Violations of a Bell inequality for entangled SU(1,1) coherent states based on dichotomic observables

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We study the violation of the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality for entangled SU(1,1) coherent states of the form proposed by Perelomov. Specifically, we examine Bell-CHSH violations by such states in the case in which distant observers Alice and Bob perform local, noncompact, SU(1,1) transformations characterized by hyperbolic angles on each of the subsystems and subsequently measure dichotomic observables, namely SU(1,1) parity operators. We find significant violations over a broad range of hyperbolic angles.

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I. INTRODUCTION

Coherent states, as a class of pure quantum states, are special kinds of states in the sense that they contain complex continuous parameters that can be assigned a wide range of values such that the amplitudes of the states can place them in either the microscopic or the macroscopic realms and anywhere in between. In the case of the canonical, or Glauber [1], coherent states, which describe a quantized single-mode field pure state as close to classical as is possible, the amplitudes of the states can be very large or very small and the states still retain their classical-like properties. The canonical coherent states carry the noise level of the quantum vacuum. On the other hand, by superposing coherent states of identical macroscopic amplitude but of maximal phase difference one obtains highly nonclassical states of the type known as Schrödinger cat states [2]. In the case of a two-mode field state, one can consider, beyond the product of coherent states and cat states of each mode, an entanglement of coherent states over the two modes [3]. If the amplitudes of the coherent states are low, the states are in the microscopic realm, but the amplitudes could be large enough to place the coherent state components into a realm that presents the prospect of violating Bell-type inequalities in a mesoscopic or even macroscopic quantum mechanical system. Violations of Bell inequalities rule out realistic local hidden-variable theories in favor of quantum mechanics [3]. It is one thing to contemplate the violation of Bell inequalities by states involving a small number of particles, as has been done in all known experiments [4], but quite another if a mesoscopic or macroscopic number of particles are involved [5]. Naively, one might expect that the violation of such inequalities might diminish and even vanish in the limit of macroscopically occupied component states, but this is generally not the case.

There are other kinds of systems and coherent states with different levels of entanglement that can be considered. The second most common type of coherent states discussed in the literature after the canonical coherent states are the spin [6], or atomic [7], coherent states, also known as the SU(2) coherent states [8]. In the context of a collection of two-level atoms, the SU(2) (atomic) coherent states contain no entanglement between the atoms. But if the SU(2) coherent states are realized as two-mode bosonic states (such as for two modes of light [9] or for a two-component Bose-Einstein condensate [10]) for a total fixed number of particles, then entanglement will exist between the two modes. In analogy to the entangled ordinary coherent states, one can consider violations of a Bell inequality by entangled SU(2) coherent states, as was done by Gerry *et al.* [11] by employing dichotomic observables, namely the SU(2) parity operators of each component of the entangled state. The SU(2) parity operator, for a given irreducible representation $\mathcal{D}^{(j)}$, wherein $J_z|j,m\rangle = m|j,m\rangle$, has the form

$$\Pi^{(j)} = e^{i\pi(j-J_z)} = \sum_{m=-j}^{j} |j,m\rangle(-1)^{j-m}\langle j,m|. \tag{1}$$

The entangled SU(2) coherent states could represent two, possibly separated, ensembles of two-level atoms similar to the kind of entangled states between atomic ensembles created experimentally by Polzik and collaborators [12]. If we instead assume the SU(2) coherent states are associated with two sets of two bosonic systems, such as a two-mode quantized field or a two-component Bose-Einstein condensate, the entangled SU(2) coherent states ultimately involve the entanglement of four bosonic modes. Regardless of the realization, the entangled SU(2) coherent state has been shown to violate a Bell-type inequality [11], specifically the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) form of the inequality [13]. In fact, it was shown that under certain circumstances, the entangled SU(2) coherent states can violate the Bell-CHSH inequality maximally. That is, the violations can reach Csirel'son bound [13], and can do so for arbitrary spins of the component SU(2) coherent states. It was already known that, counterintuitively, violations of Bell inequalities do not vanish for spin-singlet states of high spin (j) in spite of the general belief that large spin is a classical limit [14]. In fact, using dichotomic variables that amount to SU(2) parity operators, Peres [15] showed that the spin-singlet states violate the inequality by a constant amount in the limit $j \to \infty$, though this violation is not at the Csirel'son bound.

In this paper we study the violations of the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality by entangled SU(1,1) coherent states as defined by Perelomov [8] using as dichotomic variables the SU(1,1) parity operators acting upon each of the components. The entangled states considered are the noncompact analog of the entangled SU(2) coherent states studied in Ref. [11]. Here the component states are Perelomov SU(1,1) coherent states defined by the action of the SU(1,1) displacementlike operator acting on the ground state of the relevant unitary irreducible representation (UIR). The Perelomov SU(1,1) coherent states play an important role in quantum optics as realizations of them include the single-mode squeezed vacuum and squeezed one-photon states, the two-mode squeezed vacuum states [16], and squeezed number states [17,18]. By coupling together SU(1,1) states using the SU(1,1) Clebsch-Gordan coefficients [19,20] one can create new forms of Perelomov SU(1,1) coherent states with multimode entanglement, that is, states involving two, three, or four modes or even more. Aside from the quantum-optical states just mentioned, Perelomov SU(1,1) coherent states have recently been discussed in the context of matter waves where the states may be generated by Feshbach resonances in the dissociation of molecular BECs into bosons [21]. We consider the study of a Bell's inequality violations for the entangled SU(1,1) coherent state to be a natural extension of the previous work on such violations involving entangled canonical coherent states and entangled SU(2) coherent states.

Very recently, Schnabel [22] has speculated on the prospect of a macroscopic test of quantum mechanics with objects of high masses placed in nonclassical states of motion. The ideal candidate for such a demonstration would be two massive (of the order of 0.1 kg) pendulum-suspended mirrors in a Michelson interferometer setup prepared in entangled states of mechanical motion of their centers of mass. This could be done by an entanglement swapping protocol [23] that transfers the quantum state of light onto quantum states of mechanical motion of the centers of mass of the mirrors. An obvious choice for the state of nonclassical motion would be the squeezed vacuum state of the SU(1,1) type, or more accurately, two masses could conceivably be prepared in an entanglement of single-mode SU(1,1) coherent states.

Wang *et al.* [24] have studied entangled SU(2) and SU(1,1) coherent states, but our work here on SU(1,1) states and the earlier work on the SU(2) states is significantly different on several accounts. First, our states are more general and do not require that the components have identical Bargmann indices in the former case or spins in the latter. Second, as mentioned, we have used as the dichotomic observables the corresponding SU(1,1) or SU(2) parity operators, respectively. In the SU(2) case [11], the observers each perform local, compact transformations (rotations) of the states whereas in the SU(1,1) case below we consider noncompact transformations (hyperbolic "rotations"). In Ref. [24] only compact transformations were considered in both cases. Lastly, unlike in Ref. [24], we perform numerical calculations with a search algorithm to determine the maximal Bell inequality violations for a given set of state parameters as results of this nature are not analytically available for any of the SU(1,1) cases in our scheme.

The paper is organized as follows. In Sec. II we review the formalism for SU(1,1) and the corresponding Perelomov coherent states, indicating their realizations and representations relevant to quantum optics. In Sec. III we present the entangled Perelomov entangled coherent states, and in Sec. IV we develop the relevant Bell inequality and present our results. In Sec. V we briefly discuss a possible scheme for the generation of the entangled SU(1,1) coherent states, and in Sec. VI we conclude the paper with a summary of results and some indications of further work.

II. PERELOMOV SU(1,1) COHERENT STATES

We begin by reviewing the relevant unitary irreducible representations (UIRs) of SU(1,1) and the associated Perelomov coherent states. We shall be concerned mainly with the su(1,1) Lie algebra which consists of the elements (operators) K_3 and $K_\pm$ satisfying the commutation relations

$$[K_3,K_\pm] = \pm K_\pm, \quad [K_+,K_-] = -2K_3. \tag{2}$$

The operator K_3 generates compact SU(1,1) transformations of the elliptic class [16] whereas the combinations $K_1 = (K_+ + K_-)/2$ and $K_2 = (K_+ - K_-)/2i$ are generators of noncompact SU(1,1) transformations of the hyperbolic class [16]. The su(1,1) Lie algebra in terms of K_1, K_2, and K_3 is

$$[K_1,K_2] = -iK_3, \quad [K_2,K_3] = iK_1, \quad [K_3,K_1] = iK_2. \tag{3}$$

The Casimir operator is

$$C = K_3^2 - K_1^2 - K_2^3 = K_3^2 - \tfrac{1}{2}(K_+K_- + K_-K_+). \tag{4}$$

The relevant unitary irreducible representations are the positive discrete series $D^k : \{|k,m\rangle, k > 0, m = 0,1,2\cdots\}$ satisfying the relations

$$C|k,m\rangle = k(k-1)|k,m\rangle, \tag{5}$$

$$K_3|k,m\rangle = (m+k)|k,m\rangle, \tag{6}$$

$$K_+|k,m\rangle = [(m+1)(m+2k)]^{1/2}|k,m+1\rangle, \tag{7}$$

$$K_-|k,m\rangle = [m(m+2k-1)]^{1/2}|k,m-1\rangle. \tag{8}$$

The states $|k,m\rangle$ are generated from the "ground" state $|k,0\rangle$ according to

$$|k,m\rangle = \left[\frac{\Gamma(2k)}{m!\Gamma(2k+m)}\right]^{1/2}(K_+)^m|k,0\rangle. \tag{9}$$

The SU(1,1) Perelomov coherent state is defined in analogy with the displaced vacuum definition of the ordinary harmonic oscillator coherent state by applying to the ground state the SU(1,1) "displacement" operator

$$S(z) = \exp(zK_+ - z^*K_-), \quad z = -\frac{r}{2}e^{-i\phi}, \tag{10}$$

where r and ϕ are group parameters having ranges $0 < r < \infty$ and $0 \leqslant \phi \leqslant 2\pi$. The Perelomov coherent states are thus given by

$$\begin{aligned} |\xi,k\rangle &= S(z)|k,0\rangle, \\ &= (1-|\xi|^2)^k \sum_{m=0}^{\infty}\left[\frac{\Gamma(2k+m)}{m!\Gamma(2k)}\right]^{1/2}\xi^m|k,m\rangle, \end{aligned} \tag{11}$$

where $\xi = -e^{-i\phi}\tanh(r/2)$. Note that the parameter $|\xi|$ is within the unit circle on the complex plane: $0 \leqslant |\xi| < 1$.

Important physical realizations of the SU(1,1) coherent states are as follows. One is that of the single-mode squeezed vacuum and squeezed one-photon states. The elements of the Lie algebra are realized in terms of a single set of bosonic annihilation and creation operators according to

$$K_+ = \tfrac{1}{2}a^{\dagger 2}, \quad K_- = \tfrac{1}{2}a^2, \quad K_3 = \tfrac{1}{2}\left(a^\dagger a + \tfrac{1}{2}\right), \tag{12}$$

for which the Casimir operator becomes $C = -\frac{3}{16}$ indicating possible Bargmann indices $k = \frac{1}{4}$ and $\frac{3}{4}$. The usual boson number states $\{|n\rangle; n = 0,1,2\ldots\}$ map onto the UIRs of SU(1,1) with the above given Casimir operator according to

$$|n\rangle \Leftrightarrow |k,m\rangle \quad \text{for} \quad n = 2(m+k) - 1/2. \tag{13}$$

Thus for $k = \frac{1}{4}$ we have $n = 2m$ so that only the even-photon-number states are mapped onto this UIR of SU(1,1), and for $k = \frac{3}{4}$ we have the odd-number states with $n = 2m + 1$ mapping onto a different UIR. Note that the "ground" states for the respective UIRs correspond to the number states $|\frac{1}{4},0\rangle = |0\rangle$ and $|\frac{3}{4},0\rangle = |1\rangle$. The corresponding Perelomov coherent states given in terms of photon number states are

$$\left|\xi,\frac{1}{4}\right\rangle = (1-|\xi|^2)^{1/4}\sum_{m=0}^{\infty}\left[\frac{\Gamma(m+1/2)}{m!\Gamma(1/2)}\right]^{1/2}\xi^m|2m\rangle, \tag{14}$$

which is the squeezed vacuum state [16], while

$$\left|\xi,\frac{3}{4}\right\rangle = (1-|\xi|^2)^{3/4}\sum_{m=0}^{\infty}\left[\frac{\Gamma(m+3/2)}{m!\Gamma(3/2)}\right]^{1/2}\xi^m|2m+1\rangle \tag{15}$$

is the squeezed one-photon state [16]. Note that in the former only the even-photon-number states are populated while in the later only the odd are populated.

Another important physical realization is the two-mode realization of the su(1,1) Lie algebra given by [8]

$$K_+ = a^\dagger b^\dagger, \quad K_- = ab, \quad K_3 = \tfrac{1}{2}(a^\dagger a + b^\dagger b + 1), \tag{16}$$

for which the Casimir operator is

$$C = \tfrac{1}{4}[(a^\dagger a - b^\dagger b)^2 - 1]. \tag{17}$$

Here $(a,a^\dagger)$ and $(b,b^\dagger)$ are sets of boson operators. Denoting the eigenvalue of $a^\dagger a - b^\dagger b$ as q and where without loss of generality we can take q as a non-negative integer, one can show that $k = (1+q)/2$. The SU(1,1) basis maps onto the product of photon number states according to $|(1+q)/2,m\rangle = |m+q\rangle_a \otimes |m\rangle_b$. The corresponding Perelomov coherent state is [17]

$$|\xi,(1+q)/2\rangle = (1-|\xi|^2)^{(1+q)/2}\sum_{m=0}^{\infty}\left[\frac{(m+q)!}{m!q!}\right]^{1/2} \times \xi^m|m+q\rangle_a \otimes |m\rangle_b. \tag{18}$$

In the case $q = 0$ we have

$$|\xi,1/2\rangle = (1-|\xi|^2)^{1/2}\sum_{m=0}^{\infty}\xi^m|m\rangle_a \otimes |m\rangle_b, \tag{19}$$

which is the two-mode squeezed vacuum state [16,17]. Note the pairwise correlations between the number states of the two modes, these being the responsible for the entanglement inherent in the state. Gilles and Knight [18] also studied the states of Eq. (18) and went on to introduce and study the nonclassical properties of the two-mode squeezed twin-Fock state $|q\rangle_a \otimes |q\rangle_b$. Strictly speaking, though, the resulting squeezed twin-Fock states are not Perelomov SU(1,1) coherent states as they cannot be expressed in the form of Eq. (11).

III. ENTANGLED PERELOMOV SU(1,1) COHERENT STATES AND VIOLATIONS OF A BELL INEQUALITY

We assume now that we have two generic sets of SU(1,1) systems shared between Alice and Bob where Alice can perform SU(1,1) transformations with generators $\{K_1^A,K_2^A,K_3^A\}$ and Bob can perform transformations with generator $\{K_1^B,K_2^B,K_3^B\}$. Following earlier work on entangled SU(2) coherent states [11], we consider entangled SU(1,1) Perelomov coherent states as given by

$$|\Psi\rangle = \mathcal{N}(|\xi_A,k_A\rangle_A \otimes |-\xi_B,k_B\rangle_B + e^{i\Phi}|-\xi_A,k_A\rangle_A \otimes |\xi_B,k_B\rangle_B), \tag{20}$$

where the subscripts A and B stand for the components of the system possessed by Alice and Bob, respectively, and where the normalization factor $\mathcal{N}$ is given by

$$\mathcal{N} \equiv \frac{1}{\sqrt{2}}\left[1 + \cos\Phi\left(\frac{1-|\xi_A|^2}{1+|\xi_A|^2}\right)^{2k_A}\left(\frac{1-|\xi_B|^2}{1+|\xi_B|^2}\right)^{2k_B}\right]^{-1/2}. \tag{21}$$

In terms of the SU(1,1) bases the above state can be written as

$$|\Psi\rangle = \sum_{m_A}^{\infty}\sum_{m_B}^{\infty}B_{m_A m_B}^{k_A k_B}|k_A,m_A\rangle_A \otimes |k_B,m_B\rangle_B, \tag{22}$$

where

$$B_{m_A m_B}^{k_A k_B} \equiv \mathcal{N}(1-|\xi_A|^2)^{k_A}(1-|\xi_B|^2)^{k_B} \times \left\{\frac{\Gamma(m_A+2k_A)\Gamma(m_B+2k_B)}{m_A!m_B!\Gamma(2k_A)\Gamma(2k_A)}\right\}^{1/2}\xi_A^{m_A}\xi_B^{m_B} \times [(-1)^{m_B} + e^{i\Phi}(-1)^{m_A}]. \tag{23}$$

We now assume that Alice and Bob each perform SU(1,1) transformations on the components of the state of Eq. (22) in his or her possession. In the case of SU(2) states, all the transformations would be rotations which are compact transformations. In the SU(1,1) case, only those transformations generated by K_3 are compact. But, our state of Eq. (22) is decomposed into eigenkets of the two K_3 operators so that only trivial phase factors for the coherent state amplitudes are produced. On the other hand, the operators K_1 and K_2 generate noncompact SU(1,1) transformations and in that sense are analogous to the transformations produced by the usual displacement operator, $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ where α is an unrestricted complex number. For the purposes of this paper we choose transformations generated by the K_1 operators. The transformed state is then

$$|\Psi'\rangle = \exp\left(-i\theta_A K_1^A\right)\exp\left(-i\theta_B K_1^B\right)|\Psi\rangle, \tag{24}$$

where θ_A and θ_B are hyperbolic angles, $-\infty < \theta_{A,B} < \infty$. Our observable will be the dichotomic SU(1,1) parity operator for the positive discrete UIRs of Bargmann index k as given by

$$\Pi^{(k)} = \exp[i\pi(k - K_3)] = \sum_{m=0}^{\infty} |k,m\rangle(-1)^m\langle k,m|, \quad (25)$$

such that $\Pi^{(k)}|k,m\rangle = (-1)^m|k,m\rangle$. Notice, by the way, that $\Pi^{(k)}|\xi,k\rangle = |-\xi,k\rangle$ which means that the components of the state of Eq. (20) are entangled with respect to parity. Further notice that we define the correlation function [25]

$$\begin{aligned} C(\theta_A,\theta_B) &= \langle\Psi'|\Pi_A^{(k_A)} \otimes \Pi_B^{(k_B)}|\Psi'\rangle \\ &= e^{-i\pi(k_A+k_B)}\langle\Psi|e^{i\pi(K_3^A\cosh\theta_A + K_2^A\sinh\theta_A)} \\ &\quad \times e^{i\pi(K_3^B\cosh\theta_B + K_2^B\sinh\theta_B)}|\Psi\rangle, \end{aligned} \quad (26)$$

where we have used the Baker-Hausdorff relation

$$e^{i\theta K_1}K_3e^{-i\theta K_1} = K_3\cosh\theta + K_2\sinh\theta. \quad (27)$$

The operator

$$V(g) = \exp[i\pi(K_3\cosh\theta + K_2\sinh\theta)] \quad (28)$$

corresponds to an SU(1,1) group element g which can be determined from the fundamental (nonunitary) 2×2 representation of the Lie algebra given by $K_1 = i\sigma_2/2, K_2 = -i\sigma_1/2$, and $K_3 = \sigma_3/2$ where the $\{\sigma_i, i = 1,2,3\}$ are the usual Pauli matrices. In the fundamental 2×2 representation the above group element is given by [26]

$$\begin{aligned} V(g)_{2\times 2} &= \exp[i\pi(K_3\cosh\theta + K_2\sinh\theta)]|_{2\times 2} \\ &= \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \end{aligned} \quad (29)$$

where $\alpha = i\cosh\theta$ and $\beta = \sinh\theta$. The matrix on the right-hand side of the previous equation has the general form of an SU(1,1) element if the relation $|\alpha|^2 - |\beta|^2 = 1$ holds, which is clearly satisfied by the specific case here. Using the expansion in Eq. (22) we find that

$$\begin{aligned} C(\theta_A,\theta_B) = e^{-i\pi(k_A+k_B)} &\sum_{m'_A=0}^{\infty}\sum_{m'_B=0}^{\infty}\sum_{m_A=0}^{\infty}\sum_{m_B=0}^{\infty}\left(B^{k_Ak_B}_{m'_Am'_B}\right)^* \\ &\times B^{k_Ak_B}_{m_Am_B}V^{(k_A)}_{m'_A,m_A}(\alpha_A,\beta_B)V^{(k_B)}_{m'_B,m_B}(\alpha_B,\beta_B), \end{aligned} \quad (30)$$

where the V functions are the so-called Bargmann functions [16] corresponding to the above SU(1,1) group element and are defined by

$$V^{(k)}_{m',m}(\alpha,\beta) \equiv \langle k,m'|\exp[i\pi(K_3\cosh\theta + K_2\sinh\theta)]|k,m\rangle, \quad (31)$$

and are given explicitly by

$$\begin{aligned} V^{(k)}_{m',m}(\alpha,\beta) = &\frac{1}{\Gamma(1+m'-m)}\left[\frac{\Gamma(m'+1)\Gamma(m'+2k)}{\Gamma(m+1)\Gamma(m+2k)}\right]^{1/2} \\ &\times(\alpha^*)^{-m'-m-2k}\beta^{m'-m}{}_2F_1 \\ &\times(-m,1-m-2k,1+m'-m;-|\beta|^2), \end{aligned} \quad (32)$$

for $m' \leqslant m$ and for $m' \geqslant m$ by $V^{(k)}_{m',m}(\alpha,\beta) = V^{(k)}_{m,m'}(\alpha,-\beta^*)$.

IV. VIOLATION OF THE BELL-CHSH INEQUALITY

The Clauser-Horne-Shimony-Holt (CHSH) [13] form of Bell's theorem is as follows: If Alice performs measurements with her detector set at hyperbolic angles θ_1 and θ_2 and Bob sets his detector angles at θ_3 and θ_4, then for

$$S(\theta_1,\theta_2,\theta_3,\theta_4) = C(\theta_1,\theta_3) + C(\theta_1,\theta_4) + C(\theta_2,\theta_3) - C(\theta_2,\theta_4) \quad (33)$$

we have $|S| \leqslant 2$ for realistic local hidden-variable theories whereas quantum mechanics may violate the inequality in the range $2 < |S| \leqslant 2\sqrt{2}$, where $S = 2\sqrt{2}$ is Csirel'son's bound [13,14]. Violations of the Bell-CHSH inequality are possible to find only through numerical methods. We performed searches over the variables $\theta_1,\theta_2,\theta_3$, and θ_4 to maximize $|S|$ for various values of the parameters ξ_A,ξ_B,k_A, and k_B characterizing the states. These searches were performed using the MATHEMATICA algorithm called FindMaximum [27].

Roughly speaking, we find violations occurring for parameters θ_i in the range $0 < |\theta_i| \leqslant 0.200$, where θ_i can be any of the previously mentioned transformation parameters. In Table I we list the maximal values attained for the function $|S(\theta_1,\theta_2,\theta_3,\theta_4)|$ for a selection of state parameters along with corresponding maximizing values of the θ parameters. Part (a) of Table I contains results for the standard physically relevant values of the Bargmann indices, $k = \frac{1}{4},\frac{3}{4},\ \frac{1}{2}$ and in various combinations. In part (b) we show a sampling of the larger violations that are possible with other choices of the Bargmann indices. It is clear that robust violations of the SU(1,1) parity based Bell-CHSH inequality occur for a wide range of state parameters characterizing entangled SU(1,1) Perelomov coherent states of the form given in Eq. (20). We note that because the exchanges $k_A \leftrightarrow k_B$ and $r_A \leftrightarrow r_B$ preserve the value of $|S|_{\max}$ for $(\theta_1,\theta_2) \leftrightarrow (\theta_3,\theta_4)$, there is no need to include the "symmetric" results.

As just mentioned, it is clear that substantial violations occur for entanglements between SU(1,1) coherent states having different Bargmann indices. A similar situation occurred for entangled SU(2) coherent states of different j values [11] where maximal violations of the inequality were found for arbitrary spin values of the two-component systems for specific values of the coherent state parameters.

To further illustrate the nature of the Bell-CHSH violation region, we plot in Fig. 1 the surface $|S(-\theta,\varphi,\theta,-\varphi)|$ for the the SU(1,1) entangled coherent state having $k_A = k_B = 2$, $r_A = r_B = 0.6$, and $\Phi = 0$, where $r_i \equiv |\xi_i|$ for $i = A,B$. The constraints $-\theta_1 = \theta_3 \equiv \theta$ and $\theta_2 = -\theta_4 \equiv \varphi$ ensure that the apex of this surface passes through the point at which $|S(-\theta,\varphi,\theta,-\varphi)| \to |S|_{\max} \approx 2.418$ (see the emboldened entry in Table II). In Fig. 2 we plot $|S|$ along the line $\varphi = (4.0410)\theta$ for the same state parameters as in Fig. 1. This line in the θ,φ plane is the one passing through the origin and the point at which $|S(-\theta,\varphi,\theta,-\varphi)| \to |S|_{\max} \approx 2.418$. The plots shown in Figs. 1 and 2 are representative of analogous plots for all of the cases in which we have found Bell-CHSH violations.

In Table II we examine the dependence of Bell-CHSH violations on the relative phase angle, Φ, for a state having remaining parameters equal to those for the state considered in Figs. 1 and 2. We note the small but nontrivial variation of

TABLE I. Representative violations of the Bell-CHSH Inequality by entangled SU(1,1) Perelomov coherent states. For convenience the table is organized into two pieces. Part (a) shows representative Bell-CHSH violations by states for which well-known physical realizations of the implicated representations of SU(1,1) occur, as described in the narrative. Part (b) shows representative larger Bell-CHSH violations for other combinations of the Bargmann indices involved in the entangled state.

(a) Representative Bell-CHSH for entangled states realizable by one- and two-mode Bosonic systems.

k_A	k_B	$r_A = \|\xi_A\|$	$r_B = \|\xi_B\|$	θ_1	θ_2	θ_3	θ_4	$\|S\|_{max}$
$\frac{1}{4}$	$\frac{1}{4}$	0.8	0.8	0.0199	−0.1896	−0.0199	0.1896	2.055
$\frac{1}{4}$	$\frac{1}{2}$	0.6	0.6	0.0559	−0.3662	−0.0244	0.2876	2.059
$\frac{1}{4}$	$\frac{1}{2}$	0.6	0.8	0.0658	−0.3642	−0.0114	0.1399	2.077
$\frac{1}{4}$	$\frac{1}{2}$	0.8	0.8	0.0320	−0.1851	−0.0134	0.1391	2.094
$\frac{1}{4}$	$\frac{3}{4}$	0.6	0.6	0.0732	−0.3599	−0.0186	0.2374	2.081
$\frac{1}{4}$	$\frac{3}{4}$	0.6	0.8	0.0847	−0.3560	−0.0083	0.1122	2.102
$\frac{1}{4}$	$\frac{3}{4}$	0.8	0.8	0.0407	−0.1807	−0.0098	0.1114	2.122
$\frac{1}{2}$	$\frac{1}{2}$	0.6	0.6	−0.0406	0.2835	0.0406	−0.2835	2.102
$\frac{1}{2}$	$\frac{1}{2}$	0.6	0.8	0.0472	−0.2814	−0.0188	0.1374	2.128
$\frac{1}{2}$	$\frac{1}{2}$	0.8	0.8	0.0217	−0.1360	−0.0217	0.1360	2.152
$\frac{1}{2}$	$\frac{3}{4}$	0.6	0.6	0.0531	−0.2783	−0.0310	0.2337	2.136
$\frac{1}{2}$	$\frac{3}{4}$	0.6	0.8	0.0607	−0.2755	−0.0138	0.1101	2.166
$\frac{1}{2}$	$\frac{3}{4}$	0.8	0.8	0.0278	−0.1328	−0.0159	0.1091	2.191
$\frac{3}{4}$	$\frac{3}{4}$	0.6	0.6	0.0406	−0.2295	−0.0406	0.2295	2.177
$\frac{3}{4}$	$\frac{3}{4}$	0.6	0.8	0.0462	−0.2271	−0.0181	0.1079	2.209
$\frac{3}{4}$	$\frac{3}{4}$	0.8	0.8	−0.0205	0.1067	0.0205	−0.1067	2.235

(b) Representative larger Bell-CHSH for entangled states involving general choices of Bargmann indices. For computational convenience, we set $r_A = |\xi_A| = r_B = |\xi_B| = 0.6$ in this part of the table. The entry in bold corresponds to the zenith point in each of the plots shown in Figs. 1 and 2.

k_A	k_B	$r_A = \|\xi_A\|$	$r_B = \|\xi_B\|$	θ_1	θ_2	θ_3	θ_4	$\|S\|_{max}$
$\frac{1}{2}$	$\frac{3}{2}$	0.6	0.6	0.0794	−0.2640	−0.0164	0.1569	2.203
$\frac{1}{2}$	2	0.6	0.6	0.0915	−0.2565	−0.0118	0.1298	2.229
1	1	0.6	0.6	−0.0385	0.1929	0.0385	−0.1929	2.244
1	$\frac{3}{2}$	0.6	0.6	0.0488	−0.1865	−0.0262	0.1515	2.294
1	2	0.6	0.6	0.0569	−0.1812	−0.0191	0.1256	2.326
$\frac{3}{2}$	$\frac{3}{2}$	0.6	0.6	−0.0336	0.1466	0.0336	−0.1466	2.347
$\frac{3}{2}$	2	0.6	0.6	0.0395	−0.1425	−0.0247	0.1218	2.382
2	**2**	**0.6**	**0.6**	**–0.0293**	**0.1184**	**0.0293**	**–0.1184**	**2.418**

the maximum value reached for the absolute value Bell-CHSH function, $|S|$, as we vary the relative phase angle over the range $0 \leqslant \Phi \leqslant \pi$. Once again, the results presented in Table II are indicative of those for analogous variations of Φ for all of the cases that we have examined. Wang *et al.* [24] treated only cases for which $\Phi = \frac{\pi}{2}$.

As for physical representations of the entangled SU(1,1) coherent states, several possibilities are apparent. If we restrict the Bargmann indices to $k = \frac{1}{4}, \frac{3}{4}$ then we have entanglements of squeezed vacuum or squeezed one-photon states, or entanglements of a single-mode squeezed vacuum and squeezed one-photon states. For example, the state

$$|\Psi\rangle = N\big(\big|\xi_A, \tfrac{1}{4}\big\rangle \otimes \big|-\xi_B, \tfrac{3}{4}\big\rangle + \big|-\xi_A, \tfrac{1}{4}\big\rangle \otimes \big|\xi_B, \tfrac{3}{4}\big\rangle\big) \quad (34)$$

is a two-mode entangled state wherein the A mode has only even photon numbers and the B mode has only odd numbers.

On the other hand, for the cases where the two Bargmann indices are of the form $k = (1+q)/2,\ q = 0,1,2,\ldots,$ we have the situation where two two-mode Perelomov SU(1,1) coherent states are entangled, meaning that four modes in total are entangled. In the case where both Bargmann indices take the value $k = 1/2$, we have the state

$$|\Psi\rangle = N\big(\big|\xi_A, \tfrac{1}{2}\big\rangle \otimes \big|-\xi_B, \tfrac{1}{2}\big\rangle + \big|-\xi_A, \tfrac{1}{2}\big\rangle \otimes \big|\xi_B, \tfrac{1}{2}\big\rangle\big), \quad (35)$$

which, to be clear, constitutes a four-mode entangled state composed of two-mode squeezed vacuum states. Finally, we can, of course, consider the case of entanglement between single-mode and two-mode SU(1,1) coherent states. For example, we can consider the entanglement between single- and

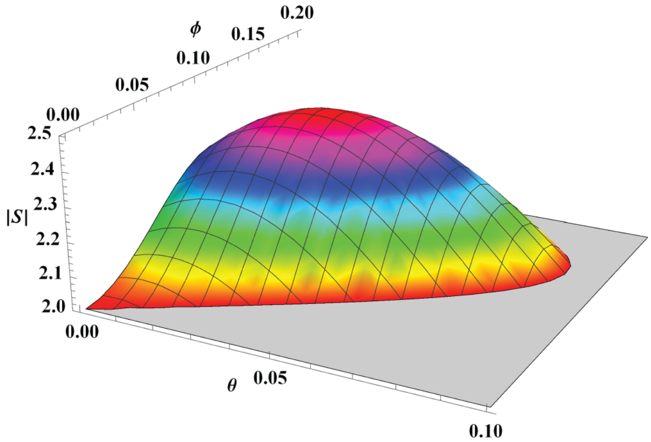

FIG. 1. Surface plot of the violation region for an entangled SU(1,1) Perelomov coherent state having $k_A = k_B = 2$, $\xi_A = \xi_B = 0.6$, and $\Phi = 0$. This plot agrees with numerical results presented in Table I(b), and it is qualitatively representative of all of the Bell-CHSH violations we have found for this system.

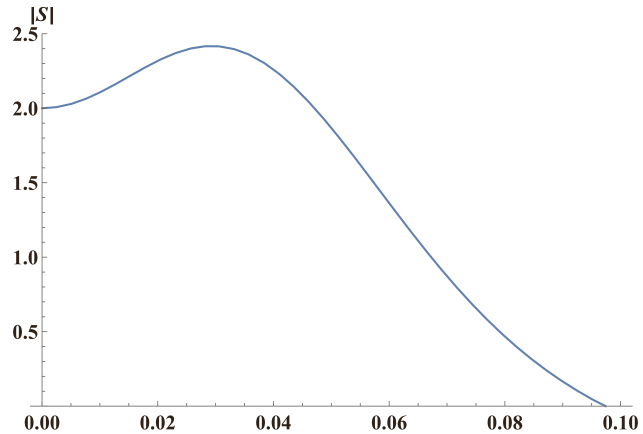

FIG. 2. The altitude above the θ-φ plane of a cut through the surface shown in Fig. 1 along the line $\varphi = (4.0410)\,\theta$. The maximal violation of the Bell-CHSH inequality for the state parameters specified in Fig. 1 occurs along this line. The shape of this curve is also indicative of the shapes of similar curves along similar cut lines within each of the violation regions we have investigated.

two-mode squeezed vacuum states such that three modes in total are entangled such as for the state

$$|\Psi\rangle = N\left(\left|\xi_A,\tfrac{1}{4}\right\rangle \otimes \left|-\xi_B,\tfrac{1}{2}\right\rangle + \left|-\xi_A,\tfrac{1}{4}\right\rangle \otimes \left|\xi_B,\tfrac{1}{2}\right\rangle\right). \quad (36)$$

This case will display a continuous variable form of the phenomenon known as *entangled entanglement* in the sense that that term is used by Walther *et al.* [28]. First, note that there are perfect correlations between the state of the field mode possessed by Alice and of the two-mode state possessed by Bob which imply that the entangled states of the two modes possessed by Bob are elements of reality in the sense of Einstein, Podolsky, and Rosen [29]. Second, notice that neither of the modes in the two-mode squeezed vacuum state will have well defined states individually. Thus the state of Bob's two field modes taken together is an element of reality but the states of the individual modes are not.

TABLE II. Results demonstrating the variation in $|S|_{\max}$ with respect to the relative phase angle Φ for the SU(1,1) entangled Perelomov coherent state with $k_A = k_B = 2$ and $r_A = r_B = 0.6$; the emboldened entry here corresponds with the emboldened entry in Table I(b). Here again we use the notation $r_i = |\xi_i|$, and we have taken ξ_i to be real, where $i = A,B$.

Φ	θ_1	θ_2	θ_3	θ_4	$\lvert S\rvert_{\max}$
0	**−0.0293**	**0.1184**	**0.0293**	**−0.1184**	**2.418**
$\pi/16$	−0.0418	0.1087	0.0418	−0.1087	2.437
$\pi/8$	−0.0544	0.0990	0.0544	−0.0990	2.446
$3\pi/16$	−0.0671	0.0895	0.0671	−0.0895	2.445
$\pi/4$	−0.0800	0.0800	0.0800	−0.0800	2.433
$5\pi/16$	−0.0931	0.0706	0.0931	−0.0706	2.411
$3\pi/8$	−0.1064	0.0612	0.1064	−0.0612	2.379
$7\pi/16$	−0.1200	0.0519	0.1200	−0.0519	2.338
$\pi/2$	−0.1339	0.0427	0.1339	−0.0427	2.289
$9\pi/16$	0.1201	−0.0519	−0.1201	0.0519	2.331
$5\pi/8$	0.1064	−0.0612	−0.1064	0.0612	2.374
$11\pi/16$	0.0931	−0.0706	−0.0931	0.0706	2.408
$3\pi/4$	0.0800	−0.0800	−0.0800	0.0800	2.432
$13\pi/16$	0.0671	−0.0895	−0.0671	0.0895	2.445
$7\pi/8$	0.0544	−0.0991	−0.0544	0.0991	2.448
$15\pi/16$	0.0418	−0.1087	−0.0418	0.1087	2.439
π	0.0293	−0.1184	−0.0293	0.1184	2.420

Note that in the state of Eq. (35), both Alice and Bob each possess two field modes where each of the states of those respective pairs of modes are elements of reality, but where none of the states of the individual field modes are. Thus the states in both (35) and (36) are examples of continuous-variable states possessing the property of entangled entanglement. The implications of these continuous-variable states will be explored elsewhere.

Finally we mention that higher values of the Bargmann indices for SU(1,1) coherent states may be possible by the pairing of bosonic modes through Feshbach resonances as discussed in Ref. [21] in the case of cold atoms, or by coupling together of SU(1,1) coherent states through the SU(1,1) Clebsch-Gordan coefficients to obtain a new set of SU(1,1) coherent states of compounded Bargmann indices [19,20,30].

V. GENERATION OF THE ENTANGLED SU(1,1) COHERENT STATES

In this section we briefly discuss possible schemes by which the entangled SU(1,1) coherent states could be generated. A number of schemes have been proposed for the generation of entangled states, some of which can be adapted for the states considered in this paper. For example, Gerry and Campos [31] studied a quantum-optical Fredkin gate [32], which amounts to a condition beam splitter, as a means for generating maximally entangled photonic states such as the (un-normalized) N00N states $|N\rangle|0\rangle + |0\rangle|N\rangle$ and entangled coherent-and-vacuum states $|\alpha\rangle|0\rangle + |0\rangle|\alpha\rangle$ for the purpose of high-sensitivity quantum-optical interferometry [33]. An earlier proposal by Sanders and Rice [34] features a nonlinear interferometer with a self-Kerr interaction in one arm as a means of generating entangled coherent states, a special case of which is a coherent state entangled with a vacuum state. This method was further discussed in connection to quantum interferometry by Gerry *et al.* [35]. Finally, Gerry and Grobe [36] discussed the nonlocal entanglement of coherent states by using a Mach-Zehnder interferometer with cross-Kerr media in each arm where the modes to be entangled never meet. Either of these schemes could be used to generate the entangled SU(1,1) coherent states assuming the availability of large enough Kerr nonlinearities, but the third method is the easiest to visualize, and it has the further advantage that the spatial locations of the field modes initially containing the states to be entangled can be far apart.

In brief, the third method goes as follows, the details of which can be found in Ref. [36]: We suppose a Mach-Zehnder interferometer with internal beam modes labeled c and d, with 50:50 beam splitters at the input and output, and with cross-Kerr media in both arms coupling each of the two interferometer modes (paths) to external modes labeled a and b each containing an SU(1,1) coherent state. The cross-Kerr interactions are of the forms $\hbar\chi a^\dagger a d^\dagger d$ and $\hbar\chi b^\dagger b c^\dagger c$ where χ is proportional to a third-order nonlinear susceptibility. In the case of entangling two two-mode SU(1,1) coherent states there will be two additional modes needed, one for each of the two-mode SU(1,1) coherent states, say the e and f modes, though these modes are not coupled to the interferometer modes or any other modes via the cross-Kerr interactions. Because of built-in correlations, only one of the modes of each of the two-mode SU(1,1) coherent states (a and b) need be coupled to the cross-Kerr media. We assume the a (and e) and b (and f) beams contain the states $|\xi_A,k_B\rangle_a$ and $|\xi_B,k_B\rangle_b$, respectively, and we assume a single photon is injected into the first beam splitter. Then the combined output state after the second beam splitter is (see Eq. (5) and Fig. 1 of Ref. [36])

$$\begin{aligned}|\text{out}\rangle = \tfrac{1}{2}[(e^{i\theta}|\xi_A,k_A\rangle_a \otimes |\xi_B e^{-i\varphi_b},k_B\rangle_b - |\xi_A e^{-i\varphi_a},k_A\rangle_a\\ \otimes|\xi_B,k_B\rangle_b)|1\rangle_c|0\rangle_d + i(e^{i\theta}|\xi_A,k_A\rangle_a\\ \otimes|\xi_B e^{-i\varphi_b},k_B\rangle_b + |\xi_A e^{-i\varphi_a},k_A\rangle_a\\ \otimes|\xi_B,k_B\rangle_b)|0\rangle_c|1\rangle_d],\end{aligned} \tag{37}$$

where $\varphi_{a,b} = \chi\tau_{a,b}$, the $\tau_{a,b}$ being the interaction times, and where the phase θ comes from the phase shift operation in the c mode, $\exp(i\theta c^\dagger c)$. The detection of the $|1\rangle_c|0\rangle_d$ or the $|0\rangle_c|1\rangle_d$ states in the c and d modes projects the a and b modes into the states

$$\begin{aligned}|\Phi_\mp\rangle = N_\mp(e^{i\theta}|\xi_A,k_A\rangle_a \otimes |\xi_B e^{-i\varphi_b},k_B\rangle_b\\ \mp |\xi_A e^{-i\varphi_a},k_A\rangle_a \otimes |\xi_B,k_B\rangle_b),\end{aligned} \tag{38}$$

respectively, where the $N_\mp$ are normalization factors. It is evident that for the appropriate choices of the angles θ and $\varphi_{a,b}$, particularly with $\varphi_{a,b} = \pi$, we can recover the states of Eq. (20).

Finally in this section we mention that a major stumbling block to schemes of the type discussed here, and which also occur in the realizations of quantum information processing devices such as the Fredkin gate, is the lack of large cross-Kerr nonlinearities to effectuate a phase shift as large as π. For recent progress on this front see the papers by Tiarks *et al.* [37] and Boddeda *et al.* [38].

VI. CONCLUSIONS

In this paper, we have extended previous work on the violation of Bell inequalities by entangled ordinary coherent states and entangled SU(2) coherent states to the case of entangled SU(1,1) coherent states. The SU(1,1) coherent states are, of course, continuous-variable states that represent many types of states of physical importance, particularly the one- and two-mode squeezed states of quantum optics. The importance of coherent states in general is that they can be mesoscopically or even macroscopically occupied by the adjustment of a single parameter. In analogy to the case of the entangled SU(2) coherent states [11], we have used as the observables the parity operators associated with the relevant unitary irreducible representations of SU(1,1) for the positive discrete series, the only representations considered here. Unlike the SU(2) case, however, we have not been able to find instances where the Bell inequality is maximally violated. This might be because we have examined cases where the Bargmann indices are relatively low, as fits for most of the known useful representations. We note from Tables I(b) and II that for Bargmann indices set at $k = 2$ we already see increasing levels of violation of the Bell inequality.

Finally, because the SU(1,1) coherent states themselves may represent systems with strong entanglement, our entangled SU(1,1) coherent states could possess the property of continuous-variable entangled entanglement. The consequences and applications of this feature will be studied elsewhere.

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