

Unitary quantum evolution for time-dependent quasi-Hermitian systems with nonobservable Hamiltonians

Andreas Fring^{1,*} and Miled H. Y. Moussa^{2,†}

¹*Department of Mathematics, City University London, Northampton Square, London EC1V 0HB, United Kingdom*

²*Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970, São Carlos, São Paulo, Brazil*

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It has been argued that it is incompatible to maintain unitary time evolution for time-dependent non-Hermitian Hamiltonians when the metric operator is explicitly time dependent. We demonstrate here that the time-dependent Dyson equation and the time-dependent quasi-Hermiticity relation can be solved consistently in such a scenario for a time-dependent Dyson map and time-dependent metric operator, respectively. These solutions are obtained at the cost of rendering the non-Hermitian Hamiltonian to be a nonobservable operator as it ceases to be quasi-Hermitian when the metric becomes time dependent.

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I. INTRODUCTION

The time evolution of Hamiltonian systems is a central and fundamental issue in quantum mechanics, especially with regard to physical applications. The key principles are very well understood for a long time for Hermitian Hamiltonian systems and can be found in almost any standard book on quantum mechanics. However, the situation is quite different for the class of non-Hermitian systems that possess real or at least partially real eigenvalue spectra. Such type of models have been investigated sporadically for a long time, but the relatively recent seminal paper [1] has initiated a more systematic study. For time-independent systems the governing principles are by now also well understood and many experiments exist to confirm the key findings, e.g. [2–4]. For recent reviews on the subject area, see for instance [5,6] or [7,8] for recent special issues.

In contrast, time-dependent non-Hermitian systems are far less well investigated and it appears that so far no consensus has been reached about a number of central issues. Whereas the treatment for systems with time-dependent non-Hermitian Hamiltonians with time-independent metric operators [9,10] is widely accepted the more general setting with a time-dependent metric is still controversially discussed [11–18]. Explicit solutions to the central equations, i.e., the time-dependent Dyson and the time-dependent quasi-Hermiticity relation, have not been reported. Instead most authors resort to a nonunitary time evolution [12,14,16–18] for these systems by insisting on a quasi-Hermiticity relation between a Hermitian and a non-Hermitian “Hamiltonian.” The main purpose of this manuscript is to demonstrate that this is in fact not necessary. We add some clarifying arguments to the central discussion, provide some analytic solutions to the key equations, and discuss some of the consequences.

Our manuscript is organized as follows. In Sec. II we state the general framework for a description of a unitary time evolution for time-dependent non-Hermitian Hamiltonians. In Sec. III we provide two explicit examples that illustrate

the working of our proposal and in Sec. IV we state our conclusions.

II. TIME-DEPENDENT DYSON AND QUASI-HERMITICITY RELATION

As our starting point we take the two time-dependent Schrödinger equations (TDSE)

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \quad H(t)\Psi(t) = i\hbar\partial_t\Psi(t). \quad (2.1)$$

Both Hamiltonians involved are explicitly time dependent, with $h(t)$ being Hermitian whereas $H(t)$ is taken to be non-Hermitian, i.e., $h(t) = h^\dagger(t)$ and $H(t) \neq H^\dagger(t)$. We also insist here that operators may only be referred to as Hamiltonians if they generate the time evolution for the system under consideration, that is if they satisfy the TDSE. Next we assume that the two solutions $\phi(t)$ and $\Psi(t)$ to (2.1) are related by a time-dependent invertible operator $\eta(t)$ as

$$\phi(t) = \eta(t)\Psi(t). \quad (2.2)$$

It then follows immediately by direct substitution of (2.2) into (2.1) that the two Hamiltonians are allied to each other as

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t). \quad (2.3)$$

Thus $h(t)$ and $H(t)$ are no longer related by a similarity transformation, or more formally by the adjoint action of the Dyson operator, as in the completely time-independent scenario [19] or the time-dependent scenario with time-independent metric, but instead their mutual dependence involves a gauge-like term as discussed in [9–11]. We emphasize, however, that although formally the last term in (2.3) resembles a gauge connection this is not the role it plays here. We refer to Eq. (2.3) as the *time-dependent Dyson relation* as it generalizes its time-independent counterpart. Taking the Hermitian conjugate of equation (2.3) and using the Hermiticity of $h(t)$ yields a relation between $H(t)$ and its Hermitian conjugate

$$H^\dagger(t)\eta^\dagger(t)\eta(t) - \eta^\dagger(t)\eta(t)H(t) = i\hbar\partial_t[\eta^\dagger(t)\eta(t)]. \quad (2.4)$$

Interpreting $\rho(t) := \eta^\dagger(t)\eta(t)$ as a metric operator this relation replaces the standard quasi-Hermiticity relation well known in the context time-independent non-Hermitian quantum mechanics [20]. The justification for this interpretation

*a.fring@city.ac.uk

†miled@ifsc.usp.br

emerges as a consistency requirement from demanding the existence of a metric operator $\rho(t)$, such that time-dependent probability densities in the Hermitian and non-Hermitian system are related as

$$\langle \phi(t) | \tilde{\phi}(t) \rangle = \langle \Psi(t) | \rho(t) \tilde{\Psi}(t) \rangle =: \langle \Psi(t) | \tilde{\Psi}(t) \rangle_{\rho}. \quad (2.5)$$

For unitary time evolution these probabilities are preserved in time such that the derivative of both sides with respect to time must vanish. For the left-hand side this is simply guaranteed by the Hermiticity of $h(t)$ and the validity of the corresponding TDSE (2.1). The right-hand side yields instead the consistency relation

$$H^{\dagger}(t)\rho(t) - \rho(t)H(t) = i\hbar\partial_t\rho(t), \quad (2.6)$$

which when compared to (2.4) allows for the aforementioned identification for $\rho(t)$ in terms of $\eta(t)$ as announced above. We refer to Eq. (2.6), which may already be found in [11], as the *time-dependent quasi-Hermiticity relation*. It is noteworthy to point out that the reverse statement also holds, i.e., metric operators that do not satisfy (2.6) do not allow for unitary time evolution.

It is now evident that in complete analogy to the time-independent scenario any self-adjoint operator $o(t)$, i.e., an observable, in the Hermitian system has an observable counterpart $\mathcal{O}(t)$ in the non-Hermitian system related to each other as $\mathcal{O}(t) = \eta^{-1}(t)o(t)\eta(t)$, since

$$\begin{aligned} \langle \phi(t) | o(t) \tilde{\phi}(t) \rangle &= \langle o(t) \phi(t) | \tilde{\phi}(t) \rangle = \langle \Psi(t) | \mathcal{O}(t) \tilde{\Psi}(t) \rangle_{\rho} \\ &= \langle \mathcal{O}(t) \Psi(t) | \tilde{\Psi}(t) \rangle_{\rho}. \end{aligned} \quad (2.7)$$

Obviously due to Eq. (2.3), the non-Hermitian Hamiltonian $H(t)$ does not belong to the set of observables in this system as it is not related to $h(t)$ by a similarity transformation, which was already pointed out in [9,10,13,15]. However, there is no compelling reason why the non-Hermitian Hamiltonian $H(t)$ ought to be observable. Nonetheless, one may easily find a closely related operator

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t), \quad (2.8)$$

which is observable as it is related to the Hermitian observable $h(t)$ by means of the aforementioned similarity transformation. In other words, $\tilde{H}(t)$ is quasi-Hermitian. However, the operator $\tilde{H}(t)$ has no obvious concrete meaning and is certainly not a Hamiltonian in the sense that it does not generate the time evolution in this system and does not satisfy the original TDSE.

The relations above are directly transferred to the time-evolution operators. Recall that for the Hermitian Hamiltonian $h(t)$, satisfying (2.1), the unitary time evolution to a state $\phi(t) = u(t,t')\phi(t')$ from a time t' to t is governed by the time-evolution operator

$$u(t,t') = T \exp \left[-i\hbar \int_{t'}^t ds h(s) \right], \quad (2.9)$$

satisfying

$$\begin{aligned} h(t)u(t,t') &= i\hbar\partial_t u(t,t'), \quad u(t,t')u(t',t'') = u(t,t''), \\ u(t,t) &= \mathbb{I}. \end{aligned} \quad (2.10)$$

As usual T denotes here time ordering. Evidently we could replace $h(t)$ by $H(t)$ or $\tilde{H}(t)$ in (2.9), with the effect that in the

former case we no longer have a unitary time evolution and in the latter we have a contradiction since $\tilde{H}(t)$ does not satisfy the TDSE for this system, i.e., it is not a Hamiltonian. However, given the time-evolution operator $u(t,t')$ for the Hermitian system it follows straightforwardly from (2.7) that the unitary time-evolution operator $U(t,t')$ for the non-Hermitian system evolving $\psi(t) = U(t,t')\psi(t')$ is given by

$$U(t,t') = \eta^{-1}(t)u(t,t')\eta(t'). \quad (2.11)$$

Thus we are in complete agreement with Mostafazadeh's conclusions in [11,13,15] that for time-dependent metric operators one cannot simultaneously have a unitary time evolution and an observable arbitrary Hamiltonian; one can only have one or the other. The treatments in [12,14,16–18] give up the possibility of a unitary time evolution by insisting on a quasi-Hermiticity relation between $H(t)$ and $h(t)$, hence leaving the role of the non-Hermitian operator $H(t)$ in an obscure state. Since it does not satisfy the TDSE it remains unclear by what kind of principle it is introduced.

Thus so far the incompatibility between the unitary time evolution and an observable Hamiltonian is left as a negative statement [11,13,15], apart from the above mentioned treatments for non-Hermitian Hamiltonians of unclear origin. It appears that no attempt has been made to solve the relations (2.3) or (2.6), apart from two examples in [21], where (2.6) has been solved but no Hermitian counterpart was constructed. A possible reason is that one may insist on the observability of the Hamiltonian. However, there is no compelling reason for such a view. In the time-independent setting it is standard procedure to commence with non-Hermitian Hamiltonians in terms of some auxiliary variables x and p , which are not observable. Here we extend this principle to the Hamiltonian itself and treat the Hamiltonian $H(t)$ as a mere auxiliary operator, which does, however, play the role as governing the time evolution.

III. SOLUTIONS TO THE TIME-DEPENDENT DYSON AND QUASI-HERMITICITY RELATION

It is of course vital to demonstrate that the above formulas are not empty and can indeed be solved consistently. As in the time-independent case we have now various options to solve these equations depending on the quantity or quantities given at the starting point. In general, we commence with the non-Hermitian Hamiltonian $H(t)$ satisfying the TDSE (2.1). One may then compute, at least in principle, the metric $\rho(t)$ from the time-dependent quasi-Hermiticity relation (2.6) as $\rho(t)$ is the only unknown quantity therein. The Dyson map $\eta(t)$ then follows directly from its relation to $\rho(t)$, in which for simplicity one may assume $\eta(t)$ to be Hermitian such that one just has to take the square root. When $\eta(t)$ and $H(t)$ are determined one can use (2.3) to compute directly the Hermitian counterpart $h(t)$. The final step then consists of solving either of the TDSE (2.1) for $\phi(t)$ or $\Psi(t)$, obtaining the counterpart simply from (2.2). Alternatively one may also make a suitable ansatz for $\eta(t)$ and compute the right-hand side of (2.3) demanding the result to be Hermitian. Let us see this in detail for two examples by solving (2.3) in the first and (2.6) in the second.

A. Non-Hermitian harmonic oscillator with linear terms

We consider first the time-dependent Hamiltonian for the harmonic oscillator with additional linear terms in the standard creation and annihilation operators a and a^\dagger , respectively,

$$H(t) = \omega(t)a^\dagger a + \alpha(t)a + \beta(t)a^\dagger, \quad \omega(t), \alpha(t), \beta(t) \in \mathbb{C}. \quad (3.1)$$

For convenience we set here and in what follows $\hbar = 1$. Evidently $H(t)$ is non-Hermitian when $\alpha(t) \neq \beta^*(t)$. Notice that when demanding \mathcal{PT} symmetry for the Hamiltonian in the time-independent setting one demands $\omega(t), \alpha(t), \beta(t) \rightarrow \omega, i\alpha, i\beta \in \mathbb{R}$, since $\mathcal{PT} : a \rightarrow -a, a^\dagger \rightarrow -a^\dagger$. However, any real-valued function $\omega(x), i\alpha(x), i\beta(x)$ may now be replaced for instance by the complex-valued functions $\omega(it), i\alpha(it), i\beta(it)$ still leaving the Hamiltonian \mathcal{PT} symmetric, since $\mathcal{PT} : t \rightarrow -t, i \rightarrow -i$. In order to solve the time-dependent Dyson relation (2.3) we make a natural ansatz for the time-dependent Dyson map

$$\eta(t) = e^{\gamma(t)a + \lambda(t)a^\dagger}, \quad \gamma(t), \lambda(t) \in \mathbb{C} \quad (3.2)$$

as being similar in form to the Hamiltonian in the argument of the exponential. Substituting $\eta(t)$ into (2.3) yields

$$h(t) = \omega(t)a^\dagger a + u(t)a + v(t)a^\dagger + f(t), \quad (3.3)$$

with the constraints

$$\begin{aligned} u &= \alpha + \omega\gamma + i\dot{\gamma}, & v &= \beta - \omega\lambda + i\dot{\lambda}, \\ f &= \frac{i}{2}(\gamma\dot{\lambda} - \dot{\gamma}\lambda) - \omega\gamma\lambda - \alpha\lambda + \beta\gamma. \end{aligned} \quad (3.4)$$

As common we denote time derivatives by an overhead dot. For $h(t)$ in (3.3) to be Hermitian we require the additional constraints $\omega(t) \in \mathbb{R}$, $u = v^*$, and $f = f^*$, which correspond to the two equations

$$\alpha - \beta^* + \omega(\gamma + \lambda^*) + i(\dot{\gamma} + \dot{\lambda}^*) = 0, \quad (3.5)$$

$$\begin{aligned} \frac{i}{2}(\gamma\dot{\lambda} - \dot{\gamma}\lambda + \gamma^*\dot{\lambda}^* - \dot{\gamma}^*\lambda^*) + \omega(\gamma^*\lambda^* - \gamma\lambda) + \alpha^*\lambda^* \\ - \alpha\lambda + \beta\gamma - \beta^*\gamma^* = 0. \end{aligned} \quad (3.6)$$

Attempting to solve these equations by assuming $\eta(t)$ to be the standard displacement operator fails, as in that case we have $\gamma = -\lambda^*$, which by (3.5) implies that $\alpha(t) = \beta^*(t)$ such that our supposedly non-Hermitian Hamiltonian $H(t)$ becomes Hermitian. Alternatively, we may take $\gamma = \lambda^*$ and $\alpha(t) = -\beta^*(t)$, which reduces the above to the simple constraint

$$\alpha + \omega\gamma + i\dot{\gamma} = 0. \quad (3.7)$$

Notice that this is just saying that u needs to vanish. We can in fact solve this equation by

$$\gamma(t) = e^{i\chi(t)} \left[\gamma(0) + i \int_0^t ds \alpha(s) e^{-i\chi(s)} \right], \quad (3.8)$$

where $\chi(t) := \int_0^t ds \omega(s)$. Thus given the model defining functions $\alpha(t)$ and $\omega(t)$ via our starting Hamiltonian $H(t)$, we can directly compute $\gamma(t)$. For the presented solution our Hermitian Hamiltonian turns out to be simply the harmonic oscillator with a time-dependent frequency and overall shift. Of course, there could be more involved solutions to (3.5)

and (3.6). The solution $\phi(t)$ to the TDSE for the Hermitian Hamiltonian $h(t)$ is then easily found as a special case of the treatment in [22], such that we have now also obtained a solution $\Psi(t) = \eta^{-1}(t)\phi(t)$ to the TDSE for the non-Hermitian Hamiltonian $H(t)$ subject to the above-mentioned constraints. For the convenience of the reader we recall the solution from [22]. The ground state $|\phi_0(t)\rangle$ was found to be a coherent state $|\theta(t)\rangle$ dressed with a time-dependent Lewis-Riesenfeld phase $\Phi_0(t)$

$$|\phi_0(t)\rangle = e^{i\Phi_0(t)} |\theta(t)\rangle, \quad (3.9)$$

given by

$$\begin{aligned} |\theta(t)\rangle &= e^{-|\vartheta(t)|^2} \sum_{n=0}^{\infty} \frac{\vartheta^n(t)}{\sqrt{n!}} |n\rangle, & \vartheta(t) &= \vartheta(0)e^{-i\chi(t)}, \\ \varphi_0(t) &= \varphi_0(0) - \int_0^t ds f(s), \end{aligned} \quad (3.10)$$

with $|n\rangle$ being a standard Fock eigenstate of the number operator $a^\dagger a$. Excited states are constructed in a similar fashion; see also [23,24] for further details.

The observables in the non-Hermitian system are easily computed. For instance, the quadratures (X, P) corresponding in the Hermitian system to the coordinate and momentum operators $x = (a^\dagger + a)/\sqrt{2}$ and $p = i(a^\dagger - a)/\sqrt{2}$, respectively, are now simply shifted operators in the original variables

$$\begin{aligned} X &= \eta^{-1} x \eta = x - i\sqrt{2} \operatorname{Im} \gamma, \\ P &= \eta^{-1} p \eta = p - i\sqrt{2} \operatorname{Re} \gamma. \end{aligned} \quad (3.11)$$

The observable operator related to the Hermitian Hamiltonian, albeit not satisfying the original TDSE, results to

$$\begin{aligned} \tilde{H}(t) &= \eta^{-1}(t) h(t) \eta(t) = \omega(t) [a^\dagger a - \gamma(t)a + \gamma^*(t)a^\dagger] \\ &+ \frac{i}{2} [\dot{\gamma}(t)\gamma^*(t) - \gamma(t)\dot{\gamma}^*(t)]. \end{aligned} \quad (3.12)$$

We notice that $\tilde{H}(t)$ and $H(t)$ have the same structure in their operator content.

B. Non-Hermitian spin chain

Next we consider a discretized lattice version of the Yang-Lee model proposed originally in [25]. The model is an Ising quantum spin chain in the presence of a magnetic field in the z direction together with a longitudinal imaginary field in the x direction

$$\begin{aligned} H_N(t) &= -\frac{1}{2} \sum_{j=1}^N [\sigma_j^z + \lambda(t)\sigma_j^x \sigma_{j+1}^x + i\kappa(t)\sigma_j^x], \\ \lambda(t), \kappa(t) &\in \mathbb{C}. \end{aligned} \quad (3.13)$$

The boundary conditions for the Pauli spin matrices are taken to be $\sigma_1 = \sigma_{N+1}$. Here we modify the model by introducing a time dependence into the coupling constants by replacing λ, κ in previous studies by time-dependent functions $\lambda(t), \kappa(t)$. The \mathcal{PT} symmetry of the Hamiltonian is $\mathcal{PT} : \sigma^x \rightarrow -\sigma^x, \sigma^z \rightarrow \sigma^z, t \rightarrow -t, i \rightarrow -i$. For small length N time-independent Dyson maps, metric operators and isospectral counterparts have been constructed in [26]. We present here the simplest example for the time-dependent scenario by taking $N = 1$,

such that the Hamiltonian acquires the form of a simple non-Hermitian 2×2 matrix

$$H_1(t) = -\frac{1}{2}[\sigma_1^z + \lambda(t)\sigma_1^x + i\kappa(t)\sigma_1^y] = -\frac{1}{2}\begin{pmatrix} 1 + \lambda(t) & i\kappa(t) \\ i\kappa(t) & \lambda(t) - 1 \end{pmatrix}. \tag{3.14}$$

Instead of solving Eq. (2.3) as in the previous subsection, we now attempt here to solve the time-dependent quasi-Hermiticity relation (2.6) for the metric operator $\rho(t)$ by assuming the most general Hermitian form as an ansatz

$$\rho(t) = \begin{pmatrix} \alpha(t) & \beta(t) + i\gamma(t) \\ \beta(t) - i\gamma(t) & \delta(t) \end{pmatrix}, \quad \alpha(t), \beta(t), \gamma(t), \delta(t) \in \mathbb{R}. \tag{3.15}$$

Taking $\lambda(t), \kappa(t) \in \mathbb{R}$, the substitution of $\rho(t)$ into (2.6) yields

$$\begin{pmatrix} \dot{\alpha} - \beta\kappa & \gamma - \frac{\kappa}{2}(\alpha + \delta) + \dot{\beta} + i\dot{\gamma} - i\beta \\ \gamma - \frac{\kappa}{2}(\alpha + \delta) + \dot{\beta} + i\beta - i\dot{\gamma} & \dot{\delta} - \beta\kappa \end{pmatrix} = 0. \tag{3.16}$$

The equations resulting from each matrix entry are solved by

$$\alpha(t) = \alpha_0 + \int_0^t ds \beta(s)\kappa(s), \quad \delta(t) = \delta_0 + \int_0^t ds \beta(s)\kappa(s), \quad \gamma(t) = \gamma_0 + \int_0^t ds \beta(s), \tag{3.17}$$

with $\beta(t)$ constraint to

$$\dot{\beta}(t) + \int_0^t ds \beta(s) - \kappa(t) \int_0^t ds \beta(s)\kappa(s) - \frac{\kappa(t)}{2}(\alpha_0 + \delta_0) + \gamma_0 = 0. \tag{3.18}$$

The latter equation is nontrivial, but we will demonstrate that it actually possesses meaningful solutions. A great simplification is achieved by assuming $\beta(t) = \dot{\kappa}(t)$, since then the two integrals may be solved easily, leaving us with a second-order differential equation for the time-dependent function $\kappa(t)$

$$\ddot{\kappa}(t) + \kappa(t)\left(1 - \frac{\alpha_0 + \delta_0}{2} + \frac{\kappa^2(0)}{2}\right) - \frac{1}{2}\kappa^3(t) + \gamma_0 - \kappa(0) = 0. \tag{3.19}$$

Given the values for the entries in the matrix ρ as in (3.17), with the above assumption and implementing (3.19) we find an additional constraint on the combination of initial values

$$|\rho(t)| = \frac{1}{4}[\kappa^2(0) - 2\alpha_0][2\delta_0 - \kappa^2(0)] - [\gamma_0 - \kappa^2(0)] > 0, \tag{3.20}$$

to guarantee a positive definite metric.

In general solution to (3.19) are Jacobi elliptic functions, that is complex, which are however excluded by the fact that $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ have to be real by assumption. Nonetheless, for special values of the elliptic modulus we may also obtain several real solutions. For instance,

$$\kappa(t) = 2 \tan(t), \quad \text{with } \gamma_0 = 0, \quad \alpha_0 = 6 - \delta_0, \quad |\rho(t)| = -4 + 6\delta_0 - \delta_0^2, \tag{3.21}$$

$$\kappa(t) = 2 \sec(t), \quad \text{with } \gamma_0 = 2, \quad \alpha_0 = 4 - \delta_0, \quad |\rho(t)| = -4 + 4\delta_0 - \delta_0^2, \tag{3.22}$$

$$\kappa(t) = 2 \tanh(t), \quad \text{with } \gamma_0 = 0, \quad \alpha_0 = -2 - \delta_0, \quad |\rho(t)| = -4 - 2\delta_0 - \delta_0^2 \tag{3.23}$$

solve the constraining equation (3.19) with δ_0 left as a free parameter. We observe that not all of these solutions are permissible as (3.22) and (3.23) will always lead to nonpositive operators $\rho(t)$. However, solution (3.21) admits the possibility $|\rho(t)| > 0$ in the range $3 - \sqrt{5} < \delta_0 < 3 + \sqrt{5}$. For convenience, we take now $\delta_0 = 1$ in what follows and analyze this solution further. Using the above values, the time-dependent metric operator is computed to

$$\rho(t) = \begin{pmatrix} 5 + 2 \tan^2(t) & 2 \sec^2(t) + 2i \tan(t) \\ 2 \sec^2(t) - 2i \tan(t) & 1 + 2 \tan^2(t) \end{pmatrix}, \tag{3.24}$$

such that $|\rho(t)| = 1$. Assuming the Dyson operator to be Hermitian we may compute it by first diagonalizing $\rho(t) = \eta^2(t) = UDU^{-1}$, with D being a diagonal matrix, and subsequently computing $\sqrt{\rho(t)} = \eta(t) = U\sqrt{D}U^{-1}$. As $\rho(t)$ is positive definite this operation is well defined. In this manner we obtain the time-dependent Dyson operator

$$\eta(t) = \frac{1}{\sqrt{\sec^2(t) + 1}} \begin{pmatrix} 2 + \sec^2(t) & \sec(t)[\sec(t) + i \sin(t)] \\ \sec(t)[\sec(t) - i \sin(t)] & \sec^2(t) \end{pmatrix}. \tag{3.25}$$

These expressions allow us to compute the Hermitian Hamiltonian $h(t)$ by means of (2.3)

$$h(t) = \frac{1}{3 + \cos(2t)} \begin{pmatrix} -\frac{1}{2}\{1 + 3\lambda(t) + [3 + \lambda(t)] \cos(2t)\} & -i \sin(2t) \\ i \sin(2t) & \frac{1}{2}\{1 - 3\lambda(t) + [3 - \lambda(t)] \cos(2t)\} \end{pmatrix}. \tag{3.26}$$

Evidently there might be many more solutions when allowing $\lambda(t), \kappa(t)$ to have nonvanishing imaginary parts or when relaxing the assumption on $\beta(t)$ in solving (3.18). Here it suffices to demonstrate that some meaningful solutions exist.

IV. CONCLUSIONS

We have demonstrated that the time-dependent quasi-Hermiticity relations (2.6) and therefore also the time-dependent Dyson relation (2.3) possess meaningful solutions. This means a consistent description of a unitary quantum time evolution with time-dependent metric is indeed possible. Unlike as in previous treatments we do not demand a quasi-Hermiticity relation between a Hermitian Hamiltonian and a non-Hermitian Hamiltonian, which inevitably leads to nonunitary quantum evolution. Instead, we do not demand the observability of the non-Hermitian Hamiltonian that satisfies the TDSE and simply treat it as an auxiliary operator. Nonetheless, the system still possesses a well-defined observable Hamiltonian in the form of $h(t)$.

Evidently there are still many open problems. Clearly more explicit solutions for concrete models would shed further light on the viewpoint we proposed. The uniqueness problem of the metric operator in the time-independent case is well known, i.e., given a non-Hermitian Hamiltonian as a starting point of the construction one obtains numerous consistent solutions for the metric operator. This issue is still unresolved to a large extent in the time-independent scenario. For the time-dependent case this difficulty appears to be much more amplified and solutions are even more ambiguous. However, more complex settings often allow one to find special criteria for very particular solutions and the hope is that one might be able to extract concrete selection criteria from these considerations.

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- [1] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, *Phys. Rev. Lett.* **80**, 5243 (1998).
- [2] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Optical Solitons in PT Periodic Potentials, *Phys. Rev. Lett.* **100**, 030402 (2008).
- [3] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, PT-symmetric optical lattices, *Phys. Rev. A* **81**, 063807 (2010).
- [4] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Observation of PT-Symmetry Breaking in Complex Optical Potentials, *Phys. Rev. Lett.* **103**, 093902 (2009).
- [5] C. M. Bender, Making sense of non-Hermitian Hamiltonians, *Rep. Prog. Phys.* **70**, 947 (2007).
- [6] A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, *Int. J. Geom. Meth. Mod. Phys.* **07**, 1191 (2010).
- [7] C. Bender, A. Fring, U. Günther, and H. Jones, Special issue: Quantum physics with non-Hermitian operators, *J. Phys. A* **45**, 440301 (2012).
- [8] M. Znojil, Special issue “Pseudo-Hermitian Hamiltonians in quantum physics in 2014”, *Int. J. Theor. Phys.* **54**, 3867 (2015).
- [9] C. Figueira de Morisson Faria and A. Fring, Time evolution of non-Hermitian Hamiltonian systems, *J. Phys. A* **39**, 9269 (2006).
- [10] C. Figueira de Morisson Faria and A. Fring, Non-Hermitian Hamiltonians with real eigenvalues coupled to electric fields: From the time-independent to the time dependent quantum mechanical formulation, *Laser Phys.* **17**, 424 (2007).
- [11] A. Mostafazadeh, Time-dependent pseudo-Hermitian Hamiltonians defining a unitary quantum system and uniqueness of the metric operator, *Phys. Lett. B* **650**, 208 (2007).
- [12] M. Znojil, Time-dependent quasi-Hermitian Hamiltonians and the unitarity of quantum evolution, [arXiv:0710.5653](https://arxiv.org/abs/0710.5653).
- [13] A. Mostafazadeh, Comment on “Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution”, [arXiv:0711.0137](https://arxiv.org/abs/0711.0137).
- [14] M. Znojil, Reply to Comment on “Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution”, [arXiv:0711.0514](https://arxiv.org/abs/0711.0514).
- [15] A. Mostafazadeh, Comment on “Reply to Comment on Time-dependent Quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution”, [arXiv:0711.1078](https://arxiv.org/abs/0711.1078).
- [16] M. Znojil, Time-dependent version of crypto-Hermitian quantum theory, *Phys. Rev. D* **78**, 085003 (2008).
- [17] J. Gong and Q.-H. Wang, Time-dependent PT-symmetric quantum mechanics, *J. Phys. A* **46**, 485302 (2013).
- [18] M. Maamache, Periodic pseudo-Hermitian Hamiltonian: Nonadiabatic geometric phase, *Phys. Rev. A* **92**, 032106 (2015).
- [19] F. J. Dyson, Thermodynamic behavior of an ideal ferromagnet, *Phys. Rev.* **102**, 1230 (1956).
- [20] F. G. Scholtz, H. B. Geyer, and F. Hahne, Quasi-Hermitian operators in quantum mechanics and the variational principle, *Ann. Phys. (NY)* **213**, 74 (1992).
- [21] H. Břila, Adiabatic time-dependent metrics in PT-symmetric quantum theories, [arXiv:0902.0474](https://arxiv.org/abs/0902.0474).
- [22] R. R. Puri and S. V. Lawande, Time-dependent invariants and stable coherent states, *Phys. Lett. A* **70**, 69 (1979).
- [23] B. Baseia, S. S. Mizrahi, and M. H. Y. Moussa, Generation of squeezing for a charged oscillator and for a charged particle in a time-dependent electromagnetic field, *Phys. Rev. A* **46**, 5885 (1992).
- [24] S. S. Mizrahi, M. H. Y. Moussa, and B. Baseia, The quadratic time-dependent Hamiltonian: Evolution operator, squeezing regions in phase space and trajectories, *Int. J. Mod. Phys. B* **08**, 1563 (1994).
- [25] G. von Gehlen, Critical and off critical conformal analysis of the Ising quantum chain in an imaginary field, *J. Phys. A* **24**, 5371 (1991).
- [26] O. A. Castro-Alvaredo and A. Fring, A spin chain model with non-Hermitian interaction: The Ising quantum spin chain in an imaginary field, *J. Phys. A* **42**, 465211 (2009).