Geometric phases and the Rabi Hamiltonian

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We address Berry phases that have been predicted to appear when a two-level system interacts with a quantized field, including the vacuum state. The issue became controversial after it was argued that the appearance of such Berry phases is an artifact of the rotating-wave approximation (RWA). As is widely believed, whenever the RWA applies, one may replace the Rabi model of a two-level system interacting with a quantized field by the analytically solvable Jaynes-Cummings model. Conflicting predictions of these two models under conditions that validate the RWA would signal a serious inconsistency of this approximation. We show that this is not the case and that claims to the contrary are inconsistent with analytical results concerning the Rabi model. We provide also numerical evidence supporting our analytical approach. Furthermore, we argue that the appearance of Berry phases in the addressed cases does not depend on adiabatic conditions nor on any particular Hamiltonian, but on the underlying vector space. Thus, the appropriate framework is given by the kinematic approach to geometric phases, which contains Berry's phase as a special case.

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I. INTRODUCTION

The Jaynes-Cummings (JC) Hamiltonian has proved to be a very useful analytically solvable approximation of the Rabi Hamiltonian. The latter governs the dynamics of a two-level atom interacting with a quantized one-mode electromagnetic field. The two Hamiltonians are given, respectively, by

$$H_{\mathrm{JC}} = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g(\sigma_+ a + \sigma_- a^\dagger) \tag{1}$$

and

$$H_R = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g(\sigma_+ + \sigma_-)(a + a^\dagger), \tag{2}$$

in units of $\hbar = 1$. Here, ω_0 is the transition frequency between the two atomic levels; ω is the single-mode frequency of the field that is described in terms of annihilation (a) and creation ($a^\dagger$) operators; and g is the atom-field dipole-coupling constant. According to common wisdom, whenever $\omega \approx \omega_0$ and $g/\omega \ll 1$, the rotating-wave approximation (RWA) applies and $H_R \to H_{\mathrm{JC}}$. Recently, however, Larson made the striking claim that there are instances for which the RWA breaks down at all orders of g/ω [1]. This is allegedly the case when an instantaneous eigenstate of H_R performs an adiabatic evolution. Such an evolution occurs, for instance, when one applies the unitary transformation $U(\varphi) = \exp(-i\varphi a^\dagger a)$. Larson claimed that there should be no Berry phase showing up in this case. If we instead apply the RWA, thereby approximating the eigenstates of H_R by those of H_{JC}, a nonvanishing and nontrivial Berry phase appears even if the field is in its vacuum state, as first predicted by Fuentes-Guridi *et al.* [2]. Larson arrived at his conclusion by first addressing the semiclassical case, i.e., a two-level atom interacting with a classical field. In such a case, the JC model leads to semiclassical energy surfaces in phase space, which contain a so-called conical intersection (CI) [3,4]. Trajectories in phase space that encircle the CI give rise to nonvanishing Berry phases. Working with Rabi's semiclassical Hamiltonian, the energy surfaces do not contain a CI. Hence, no Berry phase shows up. These semiclassical arguments suggest that the same state of affairs would arise in the fully quantized case. By numerical diagonalization of the unitarily transformed Rabi Hamiltonian, $H_R(\varphi) \equiv U(\varphi)H_R U^\dagger(\varphi)$, Larson finds that Berry phases vanish for various low-lying eigenstates and for different parameter choices, including those for which the RWA should apply.

Larson's findings have been disputed by several authors, who presented both numerical and theoretical arguments against them [5–7]. Wang *et al.* [5] object to the way in which Larson performs the semiclassical approximation, i.e., by replacing the operators a and $a^\dagger$ in $H_R(\varphi)$ by the corresponding c numbers α and α^*. They argue that when the correct semiclassical approximation is performed there is a nonvanishing Berry phase also for the Rabi Hamiltonian. Such a correct semiclassical approximation consists of applying the variational method to the effective Hamiltonian $H_v(\alpha) := \langle\alpha|H_R|\alpha\rangle$, where $|\alpha\rangle$ denotes a coherent state, i.e., one that satisfies $a|\alpha\rangle = \alpha|\alpha\rangle$. In that case, a nonvanishing Berry phase appears, both in the JC and in the Rabi case. This Berry phase is given by

$$\gamma_n = i\oint \langle\psi_n|U^\dagger(\varphi)\frac{d}{d\varphi}U(\varphi)|\psi_n\rangle d\varphi = 2\pi\langle\psi_n|a^\dagger a|\psi_n\rangle, \tag{3}$$

with $|\psi_n\rangle$ being the nth eigenstate of the considered Hamiltonian.

In the fully quantized case it is necessary to resort to numerical calculations, because a closed-form, analytical diagonalization of the Rabi Hamiltonian has not been reached to date. Now, while some authors have reported numerical calculations yielding nonvanishing Berry phases, Larson has reported conflicting results, also stemming from numerical calculations [1,8]. According to Larson, these numerical results confirm that γ_n in Eq. (3) exactly vanishes to all orders of g/ω. Even though we are not in a position to say what possibly went wrong in Larson's numerical calculations, we are confident that some flaw must be there, as Larson's results are in conflict with the following analytical argument showing that there is a nonvanishing Berry phase in Rabi's case. Indeed,

we can generally write the eigenstates of H_R in the form

$$|\psi_n\rangle = \sum_{m=0}^{\infty} A_m^n |e,m\rangle + B_m^n |g,m\rangle, \tag{4}$$

where $\{|e,m\rangle,|g,m\rangle\}_{m=0}^{\infty}$ is a basis of the Hilbert space on which H_R acts. Here, $|e,m\rangle \equiv |e\rangle \otimes |m\rangle$ is the atom-field state in which the atom is in its excited state $|e\rangle$ and the field is in a Fock state $|m\rangle$ with m photons, and similarly for the ground state $|g\rangle$ of the atom. Equation (3) yields then

$$\gamma_n = 2\pi \langle \psi_n | a^\dagger a | \psi_n \rangle = 2\pi \sum_{m=1}^{\infty} m\left[\left|A_m^n\right|^2 + \left|B_m^n\right|^2\right]. \tag{5}$$

Hence, γ_n would vanish only if $A_m^n = 0 = B_m^n, \forall m > 0$, which is clearly not the case. Our numerical results (see below) are in agreement with the above equation and also with other authors' findings, but not with Larson's. These numerical results show that the values of the geometric phase are not only nonvanishing but nontrivial as well. Moreover, very recent semianalytical results [9] show that the energy landscape of the full quantum Rabi model does contain conical intersection points. Though geometric phases associated to trajectories encircling these points have not been calculated, they are expected to be nonvanishing [9].

Now, the issue at hand goes beyond the particular result concerning a vanishing or nonvanishing Berry phase. What we are dealing with here is not Berry's but a geometric phase. We recall that Berry's phase is a particular instance of geometric phases. For a long time, the geometric phase has been "recognized to be the Hamiltonian-independent, nonintegrable component of the total phase, depending exclusively on the geometry in the ray space" [10]. Berry's phase corresponds to the special case in which an instantaneous eigenstate of a time-dependent Hamiltonian adiabatically evolves along a closed trajectory. Though Fuentes-Guridi *et al.* [2] first and Larson [1] afterwards referred to an adiabatic evolution, such a condition did not need to be invoked in their actual calculations. On the other hand, the addressed Hamiltonians play no other role than to be the providers of the quantum (eigen)states being addressed. Once the state $|\psi_n\rangle$ is chosen, its associated geometric phase is fixed by the unitary operator $U(\varphi)$ that rules its evolution [see Eq. (3)]. Hence, we could have similarly chosen other states and some other unitary or nonunitary evolution and in this way arrive at results that would be largely independent of any possible Hamiltonian to which those states might be related. Of course, the physical realization of the chosen evolution may require the action of some Hamiltonian, and this action might be only possible under adiabatic conditions. But this does not mean that the geometric phase thereby exhibited is essentially linked to the Hamiltonian or to adiabaticity, as it is linked to the geometry of ray space. The nature of this space, on the other hand, is largely fixed by the family of states being addressed, i.e., by the kind of evolutions being considered. It is thus worthwhile to consider various possible evolutions and states submitted to them, which is the actual issue of our concern here.

II. KINEMATIC APPROACH TO THE GEOMETRIC PHASE

As we said before, Berry's phase is a special instance of geometric phases. Indeed, the conditions under which Berry's phase originally arose are as follows. One considers a system that evolves under the action of a time-dependent Hamiltonian $H(R(t))$ the instantaneous eigenstates of which are $|E_n; R(t)\rangle$. Here, $R = (R^1, R^2, \ldots, R^d)$ denotes the parameters through which H depends on time. It is assumed that the initial state is one of the eigenstates of H, say $|E_n, R(0)\rangle$, and that the evolution is slow enough for the system not to jump into $|E_m, R(t)\rangle$, with $m \neq n$. As the eigenstates are generally defined up to a phase factor, the adiabatic approximation that is implied in the above assumption can be best expressed in terms of projectors [3]:

$$|\psi(t)\rangle\langle\psi(t)| \approx |E_n; R(t)\rangle\langle E_n; R(t)|. \tag{6}$$

Unless $|\psi(t)\rangle$ is a stationary state, condition (6) can hold only as an approximation [3]. Under the above conditions, and considering a cyclic process, $R(T) = R(0)$, Berry's phase is defined as

$$\begin{aligned}\gamma_n &:= i\int_0^T \langle E_n; R(t)| \frac{d}{dt} |E_n; R(t)\rangle dt \\ &= i \oint \langle E_n; R(t)| \frac{\partial}{\partial R^\mu} |E_n; R(t)\rangle dR^\mu \equiv \oint \mathbf{A}_\mu^n(R) dR^\mu,\end{aligned} \tag{7}$$

where $\mathbf{A}^n(R) := i\langle E_n; R|\nabla|E_n; R\rangle$ is the Mead-Berry connection, and a sum over repeated indices is implied. Thus, whereas Eq. (6) holds only under adiabatic conditions, γ_n depends on nothing but the geometry of the underlying vector space. What Berry uncovered by addressing the adiabatic theorem of Born and Fock was a particular manifestation of a more general concept. The essential nature of this concept is a geometric one [11]. This can be best exposed by following a more general setting, which is known as the kinematic approach.

Within the kinematic approach [12], the geometric phase is defined as

$$\gamma(\mathcal{C}) = \arg\langle\psi(0)|\psi(s)\rangle + i\int_0^s \langle\psi(s')| \frac{d}{ds'} |\psi(s')\rangle ds'. \tag{8}$$

Here, $\mathcal{C}$ denotes the path that joins the initial state $|\psi(0)\rangle$ with the final state $|\psi(s)\rangle$. This definition is independent of the dynamics that governs the evolution of the state $|\psi(s')\rangle$, as long as this evolution fixes the path $\mathcal{C} : s' \in [0,s] \to |\psi(s')\rangle$. The geometric phase $\gamma(\mathcal{C})$ is the sum of a total (Pancharatnam [13]) phase $\arg\langle\psi(0)|\psi(s)\rangle$ and a dynamic phase $i\int_0^s \langle\psi(s')|d\psi(s')\rangle$. For closed paths—the ones we focus on here—the total phase is trivial and $\gamma(\mathcal{C})$ reduces to the dynamic part. The s dependence of $|\psi(s)\rangle$ can generally arise when the parameters defining the state $|\psi(\alpha,\beta,\ldots)\rangle$ turn into functions of some parameter s: $\alpha(s),\beta(s),\ldots$. Typical examples of s are time and path length. An important feature of the geometric phase is that it is parameter independent, i.e., it is invariant under $s \to s' = s'(s)$. It is also invariant under local gauge transformations: $|\psi(s)\rangle \to |\psi'(s)\rangle = \exp[i\alpha(s)]|\psi(s)\rangle$. This property allows us to nullify either of the two contributions. The dynamic contribution can be nullified by gauge-transforming $|\psi(s)\rangle \to$

$|\psi'(s)\rangle$ so that $\langle\psi'(s)|d\psi'(s)\rangle = 0$, which is known as the parallel-transport condition [14]. Alternatively, we can nullify the Pancharatnam contribution by gauge-transforming $|\psi(s')\rangle \to |\psi'(s')\rangle = \exp[-i\arg\langle\psi(0)|\psi(s')\rangle]|\psi(s')\rangle$ [15], so that $\arg\langle\psi'(0)|\psi'(s)\rangle = 0$. Using this gauge-transformed state and reverting to unprimed notation, we see that $\gamma(\mathcal{C})$ in Eq. (8) coincides with Eq. (7), whenever $s' = t$, and $|\psi(t)\rangle$ satisfies the (gauge-transformed) Schrödinger equation with the initial condition $|\psi(0)\rangle = |E_n; R(0)\rangle$. If we further assume that the adiabatic approximation (6) holds true, then we can call $\gamma(\mathcal{C})$ a Berry phase, following common parlance. To say that $|\psi(t)\rangle$ satisfies the Schrödinger equation with said initial condition is equivalent to saying that $|\psi(t)\rangle = U_H(t)|E_n; R(0)\rangle$, where $U_H(t)$ is the evolution operator, which can be generally expressed as a Dyson series: $U_H(t) = \mathcal{T}[\exp(-i\int_0^T H(R(t))dt)]$, with $\mathcal{T}$ meaning the time-ordering operator.

As previously said, Fuentes-Guridi *et al.* [2] and Larson [1] considered a transformation of the form $|\psi_n(\varphi)\rangle = U(\varphi)|\psi_n\rangle$, with $U(\varphi) = \exp(-i\varphi a^\dagger a)$. Here, $|\psi_n\rangle$ is an eigenstate of either $H_{\rm JC}$ [see Eq. (1)] or H_R [see Eq. (2)]. Fuentes-Guridi *et al.* [2] assume that "Berry's phase" is given by Eq. (3), i.e., $\gamma_n = i\oint\langle\psi_n(\varphi)|d\psi_n(\varphi)\rangle = 2\pi\langle\psi_n|a^\dagger a|\psi_n\rangle$. They argue that the "phase shift operator" $U(\varphi)$, when "applied adiabatically to the Hamiltonian of the system, is capable of changing the state of the field" [2]. This is because $U(\varphi)H_{\rm JC}U^\dagger(\varphi) = \omega_0\sigma_z/2 + \omega a^\dagger a + g(\sigma_+ a e^{i\varphi} + \sigma_- a^\dagger e^{-i\varphi})$. We notice that the term "Berry's phase" is not properly used here. Indeed, to begin with, $H_{\rm JC}$ is time independent, so that it generates an evolution having no geometric (Berry's) phase factor attached to it. If the initial state is $|\psi_n\rangle$, it accumulates only a trivial phase factor: $|\psi_n(t)\rangle = \exp(-iH_{\rm JC}t)|\psi_n\rangle = \exp(-iE_n t)|\psi_n\rangle$. As we are dealing with a strictly stationary case, $|\psi_n(t)\rangle\langle\psi_n(t)| = |\psi_n\rangle\langle\psi_n|$, there is no Berry's phase. The unitary transformation $H_{\rm JC} \to U(\varphi)H_{\rm JC}U^\dagger(\varphi)$ is assumed to be applied "adiabatically" [1,2]. For this to make sense, i.e., for φ to vary "slowly" [1], φ must be time dependent and so thus $U(\varphi)$. If we submit the system to the time-dependent unitary transformation $U(\varphi(t))$, then the Schrödinger equation for $|\psi'\rangle = U|\psi\rangle$ reads

$$i\frac{\partial|\psi'\rangle}{\partial t} = \left(UH_{\rm JC}U^\dagger + i\frac{\partial U}{\partial t}U^\dagger\right)|\psi'\rangle \equiv H'(t)|\psi'\rangle. \quad (9)$$

A properly defined Berry's phase must be related to the evolution operator that is associated with $H'(t)$. Any phase that is defined in terms of $U(\varphi) = \exp(-i\varphi a^\dagger a)$ [1,2] is not a Berry's phase.

A *geometric* phase can be properly defined through Eq. (8), by setting $s' = \varphi$, $|\psi(\varphi)\rangle = U(\varphi)|\psi_n\rangle$, and considering a closed trajectory or else a gauge for which $\arg\langle\psi(0)|\psi(s = 2\pi)\rangle = 0$. The evolution of the state vector is in this case not assumed to be ruled by the Hamiltonian $H_{\rm JC}$ (or H_R) and for this reason any accumulated phase cannot be dubbed a Berry phase. Furthermore, the Hamiltonian is thereby limited to be just a provider of initial (eigen)states. This is the formulation we adopt here.

The parameters R^μ no longer define the Hamiltonian, but just the states being addressed, i.e., $|\psi\rangle = |\psi(R)\rangle$. Thus, whenever $R^\mu = R^\mu(s)$, we have that $\langle\psi(s)|\dot\psi(s)\rangle = \langle\psi|\partial_\mu\psi\rangle\dot R^\mu(s)$, where $|\dot\psi(s)\rangle$ stands for $d|\psi(s)\rangle/ds$ and $\partial_\mu \equiv \partial/\partial R^\mu$. It follows that, for a closed trajectory or for a gauge that nullifies Pancharatnam's phase,

$$\gamma(\mathcal{C}) = i\oint_{\mathcal{C}}\langle\psi(s)|\dot\psi(s)\rangle ds \equiv -\oint_{\mathcal{C}}\mathcal{A} = -\int_S \mathcal{F}, \quad (10)$$

where $\mathcal{A} \equiv \mathcal{A}_\mu dR^\mu$, with $\mathcal{A}_\mu = \langle\psi|\partial_\mu\psi\rangle$, while $\mathcal{F} \equiv d\mathcal{A} = (\mathcal{F}_{\mu\nu}/2)dR^\mu \wedge dR^\nu$ and S is a surface the boundary of which is the closed path $\mathcal{C}$. We can also write [11,16]

$$\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\nu\mu} = 2\,{\rm Im}\langle\partial_\mu\psi|\partial_\nu\psi\rangle. \quad (11)$$

When the parameters R^μ come from a unitary $U(R)$, i.e., $|\psi(R)\rangle = U(R)|\psi(0)\rangle$, we can alternatively write

$$\mathcal{F}_{\mu\nu} = 2\,{\rm Im}\langle\psi|(\partial_\mu U)^\dagger\partial_\nu U|\psi\rangle. \quad (12)$$

It is instructive to address some concrete examples for which the previous results can be applied. We do this next.

A. Schwinger's approach to angular momentum: I

Let us consider a Hilbert space $\mathcal{H}_T = \mathcal{H}_2 \otimes \mathcal{H}_a \otimes \mathcal{H}_b$, i.e., one which is the direct product of three Hilbert spaces the orthonormal bases of which are, respectively, $\{|+\rangle,|-\rangle\}$, $\{|n\rangle_a\}_{n=0}^\infty$, and $\{|n'\rangle_b\}_{n'=0}^\infty$, the two last ones being Fock bases associated to two different modes. We consider further the unitary transformation

$$U(\theta,\phi) = \exp(-i\phi J_z)\exp(-i\theta J_y), \quad (13)$$

where the angular momentum operators are defined in terms of the annihilation and creation operators of modes a and b, according to Schwinger's approach:

$$J_x = \frac{1}{2}(a^\dagger b + ab^\dagger), \quad J_y = \frac{1}{2i}(a^\dagger b - ab^\dagger),$$
$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b). \quad (14)$$

We will apply $U(\theta,\phi)$—which implicitly contains the identity operator of $\mathcal{H}_2$—to the states

$$|\Psi^\pm_{n,n'}\rangle = \frac{1}{\sqrt{2}}[|+\rangle|n\rangle_a \pm |-\rangle|n+1\rangle_a] \otimes |n'\rangle_b$$
$$\equiv \frac{1}{\sqrt{2}}[|+,n\rangle \pm |-,n+1\rangle]|n'\rangle. \quad (15)$$

The only nonvanishing components of $\mathcal{F}_{\mu\nu}$ [see Eq. (12)] are $\mathcal{F}_{\theta\phi} = -\mathcal{F}_{\phi\theta}$. On account of $\partial U/\partial\theta = -iUJ_y$ and $\partial U/\partial\phi = -iJ_z U$, Eq. (12) reads in this case

$$\mathcal{F}_{\phi\theta} = 2\,{\rm Im}\langle\Psi^\pm_{n,n'}|U^\dagger J_z U J_y|\Psi^\pm_{n,n'}\rangle. \quad (16)$$

A straightforward calculation yields

$$\langle U^\dagger J_z U J_y\rangle = \frac{1}{2i}\sin\theta\langle J_z\rangle, \quad (17)$$

$$J_z|\Psi^\pm_{n,n'}\rangle = \frac{1}{2\sqrt{2}}[(n-n')|+,n\rangle$$
$$\pm(n+1-n')|-,n+1\rangle]|n'\rangle, \quad (18)$$

so that $2\,\mathrm{Im}\langle\Psi^{\pm}_{n,n'}|U^{\dagger}J_zUJ_y|\Psi^{\pm}_{n,n'}\rangle = -\sin\theta[(n-n')/2+1/2]/2$. Inserting this expression in Eq. (10) we get

$$\begin{aligned}\gamma_{nn'} &= -\int_S \mathcal{F} = -\iint_\Omega \mathcal{F}_{\phi\theta}\,d\theta\,d\phi \\ &= \frac{1}{2}\left[(n-n')+\frac{1}{2}\right]\iint_\Omega \sin\theta\,d\theta\,d\phi \\ &= \left[(n-n')+\frac{1}{2}\right]\frac{\Omega}{2}, \end{aligned} \tag{19}$$

where Ω is the solid angle subtended by the closed path $\mathcal{C}$. This result has been previously obtained by Fuentes-Guridi *et al.* [2], who interpreted it in terms of a Hamiltonian H_0^{2q}, which is made of H_{JC} at resonance [see Eq. (1)], plus an additional term $\omega b^{\dagger}b$ that is related to a second field mode, i.e., $H_0^{2q} = H_{\mathrm{JC}}(\omega_0=\omega)+\omega b^{\dagger}b$. Mode b does not interact with the two-level atom at this point. The interaction comes about when instead of H_0^{2q} one considers the transformed Hamiltonian $U(\theta,\phi)H_0^{2q}U^{\dagger}(\theta,\phi)$ [2,17]. Now, as we have seen, neither the Hamiltonian nor the adiabatic condition have been invoked to obtain the geometric phase $\gamma_{nn'}$ of Eq. (19). This phase reflects the geometry of the underlying ray space alone. Reference to the Hamiltonian can sometimes obscure rather than enlighten the physical interpretation of different features that show up in geometric phases. In Ref. [2], for instance, the nonvanishing geometric phase

$$\gamma_0 = \frac{\Omega}{4}, \tag{20}$$

that arises when the field is in a vacuum state, is interpreted as a feature that has "no semiclassical correspondence" [2]. However, as we shall see next, Eq. (20) and other features of the geometric phase can be understood in terms of the underlying ray space, regardless of the quantum or classical nature of the involved phenomena.

B. Schwinger's approach to angular momentum: II

Let us recall how Schwinger's approach connects to standard angular momentum algebra. The standard basis $\{|j,m\rangle, j=0,1/2,1,\ldots; m=-j,\ldots,j\}$ consists of common eigenvectors of the commuting operators $\mathbf{J}^2$ and J_z, i.e., $\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle$ and $J_z|j,m\rangle = m|j,m\rangle$. There is a one-to-one correspondence between Schwinger's states $|n,n'\rangle$ and the standard ones, $|j,m\rangle$, which is given by $n=j+m$, $n'=j-m$. The Hilbert space $\mathcal{H}_T = \mathcal{H}_2\otimes\mathcal{H}_a\otimes\mathcal{H}_b$ that we considered before can also be defined as (is isomorph to) $\mathcal{H}_T = \mathcal{H}_2\otimes\mathcal{H}_J$, where $\mathcal{H}_2 = \mathrm{Span}\{|+\rangle,|-\rangle\}$ and $\mathcal{H}_J = \mathrm{Span}\{|j,m\rangle\}$. In short, $\mathcal{H}_T = \mathrm{Span}\{|\pm,j,m\rangle\}$. The initial states that we have previously addressed [see (15)] have the salient feature of being entangled states:

$$\begin{aligned}|\Psi^{\pm}_{n,n'}\rangle &= \frac{1}{\sqrt{2}}[|+,n,n'\rangle \pm |-,n+1,n'\rangle] \\ &= \frac{1}{\sqrt{2}}[|+,j,m\rangle \pm |-,j+1/2,m+1/2\rangle]. \end{aligned} \tag{21}$$

Written more explicitly, the transformation $U(\theta,\phi)$ [see Eq. (13)] reads $\mathbb{1}_2\otimes U(\theta,\phi)$, so that it does not affect the states $|\pm\rangle\in\mathcal{H}_2$. It is thus understandable that only those features that are exclusively related to the angular momentum subspace can show up in the geometric phase that arises when we submit $|\Psi^{\pm}_{n,n'}\rangle$ to the action of $U(\theta,\phi)$. We can therefore get rid of $\mathcal{H}_2$ and work with just $\mathcal{H}_J$. That is, for our present purposes we can consider instead of the entangled states (21) the following superposition states $|\Psi^{\pm}_{n,n'}\rangle$:

$$|\Psi^{\pm}_{j,m}\rangle = \frac{1}{\sqrt{2}}[|j,m\rangle \pm |j+1/2,m+1/2\rangle]. \tag{22}$$

Using either $|\Psi^{\pm}_{n,n'}\rangle$ or $|\Psi^{\pm}_{j,m}\rangle$, we get

$$\langle J_z\rangle_{\Psi^{\pm}} = m+\tfrac{1}{4}. \tag{23}$$

On account of Eqs. (17) and (10), we can write Eq. (19) in the form

$$\gamma_m = \left(m+\tfrac{1}{4}\right)\Omega. \tag{24}$$

The particular case $\gamma_0=\Omega/4$ is not related here to the quantum vacuum nor to any quantum feature having no classical counterpart. The phase γ_m reflects angular momentum properties, which may be exhibited within both a classical and a quantum-mechanical framework. Indeed, a case in point would be the phase accumulated by the states

$$|\Psi^{\pm}_{j,m}\rangle = \frac{1}{\sqrt{2}}[|j,m\rangle \pm |j+1,m+1\rangle]. \tag{25}$$

This phase is given by

$$\gamma_m = \left(m+\tfrac{1}{2}\right)\Omega. \tag{26}$$

Choosing the j integer, it is possible to generate states such as those given by Eq. (25) with the help of, say, classical light beams carrying orbital angular momentum. We could submit these states to unitary transformations like $U(\theta,\phi)$. Nothing needs to have a quantum nature in this physical realization and the "vacuum" geometric phase $\gamma_0=\Omega/2$ would be exhibited within a purely classical framework.

Finally, we should stress that the geometric phase is not exclusively related to closed paths [18]. We can apply the general definition, Eq. (8), to the case at hand. The calculation of the dynamic contribution requires the evaluation of the quantities $\mathcal{A}_\mu = \langle\Psi^{\pm}_{j,m}|U^{\dagger}(\theta,\phi)\partial_\mu U(\theta,\phi)|\Psi^{\pm}_{j,m}\rangle$:

$$\langle U^{\dagger}\partial_\theta U\rangle_{\Psi^{\pm}} = -i\langle J_y\rangle_{\Psi^{\pm}} = 0, \tag{27}$$

$$\begin{aligned}\langle U^{\dagger}\partial_\phi U\rangle_{\Psi^{\pm}} &= -i\langle J_z\cos\theta - J_x\sin\theta\rangle_{\Psi^{\pm}} = -i\cos\theta\langle J_z\rangle_{\Psi^{\pm}} \\ &= -i\left(m+\tfrac{1}{4}\right)\cos\theta. \end{aligned} \tag{28}$$

Hence, the dynamic contribution to the geometric phase reads

$$\begin{aligned} i\int_0^{\theta}\langle\psi(\theta')|\dot{\psi}(\theta')\rangle d\theta' &= i\int_0^{\theta}\langle U^{\dagger}\partial_\phi U\rangle_{\Psi^{\pm}}d\theta' \\ &= -\left(m+\frac{1}{4}\right)\sin\theta. \end{aligned} \tag{29}$$

As for the Pancharatnam part, it is given by

$$\begin{aligned}
&\arg\langle\Psi^{\pm}_{j,m}|U(\theta,\phi)|\Psi^{\pm}_{j,m}\rangle \\
&\quad = \arg\langle\Psi^{\pm}_{j,m}|e^{-i\phi J_z}e^{-i\theta J_y}|\Psi^{\pm}_{j,m}\rangle \\
&\quad = \arg\left[\frac{1}{2}\left(e^{-im\phi}d^{j}_{m,m} + e^{-i(m+1/2)\phi}d^{j+1/2}_{m+1/2,m+1/2}\right)\right] \\
&\quad \equiv \arg\left[\frac{e^{-im\phi}}{2}\left(d^{j}_{m} + e^{-i\phi/2}d^{j+1/2}_{m+1/2}\right)\right], \qquad (30)
\end{aligned}$$

where $d^{j}_{m',m} \equiv \langle j,m'|e^{-i\theta J_y}|j,m\rangle \in \mathbb{R}$ are Wigner's coefficients [19]. This yields

$$\begin{aligned}
&\arg\langle\Psi^{\pm}_{j,m}|U(\theta,\phi)|\Psi^{\pm}_{j,m}\rangle \\
&\quad = -m\phi - \arctan\left(\frac{d^{j+1/2}_{m+1/2}\sin\phi/2}{d^{j}_{m} + d^{j+1/2}_{m+1/2}\cos\phi/2}\right). \qquad (31)
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\gamma_{j,m} = &-m\phi - \arctan\left(\frac{d^{j+1/2}_{m+1/2}\sin\phi/2}{d^{j}_{m} + d^{j+1/2}_{m+1/2}\cos\phi/2}\right) \\
&-\left(m+\frac{1}{4}\right)\sin\theta. \qquad (32)
\end{aligned}$$

Here, $\phi = \phi(\theta)$, as we have assumed θ to be the parameter in terms of which the curve $\mathcal{C}$ is described. This was possible because $\gamma(\mathcal{C})$ is parameter-invariant. It is also invariant under local gauge transformations $|\psi(s)\rangle \to |\psi'(s)\rangle = \exp(i\alpha(s))|\psi(s)\rangle$. Thus, the geometric phase cannot keep track of such local phase changes, something it would be instructive to explore by using an approach such as the present one. Such an endeavor goes, however, beyond our present scope and is deferred to future work.

III. NUMERICAL RESULTS

Even though Eq. (5) unescapably leads to the conclusion that $\gamma_n \neq 0$, Larson has reported numerical results to the contrary. This motivated us to look for a possible explanation of such a discrepancy. We have thus committed ourselves to find out whether some subtleties of the employed numerical techniques could possible yield a result such as $\gamma_n = 0$ in special cases. None of our results explain how $\gamma_n = 0$ could have been obtained. On the contrary, we could only obtain results that are in accordance with those of other authors who got $\gamma_n \neq 0$, by either diagonalizing the Rabi Hamiltonian [see Eq. (2)] or the unitarily transformed one, $H_R(\varphi) \equiv U(\varphi)H_R U^{\dagger}(\varphi)$. Moreover, we diagonalized both versions of H_R, the one written in terms of a and $a^{\dagger}$ and the one written in terms of the quadrature operators x and p. In neither case could we reproduce Larson's results. As way of example, we show in Fig. 1 the geometric phase for four eigenvectors of the Rabi Hamiltonian being evolved with $U(\varphi) = \exp(-i\varphi a^{\dagger}a)$. That is, Eq. (3) was evaluated numerically for four eigenvectors and varying $g/\omega \in [0,1]$ for three relative detunings $\Delta' \equiv (\omega_0 - \omega)/\omega$. As can be seen, for large enough values of g/ω the geometric phase is definitely nonvanishing, no matter which case we address, the resonant case or the nonresonant (blueshifted or redshifted) one. We made sure that our calculations of the geometric phase were accurate to 99.9%, in the following sense: we used an $N \times N$ matrix representation of the Rabi Hamiltonian and fixed N so that the results obtained by increasing N to $N+1$ do not change by more than 0.1%. To reach such an accuracy in the calculation of the geometric phase, we required a rather small 20×20 matrix. Furthermore, our numerical results show good agreement with the analytically approximated solutions provided by both Liu *et al.* [6] and Deng and Li [7] (see Fig. 1). Liu *et al.* [6] give a detailed discussion about how the geometric phases obtained with Rabi eigenvectors converge to those obtained with JC eigenvectors. Their conclusions apply in our case as well, including those referring to the vacuum-induced Berry's phase. In order to further exhibit the accuracy of our numerical results, we show in Fig. 2 the numerically calculated eigenvalues of the Rabi Hamiltonian and compare them with the analytically approximated solutions given by Irish [20] (left panel) and by Zhang *et al.* [21] (right panel). Our numerical calculations are accurate to 99.9% in this case as well. A 22×22 matrix was required to reach such an accuracy, because in this case we calculated a larger number of eigenvalues than in the geometric phase calculation. We are confident that our results are representative of the general case and that the geometric phase does not vanish, in accordance with the analytical argument given in connection to Eq. (5).

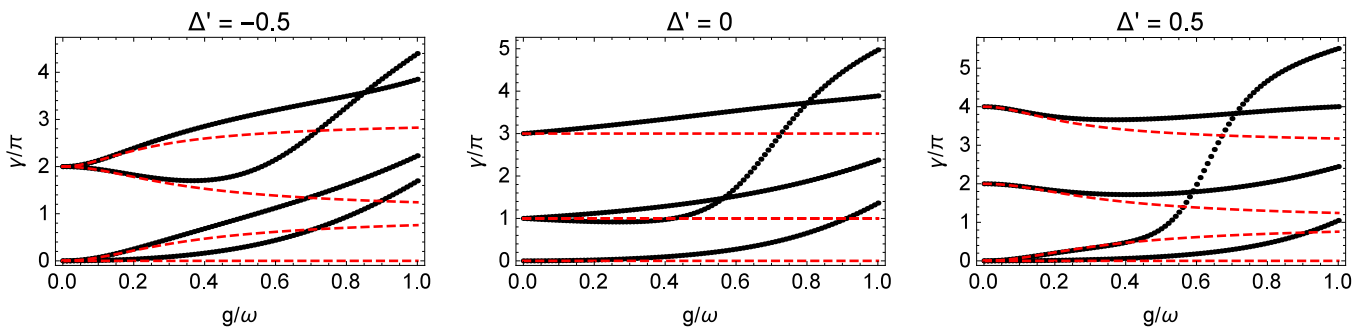

FIG. 1. Geometric phase γ (in units of π) associated to eigenvectors of the Rabi Hamiltonian, as a function of g/ω. Numerical diagonalization of the Rabi Hamiltonian shows that $\gamma \neq 0$ can be obtained in the resonant [$\Delta' \equiv (\omega_0 - \omega)/\omega = 0$] and nonresonant [$\Delta' \gtrless 0$] cases. Dashed lines correspond to phases associated to eigenvectors of the JC Hamiltonian.

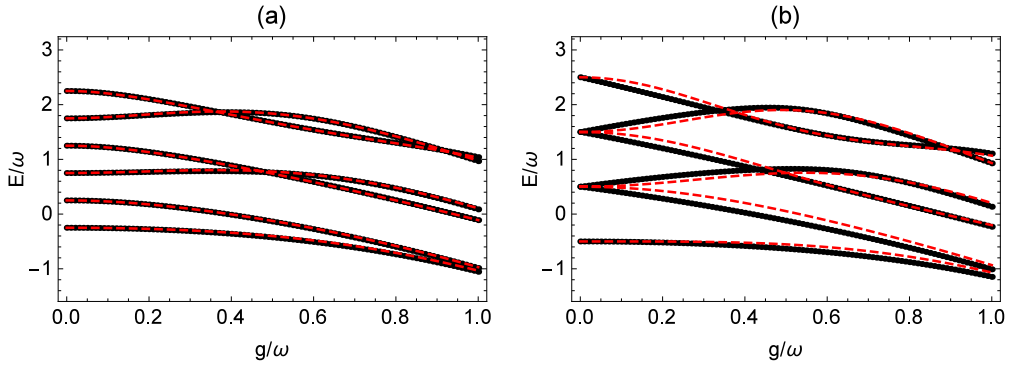

FIG. 2. Eigenvalues (in units of ω) of the Rabi Hamiltonian, as a function of g/ω. Numerical diagonalization of the Rabi Hamiltonian is compared with the analytical approximations provided by Irish [20] (left panel) and by Zhang *et al.* [21] (right panel). In the case of Irish we consider the nonresonant case ($\Delta' = -0.5$). Differences between analytical (dashed lines) and numerical results are negligible. In the case of Zhang *et al.*, we consider the resonant case ($\Delta' = 0$). Small differences can be observed for some values of g/ω.

IV. SUMMARY AND CONCLUSIONS

The results we have presented provide evidence for a generally nonvanishing geometric phase that is related to the Rabi Hamiltonian. These results should serve to settle an ongoing controversy on this matter. We have provided analytical and numerical evidence about nonvanishing geometric phases for the case in which an eigenvector of the Rabi Hamiltonian evolves under a unitary transformation of the form $U(\varphi) = \exp(-i\varphi a^\dagger a)$. We have also addressed the case of an eigenvector of an extended Rabi Hamiltonian, which is submitted to the transformation $U(\theta,\phi) = \exp(-i\phi J_z)\exp(-i\theta J_y)$. We have shown that in these cases the geometric phase reflects specific features of the underlying vector space. This space is explored by a state that evolves under some unitary U. The evolution does not need to be adiabatic and the Hamiltonian does not play any essential role in the whole, except as a provider of initial (eigen)vectors. Thus, the appropriate framework for dealing with the geometric phase in these cases is given by the kinematic approach of Mukunda and Simon [12]. This approach brings to the fore the essential features on which the geometric phase depends, thereby avoiding unwarranted interpretations of its appearance. Finally, our results obtained with the eigenvectors of the Rabi Hamiltonian do converge to the corresponding ones obtained with the Jaynes-Cummings Hamiltonian. Thus, the rotating-wave approximation consistently applies when dealing with geometric phases, as it applies in other cases as well.

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[1] J. Larson, Phys. Rev. Lett. **108**, 033601 (2012).
[2] I. Fuentes-Guridi, A. Carollo, S. Bose, and V. Vedral, Phys. Rev. Lett. **89**, 220404 (2002).
[3] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, and J. Zwanziger, *The Geometric Phase in Quantum Systems* (Springer, Berlin, 2003).
[4] J. Schön and H. Köppel, J. Chem. Phys. **108**, 1503 (1998).
[5] M. Wang, L. F. Wei, and J. Q. Liang, Phys. Lett. A **379**, 1087 (2015).
[6] T. Liu, M. Feng, and K. Wang, Phys. Rev. A **84**, 062109 (2011).
[7] W-W. Deng and G. Li, J. Phys. B **46**, 224018 (2013).
[8] J. Larson, Phys. Scr. **T153**, 014040 (2013).
[9] M. T. Batchelor, Z-M. Li, and H-Q. Zhou, J. Phys. A: Math. Theor. **49**, 01LT01 (2016).
[10] V. C. Rakhecha and A. G. Wagh, Pramana, J. Phys. **46**, 315 (1996).
[11] J. C. Budich and B. Trauzettel, Phys. Status Solidi RRL **7**, 109 (2013).
[12] N. Mukunda and R. Simon, Ann. Phys. (NY) **228**, 205 (1993).
[13] S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 247 (1956); Reprinted in *Collected Works of S. Pancharatnam* (Oxford University Press, Oxford, 1975).
[14] A. Bohm, L. J. Boya, and B. Kendrick, Phys. Rev. A **43**, 1206 (1991).
[15] E. Sjöqvist, Int. J. Quantum Chem. **115**, 1311 (2015).
[16] R. Resta, J. Phys.: Condens. Matter **12**, R107 (2000).
[17] A. Carollo, M. F. Santos, and V. Vedral, Phys. Rev. A **67**, 063804 (2003).
[18] J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339 (1988).
[19] V. Devanathan, *Angular Momentum Techniques in Quantum Mechanics* (Kluwer Academic, Dordrecht, 2002).
[20] E. K. Irish, Phys. Rev. Lett. **99**, 173601 (2007).
[21] Y-Y. Zhang, Q-H. Chen, and S-Y. Zhu, Chin. Phys. Lett. **30**, 114203 (2013).