Strong enhancing effect of correlations of photon trajectories on laser beam scintillations

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To provide a detailed description of the dynamics of laser beam propagation in the atmosphere we use the method of the photon distribution function in the phase space, which reduces the analysis to consideration of photon trajectories and their correlations. The scintillation index σ^2 is calculated for the range of moderate and strong turbulence, which is the most challenging for analytical consideration. The considerable growth of σ^2 (by two to three times) found for moderate turbulence is shown to be due to correlations between photon trajectories. Our calculations demonstrate that the maximum of σ^2 can be considerably decreased by an increase of the source aperture or the use of the fast phase diffuser.

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I. INTRODUCTION

The recent interest in beam propagation was awakened by the development of quantum communication in the free atmosphere [1,2]. Detailed studies of the effect of the turbulence-induced losses on the quantum state of light in the course of satellite-mediated communication and for realization of the entanglement transfer in the atmosphere were reported in Refs. [3,4]. Particularly, the range of moderate turbulence where the signal-to-noise ratio (SNR) can be considerably smaller than unity is of great interest. However, that range is the most demanding for analytical consideration. Currently, there are reliable analytical solutions only for the limiting cases of weak and strong turbulence.

Fluctuations (or scintillations) of the laser radiation caused by turbulence of the Earth's atmosphere limit the performance of light communication systems. Temperature inhomogeneities of the atmosphere generate turbulent eddies which are accompanied by an inhomogeneous modulation of the air density and the corresponding modulation of the refractive index [5–7]. As a result, laser beams propagate in a medium with a randomly distributed refractive index. In the course of the eddies' evolution, the random distribution of the refractive index varies synchronously with eddies. Therefore, an initially stationary laser beam has a time-varying intensity in the detector plane. Although this qualitative picture of the intensity fluctuations seems quite simple, it is a great challenge for an adequate theoretical analysis.

Any theoretical model of the intensity fluctuations should account for a very wide range of characteristic lengths of inhomogeneities, index-of-refraction structure constant C_n^2, etc. The characteristic lengths cover the interval from a few millimeters (the inner radius l_0 of eddies) to a hundred meters (the outer radius L_0). As a result, various scenarios of beam behavior can be observed. The scattering by large-size eddies results in random redirections of the beam as a whole. This process is known in the literature as a "wandering" or "dancing" of the beam [8,9]. On the other hand, the scattering by small-size eddies causes spreading of the beam. For a long-distance propagation or a strong turbulence, the beam radius becomes greater than the characteristic sizes of the inhomogeneities. In this case the probability of the beam to be redirected becomes small, and the relative value of the wandering radius decreases [10].

The beam wandering and broadening can be considered the specific manifestations of the scintillation effect. The scintillations have a tendency of saturating for a long-distance propagation [11,12] (the regime of a strong turbulence). This is because in the course of propagation the radiation acquires the properties of the Gaussian statistics, and the SNR tends to unity. The asymptotic behavior of the scintillation index, $\sigma^2 \to 1$, was explained in Refs. [13–15]. Moreover, it was shown quite generally that this property stays unchanged for any refractive index distribution, provided the response time of the recording instrument is short compared with the source coherence time. This result was confirmed analytically in [16].

At the same time, it was shown by different approaches [17–20] that there is a possibility of significant suppression of the scintillations by means of partially coherent laser beams with the coherence time shorter than the detector integration time (a slow detector). Recent theoretical and experimental developments on the propagation of partially coherent beams in a turbulent atmosphere were discussed in [21].

There are several analytical approaches explaining the behavior of the scintillation index in the case of strong turbulence [18,22,23]. Their analysis is based on the physical picture where four waves, forming the second moment of the intensity, conserve only pair correlations in the course of long-distance propagation. Two different pairs of the photon trajectories contribute to the square of the photon density at the detector. Dashen used the Feynman path integrals to prove that in a convincing manner [22].

The formalism of the photon distribution function (or the photon density in the coordinate-momentum space [24]) was applied to the problem of scintillations in [10,18,25,26]. Those papers are based on a physical picture which is similar to that described above. The method of the photon distribution function (PDF) was used for the description of both classical and quantum light including propagation of single-photon pulses (see, for example, Refs. [25,27,28]). Fluctuations of photon counts under conditions of strong and weak turbulence were investigated in Ref. [25]. The results were expressed via the scintillation index, which describes a stationary beam [see Eqs. (23) and (25) therein].

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The solution of the kinetic equation for the operator of photon density is based on the method of characteristics. The assumption of weak disturbances of photon momenta by the atmosphere (the paraxial approximation) reduces the problems of scintillations to the problem of obtaining photon trajectories and their correlations. A slowly varying fluctuating force, deflecting photon trajectories from straight lines, describes the effect of the atmospheric eddies on scintillations.

The challenging aim of this study is to develop an analytical approach and perform numerical computer calculations for the intensity fluctuations of the laser beam under moderate and strong turbulence conditions. In particular, we are going to explain the physical nature of large scintillations of laser beams. Analytical calculations using an iteration procedure within the photon-distribution-function formalism are supplemented by indispensable numerical computer many-fold integration to calculate the scintillation index.

The rest of this paper is organized as follows. In Sec. II, we review the model Hamiltonian of the system and our theoretical approach based on the dynamics of PDF. Also, the modification of the initial photon distribution caused by the phase diffuser is analyzed. In Sec. III, we explain how the correlations of photon trajectories can be accounted for by the use of the iterative scheme. In Sec. IV, the results of the calculations are analyzed. The physical nature of correlation and supercorrelation mechanisms is explained. In Sec. V, the results are briefly summarized. The Appendix provides the criteria of the applicability of our approach.

II. PHOTON-DISTRIBUTION-FUNCTION APPROACH

The photon distribution function is defined by analogy with distribution functions in solid-state physics. In particular, it is similar to the phonon distribution function. Both of them are defined as [24,29]

$$f(\mathbf{r},\mathbf{q},t) = \frac{1}{V}\sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}} b^{\dagger}_{\mathbf{q}+\mathbf{k}/2} b_{\mathbf{q}-\mathbf{k}/2}, \tag{1}$$

where $b^{\dagger}_{\mathbf{q}}$ and $b_{\mathbf{q}}$ are the bosonic creation and annihilation operators of photons or phonons with momenta $\mathbf{q}$ and $V \equiv L_x L_y L_z$ is the normalizing volume. Polarization of the corresponding modes is not specified in (1). In the paraxial approximation, assumed here, the initial polarization of the beam remains almost unchanged even for a long-distance propagation (see, for example, Ref. [30]).

The operator $f(\mathbf{r},\mathbf{q},t)$ describes the photon density in the phase $(\mathbf{r},\mathbf{q})$ space. Usually, the characteristic sizes of spatial inhomogeneities of the radiation field are much greater than the wavelength. In this case the sum in Eq. (2) can be restricted by small k. Henceforth, we consider that $k < k_0 \ll q_0$, where q_0 is the wave vector corresponding to the central frequency of the radiation, $\omega_0 = cq_0$. At the same time k_0 should be sufficiently large to provide the required accuracy of the beam profile description.

The evolution of the Heisenberg operator $f(\mathbf{r},\mathbf{q},t)$ is determined by the commutator

$$\partial_t f(\mathbf{r},\mathbf{q},t) = \frac{1}{i\hbar}[f(\mathbf{r},\mathbf{q},t),H], \tag{2}$$

where

$$H = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} b^{\dagger}_{\mathbf{q}} b_{\mathbf{q}} - \sum_{\mathbf{q},\mathbf{k}} \hbar\omega_{\mathbf{q}} n_{\mathbf{k}} b^{\dagger}_{\mathbf{q}} b_{\mathbf{q}+\mathbf{k}} \tag{3}$$

is the Hamiltonian of photons in a medium with a fluctuating refractive index $n(\mathbf{r})$ ($n_{\mathbf{k}}$ is its Fourier transform) and $\hbar\omega_{\mathbf{q}} = \hbar cq$ and $\mathbf{c}_{\mathbf{q}} = \frac{\partial\omega}{\partial\mathbf{q}}$ are the vacuum values of the photon energy and velocity, respectively.

Assuming the characteristic values of the photon momentum are much greater than the wave vectors of turbulence, the kinetic equation for the photon distribution function can be written as

$$\{\partial_t + \mathbf{c}_{\mathbf{q}}\partial_{\mathbf{r}} + \mathbf{F}(\mathbf{r})\partial_{\mathbf{q}}\} f(\mathbf{r},\mathbf{q},t) = 0, \tag{4}$$

where $\mathbf{F}(\mathbf{r}) = \omega_0 \partial_{\mathbf{r}} n(\mathbf{r})$ is the random force originating from the atmospheric turbulence. The general solution of Eq. (4) is given by

$$f(\mathbf{r},\mathbf{q},t) = \phi\left\{\mathbf{r} - \int_0^t dt' \frac{\partial \mathbf{r}(t')}{\partial t'}; \mathbf{q} - \int_0^t dt' \frac{\partial \mathbf{q}(t')}{\partial t'}\right\}, \tag{5}$$

where the function $\phi(\mathbf{r},\mathbf{q})$ is the "initial" value of $f(\mathbf{r},\mathbf{q},t)$, i.e.,

$$\phi(\mathbf{r},\mathbf{q}) = \frac{1}{V}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} (b^{+}_{\mathbf{q}+\mathbf{k}/2} b_{\mathbf{q}-\mathbf{k}/2})|_{t=0}$$
$$\equiv \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \phi(\mathbf{k},\mathbf{q}). \tag{6}$$

The derivatives $\frac{\partial \mathbf{r}(t')}{\partial t'}$ and $\frac{\partial \mathbf{p}(t')}{\partial t'}$ should satisfy the equations

$$\frac{\partial \mathbf{r}(t')}{\partial t'} = \mathbf{c}(\mathbf{q}(t'))$$
$$\frac{\partial \mathbf{q}(t')}{\partial t'} = \mathbf{F}(\mathbf{r}(t')), \tag{7}$$

completed with the boundary conditions $\mathbf{r}(t') = \mathbf{r}$ and $\mathbf{q}(t') = \mathbf{q}$ for $t = t'$. As we see, Eq. (7) coincide with the classical (Newton's) equations of motion of a point particle moving with the velocity $\mathbf{c}_{\mathbf{q}}$ and affected by an external force $\mathbf{F}(\mathbf{r})$. Formal solutions of Eq. (7) can be written as

$$\mathbf{q}(t') = \mathbf{q} + \int_t^{t'} dt'' \mathbf{F}(\mathbf{r}(\mathbf{q},t'')) \tag{8}$$

and

$$\mathbf{r}(\mathbf{q},t') = \mathbf{r} - \mathbf{c}_{\mathbf{q}}(t-t') - \frac{c}{q_0}\int_t^{t'} dt''(t''-t')\mathbf{F}(\mathbf{r}(\mathbf{q},t'')). \tag{9}$$

Equations (8) and (9) can be interpreted as the photon trajectories. To justify this terminology, two properties of the operator $f(\mathbf{r},\mathbf{q},t)$ should be mentioned. First of all, let us consider the integration over $\mathbf{r}$, which gives the total number of photons with the momentum $\mathbf{q}$: $\int d\mathbf{r} f(\mathbf{r},\mathbf{q},t) = b^{\dagger}_{\mathbf{q}} b_{\mathbf{q}}$. Second, the value $\hat{n}(\mathbf{r}) = \sum_{\mathbf{q}} f(\mathbf{r},\mathbf{q},t)$ is the operator of the photon density in the coordinate domain [31], which is similar to the well-known Mandel coarse-grained photon density $\hat{n}_M$ (see [32], Chap. 12, and [33]).

The above-mentioned properties indicate that $f(\mathbf{r},\mathbf{q},t)$ does really describe the photon density in the phase space. The formal solution (5) shows that the quantity $f(\mathbf{r},\mathbf{q},t)$ does not vary if one moves along the trajectory given by Eqs. (8) and (9). The distribution function at any set of variables $(\mathbf{r},\mathbf{q},t)$ is equal to its initial value at the beginning of the corresponding trajectory (at $t = 0$). This can be seen if the integration over t' in Eq. (5) is carried out. Equations (8) and (9) describe displacements of "small elements" of the initial distribution. The term "photon trajectories" is used here irrespective of the number of photons and their statistics. The terms "photon trajectories" and "photon paths" [34], which are not rigorous, are widely used in the literature.

We should add that Eq. (4) was derived within the assumption of $k_0 \ll q_0$. This situation can be expressed by means of the increased thickness of the line (9): it should be of the order of π/k_0. In practice, it can be only larger than several wavelengths of the light [32].

Equations (8) and (9) allow us to rewrite expression (5) as

$$f(\mathbf{r},\mathbf{q},t) = \phi\left\{\mathbf{r} - \mathbf{c_q}t + \frac{c}{q_0}\int_0^t dt' t'\mathbf{F}(\mathbf{r}(\mathbf{q},t'));\mathbf{q} - \int_0^t dt'\mathbf{F}(\mathbf{r}(\mathbf{q},t'))\right\}. \qquad (10)$$

If $\mathbf{F}(\mathbf{r})$ is a known function, an approximate value for $f(\mathbf{r},\mathbf{q},t)$ can be obtained by inserting the term $\mathbf{r}(\mathbf{q},t') \approx \mathbf{r} - \mathbf{c_q}(t - t')$ into Eq. (10). In this case the argument of the fluctuating force $\mathbf{F}(\mathbf{r}(\mathbf{q},t'))$ is replaced by a straight line, which is correct only in the absence of the turbulence. Improvement of the theory can be achieved if the argument of $\mathbf{F}$ accounts for the turbulence.

It follows from Eq. (10) that statistical properties of the radiation depend not only on the turbulence but also on the initial distribution function $\phi(\mathbf{r},\mathbf{q})$. This function is determined by the source field. Its explicit form is determined in the course of "sewing" of the near-aperture and atmospheric fields [18] given by the amplitudes $b_{\mathbf{q}}(b^\dagger_{\mathbf{q}})$. We consider light propagation in the z direction. The source field is assumed to be described by the Gaussian function, $\Phi(\mathbf{r}) = (2/\pi)^{1/2}\frac{1}{r_0}\exp(-r_\perp^2/r_0^2)$. Then the propagating amplitudes are given by

$$b_{\mathbf{q}_\perp,q_0}(t = 0) = b(2\pi/S)^{1/2}r_0\exp(-q_\perp^2 {r_0}^2/4), \qquad (11)$$

where b is the near-aperture amplitude of the laser field, the index $\perp$ means perpendicular to the z-axis components, and $S = L_xL_y$.

We take into account the effect of the phase diffuser by multiplying the distribution $\Phi(\mathbf{r})$ by the phase factor $e^{-i\mathbf{a}\mathbf{r}_\perp}$, where the quantity $\mathbf{a}$ is a random variable. In this case Eq. (11) should be modified by substituting in its right-hand side $\mathbf{q}_\perp + \mathbf{a} \equiv \mathbf{q_a}$ for $\mathbf{q}_\perp$. Such a simple modeling of the phase diffuser is justified if (i) the detection time is much longer than the characteristic time of the variation of $\mathbf{a}$ (slow detector) and (ii) there is a large root-mean-square of the phase fluctuations. (More detailed analysis is presented in [26].) This case corresponds to the Gaussian distribution of $\mathbf{a}$:

$$P(a_{x,y}) = \frac{\lambda}{2\pi^{1/2}}\exp\left(-a_{x,y}^2\lambda^2/4\right), \qquad (12)$$

with a covariance $\langle a_{x,y}^2\rangle = \lambda^{-2}$, and the transverse correlation function of the outgoing field (at $t = 0$) is given by

$$\langle E(\mathbf{r}_\perp)E(\mathbf{r}_\perp + \Delta)\rangle_{\mathbf{a}} = E_0^2\exp\left\{-[r_\perp^2 + (\mathbf{r}_\perp + \Delta)^2]r_0^{-2}\right\}\exp(-\Delta^2\lambda^{-2}). \qquad (13)$$

Here $E_0 = E(r_\perp = 0,\Delta = 0,t = 0)$ and the notation $\langle\cdots\rangle_{\mathbf{a}}$ means averaging over distribution $P(a_{x,y})$. The radiation, whose correlation properties are described by function (13), is referred to as the Gaussian Shell-model field. The parameter λ in the exponential factor describes the decrease of the transverse correlation length. It can also be said that this parameter generates a new characteristic length, $1/r_1$, in the momentum distribution (i.e., in the $\mathbf{q}$ domain). This is seen from $\phi(\mathbf{k},\mathbf{q})$, which after averaging over the fluctuations of $\mathbf{a}$ reduces to

$$\langle\phi(\mathbf{k},\mathbf{q})\rangle_{\mathbf{a}} = 2\pi\frac{b^\dagger b}{VS}r_1^2\exp\left(-q_\perp^2\frac{r_1^2}{2} - k_\perp^2\frac{r_0^2}{8}\right), \qquad (14)$$

where $r_1^2 = r_0^2(1 + 2r_0^2\lambda^{-2})^{-1}$ and variables q_z and k_z are omitted. It is seen from Eq. (14) that $q_\perp$ is distributed in the range of the order of $\sqrt{2}/r_1$, which is greater than the one for the coherent beam. In contrast, the characteristic value of $\tilde{k}$ depends only on the initial size of the beam ($\tilde{k} \sim \sqrt{8}/r_0$).

In the course of light propagation, the diffraction phenomena and scattering by atmospheric inhomogeneities broaden the beam; as a result, $\tilde{k}$ decreases. At the same time, the value of $\tilde{q}$ increases with the distance because of the Brownian-like motion of photons in the $\mathbf{q}_\perp$ domain (see Ref. [18]). Such a simple physical picture, elucidating the evolution of the beam geometry, is, however, not applicable to the description of scintillations. The phenomenon of scintillations is more complicated and can be described in terms of spatiotemporal correlations of four waves.

III. SCINTILLATION INDEX

The photon distribution function is used here to obtain the scintillation index σ^2. The definition of σ^2 is given by

$$\sigma^2 = \frac{\langle I^2(\mathbf{r})\rangle - \langle I(\mathbf{r})\rangle^2}{\langle I(\mathbf{r})\rangle^2}. \qquad (15)$$

The photon density $I(\mathbf{r},t)$ is expressed in terms of the distribution function as

$$I(\mathbf{r},t) = \sum_{\mathbf{q}} f(\mathbf{r},\mathbf{q},t) = 2\pi\frac{b^\dagger b r_0^2}{SV} \times \sum_{\mathbf{q},\mathbf{k}}\exp\left\{-i\mathbf{k}\cdot\left[\mathbf{r} - \mathbf{c_q}t + \frac{c}{q_0}\int_0^t dt' t'\mathbf{F}(\mathbf{r}(\mathbf{q},t'))\right]\right\} \times \exp\left(-Q_{\mathbf{a}}^2\frac{r_0^2}{2} - k^2\frac{r_0^2}{8}\right), \qquad (16)$$

where $\mathbf{Q_a} \equiv \mathbf{Q} + \mathbf{a} = \mathbf{q} + \mathbf{a} - \int_0^t dt'\mathbf{F}(\mathbf{r}(\mathbf{q},t'))$. The summation is taken over the $\mathbf{q}_\perp$ and $\mathbf{k}_\perp$ components, while q_z and k_z are considered to be fixed: $q_z = q_0$ and $k_z = 0$. The exponential term originates from the solution (10) of the kinetic equation (4).

To obtain $\langle I(\mathbf{r},t)\rangle$, three independent averagings are required. One of them concerns the source variables. In the case of a coherent state of the source $|\beta\rangle$, we have $\langle b^\dagger b\rangle = |\beta|^2$. The second averaging over a random phase of the diffuser should be carried out as explained by Eq. (14). The third averaging deals with the fluctuating force $\mathbf{F}$. These three actions can be performed independently, which facilitates the analysis. Also, the calculations are simplified if we use the identity

$$\exp\left(-Q^2 r_0^2/2\right) \equiv \int \frac{d\mathbf{p}}{2\pi r_0^2} \exp\left(i\mathbf{p}\cdot\mathbf{Q} - p^2/2r_0^2\right). \tag{17}$$

Because of Eq. (17), the term in the exponent of Eq. (16) reduces to the linear in $\mathbf{F}$ form. Then, considering $\mathbf{F}$ as a random Gaussian variable, the value of $\langle I(\mathbf{r},t)\rangle$ can be easily obtained in the manner explained in Ref. [18]. To calculate $\langle I(\mathbf{r},t)\rangle$, an explicit form of the refractive-index correlation function, $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$, is required. In a statistically homogeneous atmosphere it can be written as

$$\langle n(\mathbf{r})n(\mathbf{r}')\rangle = \int d\mathbf{g} \exp[-i\mathbf{g}\cdot(\mathbf{r}-\mathbf{r}')]\psi(\mathbf{g}). \tag{18}$$

The widely used von Kármán approximation for the spectrum $\psi(\mathbf{g})$ is given by

$$\psi(\mathbf{g}) = 0.033 C_n^2 \frac{\exp[-(g l_0/2\pi)^2]}{\left[g^2 + L_0^{-2}\right]^{11/6}}, \quad |\mathbf{g}| \equiv g, \tag{19}$$

where the vector $\mathbf{g}$ is defined in the three-dimensional domain.

The "source" part of $\langle I^2(\mathbf{r})\rangle$, given by $\langle b^\dagger b b^\dagger b\rangle$, is approximately equal to $\langle b^\dagger b^\dagger b b\rangle = |\beta|^4$, when the condition $|\beta|^4 \gg |\beta|^2$ is satisfied. This inequality implies that the initial laser radiation is in a multiphoton coherent state. The averaging over independent random quantities $\mathbf{a}$ and $\mathbf{a}'$ can be used instead of the time averaging of the diffuser state. Then we have

$$\langle I^2(\mathbf{r},t)\rangle = \left|\frac{2\pi\beta^2 r_0^2}{VS}\right|^2 \sum_{\substack{\mathbf{q},\mathbf{k},\\ \mathbf{q}',\mathbf{k}'}} \Big\langle e^{-i\mathbf{k}\cdot[\mathbf{r}-\mathbf{c}_\mathbf{q}t+\frac{c}{q_0}\int_0^t dt' t'\mathbf{F}(\mathbf{r}(\mathbf{q},t'))]-i\mathbf{k}'\cdot[\mathbf{r}-\mathbf{c}_{\mathbf{q}'}t+\frac{c}{q_0}\int_0^t dt' t'\mathbf{F}(\mathbf{r}(\mathbf{q}',t'))]}$$
$$\times \left\{ e^{-(Q_\mathbf{a}^2+Q_{\mathbf{a}'}^2+\frac{k^2+k'^2}{4})\frac{r_0^2}{2}} + e^{-[(\mathbf{Q}_\mathbf{a}+\frac{\mathbf{k}}{2})^2+(\mathbf{Q}'_\mathbf{a}-\frac{\mathbf{k}'}{2})^2+(\mathbf{Q}_{\mathbf{a}'}-\frac{\mathbf{k}}{2})^2+(\mathbf{Q}'_{\mathbf{a}'}+\frac{\mathbf{k}'}{2})^2]\frac{r_0^2}{4}} \right\}\Big\rangle. \tag{20}$$

There are two terms in the braces in Eq. (20). They appear only if the initial four-wave correlation reduces to the pair correlation [18]. Such a modification of the statistical properties of the radiation occurs when the waves propagate for a long time which is sufficient for randomization of the transverse photon momentum. A more general case, which includes the regime of fast detection, was analyzed in Ref. [26].

The averaging of Eq. (20) over $\mathbf{a}$ and $\mathbf{a}'$ results in

$$\langle I^2(\mathbf{r},t)\rangle = \left|\frac{2\pi\beta^2 r_1^2}{VS}\right|^2 \sum_{\substack{\mathbf{q},\mathbf{k},\\ \mathbf{q}',\mathbf{k}'}} \Big\langle e^{-i\{\mathbf{k}\cdot[\mathbf{r}-\mathbf{c}_\mathbf{q}t]+\mathbf{k}'\cdot[\mathbf{r}-\mathbf{c}'_\mathbf{q}t]+\frac{c}{q_0}\int_0^t dt' t'[\mathbf{k}\cdot\mathbf{F}(\mathbf{r}(\mathbf{q},t'))+\mathbf{k}'\cdot\mathbf{F}(\mathbf{r}(\mathbf{q}',t'))]\}}$$
$$\times \left\{ e^{-(Q^2+Q'^2)r_1^2/2-(k^2+k'^2)r_0^2/8} + e^{-[(\mathbf{Q}-\mathbf{Q}')^2+(\mathbf{k}+\mathbf{k}')^2/4]r_0^2/4-[(\mathbf{Q}+\mathbf{Q}')^2+(\mathbf{k}-\mathbf{k}')^2/4]r_1^2/4} \right\}\Big\rangle. \tag{21}$$

In the absence of a phase diffuser, $r_0 = r_1$, the summands in the last set of braces contribute equally to (21).

Similar to Eq. (17), the factor $\exp[-(Q^2 + Q'^2)\frac{r_1^2}{2}]$ in (21) can be expressed in the integral form as

$$\exp\left[-(Q^2+Q'^2)\frac{r_1^2}{2}\right] = \int \frac{d\mathbf{p}d\mathbf{p}'}{(2\pi r_1^2)^2} \exp\left[i\mathbf{p}\cdot\mathbf{Q} + i\mathbf{p}'\cdot\mathbf{Q}' - (p^2+p'^2)/2r_1^2\right]. \tag{22}$$

As we see, the exponent in the left-hand side is represented as a linear form of the force $\mathbf{F}$. A similar transform is applicable to the second term in the last set of braces in (21). As a result, the fluctuating force enters the right-hand side of (21) only via the common multiplier M, given by

$$M = \exp\left\{-i\int_0^t dt' \left[\left(\mathbf{p}+\mathbf{k}t'\frac{c}{q_0}\right)\cdot\mathbf{F}(\mathbf{r}(\mathbf{q},t'))+\left(\mathbf{p}'+\mathbf{k}'t'\frac{c}{q_0}\right)\cdot\mathbf{F}(\mathbf{r}(\mathbf{q}',t'))\right]\right\}. \tag{23}$$

The calculation of the average value of I^2 reduces to averaging M with many-fold integration. Assuming the exponent in (23) is a Gaussian random variable, we can write

$$\langle M\rangle = \exp\left(-\frac{1}{2}\left\langle\left\{\int_0^t dt'\left[\left(\mathbf{p}+\mathbf{k}t'\frac{c}{q_0}\right)\cdot\mathbf{F}(\mathbf{r}(\mathbf{q},t'))+\left(\mathbf{p}'+\mathbf{k}'t'\frac{c}{q_0}\right)\cdot\mathbf{F}(\mathbf{r}(\mathbf{q}',t'))\right]\right\}^2\right\rangle\right)$$
$$\equiv \exp\left(-\frac{1}{2}(\phi_{PP}+2\phi_{PP'}+\phi_{P'P'})\right). \tag{24}$$

Two types of correlation functions determine $\langle M\rangle$:

$$\phi_{PP'} = \int_0^t \int_0^t dt'dt''(\mathbf{p}+\mathbf{k}t'c/q_0) \cdot \langle \mathbf{F}(\mathbf{r}(\mathbf{q},t'))\mathbf{F}(\mathbf{r}(\mathbf{q}',t''))\rangle \cdot (\mathbf{p}'+\mathbf{k}'t''c/q_0), \quad (25)$$

$$\phi_{PP} = \int_0^t \int_0^t dt'dt''(\mathbf{p}+\mathbf{k}t'c/q_0) \cdot \langle \mathbf{F}(\mathbf{r}(\mathbf{q},t'))\mathbf{F}(\mathbf{r}(\mathbf{q},t''))\rangle \cdot (\mathbf{p}+\mathbf{k}t''c/q_0), \quad (26)$$

where P and P' denote sets of three vector variables, $P=\{\mathbf{q},\mathbf{p},\mathbf{k}\}$ and $P'=\{\mathbf{q}',\mathbf{p}',\mathbf{k}'\}$. The correlation functions of the forces along different $(\mathbf{q}\neq\mathbf{q}')$ and coinciding $(\mathbf{q}=\mathbf{q}')$ trajectories enter Eqs. (25) and (26), respectively. The former can be rewritten as

$$\langle F_\alpha(\mathbf{r}(\mathbf{q},t'))F_\beta(\mathbf{r}(\mathbf{q}',t''))\rangle = \langle F_\alpha(\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t''))F_\beta(0)\rangle, \quad (27)$$

where the notations α and β stand for the x and y components. The expression for (26) follows from Eq. (27) by setting $\mathbf{q}=\mathbf{q}'$.

The right-hand side of Eq. (27) is assumed to be a function of the coordinate difference, $\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'')$. It is so if the atmosphere is statistically homogeneous. In the course of averaging, the dependence of the coordinate difference on the fluctuating force should also be taken into account. This dependence is given by the relation

$$\begin{aligned}\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'') &= (\mathbf{e}_z c+\mathbf{c}_{\mathbf{q}'})(t'-t'')-\mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t-t') \\ &+\frac{c}{q_0}\int_{t'}^{t''} dt_1(t'-t_1)\mathbf{F}(\mathbf{r}(\mathbf{q}',t_1)) \\ &+\frac{c}{q_0}\int_{t}^{t'} dt_1(t'-t_1)\{\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))-\mathbf{F}(\mathbf{r}(\mathbf{q}',t_1))\},\end{aligned} \quad (28)$$

which follows from Eq. (9). The distance $|\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'')|$ should be of the order of or less than the outer radius L_0 of the turbulence for nonzero correlation of fluctuating forces. Considering that $c \gg |c_{\mathbf{q}-\mathbf{q}'}|,|c_{\mathbf{q}'}|$, we infer that $|t'-t''|\leqslant L_0/c$. This means that on the right-hand side of Eq. (28) we can set $\mathbf{c}_{\mathbf{q}'}=0$ and omit the third term, which is proportional to $(t'-t'')^2$. Then Eq. (28) reduces to

$$\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'') = \mathbf{e}_z c(t'-t'')-\mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t-t')+\frac{c}{q_0}\int_t^{t'} dt_1(t'-t_1)\{\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))-\mathbf{F}(\mathbf{r}(\mathbf{q}',t_1))\}. \quad (29)$$

The second and the third terms in Eq. (29) describe the displacement of two photons from each other because of the difference in their initial velocities. The term $-\mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t-t')$ describes the divergence of two straight-line trajectories. The last term accounts for the different impact of the atmosphere on particles moving in different spatial regions.

Averaging in Eq. (27) seems to be challenging because $\mathbf{r}(\mathbf{q},t')$ and $\mathbf{r}(\mathbf{q}',t'')$ themselves depend on fluctuating force $\mathbf{F}$. Nevertheless, the analysis simplifies if we neglect the specific correlations between the forces F_α or F_β and the forces entering $\mathbf{r}(\mathbf{q},t')$ or $\mathbf{r}(\mathbf{q}',t'')$. This simplification can be justified by the following reasoning. The explicit value of the α force is given by

$$\begin{aligned}F_\alpha(\mathbf{r}(\mathbf{q},t')) &= F_\alpha\left(\mathbf{r}-\mathbf{c}_\mathbf{q}(t-t')-\frac{c}{q_o}\int_t^{t'} dt_1(t_1-t')\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))\right) \\ &= F_\alpha\left(\mathbf{r}_\perp-\mathbf{c}_{\mathbf{q}_\perp}(t-t')+c\mathbf{e}_\mathbf{z}t'-\frac{c}{q_o}\int_t^{t'} dt_1(t_1-t')\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))\right),\end{aligned} \quad (30)$$

where the relation $z=ct$ is used.

If the above-mentioned specific correlation exists, the distance $|\mathbf{r}(\mathbf{q},t_1)-\mathbf{r}(\mathbf{q},t')|$ can be estimated by the value $c(t_1-t')\leqslant L_0$. Then the integral in Eq. (30) is of the order of $(L_0/c)^2\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))$ and is negligible as the quantity quadratic in L_0. This estimate means that the relative value of the last term in Eq. (30) is given by $\Delta q/q_0 \ll 1$, where Δq_0 is the photon momentum gained due to fluctuating force during the time L_0/c.

As we see, the correlation between $F_\alpha(\mathbf{r}(\mathbf{q},t'))$ and $\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))$ is small. Hence the averaging $\langle F_\alpha F_\beta\rangle$ can be performed in two steps. First, we obtain $\langle F_\alpha F_\beta\rangle$ considering the arguments of F_α and F_β to be fixed. Only after that should the averaging of the forces, entering the arguments, be performed. Following this rule, the term (25) can be written as

$$\phi_{PP'} = \omega_0^2\int_0^t\int_0^t dt'dt''\int d\mathbf{g}\,\psi(g)\mathbf{g}\cdot\left(\mathbf{p}+\mathbf{k}t'\frac{c}{q_0}\right)\mathbf{g}\cdot\left(\mathbf{p}'+\mathbf{k}'t''\frac{c}{q_0}\right)\langle\exp\{-i\mathbf{g}\cdot[\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'')]\}\rangle, \quad (31)$$

where the expression for fluctuating force, $\mathbf{F}(\mathbf{r})=\omega_0\partial_\mathbf{r}n(\mathbf{r})$, and the Fourier transform (18) of the correlation function $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$ were used. If $P=P'$, the last factor in Eq. (31) reduces to $\exp[-i\mathbf{g}\cdot\mathbf{e}_z c(t'-t'')]$, which follows from Eq. (29).

The first-step averaging results in the appearance of the spectral density $\psi(g)$ in (31). The second-step averaging results in the appearance of the last multiplier in Eq. (31), which describes the effect of photon correlations.

Our previous analysis mostly followed Ref. [18]. The most important contribution of the present paper is accounting for the correlation influence [described by the last term in Eq. (31)] on $\varphi_{PP'}$ and the scintillation index.

In this connection, it should be noted that in Ref. [18] the multiplicative approximation for this term [see Eq. (44)

therein]

$$\langle \exp\{-\mathbf{g}\cdot[\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'')]\}\rangle \approx \langle \exp[-i\mathbf{g}\cdot\mathbf{r}(\mathbf{q},t')]\rangle\langle \exp[i\mathbf{g}\cdot\mathbf{r}(\mathbf{q}',t'')]\rangle \qquad (32)$$

was used, where the correlation of photon trajectories, $\mathbf{r}(\mathbf{q},t')$ and $\mathbf{r}(\mathbf{q}',t'')$, was completely neglected. This can be a good approximation if one considers long-distance propagation of beams. In this case, photons move away from each other and lose mutual correlations even for a small divergence of the trajectories.

In what follows, we account for the correlation effect. It is of great importance at moderate distances, where photons are still close to each other and are affected by the same turbulent eddies. The correlated forces induce correlations within the photon system, resulting in the growth of scintillations. This has stimulated constant interest in the problem of scintillations at moderate turbulence.

Further analysis is simplified after integration in Eq. (31) over the time difference $t'-t''$. The presence of the quantity $\mathbf{e}_z c(t'-t'')$ in $\mathbf{r}(\mathbf{q},t')-\mathbf{r}(\mathbf{q}',t'')$ [see Eq. (29)] provides this favorable possibility. Integration of the fast oscillating function $\exp[i\mathbf{e}_z\mathbf{g}c(t'-t'')]$ results in

$$\int_{-\infty}^{\infty} d(t'-t'')\exp[i\mathbf{e}_z\cdot\mathbf{g}c(t'-t'')] = \frac{2\pi}{c}\delta(g_z), \qquad (33)$$

where the lower and upper limits of the integral are replaced by $\mp\infty$. This approximation is valid for the case of long propagation time, $t \gg L_0/c$. Besides that, the substitution $t''=t'$ is used in the slowly varying factor, $(\mathbf{p}'+\mathbf{k}'t''c/q_0)$.

Relation (33) means that only the $g_{x,y}$ components enter Eq. (31). In particular, the Fourier transform $\psi(g)$ should be considered as a function of the two-dimensional vector $\mathbf{g}_\perp$: $\psi(g)=\psi(\sqrt{g_x^2+g_y^2})$. This observation corresponds to the known Markov approximation [5], where it is assumed that the index-of-refraction fluctuations are δ-function correlated in the direction of propagation. In fact, our derivation, based on the paraxial approximation, supports the validity of the Markov approach, which, at first glance, may seem to be doubtful.

Using Eqs. (29) and (33), expression (31) is simplified to

$$\phi_{PP'} = \frac{2\pi\omega_0^2}{c}\int_0^t dt'\int d\mathbf{g}\psi(g)\,\mathbf{g}\cdot\left(\mathbf{p}+\mathbf{k}t'\frac{c}{q_0}\right)\mathbf{g}\cdot\left(\mathbf{p}'+\mathbf{k}'t'\frac{c}{q_0}\right)e^{i\mathbf{g}\cdot\mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t-t')} \times\left\langle \exp\left(-i\mathbf{g}\cdot\frac{c}{q_0}\int_{t'}^{t} dt_1(t_1-t')\{\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))-\mathbf{F}(\mathbf{r}(\mathbf{q}',t_1))\}\right)\right\rangle, \qquad (34)$$

where all the vectors have only the x and y components. As we see from Eq. (34), to obtain $\phi_{PP'}$ one needs to calculate the average value of the exponential function, which is similar to the function in (23). Following the same procedure, this average can be rewritten as

$$\left\langle \exp\left(-i\mathbf{g}\cdot\frac{c}{q_0}\int_{t'}^{t} dt_1(t_1-t')\{\mathbf{F}(\mathbf{r}(\mathbf{q},t_1))-\mathbf{F}(\mathbf{r}(\mathbf{q}',t_1))\}\right)\right\rangle = \exp\left(-2\pi c^3\int_{t'}^{t} dt_1(t_1-t')^2\int d\mathbf{g}'\psi(g')(\mathbf{g}\cdot\mathbf{g}')^2[1-\langle e^{-i\mathbf{g}'\cdot[\mathbf{r}(\mathbf{q},t_1)-\mathbf{r}(\mathbf{q}',t_1)]}\rangle]\right). \qquad (35)$$

Again, the same function appears in the exponent of the right-hand side of Eq. (35) after substitution of the trajectories (28). Similar steps can be undertaken many times. In this way, the time hierarchy, $0<t'\leqslant t_1\cdots\leqslant t_i\leqslant t$, is generated. If the photon-turbulence interaction time, $t-t_i$, is short, the disturbance of the trajectory is small and vanishes when $t_i\to t$. In this case both values, $\mathbf{r}(\mathbf{q},t_i)$ and $\mathbf{r}(\mathbf{q}',t_i)$, approach their initial value, $\mathbf{r}$, irrespective of the initial momenta $\mathbf{q}$ and $\mathbf{q}'$ [see the conditions after Eq. (7)]. Therefore we substitute the quantity

$$\tfrac{1}{2}\langle\{\mathbf{g}'\cdot[\mathbf{r}(\mathbf{q},t_1)-\mathbf{r}(\mathbf{q}',t_1)]\}^2\rangle \qquad (36)$$

instead of

$$1-\langle\exp\{-i\mathbf{g}'\cdot[\mathbf{r}(\mathbf{q},t_1)-\mathbf{r}(\mathbf{q}',t_1)]\}\rangle, \qquad (37)$$

assuming the exponent in Eq. (37) is small. The linear in $\mathbf{g}'$ term in the expansion of the exponential factor is ignored because of its zero-value contribution to the integral over $\mathbf{g}'$ in Eq. (35). Then the term (37) reduces to

$$\frac{1}{2}\langle\{\mathbf{g}'\cdot[\mathbf{r}(\mathbf{q},t_1)-\mathbf{r}(\mathbf{q}',t_1)]\}^2\rangle \approx \frac{(t-t_1)^2}{2}(\mathbf{c}_{\mathbf{q}-\mathbf{q}'}\cdot\mathbf{g}')^2+\frac{\pi c^3}{30}(t-t_1)^5\int d\mathbf{g}''\psi(g'')(\mathbf{c}_{\mathbf{q}-\mathbf{q}'}\cdot\mathbf{g}'')^2(\mathbf{g}'\cdot\mathbf{g}'')^2. \qquad (38)$$

To obtain Eq. (38), the approximate relation,

$$F_\alpha(\mathbf{r}(\mathbf{q},t_2))-F_\alpha(\mathbf{r}(\mathbf{q}',t_2)) \approx \mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t_2-t)\partial_{\mathbf{r}}F_\alpha(\mathbf{r}+\mathbf{c}_{\mathbf{q}}(t_2-t)), \qquad (39)$$

where $t_1\leqslant t_2\leqslant t$, was used. This approximation is in the spirit of the previous step, where the turbulence effect was assumed to be a small perturbation if t_2-t is small. Substitution of Eq. (38) into the right-hand side of Eq. (35) and integration over variables $\mathbf{g}'$, $\mathbf{g}''$, and t_1 result in

$$\exp\left\{-2.52\times10^{-3}C_n^2 l_0'^{-7/3}c^3c_{\mathbf{q}-\mathbf{q}'}^2(t-t')^5g^2\left[1+\frac{C_n^2l_0'^{-7/3}c^3(t-t')^3}{560}+\frac{\cos2\theta}{2}\left(1+\frac{C_n^2l_0'^{-7/3}c^3(t-t')^3}{2\times560}\right)\right]\right\}, \qquad (40)$$

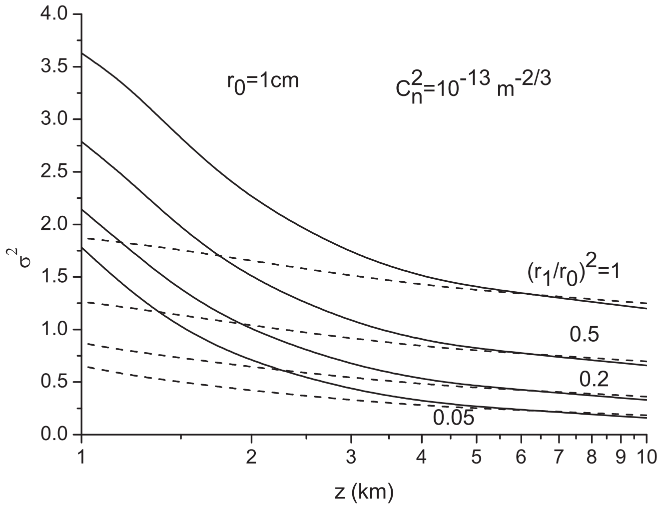

FIG. 1. Scintillation index of coherent and partially coherent beams in the atmosphere versus propagation distance z. Dashed curves correspond to the multiplicative approximation (32) for the photon correlations; solid curves are obtained within the present paper's approach [see Eqs. (35)–(40)]. $C_n^2 = 10^{-13}\,\mathrm{m}^{-2/3}$, $r_0 = 0.01$ m, $\frac{l_0}{2\pi} = 10^{-3}$ m, and $q_0 = 10^7\,\mathrm{m}^{-1}$. The upper two curves correspond to the coherent beam.

where $l_0' = l_0/2\pi$ and θ is the angle between the two-dimensional vectors $\mathbf{g}$ and $\mathbf{q} - \mathbf{q}'$.

After substitution of (40) into (35), (35) into (34), and (34) into (25), we calculate $\langle I^2(\mathbf{r},t)\rangle$. Many-fold integrations over the variables $\mathbf{q},\mathbf{q}',\mathbf{p},\mathbf{p}',\mathbf{k},\mathbf{k}',\theta$, and t' are performed both analytically and, employing a computer cluster, numerically. In the course of integration, we have used the Tatarskii modification of the refractive index spectrum, which is derived from the von Kármán form (19) by setting $L_0^{-1} = 0$. The results for σ^2 are shown in Figs. 1–3.

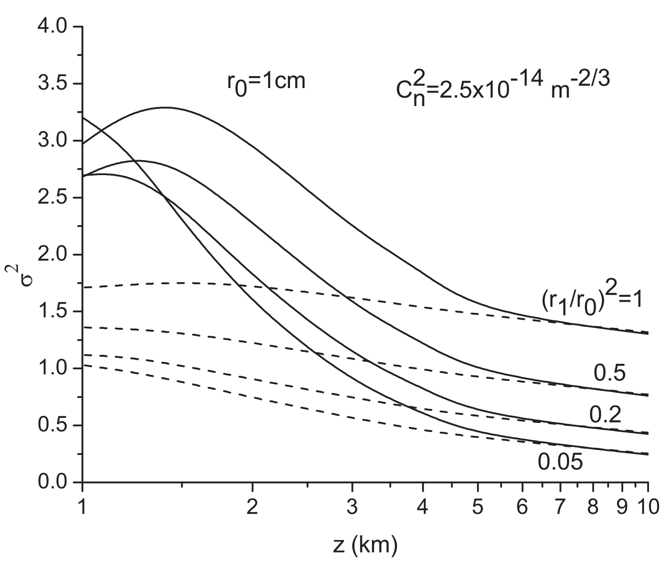

FIG. 2. The same as Fig. 1, but for a weaker turbulence strength: $C_n^2 = 2.5 \times 10^{-14}\,\mathrm{m}^{-2/3}$.

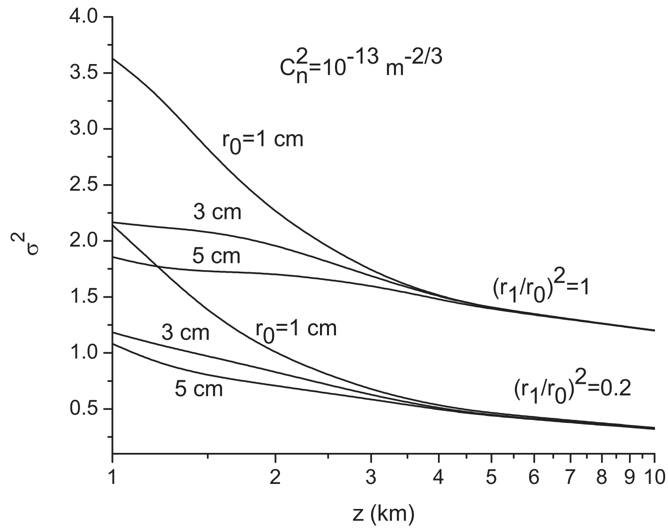

FIG. 3. Scintillation index versus propagation distance z for different initial radii of the beam: $r_0 = 0.01, 0.03, 0.05$ m. The rest of the parameters are the same as in Fig. 1.

IV. DISCUSSION

Figures 1–3 can be used to illustrate the importance of the correlations of different trajectories. To simplify our argumentations, we consider a coherent laser beam, i.e., the case with $r_0 = r_1$. Two terms in the last set of braces in Eq. (21) contribute equally to $\langle I^2(\mathbf{r},t)\rangle$. Moreover, if one sets $\phi_{PP'} = 0$ in Eq. (24), thus ignoring the correlations of photons with different initial momenta, we obtain $\langle I^2(\mathbf{r},t)\rangle = 2\langle I(\mathbf{r},t)\rangle^2$. The scintillation index σ^2 is equal to unity here.

This physical picture is realized for a long-distance propagation ($t \to \infty$) when the oscillating factor $e^{i\mathbf{g}\mathbf{c}_{\mathbf{q}-\mathbf{q}'}(t-t')}$ confines the effective volume of the integration over $\mathbf{g}$ and $\mathbf{q} - \mathbf{q}'$ to zero [see Eq. (34)]. For finite values of t, the contribution of $\phi_{PP'}$ becomes quite sizable (see Figs. 1–3), and the values of σ^2 become greater than unity.

There is a positive contribution of the $\phi_{PP'}$ term to the last exponent in Eq. (24) when the vectors $\mathbf{p},\mathbf{k}$ and $\mathbf{p}',\mathbf{k}'$ have opposite signs and the difference $|\mathbf{q} - \mathbf{q}'|$ is not too large. The most favorable conditions are realized when

$$\mathbf{p} = -\mathbf{p}', \quad \mathbf{k} = -\mathbf{k}', \quad \mathbf{q} = \mathbf{q}'. \tag{41}$$

In this case $\phi_{PP} + 2\phi_{PP'} + \phi_{P'P'} = 0$, and the value of M is equal to unity and does not depend on the turbulence. Equations (41) can be interpreted as the "supercorrelation" conditions or synchronism of four waves under which the product $\langle b^\dagger_{\mathbf{q}+\mathbf{k}/2} b_{\mathbf{q}-\mathbf{k}/2} b^\dagger_{\mathbf{q}-\mathbf{k}/2} b_{\mathbf{q}+\mathbf{k}/2}\rangle \sim \exp[i(\omega_{\mathbf{q}+\mathbf{k}/2} - \omega_{\mathbf{q}-\mathbf{k}/2} + \omega_{\mathbf{q}-\mathbf{k}/2} - \omega_{\mathbf{q}+\mathbf{k}/2})t] = 1$ does not depend on time.

The dependence of σ^2 on the initial radius r_0 can be explained as follows. The characteristic value of the initial momentum, $\tilde{q} \sim \sqrt{2}/r_0$, is greater for small r_0. Hence the volume of integration over $\mathbf{q} - \mathbf{q}'$ is also greater. At the same time the corresponding increase of $\phi_{PP'}$ occurs only for short distances z, where time intervals t are sufficiently small and the oscillating factor in Eq. (34) is close to unity.

Therefore, when r_0 decreases, there is an increase of σ^2 accompanied by the displacement of the region with enhanced fluctuations towards small z. This is clearly seen from Fig. 3.

In a similar way we can explain the considerable difference in σ^2 found for the plane-wave and spherical-wave models of radiation in Ref. [35] (their Figs. 1 and 2). It follows from the above reasoning that this effect arises due to very different initial $\mathbf{q}$ volumes in the two models.

Also, the calculations of σ^2 in Ref. [17], in which a simplified model of the turbulence was used (see Fig. 1 there), should be mentioned. The results of Ref. [17] correlate well with ours.

Comparing the results of the present paper and those based on the approximation of uncorrelated trajectories (32) (solid and dashed lines in Figs. 1 and 2, respectively), we see a more pronounced growth of σ^2 at a moderate turbulence in the former case. Figures 1–3 illustrate that this holds true for distances of 1–3 km. We attribute the evident distinction of the results to the better accuracy when accounting for the correlations of the photon trajectories. At the same time, both approaches provide the saturation effect known in the literature: $\sigma^2 \to r_1^2/r_0^2$ when $z \to \infty$.

The phase diffuser with a short characteristic time (a high-frequency diffuser) does not change qualitatively the physical picture described above. At the same time, both approaches reveal an ability of the diffuser to suppress scintillations, which is favorable for communication performances.

The effect of the phase diffuser is explained as follows. The initial phase relief, introduced by the diffuser, varies in time. The photon trajectories depend on the initial state of the radiation and varies synchronously with the diffuser state. A "slow" detector integrates the contribution of these photons. Although the atmosphere stays almost frozen during the integration time, the diffuser provides a better averaging of the propagating radiation over the refractive-index relief. Therefore the fluctuations of the detected signal decrease.

This is not a unique way to suppress fluctuations. For example, the authors of Ref. [36] proposed to use asymmetric optical vortices. The range of weak and moderate turbulence was studied. Numerical simulations of the beam propagation showed promising results. It should be emphasized that in this case the experimental setup does not require a high-frequency phase diffuser.

V. CONCLUSION

For a long time, specific statistical properties of light under conditions of moderate turbulence have attracted great interest from scientists and engineers. As an example, in the Conclusion of his widely cited paper [22], Dashen has formulated "the remaining problems" in the physics of scintillation phenomena. One of these problems is as follows: "What is the detailed behavior of E (i.e., wave field) at the boundaries between the unsaturable and saturable regimes and between the fully and partially saturated regimes?"

Up to now, this problem is still unsolved. The present paper illustrates our progress towards its solution. The method of the photon distribution function in the phase space is applied for the description of laser beam propagation through the turbulent atmosphere. Whereas the existing reliable analytical solutions are known only for the limiting cases of weak and strong turbulence, we present analytical and numerical computer calculations performed for the most challenging range of the intermediate turbulence conditions. The scintillation index σ^2 for such conditions is obtained. It is revealed that the considerable growth of σ^2 (by two to three times) for moderate turbulence is due to correlations between photon trajectories. It is shown that the scintillation index can be essentially reduced by increasing the source aperture or using a fast phase diffuser. The applicability of the distribution-function approach for short distances is analyzed in the Appendix.

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APPENDIX A: APPLICABILITY OF PDF APPROACH FOR SHORT DISTANCES

Our analysis is based on Eq. (21) obtained within the concept of photon trajectories. To consider photons as particles, whose density in the $(\mathbf{r},\mathbf{q})$ domain is defined by the distribution function $f(\mathbf{r},\mathbf{q},t)$, the uncertainty of the momentum $\mathbf{q}$ should be small. The value of the uncertainty can be estimated from the definition of the distribution function (1) as $\tilde{k}/2$. It follows from Eq. (14) that close to the source and in the absence of the diffuser the ratio $\frac{\tilde{q}}{\tilde{k}/2} \sim \frac{\langle q^2\rangle^{1/2}}{(k^2/4)^{1/2}} = \frac{(2/r_0^2)^{1/2}}{(2/r_0^2)^{1/2}} = 1$. Hence in the vicinity of the source, our calculations of σ^2 are not applicable if the light is in a coherent state.

The situation changes drastically for a remote detector. With an increase in the propagation path z, the value of $\tilde{q}$ increases. The corresponding gain of the photon momentum $\Delta\mathbf{q}$ is generated by a random force $\mathbf{F}$. Hence the average value $\langle\Delta\mathbf{q}\rangle$ is equal to zero, while the nonzero mean-square value is given by [18]

$$\langle\Delta q^2\rangle = 0.066\pi^2\Gamma(1/6)q_0^2 l_0'^{-1/3} C_n^2 z. \tag{A1}$$

In contrast to $\tilde{q}$, the value of $\tilde{k}$ decreases because of the broadening of the beam. The mean square of the beam radius is given by [8,18]

$$R^2 = \frac{r_0^2}{2}\left[1 + \frac{4z^2}{q_0^2 r_0^2 r_1^2} + \frac{8z^3 T}{r_0^2}\right], \tag{A2}$$

where $T = 0.558 l_0^{-1/3} C_n^2$. When the last term in the square brackets dominates, the ratio $\tilde{q}^2/(\tilde{k}/2)^2$ can be estimated as

$$\langle\Delta q^2\rangle R^2 \approx 15 q_0^2 l_0^{-2/3} C_n^4 z^4, \tag{A3}$$

where $\langle\Delta q^2\rangle$ is assumed to be of the order of $\tilde{q}^2$, thus ignoring the square of the initial momentum $2/r_0^2$.

Substituting $z = 10^3$ m, $q_0 = 10^7$ m^{-1}, and $l_0 = 2\pi \times 10^{-3}$ m into Eq. (A3), we obtain $\frac{\tilde{q}}{\tilde{k}/2} \sim 21$, which provides the adequacy of our approach for the whole range of z variations shown in Figs. 1–3. This range concerns not only coherent but also partially coherent beams. For partially coherent beams, the minimum z can be even smaller than for coherent beams. This is because of an additional diffuser-caused growth of $\tilde{q}^2$, which is estimated by the value $\Delta\tilde{q}^2_{\text{diffuser}} \sim 2/r_1^2$, while the initial value, $\tilde{k}^2/4 \sim 2/r_0^2$, does not depend on the diffuser effect [see Eq. (14)].

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