2,j = \pm 1\}$, where $H_2$ is the Hilbert space for a two-dimensional coin. Therefore, a QWM with two memories on $G$ corresponds to a QWoM (which updates the coin state after the shift) on the related $\overrightarrow{L}^2G$. What is more, a QWoM (which updates the coin state after the shift) on $\overrightarrow{L}^2G$ corresponds to a QWM with two memories on $G$. That means there is a one-to-one correspondence between QWM on $G$ and QWoM on the line digraph of $G$. Similarly, there is a one-to-one correspondence between QWM with $d$ memories on any regular graph $G$ and QWoM on the related line digraph $\overrightarrow{L}^dG$.

## III. QWM ON REGULAR GRAPHS

From the above discussion, we know that there is a one-to-one correspondence between QWM on a regular graph $G$ and QWoM on a line digraph of $G$. Therefore, the generic model of QWoM on a line digraph of $G$, which includes all possible QWoM (which updates the coin state after the shift) on a line digraph of $G$, is actually the generic model of QWM on $G$, which includes all possible standard QWM on $G$. Then we only need to construct the generic QWoM on the line digraph of $G$ instead of generic QWM on $G$.

Here, we introduce two definitions to prepare for constructing the model of generic QWoM on a line digraph of an $m$-regular graph $G$ (these two definitions are inspired by Ref. [28]). These two definitions are introduced to show the shift of the walker along the graph $G$.

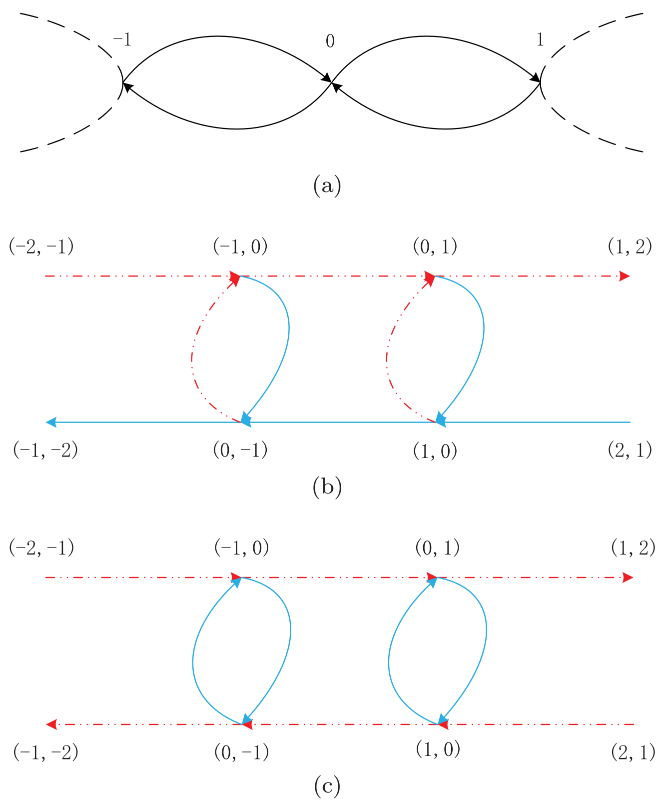


FIG. 1. The original digraph $G$ and the line digraph of $G$ denoted by $\vec{L}G$. (b) and (c) show two partitions of $\vec{L}G$, $\pi_1$ and $\pi_2$, by using different lines (dash-double-dotted red line and solid blue line) to denote $C_k$.

*Definition 1*. Let $G$ be an $m$-regular graph. Define $\pi$ as a partition of $\overrightarrow{L}^dG$ such that

$$\pi : \overrightarrow{L}^dG \to \{C_1, C_2, \ldots, C_m\}, \tag{6}$$

where $\{C_k|k = 1,\ldots,m\}$ satisfy $V(C_k) = V(\overrightarrow{L}^dG)$, $\bigcup_k E(C_k) = E(\overrightarrow{L}^dG)$ and, for every vertex $v \in V(C_k)$, the outdegree is 1. Dicycle factorization is a kind of partition which satisfies the requirement that for every vertex $v \in V(C_k)$, the outdegree and indegree are 1. We denote the set of partitions of $\overrightarrow{L}^dG$ by $\Pi_{\overrightarrow{L}^dG}$.

We show two partitions in Fig. 1. The original graph $G$ is the infinite line in Fig. 1(a). The line digraph of $G$, $\overrightarrow{L}G$, is shown in Fig. 1(b). We show partitions $\pi_1$ and $\pi_2$ in Figs. 1(b) and 1(c), respectively, by using different lines (dash-double-dotted red line and solid blue line) to denote $C_k$.

*Definition 2*. For $\pi \in \Pi_{\overrightarrow{L}^dG}$ with $\overrightarrow{L}^dG \xrightarrow{\pi} \{C_1, C_2, \cdots, C_m\}$, define

$$f_{C_k} : V(\overrightarrow{L}^dG) \to V(\overrightarrow{L}^dG) \tag{7}$$

such that for any $v \in V(\overrightarrow{L}^dG)$,

$$[v, f_{C_k}(v)] \in E(C_k). \tag{8}$$

In what follows, we construct the generic QWoM on $\overrightarrow{L}^dG$, i.e., the generic model of QWM with $d$-step memory on an $m$-regular graph $G$.

*Definition 3*. For a QWoM on the line digraph of $G$ denoted by $\overrightarrow{L}^dG$, the evolution is decomposed into two steps, $U = SC$, defined as

$$\begin{aligned} C &: |v,c\rangle \longrightarrow \sum_j A_{c,c_j}|v,c_j\rangle, \\ S &: |v,c_j\rangle \longrightarrow |f_{C_j}(v), gc(v,c_j)\rangle, \end{aligned} \tag{9}$$

where $A$ is a unitary coin matrix defining the transition amplitudes, which may be position time dependent. $v$ is the position at $\overrightarrow{L}G$. The coin $c$ decides the shift of the walker. The coin shift function $gc$ is defined as follows:

$$gc : V(\overrightarrow{L}^dG) \otimes H_m \longrightarrow H_m, \tag{10}$$

where $H_m$ is the Hilbert space for an $m$-dimensional coin, spanned by $\{c_1, c_2, \ldots, c_m\}$.

The shift operator $S$ requires the walker to walk along the subgraph $C_k$ when the coin is $c_k$. The coin shift function $gc$ updates the coin after moving. We should remind readers that QWM is decided by the coin operator, partition, and coin shift function.

Until now, we get a generic model of QWoM on the line digraph of $G$, i.e., a generic model of QWM on regular graph $G$. However, the model seems too formalized. Next, we will show the concrete form of the model. It is worth recalling that the evolution of quantum walks on an infinite graph at time $t$ equals the evolution of quantum walks on a bigger finite graph at time $t$. Therefore, we only need to consider the finite graph when we focus on the outstate after $t$ steps.

The coin operator $C$ for this model is similar to that for QWoM, which may be position history dependent. However, the shift operator $S$ for this model is more complex. Below we will show the concrete form of the shift operator.

Suppose $M(G)$ is the adjacent matrix of an $m$-regular graph $G$ on $N$ vertices. Then a partition of $G$ is actually a partition of the adjacent matrix $M(G)$, i.e.,

$$M(G) \overset{\pi}{=\!=\!=} M(D_1) + M(D_2) + \cdots + M(D_m), \tag{11}$$

where there is exactly one entry which is 1 and entries are 0 elsewhere for every row vector of $M(D_k), k = 1,\ldots,m$. Furthermore, a dicycle factorization is a special partition in which there is exactly one entry which is 1 and entries are 0 elsewhere for every column vector of $M(D_k), k = 1,\ldots,m$.

Then, we introduce an important theorem [29] about the adjacent matrix of the line digraph of $G$.

*Theorem 1*. Let $G$ be an $m$-regular digraph on $N$ vertices and let $\{D_1, D_2, \ldots, D_m\}$ be a dicycle factorization of $G$. Then there is a labeling of $V(\overrightarrow{L}G)$ such that

$$M(\overrightarrow{L}G) = \begin{pmatrix} M(D_1) & M(D_2) & \cdots & M(D_m) \\ M(D_1) & M(D_2) & \cdots & M(D_m) \\ \vdots & \vdots & \ddots & \vdots \\ M(D_1) & M(D_2) & \cdots & M(D_m) \end{pmatrix}. \tag{12}$$

With this theorem, we can get the adjacent matrix $M(\overrightarrow{L}^dG)$ for any $m$-regular graph $G$. First, we need to partition the $m$-regular graph $G$ into $\{D_1, D_2, \ldots, D_m\}$ which is a dicycle factorization of $G$. This process is very easy for a regular graph. According to Theorem 1, we can get the adjacent matrix

$M(\overrightarrow{L}G)$. Then we partition $M(\overrightarrow{L}G)$ to

$$M(\overrightarrow{L}G) = \begin{pmatrix} M(D_1) & & & \\ & M(D_2) & & \\ & & \ddots & \\ & & & M(D_m) \end{pmatrix} + \cdots + \begin{pmatrix} 0 & M(D_2) & & \\ & 0 & M(D_3) & \\ & & 0 & M(D_m) \\ M(D_1) & & & 0 \end{pmatrix}, \tag{13}$$

where the formalization of each term in the sum is similar to that of the determinant. This partition is still a dicycle factorization of $M(\overrightarrow{L}G)$. We can get the adjacent matrix $M(\overrightarrow{L}^2G)$ by using Theorem 1 again. Iterating this process, we can get the adjacent matrices of $\overrightarrow{L}^dG$ for any $d$. The vertices of $V(\overrightarrow{L}^dG)$ are labeled by $v_1,\dots,v_{N2^{d-1}},\dots v_{N2^d}$.

Until now, we can get the adjacent matrix of $\overrightarrow{L}^dG$ for any $d$ and any regular graph $G$. A partition of $\overrightarrow{L}^dG$ is actually a partition of the adjacent matrix of $\overrightarrow{L}^dG$, i.e.,

$$M(\overrightarrow{L}^dG) \overset{\pi}{=\!=\!=} M(C_1) + M(C_2) + \cdots + M(C_m), \tag{14}$$

where there is exactly one entry which is 1 and entries are 0 elsewhere for every row vector of $M(C_k), k = 1,\dots,m$. Furthermore, $M(C_k)_{(v_i,v_j)} = 1$ if and only if $f_{C_k}(v_i) = v_j$. Therefore, the shift operator $S$ is

$$S = \sum_{i,k} M(C_k)^T |v_i\rangle\langle v_i| \otimes |gc(v_i,c_k)\rangle\langle c_k|. \tag{15}$$

From the above equation, we find that in order to ensure the unitarity of the shift operator $S$, $gc$ has to follow some rules. From Theorem 1, we can easily know that $M(\overrightarrow{L}^dG)$ can be viewed as a combination of $m$ same block matrices, i.e.,

$$M(\overrightarrow{L}^dG) = \begin{pmatrix} M_1(\overrightarrow{L}^dG) \\ \vdots \\ M_m(\overrightarrow{L}^dG) \end{pmatrix}, \tag{16}$$

where each of $\{M_i | i = 1,\dots,m\}$ includes $Nm^{d-1}$ rows of $M(\overrightarrow{L}^dG)$. Furthermore, because we choose dicycle factorizations to partition the regular graph $G$ and its line digraphs, there is exactly one entry which is 1, and entries are 0 elsewhere for every column vector of $M_i(\overrightarrow{L}^dG)$. Therefore, there are $m$ $\{C_k,v_i\}$ satisfying $v = f_{C_k}(v_i)$ for any $v \in V(\overrightarrow{L}^dG)$. Under the sort order of vertices, in order to ensure the unitarity of the shift operator $S$, for the corresponding $m$ $\{c_k,v_i\}$, the set of $\{gc(v_i,c_k)\}$ has to satisfy

$$\{\overbrace{gc(v_i,c_k),\dots}^{m}\} = \{c_1,c_2,\dots,c_m\}. \tag{17}$$

Thus, we finally get the concrete form of the generic model of QWoM on the line digraph of $G$. Due to the one-to-one correspondence between QWoM on the line digraph of $G$ and QWM on $G$, this model is actually the generic model of QWM, which includes all possible standard QWM on regular graphs. More importantly, this model provides a way to build any wanted QWM directly. Furthermore, it seems the walker has to walk on a more complex graph, but the generic model does not increase or decrease the evolution space. It provides a fresh viewpoint to study the QWM by transforming the coin space to position space.

## IV. QWM WITH ONE MEMORY ON THE LINE

The generic model of QWM provides a way to design any wanted QWM. With all standard QWM on regular graphs, it is possible to study the properties of different kinds of QWM. However, it is unrealistic to study each particular situation in this paper. In this part, we focus on the properties of the simplest one, QWM with one memory on the line, i.e., $m = 2, d = 1$, and its relation to the partition and coin shift function.

In order to study in depth the properties of the generic model of QWM, we consider QWM with different partitions and coin shift functions and the standard QWoM for comparison. We denote the partitions and coin shift functions in Refs. [24,25] by $\pi_1$, $\pi_2$, $gc_1$, $gc_2$, respectively. $\pi_1$ and $\pi_2$ are shown in Figs. 1(b) and 1(c). $gc_1$ and $gc_2$ are presented in Eqs. (A4) and (A5). We also consider a random partition $\pi_3$ and a random dicycle factorization partition $\pi_4$, which are produced by MATLAB. Therefore, there are six kinds of QWM: QWM with $\pi_1$, $gc_1$; QWM with $\pi_2$, $gc_1$; QWM with $\pi_2$, $gc_2$; QWM with $\pi_3$, $gc_1$; QWM with $\pi_4$, $gc_1$; and QWM with $\pi_4$, $gc_2$. To avoid the confusion that the QWM we simulated include QWM with $\pi_2,gc_1$ but not QWM with $\pi_1,gc_2$, we remind readers that for QWM with partition $\pi_1$, the coin shift function $gc_2$ does not satisfy the constraint (17). We leave details for Appendix A. For the same reason, QWM with $\pi_3$, $gc_2$ does not exist. For the coin operator, in this paper, we only consider the Hadamard matrix as the coin operator.

QWM bring possibilities for new phenomena. However, QWM with a dicycle factorization partition and the coin shift function $gc_2$ reduce the possibilities. When the initial position state is $|0\rangle_p$, there are QWM with different dicycle factorization partitions and the coin shift function $gc_2$ which create the same probability distribution for any initial coin state. The evolution of this kind of QWM is only affected by the partition at three positions around the center, i.e., $-1$, 0, 1. Therefore, the number of different QWM with a dicycle factorization partition and coin shift function $gc_2$ is $8 = 2^3$. Luckily, this phenomenon does not appear in other kinds of QWM. For other kinds of QWM, the number of different QWM increases exponentially with time $t$, as we predicted.

Now we consider the variance of QWM. For QWoM, we usually only consider current position of the walker, and ignore the coin state. For QWM, we still only consider current position rather than the memory. Therefore, variance is defined as follows:

$$\mathrm{Var} = \sum_x p(x)x^2 - \left[\sum_x p(x)x\right]^2, \tag{18}$$

where $x$ is the current position of the walker and $p(x)$ is the norm's square of the amplitude for the walker at position $x$. Ellinas and Smyrnakis [30] have shown that the general form for the variance of a quantum walk is $\sigma(t)^2 = K_2t^2 + K_1t + K_0^2$.

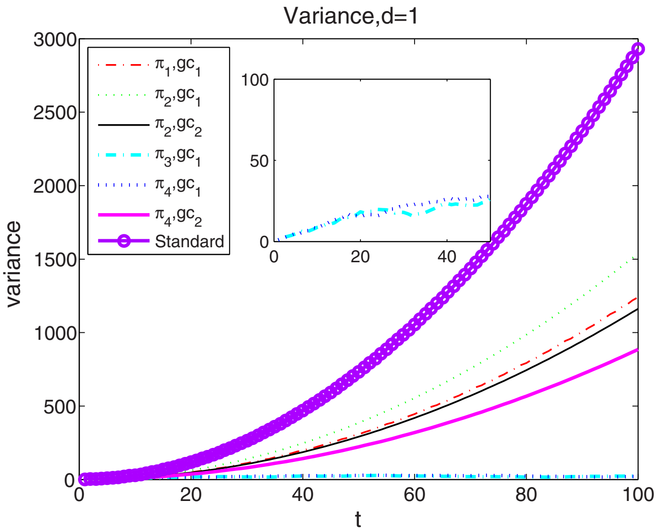


FIG. 2. Variance of six kinds of QWM and the standard QWoM, denoted by the thin dash-dotted red line, thin dotted green line, thin solid black line, thick dash-dotted cyan line, thick dotted blue line, thick solid pink line, and purple circles. The initial coin state is $\frac{1}{2}\{|-1,0\rangle|1\rangle + i|-1,0\rangle|-1\rangle + |1,0\rangle|1\rangle + i|1,0\rangle|-1\rangle\}$. The inset shows the variance of QWM with $\pi_3,gc_1$ and $\pi_4,gc_1$ once again.

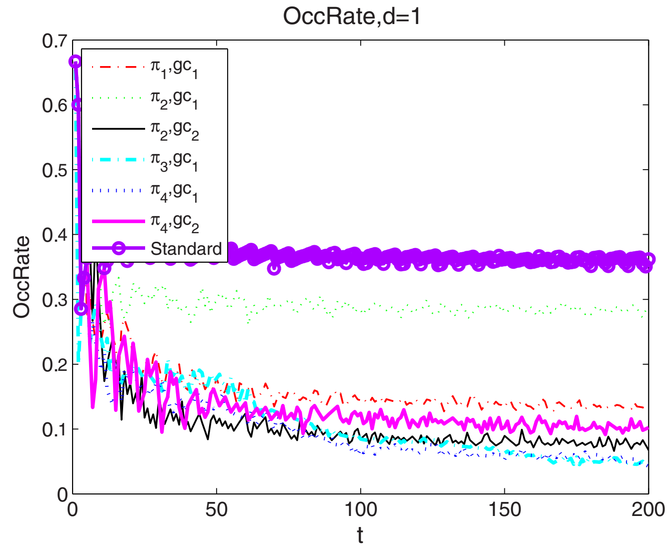


FIG. 3. Occupancy rate of six kinds of QWM and the standard QWoM, denoted by the thin dash-dotted red line, thin dotted green line, thin solid black line, thick dash-dotted cyan line, thick dotted blue line, thick solid pink line, and purple circles. The initial coin state is $\frac{1}{2}\{|-1,0\rangle|1\rangle + i|-1,0\rangle|-1\rangle + |1,0\rangle|1\rangle + i|1,0\rangle|-1\rangle\}$.

QWoM still follow the rule. From Fig. 2, we know that most QWM are ballistic, except for QWM with a random partition and the coin shift function $gc_1$. It is surprising that QWM with a random dicycle factorization partition and $gc_2$ are still ballistic while the other QWM with a random partition are diffusive. We find that the random partition destroys the ballistic nature of QWM; only QWM with a dicycle factorization and the coin function $gc_2$ remain ballistic. From another angle, QWM with the coin shift function $gc_2$ can generate ballistic behavior. For QWM with the coin shift function $gc_1$, only organized partitions (such as the partition with repetition) can help to make QWM ballistic. Luckily, considering the application of QWM, only QWM with an organized partition will gain extensive attention from researchers.

We consider the occupancy rate of QWM here. QWM with a random partition and $gc_1$ are diffusive, but that does not mean no quantum properties exist for this kind of QWM. The occupancy rate was proposed in Ref. [9] as a way of measuring the statistical property of the probability distribution. A nonzero value of the occupancy rate for a quantum walk on an infinite graph can be seen as a sign of the quantum feature. If the walker has range $N$, the occupancy rate is defined as

$$R_{\text{Occ}}(N,t) = \frac{\#\left\{x \middle| P(x,t) \geqslant \frac{1}{N}\right\}}{N}. \tag{19}$$

For QWoM on the line [9], variance has the order $O(t^2)$, and the occupancy rate has the order $O(1)$. In addition, for classical walks on the line, variance has the order $O(t)$, and the occupancy rate converges to zero. For QWM in Fig. 3, the order of the occupancy rate is still $O(1)$ for all cases, while the order of variance of QWM with a random partition and the coin shift function $gc_1$ is $O(t)$. This means that QWM do not lose all of their quantum properties even with a random partition.

Localization is an important feature of QWM (by localization, we mean the existence of position $x$ where the asymptotic probability value is nonzero). For QWM, localization is a more common property than for other kinds of quantum walks. Almost every kind of QWM we examined, except QWM with the partition $\pi_2$ and the coin shift function $gc_1$, possesses localization. For example, we show the probability at the origin at time 100 of QWM with $\pi_1$ and $gc_1$ in Fig. 4(a) and QWM with $\pi_2$ and $gc_1$ in Fig. 4(b). We choose the initial state $\alpha|-1,0\rangle|1\rangle + \sqrt{\alpha(1-\alpha)}e^{i\beta}|-1,0\rangle|-1\rangle + \sqrt{\alpha(1-\alpha)}e^{i\beta}|1,0\rangle|1\rangle + (1-\alpha)e^{2i\beta}|1,0\rangle|-1\rangle$. We find that Fig. 4(b) is basically dark while Fig. 4(a) is more bright. This implies QWM with $\pi_2$ and $gc_1$ are not localized while QWM with $\pi_1$ and $gc_1$ are localized. In fact, for other QWM we examined, the graphs of localization are also bright. Then we show the probability at the origin position with time $t$ changing in Fig. 4(c). We choose the initial state $0.1|-1,0\rangle|1\rangle + \sqrt{0.09}|-1,0\rangle|-1\rangle + \sqrt{0.09}|1,0\rangle|1\rangle + 0.9|1,0\rangle|-1\rangle$. Different initial states will not affect the subgraph drastically. From Fig. 4(c), it is easy to see that only for QWM with $\pi_2$ and $gc_1$ and standard QWoM does the probability at the original position converge to zero. This phenomenon verifies that QWM with $\pi_2$ and $gc_1$ are not localized while other QWM are localized. For QWM with a random partition and the coin shift function $gc_1$, the probability at the origin vibrates sharply because of the randomness of the partition. Nevertheless, this kind of QWM still has a high probability at the origin for large values of the time $t$. It shows again the quantum property of QWM with a random partition.

During our research, we found a rare and interesting result. Even though QWM with different partitions which could produce the same probability distribution exist, they still belong to QWM. Mostly, different kinds of quantum walks produce different probability distributions. There is only one special case. In Refs. [6,7], Di Franco *et al.* found the

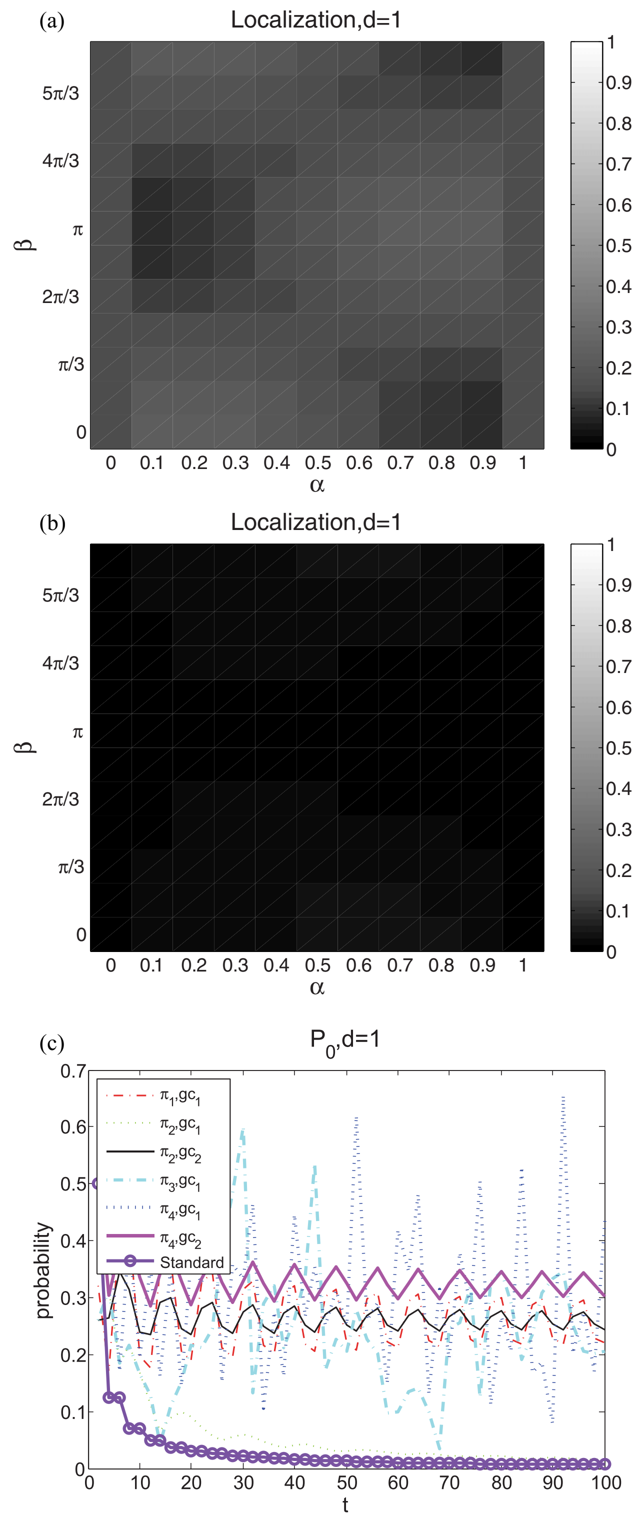


FIG. 4. (a) The probability at the origin at time 100 of QWM with $\pi_1$ and $gc_1$. (b) The probability at the origin at time 100 of QWM with $\pi_2$ and $gc_1$. The initial state is $\alpha|-1,0\rangle|1\rangle + \sqrt{\alpha(1-\alpha)}e^{i\beta}|-1,0\rangle|-1\rangle + \sqrt{\alpha(1-\alpha)}e^{i\beta}|1,0\rangle|1\rangle + (1-\alpha)e^{2i\beta}|1,0\rangle|-1\rangle$. (c) The change of probability at the origin with time $t$ changing for six kinds of QWM and the standard QWoM, denoted by the thin dash-dotted red line, thin dotted green line, thin solid black line, thick dash-dotted cyan line, thick dotted blue line, thick solid pink line, and purple circles. The initial state is $0.1|-1,0\rangle|1\rangle + \sqrt{0.09}|-1,0\rangle|-1\rangle + \sqrt{0.09}|1,0\rangle|1\rangle + 0.9|1,0\rangle|-1\rangle$.

nonlocalized case of the spatial density probability of the two-dimensional Grover walk can be obtained using only a two-dimensional coin space and a quantum walk in alternate directions. Here, we find another example. The symmetric probability distribution of the standard QWoM on the line with initial state $|0\rangle(\frac{1}{\sqrt{2}}|1\rangle + \frac{i}{\sqrt{2}}|-1\rangle)$ can be obtained by the QWM with $\pi_2$ and $gc_2$ when the initial state is $\frac{1}{2}|-1,0\rangle|1\rangle - \frac{1}{2}|-1,0\rangle|-1\rangle - \frac{1}{2}|1,0\rangle|1\rangle + \frac{1}{2}|1,0\rangle|-1\rangle$. Let us denote the coefficients in the decomposition of the standard QWoM and that of the QWM as $\alpha_{x,c}$ and $\beta_{x,x\pm1,c}$, respectively. Then we have a correspondence between them given by

$$\begin{aligned}\alpha_{x,1}^t &= (-1)^{\frac{t+x}{2}} e^{i\frac{\pi}{4}}\left(-i\beta_{x-1,x,1}^t - \beta_{x-1,x,-1}^t\right),\\ \alpha_{x,-1}^t &= (-1)^{\frac{t+x}{2}} e^{i\frac{\pi}{4}}\left(-\beta_{x+1,x,1}^t + i\beta_{x+1,x,-1}^t\right).\end{aligned} \tag{20}$$

We leave the proof for Appendix B. Therefore, these two totally different quantum walks produce same probability distribution, i.e.,

$$\begin{aligned}P_x^t &= \left|\alpha_{x,1}^t\right|^2 + \left|\alpha_{x,-1}^t\right|^2\\ &= \left|\beta_{x-1,x,1}^t\right|^2 + \left|\beta_{x-1,x,-1}^t\right|^2 + \left|\beta_{x+1,x,1}^t\right|^2 + \left|\beta_{x+1,x,-1}^t\right|^2.\end{aligned} \tag{21}$$

## V. SUMMARY

In this paper, we found the one-to-one correspondence between QWM with $d$ memory on a regular graph $G$ and QWoM on the associated line digraph $\overrightarrow{L}^d G$. Through this correspondence, we studied QWoM on the line digraph of $G$ instead of QWM on $G$. Furthermore, through this correspondence, we constructed a generic model which includes all possible standard QWM on regular graphs.

The generic model may not make the calculation of QWM simpler because it transforms the coin space to position space, which does not increase or decrease the resource to execute QWM. However, the model gives a fresh viewpoint to study QWM on regular graphs. Furthermore, this model gives a general scheme for building all possible standard QWM on regular graphs. Then we can design any required QWM on regular graphs.

What is more, with the generic model of QWM on regular graphs, it is possible to study the properties of different kinds of QWM. Because it was unrealistic to study each particular situation in this paper, we focused on the simplest case, which is QWM with one memory on the line. We paid the most attention to QWM with different kinds of partition and coin shift functions. In this paper, we have the following results:

| | | Variance | Occupancy rate | Localization |
|---|---|---|---|---|
| QWM | $\pi_1,gc_1$ | ballistic | nonzero | yes |
| | $\pi_2,gc_1$ | ballistic | nonzero | no |
| | $\pi_2,gc_2$ | ballistic | nonzero | yes |
| | $\pi_3,gc_1$ | diffusive | nonzero | yes |
| | $\pi_4,gc_1$ | diffusive | nonzero | yes |
| | $\pi_4,gc_2$ | ballistic | nonzero | yes |
| Standard QWoM | | ballistic | nonzero | no |

(1) QWM with a sorted partition have ballistic evolution, while QWM with a random partition may become diffusive. At the same time, QWM with a random dicycle factorization partition and $gc_2$ (which number eight when we fix the initial position state to $|0\rangle$) are still ballistic. With this result, we can build a ballistic QWM as desired by choosing the appropriate partition and coin shift function. We also know that we do not need to study all QWM with a dicycle factorization partition and the coin shift function $gc_2$ because they can be reduced to eight QWM. Therefore, researching those eight QWM is enough.

(2) QWM have a nonzero value of the occupancy rate, even for QWM with a random partition. This means QWM still have quantum properties even with a random partition.

(3) Localization is a common feature for QWM, but it does not necessarily occur for all QWM. QWM with the partition $\pi_2$ and the coin shift function $gc_1$ do not possess the localization property. Our results tell us which kind of partition and coin shift function we should choose if we want to build a QWM with or without localization.

(4) A QWM could produce the same probability distribution as that of a standard QWoM on the line when the initial state is $\sqrt{\frac{1}{2}}|0\rangle_p(|1\rangle_c + i|-1\rangle_c)$. This result may be not useful but is interesting considering how rare it is.

Our work extends current research on quantum walks by showing a generic model of QWM. Furthermore, the generic model opens the door to the research of QWM by giving a general scheme for constructing all possible standard QWM. We anticipate that the abundant phenomena of QWM will be useful in quantum computation and quantum simulation.

## ACKNOWLEDGMENTS

This work is supported by NSFC (Grants No. 61272057 and No. 61572081), the Beijing Higher Education Young Elite Teacher Project (Grants No. YETP0475 and No. YETP0477), the BUPT Excellent Ph.D. Students Foundation (Grant No. CX201326), and the China Scholarship Council (Grant No. 201306470046).

## APPENDIX A

First, we show the partitions of the two kinds of QWM [24,25], labeled by $\pi_1$ and $\pi_2$, in Figs. 1(b) and 1(c), respectively. The essential difference between the two partitions is that the partition $\pi_2$ is a dicycle factorization of the line digraph in Fig. 1(a), which means that for every vertex $v \in V(C_k)\,(k=1,2)$, the outdegree and indegree of $v$ are both 1. This difference makes the coin shift function $gc$ for different QWM take on different forms.

We denote the vertices as $v_1,\dots,v_N,\dots v_{2N}$ (the sort of order should obey Theorem 1 in Sec. III). According to Eqs. (16), for any $v \in \overrightarrow{L}G$, $M(\overrightarrow{L}G)_{(v_i,v)} = 1$ if and only if $M(\overrightarrow{L}G)_{(v_{i+N},v)} = 1$.

If the partition is a dicycle factorization, because the indegree of any $v$ is 1, there is no $(v_j,C_k)$ that satisfies $f_{C_k}(v_j) = f_{C_k}(v_{j+N})$. Therefore, the constraint (17) can be written as

$$\begin{aligned} gc(v_i,1) + gc(v_{i+N},-1) &= 0,\\ gc(v_i,-1) + gc(v_{i+N},1) &= 0, \end{aligned} \tag{A1}$$

where $i \leqslant N2^{d-1}$.

If the partition is not a dicycle factorization, there exist $(v_j,C_k)$ that satisfy $f_C(v_j) = f_C(v_{j+N})$. Therefore,

$$\begin{aligned} gc(v_i,1) + gc(v_{i+N},-1) = 0 \quad (i \neq j),\\ gc(v_i,-1) + gc(v_{i+N},1) = 0 \quad (i \neq j),\\ gc(v_j,1) + gc(v_{j+N},1) = 0,\\ gc(v_j,-1) + gc(v_{j+N},-1) = 0. \end{aligned} \tag{A2}$$

Summarizing the above conditions, the choice

$$\begin{aligned} gc(v_i,1) = k_i, \quad gc(v_{i+N},1) = -k_i,\\ gc(v_i,-1) = k_i, \quad gc(v_{i+N},-1) = -k_i, \end{aligned} \tag{A3}$$

with $k_i = \pm 1$. This choice befits any partition.

For QWM in Ref. [24], for any $v_j$, there exist $C_k$ such that $f_{C_k}(v_j) = f_{C_k}(v_{j+N})$, and the coin state takes values $\pm 1$. The coin shift function $gc_1$ in Ref. [24] is

$$\begin{aligned} gc_1(v_i,1) = 1, \quad gc_1(v_{i+N},1) = -1,\\ gc_1(v_i,-1) = 1, \quad gc_1(v_{i+N},-1) = -1, \end{aligned} \tag{A4}$$

which satisfies Eqs. (A3) and befits any partition.

For QWM in Ref. [25], because $\pi_2$ is a dicycle factorization, there do not exist $(v_j,C_k)$ such that $f_{C_k}(v_j) = f_{C_k}(v_{j+N})$. The coin shift function $gc_2$ in Ref. [25] is

$$\begin{aligned} gc_2(v_i,1) = 1, \quad gc_2(v_{i+N},1) = 1,\\ gc_2(v_i,-1) = -1, \quad gc_2(v_{i+N},-1) = -1, \end{aligned} \tag{A5}$$

where $i \leqslant N$. Here, $gc_2$ only satisfies Eqs. (A1) rather than Eqs. (A3), which means $gc_2$ only work for QWM with a dicycle factorization partition.

## APPENDIX B

For QWM with $\pi_2$ and $gc_2$,

$$\begin{aligned} \beta^{t+1}_{x,x+1,1} &= \frac{1}{\sqrt{2}}\left(\beta^t_{x+1,x,1} + \beta^t_{x+1,x,-1}\right),\\ \beta^{t+1}_{x,x-1,1} &= \frac{1}{\sqrt{2}}\left(\beta^t_{x-1,x,1} + \beta^t_{x-1,x,-1}\right),\\ \beta^{t+1}_{x,x-1,-1} &= \frac{1}{\sqrt{2}}\left(\beta^t_{x+1,x,1} - \beta^t_{x+1,x,-1}\right),\\ \beta^{t+1}_{x,x+1,-1} &= \frac{1}{\sqrt{2}}\left(\beta^t_{x-1,x,1} - \beta^t_{x-1,x,-1}\right). \end{aligned} \tag{B1}$$

We first want to prove the amplitudes satisfy the constraint

$$\begin{aligned} \beta^t_{x,x+1,1} + \beta^t_{x,x+1,-1} + \beta^t_{x,x-1,1} + \beta^t_{x,x-1,-1} &= 0,\\ \beta^t_{x+1,x,1} + \beta^t_{x-1,x,1} &= 0. \end{aligned} \tag{B2}$$

Our proof works by induction on $t$. When the QWM begin with the initial state $\frac{1}{2}|-1,0\rangle|1\rangle - \frac{1}{2}|-1,0\rangle|-1\rangle - \frac{1}{2}|1,0\rangle|1\rangle + \frac{1}{2}|1,0\rangle|-1\rangle$, i.e.,

$$\begin{aligned} \beta^0_{-1,0,1} = \tfrac{1}{2}, \quad \beta^0_{-1,0,-1} = -\tfrac{1}{2},\\ \beta^0_{1,0,1} = -\tfrac{1}{2}, \quad \beta^0_{1,0,-1} = \tfrac{1}{2}, \end{aligned} \tag{B3}$$

it is easy to verify, by means of a direct calculation, that Eqs. (B2) are satisfied at $t = 0$. Now, we assume Eqs. (B2) are true for any $x$ at time $t$; then we prove that they hold at time $t + 1$.

$$\begin{aligned}
&\beta^{t+1}_{x,x+1,1} + \beta^{t+1}_{x,x+1,-1} + \beta^{t+1}_{x,x-1,1} + \beta^{t+1}_{x,x-1,-1}\\
&= \sqrt{2}\big(\beta^{t}_{x+1,x,1} + \beta^{t}_{x-1,x,1}\big)\\
&= \beta^{t-1}_{x,x+1,1} + \beta^{t-1}_{x,x+1,-1} + \beta^{t-1}_{x,x-1,1} + \beta^{t-1}_{x,x-1,-1}\\
&= \sqrt{2}\big(\beta^{t-2}_{x+1,x,1} + \beta^{t-2}_{x-1,x,1}\big).
\end{aligned} \tag{B4}$$

Therefore, Eqs. (B2) are satisfied at any time $t$.

For the standard QWoM on the line,

$$\begin{aligned}
\alpha^{t+1}_{x,1} &= \frac{1}{\sqrt{2}}\big(\alpha^{t}_{x-1,1} + \alpha^{t}_{x-1,-1}\big),\\
\alpha^{t+1}_{x,-1} &= \frac{1}{\sqrt{2}}\big(\alpha^{t}_{x+1,1} - \alpha^{t}_{x+1,-1}\big).
\end{aligned} \tag{B5}$$

We want to prove the relation

$$\begin{aligned}
\alpha^{t}_{x,1} &= (-1)^{\frac{t+x}{2}} e^{i\frac{\pi}{4}}\big(-i\beta^{t}_{x-1,x,1} - \beta^{t}_{x-1,x,-1}\big),\\
\alpha^{t}_{x,-1} &= (-1)^{\frac{t+x}{2}} e^{i\frac{\pi}{4}}\big(-\beta^{t}_{x+1,x,1} + i\beta^{t}_{x+1,x,-1}\big),
\end{aligned} \tag{B6}$$

with the initial condition for the standard QWoM given by $\alpha_{0,1} = \frac{1}{\sqrt{2}}, \alpha_{0,-1} = \frac{i}{\sqrt{2}}$. Again, we proceed by induction in $t$. When $t = 0$, Eqs. (B6) are satisfied by means of a direct calculation. Now, we assume Eqs. (B6) are true for any $x$ at time $t$; then we prove that they hold at time $t + 1$.

$$\begin{aligned}
\alpha^{t+1}_{x,1} &= \frac{1}{\sqrt{2}}\big(\alpha^{t}_{x-1,1} + \alpha^{t}_{x-1,-1}\big)\\
&= \frac{1}{\sqrt{2}}(-1)^{\frac{t+x-1}{2}} e^{i\frac{\pi}{4}}\big(-i\beta^{t}_{x-2,x-1,1} - \beta^{t}_{x-2,x-1,-1}\\
&\quad - \beta^{t}_{x,x-1,1} + i\beta^{t}_{x,x-1,-1}\big)\\
&= \frac{1}{\sqrt{2}}(-1)^{\frac{t+x-1}{2}} e^{i\frac{\pi}{4}}\big(i\beta^{t}_{x,x-1,1} - \beta^{t}_{x-2,x-1,-1}\\
&\quad + \beta^{t}_{x-2,x-1,1} + i\beta^{t}_{x,x-1,-1}\big)\\
&= (-1)^{\frac{t+x+1}{2}} e^{i\frac{\pi}{4}}\big(-i\beta^{t+1}_{x-1,x,1} - \beta^{t+1}_{x-1,x,-1}\big),
\end{aligned} \tag{B7}$$

$$\begin{aligned}
\alpha^{t+1}_{x,-1} &= \frac{1}{\sqrt{2}}\big(\alpha^{t}_{x+1,1} - \alpha^{t}_{x+1,-1}\big)\\
&= \frac{1}{\sqrt{2}}(-1)^{\frac{t+x+1}{2}} e^{i\frac{\pi}{4}}\big(-i\beta^{t}_{x,x+1,1} - \beta^{t}_{x,x+1,-1}\\
&\quad + \beta^{t}_{x+2,x+1,1} - i\beta^{t}_{x+2,x+1,-1}\big)\\
&= \frac{1}{\sqrt{2}}(-1)^{\frac{t+x+1}{2}} e^{i\frac{\pi}{4}}\big(i\beta^{t}_{x+2,x+1,1} - \beta^{t}_{x,x+1,-1}\\
&\quad - \beta^{t}_{x,x+1,1} - i\beta^{t}_{x+2,x+1,-1}\big)\\
&= (-1)^{\frac{t+x+1}{2}} e^{i\frac{\pi}{4}}\big(-\beta^{t+1}_{x+1,x,1} + i\beta^{t+1}_{x+1,x,-1}\big).
\end{aligned} \tag{B8}$$

Therefore, Eqs. (B6) are satisfied at any time $t$.


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