Phase transitions of energy and wave functions and bound states in the continuum

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This paper studies a particle subjected to an infinite potential well in the circumstance of a fractional dimensional Lévy path. To obtain analytic expression for the wave functions and energy levels, we introduce the fractional corresponding operator and a generalized de Moivre's theorem. Phase transitions of the energy and wave functions are found when the Lévy path dimension changes from integer to noninteger in nature. More importantly, we demonstrate the existence of stable bound states in the continuum in a simple potential. The results predict a phenomenon in which all bound states energy levels of the particle are continuous and the particle remains in bound states. This phenomenon can be demonstrated that this is a characteristic phenomenon of a fractional system. This phenomenon provides both an *a priori* criterion for theoretically describing an unknown quantum system with fractional derivatives and a sufficient condition for verifying the preparation of a fractional quantum system in experiment. Finally, we compare our results for fractional quantum systems with the existing results and explain the cause of the reported phenomenon.

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I. INTRODUCTION

Quantum mechanics has become one of the most successful theories in the history of science. Over the past few decades, fractional calculus has also become a very popular mathematical tool [1]. Fractional derivatives can describe non-Markovian evolution with a memory effect and nonlocal quantum phenomena [2] and enable the study of path integrals for Lévy flights and paths of noninteger fractal dimensions [3,4]. The use of fractional calculus for this purpose in quantum mechanics was firstly introduced by Laskin. In Refs. [3,4], he proposed a fractional quantization method and constructed a fractional Schrödinger equation with the Rizes fractional derivative operator by introducing a Lévy path integral. Subsequently, many other forms of fractional Schrödinger equations have been presented [2,5–10], and they have been applied to a large numbers of cases, such as one-dimensional (1D) Lévy crystals [11,12], the Thomas-Fermi model and Hohenberg-Kohn theorems [13], various fractional Schrödinger equations with time-fractional derivatives [2,6–10], profile decompositions with angularly regular data [14], optimal controls [15], transmission through locally periodic potentials [16], and various potential fields [17–22].

However, to date, there is still no criterion for determining in advance whether a phenomenon should be described using a fractional model. A typical strategy is to addressed a fractional problem either by directly replacing classical derivatives with fractional derivatives or by subjectively applying a suitable fractional model to the system. However, these approaches all follow an *a posteriori* strategy. The first approach is simple but may sometimes yield undesired results, for example, energy nonconservation [8,9]. The latter approach is able to guarantee correct results, but it can be quite difficult to construct a suitable model of this type for a given fractional system. Moreover, there are many cases in which no such models have been found, such as scenarios with fractional time. Therefore, an *a priori* strategy that is both simple and guaranteed to yield correct results would be a far preferable option when considering a fractional quantum system.

In this paper, we consider a nonrelativistic particle subjected to a 1D infinite potential well, which is a good description for 1D bound state problems [23,24] and can be created in the laboratory [24–28]. We introduce our fractional quantum quantization method based on a fractional corresponding operator with a power-law memory kernel and also derive a general de Moivre's theorem to modify the method for solving a specific kind of fractional differential equation to study the problem of interest. Using this method, we can choose the most convenient fractional derivative to describe a particular problem depending on the specific characteristics of the problem, rather than only the Rizes fractional derivative operator [3,4,13,16,17,19–22,29]. Although in recent years many authors have studied the problem of a 1D infinite potential well with a fractional Lévy path [17–19,21,22], their results are similar to the results obtained in classical quantum mechanics: the energy spectrum of a moving particle in a 1D infinite potential well is discrete and nondegenerate [30–32]. However, their results for a fractional system are incomplete because they have not considered the effects of a fractional fractal dimension on a fractional system; i.e., they have considered only a subset of the solutions of the system. In our approach, the results are remarkably different in that the particle has continuous energy levels and the degeneracy is strongly related to the fractal dimension. We deduce this phenomenon and obtain the relations between the fractal dimension and the continuity and degeneracy of the energy levels in three steps:

(1) Starting from the fractional corresponding operator T_α [5] and its general classical-to-quantum rules as given in Eq. (1), we define a operator. After discussing its applicability,

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we establish the general fractional Schrödinger equation [Eq. (4)] based on the general classical-to-quantum rules [Eq. (1)].

(2) Analytical expressions for the bound states and energy levels are obtained for a particle in a 1D infinite potential well in the circumstance of a Lévy path with a fractional fractal dimension 2α. Then we show how a fractional fractal dimension affects the energy levels and wave functions.

(3) We compare our results with the results of previous investigations regarding fractional quantum mechanics [17–19,21,22]. In our approach, if we let certain parameters take specific values, we can obtain these results reported in the cited articles. If we let the fractal dimension of the Lévy path be $2\alpha \to 2$, which corresponds to the case of classical quantum mechanics, our results will degenerate to the classical results.

Our results demonstrate the existence of bound states in the continuum (BICs). The possibility has previously been considered by many authors [33–41]. These BICs are in two cases: One kind of BICs are fragile and have been realized only in tailored potentials without interactions between particles [32,42,43]. The other kind of BICs are realized by the interactions between particles [44–46]. However, our results indicate a phenomenon in which all the bound states energy levels of the particle are continuous and the particle remains in bound states which can be realized in simple potential. Moreover, our approach provides an *a priori* strategy. This physical phenomenon can only be exactly described by fractional derivative models. Thus, when we are considering an unknown physical model, this predicted phenomenon provides a criterion for determining in advance whether fractional derivatives should be used to describe it. This may offer a new way of studying fractional quantum systems. Moreover, because no experimental realization or observation of a fractional quantum mechanics system has been reported [11], as a corollary, we propose this phenomenon as a sufficient condition to verify whether a fractional quantum mechanics system has been successfully prepared in an experiment.

Finally, we explain the cause of the above phenomenon based on a spectral analysis of the fractional Schrödinger operator.

II. FRACTIONAL CORRESPONDING OPERATOR

As mentioned above, in order to study BICs and the relationship between the fractal dimension and the degeneracy of a particle in the circumstance of a Lévy path with a fractional 2α dimension, we should introduce the fractional corresponding operator T_α, which has the corresponding rules

$$p^\alpha \to (-i\hbar)^\alpha T_\alpha, \tag{1}$$

where $2\alpha \in (1,2]$ is the fractal dimension of the Lévy path and p is the momentum of the quantum system [5]. When $\alpha = 2$, Eq. (1) takes the form of the classical corresponding relation $p^2 \to -\hbar^2\partial^2/\partial x^2$. To obtain the analytical expressions for the bound states and energy levels of a particle subjected to a 1D infinite potential well in the circumstance of a Lévy path with a fractional 2α dimension, we should first modify the fractional corresponding operator accordingly. Obviously, we will prove that this modification does not change the operator's physics properties.

The underlying concept of the fractional corresponding operator as follows: A linear operator T_α of order $\alpha \in (0,+\infty)$ with respect to x is called a fractional corresponding operator if it satisfies the following conditions:

(1)

$$T_\alpha \delta\left(\frac{x+y}{c}\right) = \frac{i^\alpha}{2\pi c^\alpha}\int_{-\infty}^{+\infty} k^\alpha \exp\left[ik\left(\frac{x+y}{c}\right)\right] dk, \tag{2}$$

for any $x,y \in \mathbb{R}$, where $c \neq 0$ is a constant.

(2)

$$T_\alpha[g(x)l(y)] = l(y)T_\alpha g(x), \tag{3}$$

for any $g(x)$ and $l(y)$ are continuous functions.

(3)

$$T_\alpha \int_{-\infty}^{+\infty} f(x,t)\, dt = \int_{-\infty}^{+\infty} T_\alpha f(x,t)\, dt,$$

for any $f(x,t) \in L^2(\mathbb{R}^2)$.

(4) When α approaches n, where n is a positive integer, T_n is a classical derivative operator.

If we replace condition (1) with the stronger condition $T_\alpha e^{kx} = k^\alpha e^{kx}$ for any $k \neq 0$, we obtain another operator, which we call the strong fractional corresponding operator. The two results presented below (see Appendix A for the proofs) indicate that compared with the fractional corresponding operator, the physical properties of the strong fractional corresponding operator remain unchanged. Note the following:

(1) The strong fractional corresponding operator is a special case of the fractional corresponding operator.

(2) The popular fractional derivative operators coincide with the strong fractional corresponding operator, such as the R-L fractional derivative operator, the G-L fractional derivative operator, the Caputo fractional derivative operator, and the fractional derivative operator based on generalized functions.

For α on the interval (0.5,1], we can use the methods presented in Refs. [4,5] to build the general fractional Schrödinger equation in one dimension as follows:

$$i\hbar\frac{\partial\varphi(x,t)}{\partial t} = D_{2\alpha}(-i\hbar)^{2\alpha}T_{2\alpha}\varphi(x,t) + V(x,t)\varphi(x,t). \tag{4}$$

A concrete instance of this equation is as follows:

$$i\hbar\frac{\partial\varphi(x,t)}{\partial t} = D_{2\alpha}(-i\hbar)^{2\alpha}\,{}^{G}_{-\infty}D^{2\alpha}_{x}\varphi(x,t) + V(x,t)\varphi(x,t),$$

which is a special case, where ${}^{G}_{-\infty}D^{2\alpha}_{x}$ is the G-L fractional derivative operator.

III. "*A PRIORI* PREDICTED" PHENOMENON

In this section, we demonstrate how a fractional fractal dimension leads to the continuity and degeneracy of the energy levels and present the formula for the relation between the energy degeneracy and the fractal dimension. In addition, a phenomenon is predicted in which all bound states energy levels of the particle are continuous and the particle remains in bound states, thereby providing an *a priori* prediction-based strategy. At the end of this section, we will compare our results with those reported in Refs. [17–19,21,30–32]. For clarity, let us consider a nonrelativistic particle moving in a 1D infinite potential well denoted by $V(x)$:

$$V(x) = \begin{cases} 0 & -a < x < a \\ +\infty & x \leqslant -a, x \geqslant a \end{cases}, \tag{5}$$

where $a > 0$ is a constant.

A. Phase transition of energy

In the region $-a < x < a$, by substituting Eq. (5) into Eq. (4), we can obtain the general fractional Schrödinger equation,

$$i\hbar\frac{\partial\varphi(x,t)}{\partial t} = D_{2\alpha}(-\hbar^2)^\alpha T_{2\alpha}\varphi(x,t), \tag{6}$$

where $T_{2\alpha}$ of order $\alpha \in (0.5,1]$ with respect to x is the strong fractional corresponding operator.

The boundary conditions are

$$\varphi(-a,t) = 0$$
$$\varphi(a,t) = 0. \tag{7}$$

By the separation of variables $\varphi(x,t) = f(t)\psi(x)$ and Eqs. (3) and (6) become

$$f(t) = \exp\frac{iEt}{\hbar}, \tag{8}$$

$$D_{2\alpha}(-\hbar^2)^\alpha T_{2\alpha}\psi(x) = E\psi(x). \tag{9}$$

If we let

$$c = \frac{E}{D_{2\alpha}(-\hbar^2)^\alpha},$$

then by applying the modified method of solving fractional differential equations (see Appendix B for a description of this method and related mathematical deductions), we can obtain the solutions to Eq. (9):

$$\psi(x) = \sum_{k\in K} B_k \exp(\beta_k x)\cos(\gamma_k x) + C_k \exp(\beta_k x)\sin(\gamma_k x), \tag{10}$$

where $\beta_k = \cos[(\theta + 2k\pi)/(2\alpha)]|c|^{1/2\alpha}$, $\gamma_k = \sin[(\theta + 2k\pi)/(2\alpha)]|c|^{1/2\alpha}$, $\theta = \arg c$, and $k \in K$ where K is an index set.

Note the following facts: (1) If α is irrational, then according to the conclusions of spectral analysis of the fractional Schrödinger operator (see Appendix F for more details), there are countably infinitely many undetermined parameters in Eq. (10). Therefore, we need at least countably infinitely many conditions to determine all of the parameters. However, Eq. (7) combined with the normalizing conditions is not sufficient enough to determine all the parameters $\{B_k, C_k\}_{k\in K}$; thus, the energy E is arbitrary. Therefore, the energy levels are continuous because of the freedom of the parameters. (2) If $\alpha \neq 1$ is a rational number, then we can assume that $\alpha = n/m$. Because $\alpha \in (0.5,1)$, the numerator n must satisfy $n \geqslant 2$. We know that there are $2n(\geqslant 4)$ undetermined parameters in Eq. (10). Therefore, we need at least $2n$ conditions to determine all of the parameters. However, including Eq. (7) and the normalizing conditions, there are only $3(< 4 \leqslant 2n)$ conditions; thus, the energy E is arbitrary. Again, the energy levels are continuous. (3) If $\alpha = 1$, we recover the classical case. We know that there are two undetermined parameters in Eq. (10), and we have a sufficient number of conditions to determine all of the unknown parameters; therefore, the energy E is not arbitrary, which implies that the energy levels are discrete.

As a result, a change in the Lévy path dimension from integer to noninteger in nature can cause the energy levels change from discrete to continuous.

B. Phase transition of wave functions

For convenience, we assume that $W = \{\exp(\beta_k x)\cos(\gamma_k x), \exp(\beta_k x)\sin(\gamma_k x)\}_{k\in K}$ and that $w_i(x)$ is an element of W, and we use the position-space representation. In the following, we will divide our analysis into three parts to demonstrate how to solve for the wave functions of the 1D infinite potential well and show how the introduce of a fractional dimension causes a phase transition.

Part I (Certain the special energies)

Choosing an arbitrary element $w_i(x)$ in W, without loss of generality, we suppose that $w_i(x) = \exp(\beta_k x)\sin(\gamma_k x)$. When $w_i(x)$ satisfies Eq. (7), we have

$$\exp(\beta_k a)\sin(\gamma_k a) = 0$$
$$\exp[\beta_k(-a)]\sin[\gamma_k(-a)] = 0. \tag{11}$$

Because $\exp[\beta_k(\pm a)] \neq 0$, $\sin(\gamma_k a) = 0$. We can obtain a sequence of energies that satisfies the above conditions:

$$E = |D_{2\alpha}(-\hbar^2)^\alpha|\left\{\frac{g\pi}{2a\sin[(\theta + 2k\pi)/(2\alpha)]}\right\}^{2\alpha}.$$

When the particle is at one of these energies, the corresponding $w_i(x)$ is its wave function

$$w_i(x) = \frac{1}{\sqrt{l}}\exp\left(\cos\frac{\theta + 2k\pi}{2\alpha}\right)\sin\left(\frac{g\pi x}{2a} + \frac{g\pi}{2}\right),$$

where $l = \int_{-a}^{a}(\exp\{\cos[(\theta + 2k\pi)/(2\alpha)]\}\sin[g\pi x/2a + g\pi/2])^2\,dx$, $\theta = \arg 1/[(-\hbar^2)^\alpha]$, and $k \in K$ where the index set K is defined as in the generalized de Moivre's theorem (see Appendix B for more details). We refer to such an energy E as a Simple Energy and to the corresponding $w_i(x)$ as a Simple Element. Because a Simple Element is itself a wave function, we will refer to such a wave function as a Simple Wave function.

Let $\alpha \to 1$; for a Simple Energy E_g, we have

$$E_g = |D_{2\alpha}(-\hbar^2)^\alpha|\left\{\frac{g\pi}{2a\sin[(\theta + 2k\pi)/(2\alpha)]}\right\}^{2\alpha} \to \frac{g^2\pi^2\hbar^2}{8ma^2}, \tag{12}$$

where $D_2 = 1/(2m)$.

The corresponding Simple Wave function is

$$w_i(x) = \frac{1}{\sqrt{l}}\exp\left(\cos\frac{\theta + 2k\pi}{2\alpha}\right)\sin\left(\frac{g\pi x}{2a} + \frac{g\pi}{2}\right) \to \frac{1}{\sqrt{a}}\sin\left(\frac{g\pi x}{2a} + \frac{g\pi}{2}\right), \tag{13}$$

where $D_2 = 1/(2m)$ and $K = \{0\}$.

We note that the limits of Eqs. (12) and (13) represent the energy levels and wave functions, respectively, of a particle

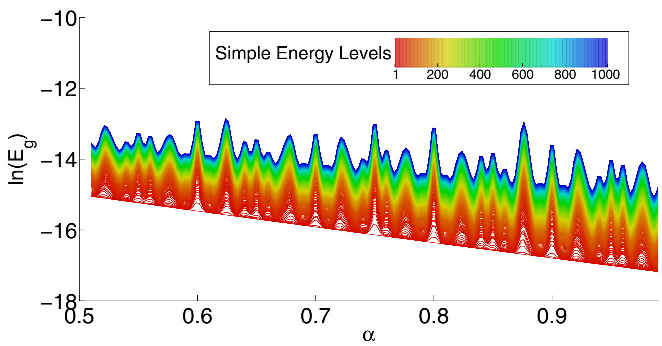


FIG. 1. Simple Energies vs α, where $a = 10^{-10}m$, $D_{2\alpha} = c^{2-2\alpha}/(2\alpha m^{2\alpha-1})$, and $m = 9.1094\times10^{-31}$ kg.

moving in a 1D infinite potential well in the classical case. In other words, for arbitrary energy levels or wave functions in the classical scenario, we can find a sequence of Simple Energies or Simple Wave functions, respectively, in different dimensions that approaches them. A plot of Simple Energies of different fractal dimensions (2α) versus α is illustrated in Fig. 1.

If $\alpha \in (0.5,1)$, then for each $w_i(x)$, the number of Simple Energies is either countable (if α is irrational) or finite (if α is rational). Moreover, W is countable; therefore, the total number of Simple Energies is still countable. However, because of the continuity of the energy levels, there are uncountably many Nonsimple Energies. From the discussion above, we know that in the fractional case, both Nonsimple Energies and Nonsimple Elements must exist, whereas in the classical scenario, only Simple Energy levels and Simple Wave functions exist.

From Part I, we know that the Nonsimple Elements are not wave functions because they do not satisfy Eq. (7). However, in Part II, we will show how the Nonsimple Elements form wave functions, which we call Nonsimple Wave functions.

Part II (Nonsimple Wave functions)

Nonsimple Wave functions are a new type of wave function that has not been reported in Refs. [17–19,21,30–32]. Such a particle wave function is produced by a fractal power. The following two results (see Appendix C for the rigorous derivations) reveal the form of Nonsimple Wave functions.

(1) A linear combination of any two Nonsimple Elements in W cannot form a wave function that satisfies the boundary conditions.

(2) A specific linear combination of any three Nonsimple Elements in W can form a Nonsimple Wave function.

In general, the Nonsimple Elements have previously been neglected when considering the problem of a 1D infinite potential well. In the classical case, it is correct that the dimension of the Lévy path is integer in nature. However, in the fractional case, the fractal will strongly influence the particle, and thus, the Nonsimple Elements must be considered; in addition, the energies corresponding to the Nonsimple Wave functions will fill the gap between any two Simple Energies, thereby making the energy continuous.

A more precise description for item (2) is given as follows.

Choose three arbitrary Nonsimple Elements, $w_b(x)$, $w_c(x)$, and $w_d(x)$, in W. Then the Nonsimple Wave function that they form is

$$\psi(x) = Bw_b(x) + Cw_c(x) + Dw_d(x), \tag{14}$$

where

$$\begin{aligned} B &= [w_d(a)w_c(-a) - w_d(-a)w_c(a)]t \\ C &= [w_d(-a)w_b(a) - w_d(a)w_b(-a)]t \\ D &= [w_c(a)w_b(-a) - w_c(-a)w_b(a)]t \end{aligned} \tag{15}$$

and the parameter t is determined by

$$t = \pm(\widetilde{B}^2F_{b,b} + \widetilde{C}^2F_{c,c} + \widetilde{D}^2F_{d,d} + \widetilde{B}\widetilde{C}F_{b,c} + \widetilde{C}\widetilde{B}F_{c,b} + \widetilde{C}\widetilde{D}F_{c,d} + \widetilde{D}\widetilde{C}F_{d,c} + \widetilde{B}\widetilde{D}F_{b,d} + \widetilde{D}\widetilde{B}F_{d,b})^{-0.5},$$

where $F_{i,j} = \int_{-a}^{a} w_i(x)w_j^*(x)\,dx$ and

$$\begin{aligned} \widetilde{B} &= w_d(a)w_c(-a) - w_d(-a)w_c(a) \\ \widetilde{C} &= w_d(-a)w_b(a) - w_d(a)w_b(-a) \\ \widetilde{D} &= w_c(a)w_b(-a) - w_c(-a)w_b(a). \end{aligned} \tag{16}$$

Figure 2 shows how three Nonsimple Elements form a wave function for a fractal dimension of $2\alpha = 1.5$. The three Nonsimple Elements (f), (g), and (h), with coefficients f,g, and h determined by Eq. (15), form the Nonsimple Wave function (b). The three Nonsimple Elements (g), (h), and (i), with coefficients g,h and i determined by Eq. (15), form the Nonsimple Wave function (c). The three Nonsimple Elements (h), (i), and (j), with coefficients h,i, and j determined by Eq. (15), form the Nonsimple Wave function (d).

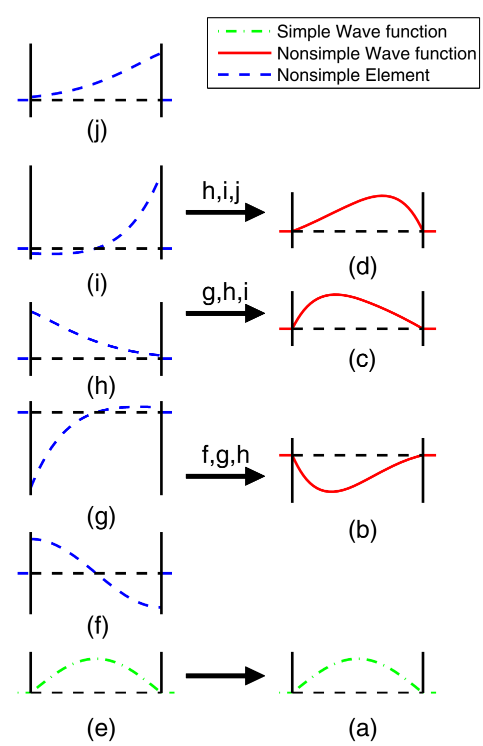


FIG. 2. An illustration of a fractional fractal dimension causes degeneracy, where we have chosen $\alpha = 0.75$, $a = 10^{-10}m$, $D_{2\alpha} = c^{2-2\alpha}/(2\alpha m^{2\alpha-1})$, $m = 9.1094\times10^{-31}$ kg, and $E = 7.2931\times 10^{-17}$ J.

Part III (Degree of degeneracy)

Let W_s be the set consisting of all Simple Wave functions in W with index set J.

Suppose that $W/W_s = \{\phi_1(x),\phi_2(x),\ldots,\phi_n(x),\ldots\}$. choose any three adjacent elements in W/W_s, denoted by $\phi_i(x)$, $\phi_{i+1}(x)$, and $\phi_{i+2}(x)$, and let $\psi_i(x)$ be a linear combination of them:

$$\psi_i(x) = B_i\phi_i(x) + C_i\phi_{i+1}(x) + D_i\phi_{i+2}(x). \tag{17}$$

The normalized $\psi_i(x)$ is a particle wave function with coefficients B_i, C_i, and D_i determined as derived in Part II.

Let $W_w = \{\psi_i(x), i \in I\}$, where I is an index set.

We obtain the following two results (see Appendix D for the rigorous derivations):

(1) All elements in $W_w \cup W_s$ are linearly independent.

(2) All states in the position-space representation at energy E can be represented by linearly combinations of elements in $W_w \cup W_s$.

Therefore, all wave functions can be represented by linearly combinations of elements in $W_w \cup W_s$. Thus, we can formulate the relation between the energy degeneracy and the fractal dimension 2α as follows: (1) If α is an irrational number, then the energy is degenerate to an infinite degree. (2) If $\alpha = n/m$ is a rational number, then the degree of energy degeneracy is $2n-2$. (3) In the particular case of $\alpha = 1$, i.e., $n = m = 1$, the degree of energy degeneracy is $2n - 2|_{n=1} = 0$, which is identical to the result in the classical scenario. Figure 2 illustrates how the elements of W form wave functions and give rise to energy degeneracy for $\alpha = 0.75$. From curves (a), (b), (c), and (d) in Fig. 2, we can see that the energy level $E = 7.2931\times10^{-17}$ J is fourfold degenerate.

As a result, a change in the Lévy path dimension from integer to noninteger in nature can cause the wave functions to change from nondegenerate to degenerate.

C. Comparisons

1. Fractional case

In our approach, if we assign special values to certain parameters, we can recover the same results for a 1D infinite potential well that are reported in Refs. [17–19,21,22]. We let α be a rational number such that $\alpha = n/m \in \mathbb{Q}^+$ for odd n, m, and the parameters in Eq. (10) satisfy

$$B_k, C_k \neq 0, \quad k = \frac{m-1}{2}, \quad \frac{3m-1}{2};$$
$$B_k, C_k = 0, \quad \text{others}. \tag{18}$$

Under these conditions, if we let $\sqrt[m]{-1} = -1$, we have $\theta = \pi$.

Then, the energies (see Appendix E for the calculation) are

$$E_g = -\left(\frac{g\pi}{2a}\right)^{2\alpha}(-\hbar^2)^\alpha D_{2\alpha} = \left(\frac{g\pi}{2a}\right)^{2\alpha}(\hbar^2)^\alpha D_{2\alpha}, \tag{19}$$

and the wave functions (see Appendix E for the calculation) are

$$\psi(x)_g = \begin{cases}\sqrt{\frac{1}{a}}\sin\left(\frac{g\pi x}{2a} + \frac{g\pi}{2}\right) & -a < x < a \\ 0 & x \leqslant -a, x \geqslant a\end{cases}. \tag{20}$$

We can see that the results are identical to those given in Refs. [17–19,21,22]. It is obvious that the energy levels given in Eq. (E5) and the wave functions given in Eq. (E6) are Simple Energies and Simple Wave functions. They are an incomplete set of energy levels and wave functions because the effect of a fractional fractal dimension is neglected. In fact, Nonsimple Energies and Nonsimple Wave functions also exist in this case. Because a comparison of Nonsimple Wave functions is presented in Fig. 2, here we compare only the Simple Wave functions form our results with those presented in Refs. [17–19,21,22], as shown in Fig. 3(a) for $\alpha = 0.75$. We obtain three Simple Wave functions using the method described in this article corresponding to $k = 0,1,2$, for each g; in particular, the red curves in Fig. 3(a) correspond to the incomplete solutions [Eq. (E6)].

2. Classical case

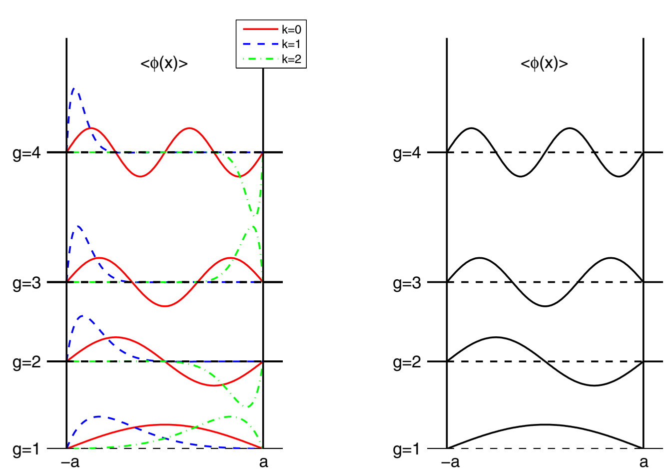
If $\alpha \to 1$, which corresponds to classical case, we can directly obtain the energy levels and wave functions from Eqs. (12) and (13):

$$E_g = \left(\frac{g\pi}{2a}\right)^{2\alpha}(\hbar^2)^\alpha D_{2\alpha}|_{\alpha=1} = \frac{g^2\pi^2\hbar^2}{8ma^2} \tag{21}$$

and

$$\psi(x)_g = \begin{cases}\sqrt{\frac{1}{a}}\sin\left(\frac{g\pi x}{2a} + \frac{g\pi}{2}\right) & -a < x < a \\ 0 & x \leqslant -a, x \geqslant a\end{cases}. \tag{22}$$

This result is identical to that obtained in classical quantum mechanics [30,31]. All of the energy levels given in Eq. (21) and the wave functions given in Eq. (22) are Simple Energies and Simple Wave functions. Because the dimension of the Lévy path is 2, there is no fractional dimensional influence on the energies and wave functions. Thus, our results are actually a generalization of the classical case. For convenience, here we compare only the Simple Wave functions from our results with those found in classical scenario, as shown in Fig. 3. We obtain



(a) $\alpha = 0.75$ (b) $\alpha = 1$

FIG. 3. The wave functions in different dimensions, where we have chosen $a = 10^{-10}m$, $D_{2\alpha} = c^{2-2\alpha}/(2\alpha m^{2\alpha-1})$, $m = 9.1094\times10^{-31}$ kg, and $E = 7.2931\times10^{-17}$ J.

three Simple Wave functions using the method described in this article, corresponding to $k = 0,1,2$, for each g; the black curves in Fig. 3(b) are the solutions obtained in the classical scenario, which are the same as the red curves in Fig. 3(a).

D. Discussion

In the above sections, we proved that all of the energy levels of a moving particle subjected to a 1D infinite potential well are continues in the circumstance of a fractional fractal dimension. At these energies, the states are always bound; thus even when perturbations to the system inevitably occur, the particle can always remain in a bound state. In other words, we have theoretically proven the existence of this phenomenon in fractional quantum mechanics. Now, we will explain why it can exist only in fractional quantum mechanics and not in classical quantum mechanics.

In classical quantum mechanics, Landau noted that a bound state solution with positive continuous energies exists in certain specific potentials [32]. However, the particle does not always remain bound, because these positive energy bound states are nearly isolated and there are few energy gaps between them; even a slight perturbation will cause wave function to shift to a nearby nonbound states [42].

As a result, our phenomenon has the three features simultaneously: all bound states energy levels of the particle are continuous; the particle can always remain in bound state; it can realized in simple potential, rather than specially tailored potentials or the interactions between particles. Therefore, our phenomenon is a feature of fractional quantum mechanics, and we can regard it as a criterion for determining whether fractional calculus should be used to describe a particle quantum system. Moreover, it also provides an experimental means of verifying the successful preparation of a fractional quantum system.

We can use the results of a spectral analysis to explain the cause of the phase transition of the energy levels between the classical and fractional cases; for convenience, we assume that the fractal dimension of the Lévy path is irrational.

In classical quantum mechanics, $\alpha = 1$, i.e., $T_2 = d^2/dx^2$. According to Ref. [47], we can easily obtain the following two conclusions regarding the discrete spectrum σ_d and the essential spectrum σ_e of the Schrödinger operator H_0 in the Hilbert space $L^2[-a,a]$:

(1) $\sigma_d(H_0) = \{\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots\}$;

(2) $\forall\lambda \in \sigma_e(H_0)$, λ is not an eigenvalue.

Because $\sigma(H_0^{2\alpha}) = \sigma_d(H_0^{2\alpha}) \cup \sigma_e(H_0^{2\alpha})$ and $\sigma_p(H_0^{2\alpha}) \subset \sigma(H_0^{2\alpha})$, where $\sigma_p(H_0^{2\alpha})$ is the characteristic spectrum, we know that the eigenvalue spectrum $\sigma_p(H_0^{2\alpha})$ consists only of isolated points. Because the eigenvalues of the Schrödinger operator are the energy levels, the particle has only discrete energy levels.

However, in fractional quantum mechanics, we can also obtain the following corresponding results (see Appendix F for more details and mathematical derivations) concerning the fractional Schrödinger operator H_0 in the Hilbert space $L^2[-a,a]$:

(1) $\sigma_d(H_{2\alpha}^0) = \emptyset$;

(2) $\forall\lambda \in \sigma_e(H_{2\alpha}^0)$, λ is a limit point.

By comparision with the results for the classical case, we can see that in fractional quantum mechanics, all elements in $\sigma_p(H_0^{2\alpha})$ are limit points. Because limit points are not isolated points, the energy of the particle is continuous. This finding is also consistent with the results derived in Sec. III A.

IV. CONCLUSION

We obtain the analytical expressions for the wave functions and energy levels of the particle subjected to a 1D infinite potential well in the circumstance of a fractional dimensional Lévy path. Our results are complete, and they reflect the fact that a fractional fractal dimension can cause phase transitions of the energy levels and wave functions: when the fractal dimension of the Lévy path changes from integer to noninteger in nature, the energy changes from discrete to continuous and the wave functions change from nondegenerate to degenerate. We introduced the concepts of Simple Energies and Simple Wave functions to explain these phase transitions and the relationships between them and the classical quantum phenomenon. In the fractional case, the energy levels include both Simple Energies and Nonsimple Energies. Nonsimple Energies are created by the existence of a fractional fractal dimension, and they fill in the gaps between the Simple Energies to make the energy continuous. Thus, there are always further bound states in the neighborhood of any bound state, meaning that the particle remains in a bound state even when perturbations to the system inevitably occur. The wave functions also include both Nonsimple Wave functions and Simple Wave functions which converge to the classical wave functions as the path dimension approach 2. In a classical quantum system, however, we can obtain only Simple Energies and Simple Wave functions.

Thus, we can predict a phenomenon in which all bound states energy levels of the particle are continuous and the particle remains in bound states, which can be regarded as an "*a priori* predicted" phenomenon. We have proven that this phenomenon is a characteristic phenomenon of a fractional system. Thus, when considering an unknown quantum system, we can determine in advance whether a fractional model should be used to describe the corresponding quantum problem by checking for the existence of stable bound states with a positive, continuous energy spectrum. In the best case scenario, this could offer a new means of verifying the successful preparation of a fractional quantum system in an experiment and provide a simple method of theoretically describing a fractional system in quantum mechanics.

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APPENDIX A: PROOFS FOR TWO RESULTS ABOUT STRONG FRACTIONAL CORRESPONDING OPERATOR

The proof for the result (1):

Assume an operator T_α with respect to x is the strong fractional corresponding operator, it satisfies the stronger condition $T_\alpha e^{kx} = k^\alpha e^{kx}$ for any $k \neq 0$.

By the following direct transformations:

$$\begin{aligned}&\frac{i^\alpha}{2\pi c^\alpha}\int_{-\infty}^{+\infty}k^\alpha\exp\left[ik\left(\frac{x+y}{c}\right)\right]dk\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{ik}{c}\right)^\alpha\exp\left[\frac{ik(x+y)}{c}\right]dk\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}T_\alpha\exp\left(ik\frac{x+y}{c}\right)dk\\&=T_\alpha\delta\left(\frac{x+y}{c}\right),\end{aligned}$$

we can know that T_α also satisfies the condition (a) in the concept of the fractional corresponding operator. Thus, T_α is the fractional corresponding operator.

The proof for the result (2) is as follows: According to Ref. [5], ${}^G_{-\infty}D^\alpha_x$, ${}^R_{-\infty}D^\alpha_x$, ${}^C_{-\infty}D^\alpha_x$, and ${}_{-\infty}\tilde{D}^\alpha_x$ have already been the fractional corresponding operators. Thus, we need only to prove that they satisfy the stronger condition $T_\alpha e^{kx}=k^\alpha e^{kx}$ for any $k\neq 0$.

First, we state that ${}^G_{-\infty}D^\alpha_x$ satisfies the stronger condition.

By the definition of the G-L fractional derivative operator [1,48], we have

$$ {}^G_{-\infty}D^\alpha_x e^{kx}=\frac{1}{\Gamma(-\alpha+m+1)}\int_{-\infty}^{x}(x-u)^{m-\alpha}k^{m+1}e^{ku}\,du. \tag{A1}$$

Let $y=k(x-u)$; Eq. (A1) becomes

$$\begin{aligned}{}^G_{-\infty}D^\alpha_x e^{kx}&=\frac{k^{m+1}e^{kx}}{k\Gamma(-\alpha+m+1)}\int_0^{+\infty}\left(\frac{y}{k}\right)^{m-\alpha}e^{-y}\,dy\\&=\frac{k^\alpha e^{kx}}{\Gamma(-\alpha+m+1)}\int_0^{+\infty}y^{m-\alpha+1-1}e^{-y}\,dy.\end{aligned}$$

By the definition of the Gamma function [1], we prove the result:

$$\begin{aligned}{}^G_{-\infty}D^\alpha_x e^{kx}&=\frac{k^\alpha e^{kx}}{\Gamma(-\alpha+m+1)}\Gamma(-\alpha+m+1)\\&=k^\alpha e^{kx}.\end{aligned}$$

We note that e^{kx} is a continuous function for any $k\in(0,+\infty)$ and the sth derivatives $(e^{kx})^{(s)}$ (for $s=1,2,\ldots$) is continuous in the interval $(-\infty,x)$, and if $x\to-\infty$, $(e^{kx})^{(s)}\to 0$. Under these conditions, the R-L fractional derivative, the G-L fractional derivative, and the Caputo fractional derivative are equivalent [1]:

$$ {}^G_{-\infty}D^\alpha_x e^{kx}={}^R_{-\infty}D^\alpha_x e^{kx}={}^C_{-\infty}D^\alpha_x e^{kx}=k^\alpha e^{kx}.$$

Second, we state that ${}_{-\infty}\tilde{D}^\alpha_x$ satisfies the stronger condition.

By the definition of $\Phi_{-\alpha}(x)$ [1], we have

$$\begin{aligned}{}_{-\infty}\tilde{D}^\alpha_x e^{kx}&=\int_{-\infty}^{x}e^{ku}\Phi_{-\alpha}(x-u)\,du\\&=\int_{-\infty}^{x}e^{ku}\frac{(x-u)^{-\alpha-1}}{\Gamma(-\alpha)}\,du.\end{aligned}\tag{A2}$$

Let $y=k(x-u)$; Eq. (A2) becomes

$$\begin{aligned}{}_{-\infty}\tilde{D}^\alpha_x e^{kx}&=\int_0^{+\infty}e^{kx-y}\frac{1}{\Gamma(-\alpha)}\left(\frac{y}{k}\right)^{-\alpha-1}d\left(\frac{y}{k}\right)\\&=\frac{e^{kx}k^\alpha}{\Gamma(-\alpha)}\int_0^{+\infty}e^{-y}y^{-\alpha-1}\,dy\\&=k^\alpha e^{kx}.\end{aligned}$$

Thus, the conclusion has been proved.

APPENDIX B: GENERALIZED DE MOIVRES THEOREM AND MODIFIED METHOD OF SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

The classical de Moivre's theorem has presented explicit expressions for the nth roots of $x^n=c$ for any integer n and complex number c. The case where n is a rational number or irrational number has been also reported in Ref. [49]. We put forward a more precise and suitable conclusion for quantum mechanics, which presents uniform set expressions.

Let

$$\lambda^\alpha=c, \tag{B1}$$

where $\alpha\in\mathbb{R}^+$ and $c\in\mathbb{C}$, then the roots of c are

$$\lambda_k=\beta_k+i\gamma_k, \tag{B2}$$

where $\beta_k=\cos[(\theta+2k\pi)/\alpha]|c|^{1/\alpha}$, $\gamma_k=\sin[(\theta+2k\pi)/\alpha]|c|^{1/\alpha}$, $\theta=\arg c$, and $k\in K\subseteq\mathbb{Z}$ where K is an index set such that if $k_1\neq k_2\in K$, then

$$\frac{k_1-k_2}{\alpha}\notin\mathbb{Z}. \tag{B3}$$

It is equivalent to say that

(1) If $\alpha\in\mathbb{N}^+$, then $K=\{0,1,\ldots,\alpha-1\}$;

(2) If $\alpha\in\mathbb{Q}^+/\mathbb{N}^+$, let $\alpha=n/m$, then $K=\{0,1,\ldots,n-1\}$;

(3) If $\alpha\in\mathbb{R}^+/\mathbb{Q}^+$, then $K=\mathbb{Z}$.

Next we will divide the proof into three steps:

Step I (Verify the expression of the roots)

Note that

$$\lambda_k=\beta_k+i\gamma_k=\exp\left(i\frac{\theta+2k\pi}{\alpha}\right)|c|^{\frac{1}{\alpha}},$$

then

$$(\lambda_k)^\alpha=\exp(i\theta+2k\pi i)|c|=\exp(i\theta)|c|=c.$$

Thus, we have verified Eq. (B2).

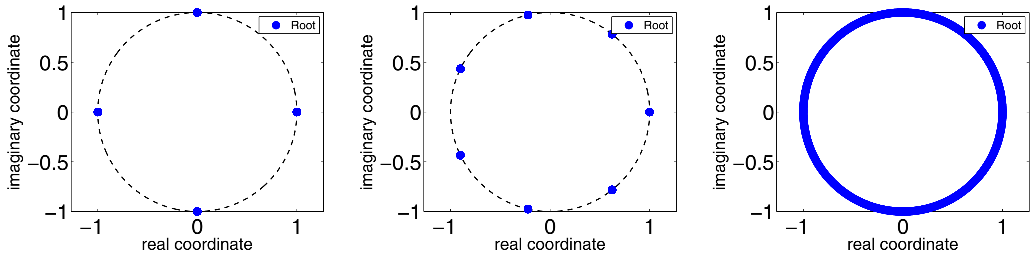
Step II (Verify that $\lambda_{k_1}\neq\lambda_{k_2}$ if $k_1\neq k_2$)

Assume that $\lambda_{k_1}=\lambda_{k_2}$:

$$\begin{aligned}\beta_{k_1}&=\beta_{k_2}\\\gamma_{k_1}&=\gamma_{k_2}.\end{aligned}\tag{B4}$$

By the sum-to-product identities, we have

$$\begin{aligned}\beta_{k_1}-\beta_{k_2}=&-2\sin\left(\frac{2\theta+2k_1\pi+2k_2\pi}{2\alpha}\right)\\&\times\sin\left(\frac{2k_1\pi-2k_2\pi}{2\alpha}\right)|c|^{\frac{1}{\alpha}}\end{aligned}$$



(a) $\alpha = 4$ (b) $\alpha = \frac{7}{3}$ (c) $\alpha = \sqrt{2}$

FIG. 4. The roots of Eq. (B1) for different α, where we have chosen $c = 1$. When $\alpha = 4$, the result is the same as it in the classical de Moivre's theorem.

and

$$\gamma_{k_1} - \gamma_{k_2} = 2\cos\left(\frac{2\theta + 2k_1\pi + 2k_2\pi}{2\alpha}\right) \times \sin\left(\frac{2k_1\pi - 2k_2\pi}{2\alpha}\right)|c|^{\frac{1}{\alpha}}.$$

Note that

$$\sin\left(\frac{2\theta + 2k_1\pi + 2k_2\pi}{2\alpha}\right) = 0 \Leftrightarrow \frac{\theta + (k_1 + k_2)\pi}{\alpha\pi} \in \mathbb{Z}, \tag{B5}$$

$$\cos\left(\frac{2\theta + 2k_1\pi + 2k_2\pi}{2\alpha}\right) = 0 \Leftrightarrow \frac{\theta + (k_1 + k_2)\pi}{\alpha\pi} - \frac{1}{2} \in \mathbb{Z}. \tag{B6}$$

Obviously, $\forall k_1$ and k_2, Eqs. (B5) and (B6) cannot hold simultaneously. If Eq. (B4) holds, we must have $\sin[(2k_1\pi - 2k_2\pi)/(2\alpha)] = 0$, meaning that $(k_1 - k_2)/\alpha \notin \mathbb{Z}$ if $k_1 \neq k_2$.

Thus, we obtain $k_1 = k_2$.

Step III (Prove the conclusion about the index set K)

If $\alpha \in \mathbb{N}^+$, by fundamental theorem of algebra [50], equation $\lambda^\alpha = c$ has α roots, so $K = \{0,1,\ldots,\alpha - 1\}$.

If $\alpha \in \mathbb{Q}^+/\mathbb{N}^+$, then $\forall k_j \in K$ and $\alpha = n/m$, we have

$$\frac{\theta + 2(k_j + n)\pi}{\alpha} = \frac{\theta + 2k_j\pi}{\alpha} + 2m\pi, \tag{B7}$$

thus,

$$\sin\left(\frac{\theta + 2k_j\pi}{\alpha}\right) = \sin\left[\frac{\theta + 2(k_j + n)\pi}{\alpha}\right]$$

$$\cos\left(\frac{\theta + 2k_j\pi}{\alpha}\right) = \cos\left[\frac{\theta + 2(k_j + n)\pi}{\alpha}\right],$$

which imply that $\lambda_{k_j} = \lambda_{k_j+n}$.

Obviously, $0 \in K$, so we have proved (2).

If $\alpha \in \mathbb{R}^+/\mathbb{Q}^+$, $\forall z_1 \neq z_2 \in \mathbb{Z}$, they satisfy Eq. (B3) because α is an irrational number. Thus, we have $z_1, z_2 \in K$.

With $0 \in K$, we obtain $K = \mathbb{Z}$.

Figure 4 illustrates the roots of Eq. (B1) and their positions in the complex plane for three specific cases depending on the generalized de Moivre's theorem.

Then, we will use the strong fraction corresponding operator and the generalized de Moivre's theorem to present a new method of solving fractional differential equations in the form

$${}^{R}_{-\infty}D^{\alpha}_{x} f(x) = cf(x), \tag{B8}$$

where ${}^{R}_{-\infty}D^{\alpha}_{x}$ is the R-L fractional derivative operator and c is a constant.

We consider a general case of Eq. (B8) as

$$T_\alpha f(x) = cf(x), \tag{B9}$$

where T_α is the strong fractional corresponding operator. If $\alpha \to n$, where n is a positive integer, Eq. (B9) becomes

$$\frac{d^n}{dx^n} f(x) = cf(x). \tag{B10}$$

Thus, Eq. (B9) is a generalization of integer order differential equations.

There exists a constant c such that

$$(T_\alpha - c)e^{\lambda x} = 0, \tag{B11}$$

by the result about the spectrum of T_α, which has been reported in Appendix F. Thus, Eq. (B11) becomes $(\lambda^\alpha - c)e^{\lambda x} = 0$, meaning that $\lambda^\alpha - c = 0$. By the conclusion about the spectrum of the strong fractional Schrödinger operator, the solutions of Eq. (B9) are

$$f(x) = \sum_{k\in K} B_k \exp(\beta_k x)\cos(\gamma_k x) + C_k \exp(\beta_k x)\sin(\gamma_k x), \tag{B12}$$

where $\beta_k = \cos[(\theta + 2k\pi)/\alpha]|c|^{1/\alpha}$, $\gamma_k = \sin[(\theta + 2k\pi)/\alpha]|c|^{1/\alpha}$, $\theta = \arg c$, and K is an index set.

In fact, Eq. (B12) has included the case of integer order as a special case; in other words, if α is a positive integer, Eq. (B12) presents the solutions of the integer order differential equations Eq. (B10) [51].

APPENDIX C: DEDUCTIONS FOR NONSIMPLE WAVE FUNCTION

For the result (1):

We will discuss it in four different cases.

Case 1

Let $w_i(x) = \exp(\beta_k x)\cos\gamma_k x$ and $w_j(x) = \exp(\beta_k x)\sin\gamma_k x$ be two Nonsimple Elements, where $k \in K$.

Assume that the wave function they form is

$$\psi(x) = A\exp(\beta_k x)\cos\gamma_k x + B\exp(\beta_k x)\sin\gamma_k x. \tag{C1}$$

Plugging Eq. (C1) into the bound conditions, we have

$$\begin{gathered} A\exp(\beta_k a)\cos\gamma_k a + B\exp(\beta_k a)\sin\gamma_k a = 0 \\ A\exp(-\beta_k a)\cos(-\gamma_k a) + B\exp(-\beta_k a)\sin(-\gamma_k a) = 0. \end{gathered} \tag{C2}$$

Considering the coefficient matrix of Eq. (C2), if the determinant of the matrix is zero,

$$\begin{vmatrix} \exp(\beta_k a)\cos\gamma_k a & \exp(\beta_k a)\sin\gamma_k a \\ \exp(-\beta_k a)\cos(-\gamma_k a) & \exp(-\beta_k a)\sin(-\gamma_k a) \end{vmatrix} = -2\cos\gamma_k a\sin\gamma_k a = 0, \tag{C3}$$

we must have $\cos\gamma_k a = 0$ or $\sin\gamma_k a = 0$. It means that at least one of $w_i(x)$ and $w_j(x)$ is a Simple Wave function, which contradicts the assumption. Thus, Eq. (C3) cannot be satisfied; that is, the system of linear equations Eq. (C2) does not have nonzero solution.

Thus, $A = B = 0$.

Case 2

Let $w_i(x) = \exp(\beta_k x)\cos\gamma_k x$ and $w_j(x) = \exp(\beta_l x)\sin\gamma_l x$ be two Nonsimple Elements, where $k \neq l \in K$.

Similarly, assume that $\psi(x) = A\exp(\beta_k x)\cos\gamma_k x + B\exp(\beta_l x)\sin\gamma_l x$. Then substituting it into boundary conditions, we have

$$\begin{gathered} A\exp(\beta_k a)\cos\gamma_k a + B\exp(\beta_l a)\sin\gamma_l a = 0 \\ A\exp(-\beta_k a)\cos(-\gamma_k a) + B\exp(-\beta_l a)\sin(-\gamma_l a) = 0. \end{gathered} \tag{C4}$$

If there exist nontrivial solutions, then

$$\begin{vmatrix} \exp(\beta_k a)\cos\gamma_k a & \exp(\beta_l a)\sin\gamma_l a \\ \exp(-\beta_k a)\cos(-\gamma_k a) & \exp(-\beta_l a)\sin(-\gamma_l a) \end{vmatrix} = -\cos\gamma_k a\sin\gamma_l a\{\exp[(\beta_k-\beta_l)a] + \exp[(\beta_l-\beta_k)a]\} = 0.$$

Because $\exp[(\beta_k - \beta_l)a] + \exp[(\beta_l - \beta_k)a] > 0$, we have $\cos\gamma_k a = 0$ or $\sin\gamma_l a = 0$. Again, it means that $w_i(x)$ or $w_j(x)$ is a Simple Wave function, which contradicts with the assumption.

Thus, the system of linear equations (C4) has only a zero solution: $A = B = 0$.

Case 3

When $w_i(x) = \exp(\beta_k x)\cos\gamma_k x$ and $w_j(x) = \exp(\beta_l x)\cos\gamma_l x$ are two Nonsimple Elements, where $k \neq l \in K$.

Repeat the same proof of the Case 2, we obtain that $w_i(x)$ and $w_j(x)$ cannot form a wave function.

Case 4

When $w_i(x) = \exp(\beta_k x)\sin\gamma_k x$ and $w_j(x) = \exp(\beta_l x)\sin\gamma_l x$ are two Nonsimple Elements, where $k \neq l \in K$

Repeating the same proof of Case 2 again, we can obtain the same result as it in Case 2.

The above four cases cover all the possibilities of linear combinations of any two Nonsimple Elements. They cannot form wave functions.

For the result (2), choose three arbitrary elements $w_b(x)$, $w_c(x)$, and $w_d(x)$ in W. Assume that the wave function they form is

$$\psi(x) = Bw_b(x) + Cw_c(x) + Dw_d(x). \tag{C5}$$

Plugging Eq. (C5) into boundary conditions, we have

$$\begin{gathered} Bw_b(a) + Cw_c(a) + Dw_d(a) = 0 \\ Bw_b(-a) + Cw_c(-a) + Dw_d(-a) = 0. \end{gathered}$$

by

$$\begin{vmatrix} w_b(a) & w_c(a) \\ w_b(-a) & w_c(-a) \end{vmatrix} \neq 0; \tag{C6}$$

i.e., the rank of the coefficient matrix of this linear system is 2. Thus, the system has nonzero general solutions:

$$\begin{aligned} B &= [w_d(a)w_c(-a) - w_d(-a)w_c(a)]t \\ C &= [w_d(-a)w_b(a) - w_d(a)w_b(-a)]t \\ D &= [w_c(a)w_b(-a) - w_c(-a)w_b(a)]t, \end{aligned} \tag{C7}$$

where t is an arbitrary constant.

In order to determine t, we assume that

$$\begin{aligned} \widetilde{B} &= w_d(a)w_c(-a) - w_d(-a)w_c(a) \\ \widetilde{C} &= w_d(-a)w_b(a) - w_d(a)w_b(-a) \\ \widetilde{D} &= w_c(a)w_b(-a) - w_c(-a)w_b(a), \end{aligned} \tag{C8}$$

and

$$F_{i,j} = \int_{-a}^{a} w_i(x)w_j^*(x)\,dx. \tag{C9}$$

Then, plug Eq. (C8), Eq. (C9) in the normalization condition of $\psi(x) = Bw_b(x) + Cw_c(x) + Dw_d(x)$, we have

$$\begin{aligned} 1 &= \int_{-a}^{a} \psi(x)\psi^*(x)\,dx \\ &= (\widetilde{B}^2F_{b,b} + \widetilde{C}^2F_{c,c} + \widetilde{D}^2F_{d,d} + \widetilde{B}\widetilde{C}F_{b,c} + \widetilde{C}\widetilde{B}F_{c,b} \\ &\quad + \widetilde{C}\widetilde{D}F_{c,d} + \widetilde{D}\widetilde{C}F_{d,c} + \widetilde{B}\widetilde{D}F_{b,d} + \widetilde{D}\widetilde{B}F_{d,b})t^2. \end{aligned} \tag{C10}$$

Solving Eq. (C10), we obtain

$$\begin{aligned} t = \pm(&\widetilde{B}^2F_{b,b} + \widetilde{C}^2F_{c,c} + \widetilde{D}^2F_{d,d} + \widetilde{B}\widetilde{C}F_{b,c} + \widetilde{C}\widetilde{B}F_{c,b} \\ &+ \widetilde{C}\widetilde{D}F_{c,d} + \widetilde{D}\widetilde{C}F_{d,c} + \widetilde{B}\widetilde{D}F_{b,d} + \widetilde{D}\widetilde{B}F_{d,b})^{-\frac{1}{2}}. \end{aligned}$$

We determine the coefficients B, C, and D, and find such a linear combination that satisfies the boundary conditions. By Eq. (B12), we know that Eq. (C5) is a wave function. Thus, the three Nonsimple Element $w_b(x), w_c(x)$, and $w_d(x)$ form a wave function.

APPENDIX D: DEDUCTIONS FOR DEGENERACY DEGREE

For the result (1):

Assume that there exists a sequence of constants $\{k_{1,j}, k_{2,i}\}$, where $j \in J$ and $i \in I$, such that

$$\sum_j k_{1,j}w_j(x) + \sum_i k_{2,i}\psi_i(x) = 0. \tag{D1}$$

Plugging $\psi_i(x) = B_i\phi_i(x) + C_i\phi_{i+1}(x) + D_i\phi_{i+2}(x)$ into Eq. (D1), we have

$$\begin{aligned} &\sum_j k_{1,j}w_j(x) + k_{2,1}B_1\phi_1(x) + (k_{2,1}C_1 + k_{2,2}B_2)\phi_2(x) \\ &\quad + \sum_i (k_{2,i-2}D_{i-2} + k_{2,i-1}C_{i-1} + k_{2,i}B_i)\phi_i(x) = 0, \end{aligned} \tag{D2}$$

where $i \in I/\{1,2\}$ and I has infinite elements.

If I has finite elements, we assume $I = \{1,2,\ldots,n+2\}$, then Eq. (D2) becomes

$$\sum_j k_{1,j} w_j(x) + k_{2,1} B_1 \phi_1(x) + (k_{2,1} C_1 + k_{2,2} B_2)\phi_2(x) + \sum_{i=3}^{n} (k_{2,i-2} D_{i-2} + k_{2,i-1} C_{i-1} + k_{2,i} B_i)\phi_i(x) + (k_{2,n-1} D_{n-1} + k_{2,n} C_n)\phi_{n+1}(x) + k_{2,n} D_n \phi_{n+2}(x) = 0. \quad \text{(D3)}$$

Thus, without loss of generality, we assume that I has infinite elements.

Because $\{w_j(x), \phi_i(x)\}$ is linearly independent, we must have

$$k_{1,j} = 0, j \in J;$$
$$k_{2,1} B_1 = 0;$$
$$k_{2,1} C_1 + k_{2,2} B_2 = 0;$$
$$k_{2,i-2} D_{i-2} + k_{2,i-1} C_{i-1} + k_{2,i} B_i = 0, \quad i \in I/\{1,2\}.$$

Because B_i, C_i, and D_i are nonzero, we obtain $k_{2,i} = 0$ for any $i \in I$. Thus, $W_w \cup W_s$ is linearly independent.

For the results (2), by Eq. (B12), we know

$$\psi(x) = \sum_k e_k w_k(x).$$

Using the notations in Step I, the above equation becomes

$$\psi(x) = \sum_j k_{1,j} w_j(x) + \sum_i k_{2,i} \phi_i(x), \quad \text{(D4)}$$

where $w_j(x) \in W_s$ and $\phi_i(x) \in W/W_s$.

Obviously we need only to prove that the second term can be linearly represented by the elements in $W_w \cup W_s$.

Consider a sequence $\{G_l\}$ as follows, for any $l \in L$:

$$G_1 = \frac{k_{2,1}}{B_1};$$
$$G_2 = \frac{k_{2,2} - G_1 C_1}{B_2};$$
$$G_l = \frac{k_{2,l} - G_{l-2} D_{l-2} - G_{l-1} C_{l-1}}{B_l}, \quad l \geqslant 3,$$

where B_l, C_l, and D_l are presented by Eq. (C7).

Letting $\{G_l\}$ represent the coefficients of the second term in Eq. (D4), we have

$$\sum_i k_{2,i} \phi_i(x) = \sum_l G_l \psi_l(x),$$

where $\psi_l(x) \in W_w$.

Thus, we have completed the proof.

APPENDIX E: CALCULATE FOR FRACTIONAL CASE

We let $\alpha = m/n \in \mathbb{Q}^+$ for odd n, m and the parameters $\{B_k, C_k\}$ satisfy

$$B_k, C_k \neq 0, \quad k = \frac{m-1}{2}, \frac{3m-1}{2};$$
$$B_k, C_k = 0, \quad \text{others}. \quad \text{(E1)}$$

Under these conditions, if we let $\sqrt[m]{-1} = -1$, we have $\theta = \pi$.

Then

$$\beta_k = \cos \frac{n\pi + 2kn\pi}{2m} |c|^{\frac{1}{2\alpha}}.$$

When $k = (m-1)/2$ or $(3m-1)/2$, n is odd, we have

$$\beta_{\frac{m-1}{2}} = \cos \frac{n\pi}{2} |c|^{\frac{1}{2\alpha}} = 0$$
$$\beta_{\frac{3m-1}{2}} = \cos \frac{3n\pi}{2} |c|^{\frac{1}{2\alpha}} = 0$$
$$\gamma_{\frac{m-1}{2}} = \sin \frac{n\pi}{2} |c|^{\frac{1}{2\alpha}} = |c|^{\frac{1}{2\alpha}}$$
$$\gamma_{\frac{3m-1}{2}} = \sin \frac{3n\pi}{2} |c|^{\frac{1}{2\alpha}} = -|c|^{\frac{1}{2\alpha}}. \quad \text{(E2)}$$

By Eqs. (E1) and (E2), we obtain

$$\psi(x) = B_{\frac{m-1}{2}} \cos\left(|c|^{\frac{1}{2\alpha}} x\right) + C_{\frac{m-1}{2}} \sin\left(|c|^{\frac{1}{2\alpha}} x\right) + B_{\frac{3m-1}{2}} \cos\left(|c|^{\frac{1}{2\alpha}} x\right) - C_{\frac{3m-1}{2}} \sin\left(|c|^{\frac{1}{2\alpha}} x\right). \quad \text{(E3)}$$

Let $B = B_{\frac{m-1}{2}} + B_{\frac{3m-1}{2}}$ and $C = C_{\frac{m-1}{2}} - C_{\frac{2m-1}{2}}$, then Eq. (E3) becomes

$$\psi(x) = B \cos\left(|c|^{\frac{1}{2\alpha}} x\right) + C \sin\left(|c|^{\frac{1}{2\alpha}} x\right).$$

By the boundary conditions, we obtain

$$B \cos\left(|c|^{\frac{1}{2\alpha}} a\right) + C \sin\left(|c|^{\frac{1}{2\alpha}} a\right) = 0$$
$$B \cos\left[|c|^{\frac{1}{2\alpha}}(-a)\right] + C \sin\left[|c|^{\frac{1}{2\alpha}}(-a)\right] = 0. \quad \text{(E4)}$$

Similarly with the case in classical quantum mechanics, we obtain the solutions of Eq. (E4):

(a) $B = 0, \sin(|c|^{\frac{1}{2\alpha}} a) = 0$;
(b) $C = 0, \cos(|c|^{\frac{1}{2\alpha}} a) = 0$.

That is,

$$|c|^{\frac{1}{2\alpha}} = \frac{g\pi}{2a}, \quad g \text{ is odd};$$
$$|c|^{\frac{1}{2\alpha}} = \frac{g\pi}{2a}, \quad g \text{ is even}.$$

Because $\theta = \pi$, the energy levels are

$$E_g = -\left(\frac{g\pi}{2a}\right)^{2\alpha} (-\hbar^2)^\alpha D_{2\alpha} = \left(\frac{g\pi}{2a}\right)^{2\alpha} (\hbar^2)^\alpha D_{2\alpha}, \quad \text{(E5)}$$

and the wave functions are

$$\psi(x)_g = \begin{cases} \sqrt{\frac{1}{a}} \sin\left(\frac{g\pi x}{2a} + \frac{g\pi}{2}\right) & -a < x < a \\ 0 & x \leqslant -a, x \geqslant a \end{cases}. \quad \text{(E6)}$$

APPENDIX F: CALCULATE SPECTRUM OF A FRACTIONAL SCHRÖDINGER OPERATOR

If a physical system under consideration is a nonrelativistic point particle of mass $m > 0$ in a real potential V, we can obtain the fractional Schrödinger operator in one dimension,

$$H_{2\alpha}^V = D_{2\alpha}[(-i\hbar)^\alpha T_\alpha]^2 + V = D_{2\alpha}(-\hbar^2)^\alpha T_{2\alpha} + V, \quad \text{(F1)}$$

where T_α of order $\alpha \in (0.5,1]$ with respect to x is the fractional corresponding operator.

By Eq. (F1) and Refs. [3–5], we know that $(-i\hbar)^{2\alpha} = (-\hbar^2)^\alpha$, meaning that $D_{2\alpha}p^{2\alpha} + V$ and $D_{2\alpha}|p|^{2\alpha} + V$ are able to be shown as $D_{2\alpha}(p^2)^\alpha + V$ and $D_{2\alpha}(|p|^2)^\alpha + V$ in one dimension. Then, by $D_{2\alpha}(p^2)^\alpha + V = D_{2\alpha}(|p|^2)^\alpha + V$, we obtain that the fractional Schrödinger operator is the self-adjoint operator because the expectation of energy $\langle E\rangle$ is real [3].

Obviously, if the operator T_α is a strong fractional corresponding operator, the above conclusions are still true depending on the result (1) of the strong fractional corresponding operator.

We will concentrate our following discussion on the case where $V = 0$. We calculate the discrete spectrum and the essential spectrum of the fractional Schrödinger operator $H_{2\alpha}^0$ in $L^2[-a,a]$.

If T is a linear operator, the set consisting of all isolated eigenvalues with infinite multiplicity and all limit points in the spectrum set $\sigma(T)$ is called the essential spectrum of T, denoted as $\sigma_e(T)$, and $\sigma_d(T) = \sigma(T)/\sigma_e(T)$ is called the discrete spectrum of T [52].

Because all isolated points in the spectrum of the self-adjoint operator are eigenvalues of itself, we know $\sigma_d(H_{2\alpha}^0)$ is actually the set of all isolated eigenvalues with finite multiplicity in $\sigma(H_{2\alpha}^0)$ [52].

Thus, we states the conclusions about the discrete spectrum and the essential spectrum of $H_{2\alpha}^0$ in $L^2[-a,a]$ as follows:

(1) $\sigma_d(H_{2\alpha}^0) = \emptyset$

(2) $\forall\lambda \in \sigma_e(H_{2\alpha}^0)$, λ is a limit point.

Next, we give the proof of the two conclusions. First, we prove the conclusion (1).

Because $\sigma_d(H_{2\alpha}^0)$ is actually a set of all isolated eigenvalues with finite multiplicity in $\sigma(H_{2\alpha}^0)$, we need only to prove that $H_{2\alpha}^0$ does not have isolated eigenvalues with finite multiplicity.

$\forall\lambda \neq 0$, let c_{λ_k} be the roots of equation $D_{2\alpha}(-\hbar^2)^\alpha {c_{\lambda_k}}^{2\alpha} = \lambda$, where $k \in K_\lambda$. By the generalized de Moivre's theorem, we have $K_\lambda = \mathbb{Z}$.

Now consider the function $\exp(c_{\lambda_k}x)$. Letting $H_{2\alpha}^0$ act on it, we obtain

$$H_{2\alpha}^0 \exp(c_{\lambda_k}x) = D_{2\alpha}(-\hbar^2)^\alpha c_{\lambda_k}^{2\alpha} \exp(c_{\lambda_k}x) = \lambda \exp(c_{\lambda_k}x), \tag{F2}$$

meaning that λ is an eigenvalue and $\{\exp(c_{\lambda_k}x)|k \in K_\lambda\}$ is its eigenspace. Because the index set K_λ has infinite elements, eigenvalue λ has infinite multiplicity. Thus, $\sigma_d(T_\alpha) = \emptyset$.

Second, we prove the conclusion (2).

As we have reported above, the nonzero eigenvalues of $H_{2\alpha}^0$ have infinite multiplicities.

$\forall\varepsilon > 0$, choosing an arbitrary element $\widetilde{\lambda}$ in $O(\lambda,\varepsilon)$, we repeat the proof process of the conclusion (1). This shows that $\widetilde{\lambda}$ is also an eigenvalue of infinite multiplicity. So λ is a limit point.

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