Revivals in quantum walks with a quasiperiodically-time-dependent coin

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We provide an explanation of recent experimental results of Xue *et al.* [P. Xue, R. Zhang, H. Qin, X. Zhan, Z. H. Bian, J. Li, and B. C. Sanders, Phys. Rev. Lett. **114**, 140502 (2015)], where full revivals in a time-dependent quantum walk model with a periodically changing coin are found. Using methods originally developed for "electric" walks with a space-dependent, rather than a time-dependent, coin, we provide a full explanation of the observations of Xue *et al.* We extend the analysis from periodic time dependence to quasiperiodic behavior with periods incommensurate to the step size. Spectral analysis, one of the principal tools for the study of electric walks, fails for time-dependent systems, but we find qualitative propagation behavior of the time-dependent system in close analogy to the electric case.

I. INTRODUCTION

Quantum walks are fundamental dynamical systems, involving a walking particle with internal degrees of freedom moving in discrete time steps on a lattice [1–4]. They have become an important test bed for many complex quantum phenomena, being well accessible both to experimental and theoretical investigation. In particular, they have recently attracted much attention as a computational resource [5–10]. They exhibit a rich variety of quantum effects such as Landau-Zener tunneling [11], the Klein paradox [12], and Bloch oscillations [13]. By taking into account on-site interactions between two particles performing a quantum walk, the formation of molecules has also been established [14–16]. Recently, a complete topological classification of quantum walks obeying a set of discrete symmetries has been obtained [17–25]. Quantum walks have been experimentally realized in such diverse physical systems as neutral atoms in optical lattices [26], trapped ions [27,28], waveguide lattices [29,30] and light pulses in optical fibers [31,32], as well as single photons in free space [33]. On the other hand, one can also observe a growing interest in quantum walks in mathematical literature where they are viewed as physical realizations of the more abstract concept of CMV matrices, the unitary analog to Jacobi matrices [34–38].

It is easy to make the coin operation of a quantum walk depend on either the location of the walker or the time (number of time step) or both. A complete analysis for the randomly time-dependent case was given in Ref. [39], even when the coin choice is driven by an external Markov process, hence allowing for correlated coin choices. In this case, the ballistic propagation [$x(t) \sim t$] of the nonrandom system reverts to diffusive propagation $x(t) \sim t^{1/2}$, i.e., Gaussian spreading with a momentum-dependent diffusion constant. In sharp contrast, disorder in space (i.e., a space-dependent set of coins fixed throughout the evolution) leads to Anderson localization [40,41], i.e., a purely discrete spectrum with exponentially localized eigenfunctions, and no propagation. Combining both kinds of disorder [39] again leads to diffusive scaling, so adding temporal disorder will slow down propagation in a nonrandom system but will speed up an Anderson localized one.

It is well known that quasiperiodic potentials share the possibility of Anderson localization with disordered ones. For quantum walks this has been analyzed in detail in Refs. [42,43]. In Ref. [42] the critical parameter is the electric field Φ. For rational fields $\Phi = 2\pi n/m$ one observes sharp revivals after m or $2m$ steps, which are exponentially sharp as a function of m. Hence, somewhat paradoxically, the revival is the sharper the longer it takes. On the other hand, the evolution does not become exactly periodic, and small errors accumulate over revival cycles leading ultimately to ballistic transport. For irrational fields sufficiently close to a rational field, i.e., up to $O(1/m^2)$ as for the continued fraction convergents, one also sees the revivals. However, depending on the infinite sequence of convergents, the long term evolution may be quite different. It may involve further yet sharper revivals on larger time scales, but typically (with probability 1) localizationlike behavior.

Looking at quasiperiodic temporal modulations of the dynamics as in Ref. [44] is a natural question. However, judging from the experience with random choices one would hardly expect the same methods to apply. Yet this is the case, as we show in this paper. The core of the argument is again a revival statement for the rational case, based on a trace formula established in Ref. [42]. However, the reason for the appearance of continued fraction approximations is different in the two cases. For example, in the temporal case analyzed in this paper the revival statement holds uniformly for all initial wave functions, whereas in Ref. [42] we had to restrict the initial support region. Another marked difference is that in the temporal case we are not repeatedly applying the same unitary operator, so there is no operator of which we could gather (discrete versus continuous) spectral information. Therefore, the interplay between spectral properties and propagation properties, which is typical for autonomous (not explicitly time-dependent) evolutions, has no analog in the temporal case.

II. THE SYSTEM

We consider, as in Ref. [44], a translation invariant quantum walk on the one-dimensional lattice $\mathbb{Z}$ with local spin-1/2 degree of freedom. Basis vectors of the system Hilbert space are thus denoted by $|x,s\rangle$ with $x \in \mathbb{Z}$ and $s = \pm 1$. The standard state-dependent shift S acts as $S|x,s\rangle = |x+s,s\rangle$ and $C(t)$ is a time-dependent coin acting solely on s, the internal degree

of freedom. The concrete model given in Ref. [44] is given by

$$C(t) = R_x(t\Phi)R_y(\theta), \tag{1}$$

with $R_{x,y}(\theta)$ the rotation around the x and y axis in spin space, respectively. The tth time step of the walk is then given by the unitary operator $W_\Phi(t) = SC(t)$. Note that when $\Phi/(2\pi) = n/m$ is rational we have $W(t + k*m) = W(t)$ so the evolution is periodic. Otherwise, it is quasiperiodic. In either case we use

$$W_\Phi^{[m,1]} = W_\Phi(m)W_\Phi(m-1)\cdots W_\Phi(1) \tag{2}$$

as a short hand for the first m steps of the walk. For the rest of the paper we will generalize the above coin operator (1) by allowing instead of $R_y(\theta)$ a slightly more general unitary coin such that the system under consideration becomes

$$W_\Phi(t) = SR_x(t\Phi)\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{3}$$

where $|a|^2 + |b|^2 = 1$. All results remain valid if instead of $R_x(t\Phi)$ we would have chosen any unitary R with $R^m = \mathbb{1}$ for some $m \in \mathbb{N}$. The choice $R = R_x(\Phi)$ is made to retain analogy with Ref. [44].

The electric walk turns out to be closely related. It has no time dependence in the coin operation [so $\Phi = 0$ in Eq. (1)]. Instead, after each step the wave function is multiplied by the x-dependent phase $\exp(i\Phi x)$, where Φ plays the role of an electric field [42].

The basic observation in Ref. [44] is that for certain fine-tuned rational values of Φ, θ and a specific initial state there are revivals (see Tables I and II in Ref. [44] for theoretical and experimental results, respectively). This observation will be generalized in this manuscript to almost all values of Φ, θ. These revivals will no longer be exact, so that in contrast to Ref. [44] the evolved state will not be exactly periodic. When $\Phi/(2\pi)$ is rational the denominator of $\Phi/(2\pi)$ sets the time for the revival, which will even be exponentially sharp in the denominator. Hence even for a moderately large denominator the time evolution will be periodic for all practical purposes. For irrational parameters one typically still gets an infinite hierarchy of revivals governed by the continued fraction expansion of $\Phi/(2\pi)$. Remarkably, these qualitative features are independent of θ and the initial state. Similar behavior is known in the case where Φ is an electric field [42], and indeed the analysis of the rational case uses a formula originally developed for that case.

III. REVIVALS AND A TRACE FORMULA

We begin the analysis with the rational case $\Phi = 2\pi n/m$ with n and m coprime. In this case $R_x^t(\Phi)$ is periodic in t with $R_x^m(\Phi) = \mathbb{1}$. Our main result is the so called *revival theorem* similar to that in Ref. [42]. It states that for m odd $W_\Phi^{[2m,1]}$ and for m even $W_\Phi^{[m,1]}$ are norm-close to the identity with quality exponentially good in m:

$$\| W_\Phi^{[2m,1]} + \mathbb{1} \| = O(|\widetilde{\alpha}|^m) \qquad m \text{ odd}, \tag{4}$$

$$\| W_\Phi^{[m,1]} + (-1)^{m/2}\mathbb{1} \| = O(|\widetilde{\alpha}|^m) \qquad m \text{ even} \tag{5}$$

where $\widetilde{\alpha}$ depends solely on the coin parameters. The exponentially good quality of the estimates depends on $|\widetilde{\alpha}| < 1$.
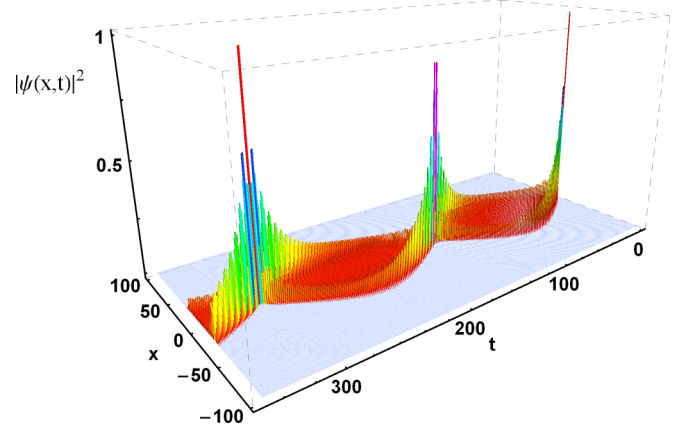

FIG. 1. Position distributions for $\Phi = 2\pi/155$ and $\phi = \pi/4$. Note the expected revival at $t = 310$ and the Bloch oscillations described in [13].

For the Hadamard walk with $a = b = 1/\sqrt{2}$ the deviation from a perfect reproduction of the initial state is $2^{-m/2+1}$. The exponentially good quality of the revivals for this choice of coin are illustrated in Fig. 1 for $m = 310$. In sharp contrast, for coin parameters $a = i/\sqrt{2}$ and $b = 1/\sqrt{2}$ we find $|\widetilde{\alpha}| = 1$ and hence no revival predictions at all. Also, the difference in behavior for m even and odd is understood intuitively, since the probability to find the particle at the origin is nonzero only after an even number of steps. To prove the above revival theorem note that the "temporally regrouped" walk $W_\Phi^{[m,1]}$ is independent of time. Hence we can apply the standard theory of translation invariant walks (see, e.g., Refs. [2,3,39]) and consider the Fourier-transformed operator

$$W_\Phi^{[m,1]}(k) = S(k)R_x(\Phi)C S(k)R_x^2(\Phi)\dots R_x^m(\Phi)C \tag{6}$$

in momentum space with dispersion relation

$$2\cos\omega_\pm(k) = \mathrm{tr}\, W_\Phi^{[m,1]}(k). \tag{7}$$

To get an explicit expression for $\omega_\pm$ we thus need to evaluate the trace $\mathrm{tr}\, W_\Phi^{[m,1]}(k)$ for which we adapt a result from Ref. [42], called the *trace formula*. This provides an expression for traces of the type $\tau_m(M) = \mathrm{tr}(M R^0 M R^1 \cdots M R^{m-1})$ where $R^m = \mathbb{1}$ with R unitary and $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a general 2×2 matrix. For m odd we get

$$\tau_m(M) = \widetilde{\alpha}^m + \widetilde{\delta}^m, \tag{8}$$

and for m even

$$\tau_m(M) = -(\widetilde{\alpha}^m + \widetilde{\delta}^m) + 2(-1)^{m/2}[(\widetilde{\alpha}\widetilde{\delta})^{m/2} - \det(M)^{m/2}]. \tag{9}$$

Here, $\widetilde{\alpha} = (BMB^*)_{11}$ and $\widetilde{\delta} = (BMB^*)_{22}$ and B is the unitary diagonalizing R.

We apply the trace formula to the walk (6) with rational $\Phi/(2\pi) = \frac{n}{m}$ such that $R_x^m(\Phi) = \mathbb{1}$. In Eqs. (8) and (9) we let $M \mapsto C S(k)$, which corresponds to replacing $\alpha \mapsto ae^{ik}, \beta \mapsto be^{-ik}, \gamma \mapsto -b^*e^{ik}, \delta \mapsto a^*e^{-ik}$ and $\det M \mapsto 1$. Note that $R \propto R_x = \exp(i\sigma_x)$ in the trace formula requires $B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which yields $2\widetilde{\alpha} = \alpha + \beta + \gamma + \delta$ and $2\widetilde{\delta} = \alpha - \beta - \gamma + \delta$. Writing $a = |a|e^{ik_a}$ and $b = |b|e^{ik_b}$ in polar
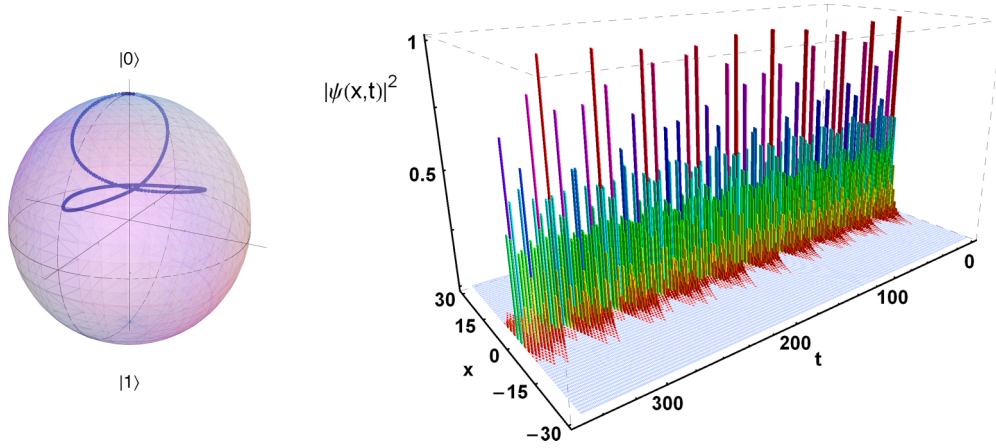
FIG. 2. Left: Bloch sphere representation of the $x = 0$ component of the wave function $\psi(x,t)$ as a function of t with t up to 1000 with $\Phi = 2\pi\varphi$ and the coin corresponding to the standard Hadamard coin. Despite the appearance of a smooth curve, successive values of t are not close to each other, but rather the curve is sampled in an incommensurate way. Right: Position distributions for the Golden Ratio $\Phi = 2\pi\varphi$ and $\phi = \pi/4$. Erratically appearing revivals not showing any periodicity are clearly visible. Also, the initial state does not spread more than approximately 15 sites, which suggests that systems with quasiperiodic time dependence do not exhibit any transport.

form then results in

$$\widetilde{\alpha} = \widetilde{\delta}^* = |a|\cos(k_a + k) + i|b|\sin(k_b - k)$$

such that by writing $\widetilde{\alpha} = |\widetilde{\alpha}|\exp(i\theta_{\widetilde{\alpha}})$ the dispersion relation (7) reads

$$\cos\omega_\pm(k)$$
$$= \begin{cases} |\widetilde{\alpha}|^m \cos(m\theta_{\widetilde{\alpha}}) & m \text{ odd} \\ -|\widetilde{\alpha}|^m \cos(m\theta_{\widetilde{\alpha}}) + 2(-1)^{\frac{m}{2}+1}(1 - |\widetilde{\alpha}|^m) & m \text{ even} \end{cases}.$$

$$(10)$$

Using this expression in the proof of the revival theorem in Ref. [42] yields Eqs. (4) and (5).

IV. THE IRRATIONAL CASE

The next step is to consider irrational values for $\Phi/(2\pi)$. Here, the spectral picture breaks down due to the time dependence of the operator $W_\Phi(t)$ as there is no concatenation of $W_\Phi(t)$ which is periodic in t. However, one may still classify such systems by their long-time propagation behavior. We distinguish between two different regimes of irrationality depending on the approximability of $\Phi/(2\pi)$ by continued fractions. Denoting by c_i the continued fraction coefficients of an irrational number x and by n_i/m_i its continuants we have $|x - n_i/m_i| < c_{i+1}^{-1}m_i^{-2}$ [45] and it is this quadratic quality of approximation in m_i which is crucial for our result. We then distinguish two regimes by how rapidly we can approximate x depending on the sequence of continued fraction coefficients c_i. The two regimes are irrationals the sequence of c_i of which is bounded or unbounded, respectively. Independent of this distinction we may estimate the norm difference of two time-dependent walks (3) with fields Φ,Φ' by $\|W_\Phi(t) - W_{\Phi'}(t)\| \leqslant t|\Phi - \Phi'|$ such that, irrespective of the initial state, after t steps we find

$$\left\| W_\Phi^{[t,1]} - W_{\Phi'}^{[t,1]} \right\| \leqslant \frac{t}{2}(t+1)|\Phi - \Phi'|.$$

Taking $\Phi'/(2\pi) = n_i/m_i$ to be a continuant of Φ we find, due to the quadratic quality of the approximation of Φ in m_i,

$$\left\| W_\Phi^{[2m_i,1]} + \mathbb{1} \right\| \leqslant \frac{4\pi}{c_{i+1}} + O\left(\frac{1}{m_i}\right) \quad m_i \text{ odd},$$
$$\left\| W_\Phi^{[m_i,1]} + (-1)^{m_i/2}\mathbb{1} \right\| \leqslant \frac{\pi}{c_{i+1}} + O\left(\frac{1}{m_i}\right) \quad m_i \text{ even}.$$

$$(11)$$

Thus, for irrational numbers the sequence of continued fraction coefficients of which diverges, i.e., $c_i \to \infty$, we get an infinite sequence of sharper and sharper revivals followed by farther and farther excursions. These revivals are, in contrast to Ref. [42], independent of the initial state. Depending on the parity of the denominator of the continuants of $\Phi/(2\pi)$, these revivals occur at times m_i and $2m_i$, which grow at least exponentially [45].

For numbers the sequence of continued fraction coefficients of which is bounded, however, Eq. (11) does not predict any revivals. The best known and worst approximable irrational is the Golden Ratio $\Phi/(2\pi) = \varphi = (\sqrt{5} - 1)/2$, which has constant continued fraction coefficients $c_i = 1$. Numerical simulations suggest that such systems do not show transport at all—a conjecture similar to that in Ref. [42], the analytical proof of which is a work in progress. Figure 2 shows the trajectory of the initial vector $\psi = |0\rangle \otimes |0\rangle$ on the Bloch sphere and the position distribution for $\Phi/(2\pi) = \varphi$. The trajectory of the Bloch vector at $x = 0$ follows a closed curve which, after delving into the interior of the Bloch sphere, approaches the initial vector ψ arbitrarily close at times independent of any continued fraction of φ due to the quasiperiodic dependence of the walk on Φ. These erratic revivals of arbitrarily good quality (Fig. 2, left) and such behavior in time-independent systems would be a signature of a pure point spectrum with exponentially decaying eigenfunctions, in the literature referred to as *Anderson localization*. However, as noted above, such a spectral treatment is meaningless in the time-dependent case. Additionally, irrationals with bounded continued fraction coefficients are of measure zero [45], such that even in the
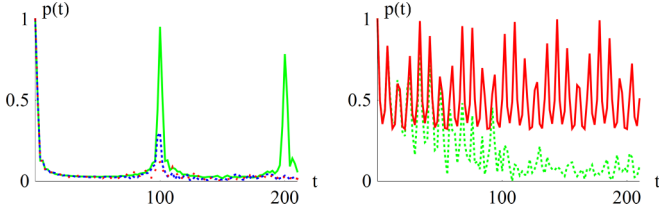
FIG. 3. Return probabilities $p(t)$ for Φ_ϵ chosen randomly in each time step. Left: $\Phi = 2\pi/100$ where the green (solid), blue (dashed), and red (dotted) lines correspond to noise $\epsilon = 10^{-4}, 5 \times 10^{-4}$, and 10^{-3}, respectively. The revivals at $t = 100$ in the first two cases as well as the absence of any revival for $\epsilon = 10^{-3} > O(100^{-2})$ are predicted by Eq. (12). Right: $\Phi = 2\pi(\sqrt{5} - 1)/2$. Whereas $p(t)$ for $\epsilon = 0$ (red, solid) is bounded from below, it converges to zero for $\epsilon = 10^{-3}$ (green, dashed), thereby implying transport.

irrational case we proved the occurrence of revivals at times equal to the denominators of the continuants for almost all Φ.

V. IMPURITIES IN THE CHOICE OF Φ

As experimental implementations never are perfect but always contain impurities, let us briefly comment on the validity of the results above in the presence of (linear) noise in Φ. Exact rationality of this parameter seems necessary for the appearance of the revivals as this exactness is indeed for stating the revival theorem.

However, as already may be inferred from the estimate for irrational values of Φ, there is some stability of the revivals against noise in Φ. Let us model fluctuations by

$$\Phi_\epsilon(x_t) = \Phi + \epsilon x_t, \qquad x_t \in [0,1],$$

where $x_t \in [-1,1]$ is chosen randomly in each time step. Using the approximation above with $\Phi' = \Phi_\epsilon(x_t)$ we find

$$\left\| W_\Phi^{[t,1]} - W_{\Phi'}^{[t,1]} \right\| \leq \frac{t}{2}(t+1)\epsilon.$$

Thus in analogy with Eq. (11) for $\Phi/(2\pi) = n/m$

$$
\begin{aligned}
\left\| W_{\Phi'}^{[2m,1]} + \mathbb{1} \right\| &\leq m(2m+1)\epsilon + O(\widetilde{\alpha}^m), \\
\left\| W_{\Phi'}^{[m,1]} + (-1)^{m/2}\mathbb{1} \right\| &\leq \frac{m}{2}(m+1)\epsilon + O(\widetilde{\alpha}^m)
\end{aligned}
\tag{12}
$$

for m odd and even, respectively. Hence if random fluctuations can be controlled on the order of $\epsilon = O(m^{-2})$ signatures of revivals are found (see Fig. 3).

Quite striking is the observation that for irrational values of Φ for which no transport is observed in clean systems, such as the Golden Ratio, the presence of noise makes the walk propagate. This "noise-induced transport" is reminiscent of the fact that the set of values showing no transport has measure zero such that independent of $\epsilon > 0$ with probability 1 in each step the random variable Φ_ϵ induces transport (Fig. 3, right).

VI. GAUGE EQUIVALENCE BETWEEN WALKS QUASIPERIODIC IN SPACE AND TIME

The similarity of the results in the body of the paper and those for electric walks [42] strongly suggest a relation between the two models. In continuous time on the lattice,

systems with linear potential and systems with a uniform time-dependent vector potential are gauge equivalent. Led by this example we here examine the possibility of transforming the spatial dependence of electric walks to a temporal one by a local gauge transformation $G_t = \bigoplus_x G_{x,t}$. The electric walk model is defined by

$$W_\Phi^E = e^{i\Phi\hat{x}} C S =: \bigoplus_x C_x S,$$

where $\hat{x}$ denotes the position operator. To establish a gauge equivalence between W_Φ^E and an explicitly time-dependent walk $W(t)$ [like the one in Eq. (3)] we have to find $G_{x,t}$ such that

$$W(t) := G_t W_\Phi^E G_{t-1}^*$$

is uniform in space, i.e., translation invariant, but explicitly time dependent. By unitarity of G_t we find

$$W^{[t,1]} = W(t)\ldots W(1) = G_t \left(W_\Phi^E\right)^t G_0.$$

Demanding locality of the $G_{x,t}$ and a shift-coin decomposition of $W(t)$ forces G_t and S to commute, which is guaranteed by choosing the ansatz $G_{x,t} = e^{-i\Phi tx}\mathbb{1}$. Then we find the time-dependent and translation invariant walk:

$$W(t) = C e^{-i\Phi(t-1)\sigma_z} S. \tag{13}$$

Comparing this time-dependent walk with Eq. (3) we find that though the models are not exactly gauge equivalent the operators implementing time dependence $e^{-i\Phi(t-1)\sigma_z}$ in Eq. (13) and $R_x(t\Phi)$ in Eq. (3) are unitarily equivalent, since the occurrence of revivals qualitatively does not depend on the explicit form of the time-dependent operator R but only on the condition $R^m = \mathbb{1}$ for some $m \in \mathbb{N}$. Quantitatively, however, the revival structure does depend on R via $\widetilde{\alpha}$. The revival predictions for electric walks hence agree with those for Eq. (13) but not for those for Eq. (3). Note that these revivals for Eq. (13) in the irrational case become independent of the support of initial states in sharp contrast to Ref. [42]. This is reminiscent of the spatial dependence of $G_{x,t}$.

VII. CONCLUSION AND OUTLOOK

In spite of the fact that the modification of quantum walks by randomness in time and by randomness in space leads to qualitatively very different phenomena we have established a case where quasiperiodic modifications in space and time, respectively, lead to very similar behavior, especially with regard to revivals. This holds for both the commensurate and the incommensurate case and revival signatures are shown to be stable with respect to noise in the quasiperiodic parameter. The models studied here are based on coined quantum walks with qubit coins, a fact that is used in an essential way. There appears to be no general mapping allowing such conclusions in more general cases. Indeed some similarities observed numerically, such as the appearance of a "smooth" trajectory in Fig. 2 (left), which is well understood in the space-quasi-periodic case, await an analytic explanation. It would also be very interesting to establish connections for systems with higher dimensional lattices and coin spaces.

APPENDIX: DETAILED INVESTIGATION OF TABLE I FROM REF. [44]

We discuss the findings of Table I from Ref. [44] by means of the more general theory laid out in the body of the paper. In Ref. [44] the coin is given by $C = R_y(\theta)$ and results are given for $\theta = 0$ and $\theta = \pi/2$ [[44], Table I]. These choices lead to $C = \mathbb{1}$ and $-i\sigma_y$, respectively, for which the revival theorem predicts perfect revivals as we show below. Let us first examine the case $C = \mathbb{1}$. Then

$$|\widetilde{\alpha}|^2 = \cos^2(k), \qquad \cos(m\theta_{\widetilde{\alpha}}) = 1$$

such that the revival theorem (4) and (5) results in

$$\left\| W_\Phi^{2m} + \mathbb{1} \right\| = 2, \quad m \text{ odd},$$

$$\left\| W_\Phi^{m} + (-1)^{m/2}\mathbb{1} \right\| = 0, \quad m \text{ even}.$$

Hence, contrary to the electric case where $C = \mathbb{1}$ shows no revivals at all, for m even periodically occurring perfect revivals are predicted. For m odd no revivals are predicted, which agrees with Table I from Ref. [44] where only even values for m are considered. Choosing $C = i\sigma_y$ yields

$$|\widetilde{\alpha}|^2 = \sin^2(k), \quad \cos(m\theta_{\widetilde{\alpha}}) = \begin{cases} 1 & m \bmod 4 = 0 \\ 0 & m \text{ odd} \\ -1 & m \bmod 4 = 2 \end{cases}$$

such that in either case

$$\left\| W_\Phi^{2m} + \mathbb{1} \right\| = 0, \quad m \text{ odd},$$

$$\left\| W_\Phi^{m} + (-1)^{m/2}\mathbb{1} \right\| = 0, \quad m \text{ even},$$

giving revival predictions independent of the parity of m. This agrees with Table I from Ref. [44] where for $\theta = \pi/2$ also odd values for m are admissible.

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