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Maximally coherent states and coherence-preserving operations

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We investigate maximally coherent states to provide a refinement in quantifying coherence and give a measure-independent definition of the coherence-preserving operations. A maximally coherent state can be considered a resource to create arbitrary quantum states of the same dimension by merely incoherent operations. We propose that only maximally coherent states should achieve the maximal value for a coherence measure and use this condition as an additional criterion for coherence measures to obtain a refinement in quantifying coherence that excludes invalid and inefficient coherence measures. Under this criterion, we then give a measure-independent definition of the coherence-preserving operations, which play a role in quantifying coherence similar to that played by the local unitary operations in the scenario of studying entanglement.

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I. INTRODUCTION

Coherence can be considered as one of the most distinctive features of quantum mechanics. Along with quantum entanglement, quantum discord, etc., coherence is viewed as a valuable resource for quantum information processing tasks [1–4], which otherwise could not be achieved efficiently or at all by classical methods. Great progress has been made in quantifying entanglement and other quantum correlations from different viewpoints [5–12]. However, a rigorous framework for quantifying coherence was proposed only recently in Ref. [13]. Following this seminal work, fruitful research has been done, some of which was mainly devoted to studying the properties of specific coherence measures [14–19] or exploring new possible coherence measures [20–22]. There are also many considerations about the manipulation of coherence [19,21–24] and the connections of coherence with quantum entanglement, quantum discord, quantum deficit, and critical phenomena of many-body systems [22,23,25,26].

In this work we present a thorough study of maximally coherent states (MCSs), give a refinement of quantifying coherence by adding a criterion for valid coherence measures, and define the coherence-preserving operations (CPOs). It should be noted that these three main results are closely related. The MCSs are defined as states that can be used as resources to produce any other states of the same dimension by merely the incoherent (free) operations [13]. A valid coherence measure C fulfilling the four criteria in Ref. [13] would assign a maximal value to a set of states that we call the maximal-coherence-value states (MCVSs) with respect to C. These four criteria ensure that a MCS is a MCVS for any valid coherence measures. However, for an arbitrary valid coherence measure, a MCVS is not necessarily a MCS. While one may expect CPOs in quantifying coherence to play a role analogous to that of the local unitary operations in studying entanglement, there is no measure-independent definition for CPOs like that of MCSs. Instead, for a specific coherence measure C we can find a set of incoherent operations under which the value of the coherence measure of an arbitrary state would be conserved. We call these operations coherence-value-preserving operations (CVPOs) with respect to C. Unfortunately, the different sets of CVPOs under different valid coherence measures are not always the same. We find that the mismatch between MCSs and MCVSs happens to many inefficient coherence measures and therefore propose a criterion that the MCVSs should be MCSs to exclude these inefficient coherence measures and thus give a refinement of quantifying coherence. This criterion also makes the different groups of CVPOs of different coherence measures converge to the unitary incoherent operations and makes it reasonable to define the unitary incoherent operations as CPOs. One operational implication of this result is that coherence of arbitrary states is impossible to protect in a task without knowledge of the state to be protected and the quantum channel it would endure.

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II. REVIEW OF QUANTIFYING COHERENCE

In quantifying coherence [13], a base $\mathcal{B} := \{|i\rangle\}$ has been chosen and fixed, which would usually be composed of eigenstates of some conserved quantity such as the Hamiltonian of the system of interest. The quantitative theory of coherence mainly consists of three basic definitions and four criteria. The three definitions are as follows.

Definition 1: Incoherent states. The diagonalized states in $\mathcal{B}$ are incoherent for $\mathcal{B}$. We denote the set of incoherent states by $\mathcal{I}$.

Definition 2: Incoherent operations. Operations mapping incoherent states onto incoherent states either with or without subselections are incoherent. An incoherent operation Φ_{ICPTP} can be specified by a set of Kraus operators $\{\boldsymbol{K}_n\}$ with $\sum_n \boldsymbol{K}_n^\dagger \boldsymbol{K}_n = \boldsymbol{I}_d$ and $\boldsymbol{\rho}_n \in \mathcal{I}$. We have the definitions $\boldsymbol{\rho}_n := \boldsymbol{K}_n \boldsymbol{\rho} \boldsymbol{K}_n^\dagger / p_n$ and $p_n := \mathrm{tr}(\boldsymbol{K}_n \boldsymbol{\rho} \boldsymbol{K}_n^\dagger)$ for all n. (In this work we continue to use this notation for the incoherent operation Φ_{ICPTP}.)

Definition 3: Maximally coherent states. A MCS is one that can be used as a resource for deterministic construction of any other state of the same dimension by incoherent operations only. It was proven in Ref. [13] that $|\Psi_d\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$ is a MCS.

The MCSs and the incoherent states set the upper and lower bounds for coherence measures, while the incoherent operations puts gradient in between. To obtain reasonable coherence measures, four criteria were proposed in Ref. [13].

Criterion 1. $C(\boldsymbol{\rho}) = 0$ if and only if $\boldsymbol{\rho}$ is incoherent.

Criterion 2. $C(\Phi_{\mathrm{ICPTP}}(\boldsymbol{\rho})) \leqslant C(\boldsymbol{\rho})$ for arbitrary Φ_{ICPTP} and $\boldsymbol{\rho}$.

Criterion 3. $\sum_n p_n C(\boldsymbol{\rho}_n) \leqslant C(\boldsymbol{\rho})$ for all Φ_{ICPTP} and $\boldsymbol{\rho}$.

Criterion 4. The coherence measure should not increase under the mixing processes of the states.

Any coherence measure satisfying the four criteria is considered valid. This gives some good coherence measures such as the relative coherence measure of coherence C_{RE} and the ℓ_1-norm coherence measure C_{ℓ_1}.

III. UNITARY INCOHERENT OPERATIONS

By Definition 2 of the incoherent operations, a CVPO of an arbitrary coherence measure is incoherent. Of all the incoherent operations, the unitary incoherent operations are the simplest. It would be useful and easier to examine them first.

Lemma 1. All the unitary incoherent operations take the form

$$\boldsymbol{U}_{\mathrm{I}} := \sum_{j=0}^{d-1} e^{i\theta_j} |\alpha_j\rangle\, \langle j|, \tag{1}$$

where $\{\alpha_j\}$ is a relabeling of $\{j\}$. They are CVPOs admitted by all the valid coherence measures.

Proof. We first prove the explicit expression of the unitary incoherent operations. Since the unitary operations transform pure states into pure states, it is obvious that the output state should be one of the base vector states, given the input is from $\mathcal{B}$. That means $\boldsymbol{U}_{\mathrm{I}}$ should only be a relabeling of the base vectors up to some phases, namely, be of the form presented in (1). Here we complete the proof of the first portion of Lemma 1 and start to prove the rest by utilizing the just proven part. One may soon realize that the inverse $\boldsymbol{U}_{\mathrm{I}}^\dagger$ is also unitary and incoherent. Therefore, for any valid coherence measure C and state $\boldsymbol{\rho}$, we can obtain $C(\boldsymbol{\rho}) \geqslant C(\boldsymbol{U}_{\mathrm{I}}\boldsymbol{\rho}\boldsymbol{U}_{\mathrm{I}}^\dagger)$ and conversely $C(\boldsymbol{U}_{\mathrm{I}}\boldsymbol{\rho}\boldsymbol{U}_{\mathrm{I}}^\dagger) \geqslant C(\boldsymbol{U}_{\mathrm{I}}^\dagger(\boldsymbol{U}_{\mathrm{I}}\boldsymbol{\rho}\boldsymbol{U}_{\mathrm{I}}^\dagger)\boldsymbol{U}_{\mathrm{I}}) = C(\boldsymbol{\rho})$, namely, $C(\boldsymbol{\rho})$ and $C(\boldsymbol{U}_{\mathrm{I}}\boldsymbol{\rho}\boldsymbol{U}_{\mathrm{I}}^\dagger)$ are of the same value. Thus $\boldsymbol{U}_{\mathrm{I}}$ is a CVPO for every valid coherence measure. ■

IV. MAXIMAL-COHERENCE-VALUE STATES

Using Lemma 1, we can obtain a set of MCVSs for every valid coherence measure by applying the unitary incoherent operations on $|\Psi_d\rangle$:

$$S_{\mathrm{MCS}} := \left\{ \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\theta_j}\, |j\rangle \,|\theta_1, \ldots ,\theta_{d-1} \in [0,2\pi) \right\}. \tag{2}$$

Notice that we have used MCS as the subscript here because we will prove in Theorem 2 that S_{MCS} is the set of MCSs too. It is very interesting but not surprising to find that this set S_{MCS} of states has its special position in the quantitative theory of coherence as a resource.

Theorem 1. S_{MCS} is the complete collection of MCVSs recognized by all the valid coherence measures, as can be shown in Fig. 1.

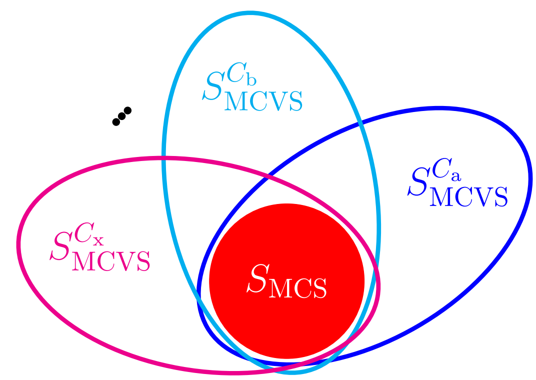

FIG. 1. Relation between MVCSs of different valid coherence measures and S_{MCS}. Here S^{C}_{MCVS} represents the full set of MCVSs with respect to a specific valid coherence measure C.

Proof. We denote by S_{MCVS} the complete collection of MCVSs granted by all the valid coherence measures. It is always true for any valid coherence measure C such that $S_{\mathrm{MCS}} \subseteq S_{\mathrm{MCVS}} \subseteq S^{C}_{\mathrm{MCVS}}$, while in Ref. [15] it was shown that $S^{C_{\mathrm{RE}}}_{\mathrm{MCVS}}$ coincides with S_{MCS}. Hence S_{MCVS} should be identical to S_{MCS}. ■

However, this does not mean that every valid coherence measure approves the states of S_{MCS} as the sole kind of MCVSs. One can find many valid coherence measures whose MCVSs include more than just the states from S_{MCS}. A specific example is C_{trivial}, which is defined as a measure of the value zero if and only if its input state is incoherent; otherwise it is always one. Another example is a continuous coherence measure C_f presented in Ref. [15] for $d = 4$. We can follow the same way to construct a family of C_f for arbitrary dimension d. Also, the S^{C}_{MCVS} could be different from one another for different C.

Another fact that makes the states of S_{MCS} special is that they are difficult to generate if we are constrained to using only incoherent channels.

Lemma 2. $\Phi_{\mathrm{ICPTP}}(\boldsymbol{\rho})$ is a state of S_{MCS} if and only if Φ_{ICPTP} is unitary and $\boldsymbol{\rho}$ itself is a state in S_{MCS}.

Proof. Given Φ_{ICPTP} is unitary and $\boldsymbol{\rho}$ belongs to S_{MCS}, it is apparent from Lemma 1 that $\Phi_{\mathrm{ICPTP}}(\boldsymbol{\rho})$ is one of the states in S_{MCS}. Next we presume that $\Phi_{\mathrm{ICPTP}}(\boldsymbol{\rho})$ belongs to S_{MCS}. Then $\Phi_{\mathrm{ICPTP}}(\boldsymbol{\rho})$ should be a pure state, since S_{MCS} contains only pure states. This means that $\boldsymbol{\rho}_n = \boldsymbol{\rho}_{n'} \in S_{\mathrm{MCS}}$ for all the different n and n' if there are any. By clinging to this fact and using the spectral expression of $\boldsymbol{\rho}$, we can finally see that Φ_{ICPTP} is unitary and $\boldsymbol{\rho}$ is from S_{MCS}. To obtain this result, one may find it very helpful to utilize a specific property of the Kraus operator $\boldsymbol{K}_n$ of Φ_{ICPTP} for which has been stated in Ref. [23] that there is at most one nonzero entry in every column of $\boldsymbol{K}_n$. For a detailed derivation, refer to Appendix A. ■

V. MAXIMALLY COHERENT STATES

In the following we will present an important result about the MCSs. We have shown that the aforementioned states of S_{MCS} are special as described in Theorem 1 and Lemma 2. The reason behind this is the following.

Theorem 2. S_{MCS} is the complete set of MCSs.

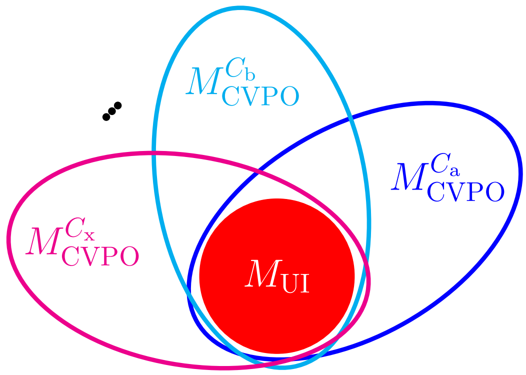

FIG. 2. Relation between CVPOs of different valid coherence measures and unitary incoherent operations. Here we denote the complete collection of unitary incoherent operations by $M_{\rm UI}$ and the CVPOs for coherence measure C by $M_{\rm CVPO}^{C}$.

Proof. First, we show that $\boldsymbol{\rho}$ is a MCS if $\boldsymbol{\rho}$ belongs to $S_{\rm MCS}$. Since the case of $|\Psi_d\rangle$ was proven explicitly in Ref. [13], we consider a state $\boldsymbol{\rho}$ that is physically different from $|\Psi_d\rangle$ but still belongs to $S_{\rm MCS}$. For such a state we can transform it to $|\Psi_d\rangle$ by exploiting a unitary incoherent operation. Then we use a set of incoherent operations to generate all the other states of the same dimension, as was done in Ref. [13]. The combination of two incoherent operations can still be counted as one incoherent operation. Therefore, $\boldsymbol{\rho}$ is indeed a MCS if $\boldsymbol{\rho} \in S_{\rm MCS}$. Second, we prove that $\boldsymbol{\rho}$ belongs to $S_{\rm MCS}$ provided $\boldsymbol{\rho}$ is a MCS. If $\boldsymbol{\rho}$ can be exploited to generate any other d-dimensional state by incoherent processes, i.e., a MCS, we can find some $\Phi_{\rm ICPTP}$ to transform it into a state of $S_{\rm MCS}$. That means $\boldsymbol{\rho}$ should be within $S_{\rm MCS}$ according to Lemma 2. In conclusion, $\boldsymbol{\rho}$ would fulfill Definition 3 of MCSs if and only if $\boldsymbol{\rho} \in S_{\rm MCS}$. ■

VI. COHERENCE-VALUE-PRESERVING OPERATIONS

We can also find all the CVPOs admitted by every valid coherence measure.

Theorem 3. The complete collection of CVPOs approved by every valid coherence measure should be the full set of unitary incoherent operations. This is expressed in Fig. 2.

Proof. First, all the unitary incoherent operations are CVPOs of an arbitrary valid coherence measure according to Lemma 1. Second, if $\Phi_{\rm ICPTP}$ is a CVPO admitted by every valid coherence measure and $|\Psi\rangle$ is a state of $S_{\rm MCS}$, $\Phi_{\rm ICPTP}(|\Psi\rangle)$ would be a MCVS under any measure and therefore belongs to $S_{\rm MCS}$. By utilizing Lemma 2, it is clear that $\Phi_{\rm ICPTP}$ is unitary. ■

VII. REFINEMENT OF QUANTIFYING COHERENCE

From Theorems 1 and 2 we can see that though the coherence measure satisfying the original four criteria of Ref. [13] would count any MCS as a MCVS, many of them also give other states maximal coherence values. A typical example of such an inefficient but valid coherence measure is $C_{\rm trivial}$, which as mentioned is zero for all incoherent states and one for all coherent states and similarly for valid measures C_f [15], which are continuous but still inefficient. Additionally, Theorem 1 indicates that there could be differences among the sets of MCVSs of different measures. Similar disagreements exist among the sets of CVPOs of different coherence measures according to Theorem 3. The latter would further make it difficult to obtain a coherence-independent definition of the CPOs. As we will see, all these problems happen to inefficient measures such as $C_{\rm trivial}$ and C_f. We therefore propose a criterion for valid coherence measures to give quantifying coherence a refinement.

Criterion 5. A valid coherence measure should only assign a maximal value to the MCSs.

This ensures that all MCVSs are MCSs and it is the same for every coherence measure. More importantly, inefficient coherence measures such as $C_{\rm trivial}$ and C_f would be excluded by this additional criterion. Some well-defined coherence measures such as the relative entropy measure, ℓ_1-norm measure, and intrinsic randomness measure [21] fulfill not only the original four criteria but also the additional criterion. The explicit proof of Criterion 5 for these three coherence measures is provided in Appendix B.

Given that C fulfill all five criteria, we can use the same argument for Theorem 3 to show the following.

Criterion 5′. The complete collection of CVPOs with respect to C is the full set of unitary incoherent operations.

Therefore, the disagreements between the CVPOs of different measures would vanish too. Moreover, Criterion 5′ is a necessary condition for all five criteria to be fulfilled. It can be used to test if a measure can satisfy the five criteria simultaneously. A typical example is the skew information measure of coherence studied in Ref. [27]. The skew information measure would actually violate not only Criterion 5′ but also Theorem 3. This indicates that both Criterion 2 and Criterion 5 are violated. Our result agrees with that presented in Ref. [18]. See Appendix B for a detailed analysis.

VIII. COHERENCE-PRESERVING OPERATIONS

An additional benefit that Criterion 5 provides is a natural way to define the CPOs. Criterion 5′, which is a consequence of Criterion 5 tells us that the set of CVPOs of any valid coherence measure C satisfying the five criteria is independent of C. Furthermore, Theorems 1 and 3 indicate that, for all the coherence measures satisfying the original four criteria, the relation between the set of unitary incoherence operations and the sets of CVPOs is structurally similar to that between $S_{\rm MCS}$ and the different $S_{\rm MCVS}^{C}$. One can get a clear view of this by comparing Figs. 1 and 2. For these reason we propose a definition of the CPOs.

Definition 4. An operation is coherence preserving if and only if it is unitary and incoherent.

This definition of the CPOs is measure independent. The CPOs defined in this way are CVPOs for every coherence measure satisfying the original four criteria and would make a full collection of the CVPOs if the coherence measure additionally satisfies Criterion 5.

This result about the CPOs has one important physical implication for the general coherence-preserving tasks. For an arbitrary coherence measure C satisfying the five criteria, one may notice that the physical process conserving the coherence values of all the d-dimensional states could only be the process of relabeling of the base $\mathcal{B}$. In other words, there is no physically nontrivial process under which the coherence value of an arbitrary d-dimensional state with respect to the measure C can be conserved. However, as it is shown in Ref. [23], we may find that the coherence value of some states with respect to C could be frozen (conserved) under specific physically nontrivial processes while that of the other states could not. That means if we want the coherence value of some state to be protected, some information about this state and the quantum channel should be provided. Complete ignorance of the state to be protected (frozen) or the quantum channel lying ahead would make the protecting task impossible to achieve in principle. Moreover, by reexamining Lemma 2, we may say that MCSs are actually the most fragile. By that we mean that the maximal coherence is the most difficult to preserve, since the only type of incoherent process-preserving MCSs is relabeling.

IX. CONCLUSION

In this work we have provided a full collection of MCSs in (2), a reasonable criterion (Criterion 5) for valid coherence measures, and a measure-independent definition (Definition 4) of the CPOs. It is understandable that the states presented in (2) are MCSs. However, a valid coherence measure satisfying the original four criteria could assign a maximal value to other states which are not MCSs. We therefore proposed a criterion to make a valid coherence measure assign only the MCSs a maximal value and therefore excludes some inefficient coherence measures. In addition, it is apprehensible that the unitary incoherent operations defined in (1) are CPOs since they are CVPOs to any coherence measure fulfilling the original four criteria. Similarly, other incoherent operations could be CVPOs for some measures satisfying the original four criteria, especially those with larger sets of MCVSs. With our criterion for coherence measures, we found that only the unitary incoherent operations are CVPOs with respect to any valid measure. We identified in Definition 4 the unitary incoherent operations as the only CPOs. Our study of the CPOs has a very significant implication that the coherence of a state is intrinsically hard to preserve when there is a lack of information about the state and the form of quantum channel it would undergo.

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APPENDIX A: DETAILED PROOF OF LEMMA 2

Here we give a detailed proof of the *only if* part of Lemma 2.

Proof. It has been claimed in the main text that $\Phi_{\text{ICPTP}}(\boldsymbol{\rho}) \in S_{\text{MCS}}$ means $\boldsymbol{\rho}_n = \boldsymbol{\rho}_{n'} \in S_{\text{MCS}}$ for all the different n and n' if there are any. Notice that

$$\boldsymbol{\rho}_n = \sum_k (q_k/p_n)\boldsymbol{K}_n|\varphi_k\rangle\,\langle\varphi_k|\boldsymbol{K}_n^\dagger, \tag{A1}$$

where q_k are the eigenvalues of $\boldsymbol{\rho}$ and $|\varphi_k\rangle$ the corresponding eigenstates. One would further obtain

$$(\boldsymbol{K}_n/\sqrt{p_n})\,|\varphi_k\rangle = (\boldsymbol{K}_n/\sqrt{p_n})\,|\varphi_{k'}\rangle \in S_{\text{CMS}}. \tag{A2}$$

Here we have ignored the global phase difference and will do the same in the following. This relation should be true for all k and k' if both q_k and $q_{k'}$ are nonvanishing. Thus

$$|\langle\varphi_k|(\boldsymbol{K}_n^\dagger/\sqrt{p_n})|i\rangle| = 1/\sqrt{d}, \tag{A3}$$

where $|i\rangle$ is an arbitrary base vector of $\mathcal{B}$. This indicates that $\boldsymbol{K}_n^\dagger\,|i\rangle$ should not be a null vector for any $|i\rangle$. According to Ref. [23], if Φ_{ICPTP} is incoherent we can write $\boldsymbol{K}_n$ as

$$\boldsymbol{K}_n = \sum_{j=0}^{d-1} \sqrt{p_n} K_{nj} e^{i\gamma_{nj}} |\lambda_{nj}\rangle\,\langle j|, \tag{A4}$$

where $\{\lambda_{ni}\} = \{i\}$ and K_{nj} should all be nonzero to ensure $\boldsymbol{K}_n^\dagger\,|i\rangle \neq 0$. This makes $\boldsymbol{K}_n$ invertible. Hence $\boldsymbol{K}_n\,|\varphi_k\rangle$ would be different from $\boldsymbol{K}_n|\varphi_{k'}\rangle$ if $|\varphi_k\rangle$ differs from $|\varphi_{k'}\rangle$. Applying this to Eq. (A1), we can see that $\boldsymbol{\rho}_n$ being a pure state implies that $\boldsymbol{\rho}$ should also be a pure state

$$|\varphi\rangle = \sum_{j=0}^{d-1} \varphi_j e^{i\vartheta_j}|j\rangle, \tag{A5}$$

where φ_j are all non-negative and satisfy the normalization condition of $|\varphi\rangle$. We can then rewrite $\boldsymbol{\rho}_n$ as

$$\frac{1}{\sqrt{p_n}}\boldsymbol{K}_n|\varphi\rangle = \sum_{j=0}^{d-1} K_{nj}\varphi_j e^{i(\gamma_{nj}+\vartheta_j)}|\lambda_{nj}\rangle \in S_{\text{MCS}}. \tag{A6}$$

From this expression we know that there is no null φ_j and $K_{nj} = 1/\sqrt{d}\varphi_j$. Thus K_{nj} is independent of n. Also, $\gamma_{nj} - \gamma_{nj'}$ should be independent of n for every j and j' because $\boldsymbol{\rho}_n = \boldsymbol{\rho}_{n'}$, that is, $(\boldsymbol{K}_n/\sqrt{p_n})|\varphi\rangle = (\boldsymbol{K}_{n'}/\sqrt{p_{n'}})|\varphi\rangle$. Therefore, $\boldsymbol{K}_n/\sqrt{p_n}$ and $\boldsymbol{K}_{n'}/\sqrt{p_{n'}}$ are mutually equivalent up to some global phase. One may notice that the Kraus operators $\boldsymbol{K}_n$ that have been considered are those with nonzero p_n. It is enough, though. Given the facts that diagonal entries of the sum of $\boldsymbol{K}_n^\dagger\boldsymbol{K}_n$ with nonvanishing p_n should never exceed one and there is a normalization constraint on φ_j, we can obtain that $\boldsymbol{\rho}$ should belong to S_{MCS}. And Φ_{ICPTP} would be an unitary operation provided further the completeness relation of the Kraus operators. ■

APPENDIX B: ANALYSIS OF SPECIFIC COHERENCE MEASURES

In this Appendix we first analyze some coherence measures satisfying Criteria 1–4 and show that they satisfy also Criterion 5. Among them, the relative entropy measure and ℓ_1-norm measure have been proven in Ref. [13] and the intrinsic randomness measure was proven in Ref. [21] to fulfill the original four criteria. We also discuss the skew information that was claimed to satisfy the original four criteria in Ref. [27].

The skew information measure turns out to violate not only Criterion 5 but also Criterion 2 in the general case of $d \geqslant 3$.

1. Relative entropy coherence measure

The C_{RE} can certainly fulfill Criterion 5 because the full set of maximal-coherence-value states $S_{\mathrm{MCVS}}^{C_{\mathrm{RE}}}$ is identical to S_{MCS}, as has been presented in Ref. [15].

2. The ℓ_1-norm coherence measure

We show that the ℓ_1 coherence measure also satisfies Criterion 5. We obtain the maximal value of the ℓ_1-norm measure of coherence $C_{\ell_1}(|\Psi_d\rangle\langle\Psi_d|) = d-1$, given $|\Psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle$ and

$$C_{\ell_1}(\boldsymbol{\rho}) = \sum_{\substack{i,j=0\\ i\neq j}}^{d-1} |\langle i|\boldsymbol{\rho}|j\rangle|. \tag{B1}$$

One may consider an arbitrary state

$$\boldsymbol{\rho} = \sum_k q_k |\varphi_k\rangle\langle\varphi_k|, \tag{B2}$$

where all the q_k are positive and fulfill the trace normalization condition. It can be derived that

$$\begin{aligned} C_{\ell_1}(\boldsymbol{\rho}) &= \sum_{j,j'=0}^{d-1} |\langle j|\boldsymbol{\rho}|j'\rangle| - 1 \\ &= \sum_{j,j'=0}^{d-1}\left|\sum_k q_k \langle j|\varphi_k\rangle\langle\varphi_k|j'\rangle\right| - 1 \\ &\leqslant \sum_{j,j'=0}^{d-1}\sum_k q_k |\langle j|\varphi_k\rangle||\langle\varphi_k|j'\rangle| - 1 \qquad (B3) \\ &= d^2\sum_k q_k\left(\sum_{j=0}^{d-1}\frac{1}{d}|\langle j|\varphi_k\rangle|\right)^2 - 1 \\ &\leqslant d^2\sum_k q_k \sum_{j=0}^{d-1}\frac{1}{d}|\langle j|\varphi_k\rangle|^2 - 1 \qquad (B4) \\ &= d-1. \qquad (B5) \end{aligned}$$

As we will see, to make the equality in (B4) hold true, it is required that $|\langle j|\varphi_k\rangle|$ must be of the same value $1/\sqrt{d}$. Therefore, $|\varphi_k\rangle$ can be expressed as

$$|\varphi_k\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{i\theta_{kj}}|j\rangle. \tag{B6}$$

To further reduce the inequality (B3) into an equality, we must make sure that either there is only one nonzero q_k or $\langle j|\varphi_k\rangle\langle\varphi_k|j'\rangle = e^{i(\theta_{kj}-\theta_{kj'})}/d$ is independent of k. That means $\boldsymbol{\rho}$ is a pure state and must come from S_{MCS}. Further, one may notice that the ℓ_1-norm coherence measure of a state from S_{MCS} would always be $d-1$. Therefore, the ℓ_1-norm measure of coherence also satisfies Criterion 5.

3. Intrinsic randomness

The so-called intrinsic randomness has been defined in Ref. [21] as

$$C_{\mathrm{IR}}(\boldsymbol{\rho}) := \begin{cases} C_{\mathrm{RE}}(\boldsymbol{\rho}) & \text{if } \boldsymbol{\rho} \text{ is pure} \\ \min\limits_{q_k,\boldsymbol{\rho}_k}\sum_k q_k C_{\mathrm{RE}}(\boldsymbol{\rho}_k) & \text{otherwise.}\end{cases} \tag{B7}$$

Now we set to prove that it also satisfies Criterion 5. When $\boldsymbol{\rho}$ is pure, the intrinsic randomness measure coincides with the relative entropy measure. Therefore $C_{\mathrm{IR}}(\boldsymbol{\rho})$ can achieve the maximal value if and only if $\boldsymbol{\rho}$ is within S_{MCS}. In the case that $\boldsymbol{\rho}$ is a mixed state, $C_{\mathrm{IR}}(\boldsymbol{\rho})$ could be of that maximal value only if $\boldsymbol{\rho}$ can be decomposed solely into a statistical mixture of states from S_{MCS}, which, however, is not possible because a mixed state always has at least two distinct eigenvectors $|\varphi_0\rangle$ and $|\varphi_1\rangle$ with nonvanishing eigenvalues q_0 and q_1. For convenience we can assume $q_0 \leqslant q_1$ without loss of generality. One may realize that $(q_0|\varphi_0\rangle\langle\varphi_0| + q_1|\varphi_1\rangle\langle\varphi_1|)$ can be replaced by $[q_0|\varphi_+\rangle\langle\varphi_+| + q_0|\varphi_-\rangle\langle\varphi_-| + (q_1-q_0)|\varphi_1\rangle\langle\varphi_1|]$. The states $|\varphi_\pm\rangle$ are defined as superpositions of $|\varphi_0\rangle$ and $|\varphi_1\rangle$ and are designed to be mutually orthogonal. By choosing the superposition parameters carefully, we can keep $|\varphi_\pm\rangle$ out of S_{MCS} even if $|\varphi_0\rangle$ and $|\varphi_1\rangle$ belong to S_{MCS}. That means a mixed state can never have only decompositions of states from S_{MCS}. Thus, $\boldsymbol{\rho}$ is a MCVS with respect to the intrinsic randomness measure of coherence if and only if $\boldsymbol{\rho} \in S_{\mathrm{MCS}}$.

4. Skew information

The skew information [27,28] is defined as

$$C_{\mathrm{skew}}(\boldsymbol{\rho},\boldsymbol{K}) := -\tfrac{1}{2}\mathrm{tr}([\sqrt{\boldsymbol{\rho}},\boldsymbol{K}]^2), \tag{B8}$$

where $\boldsymbol{K} := \sum_{i=0}^{d-1} k_i|i\rangle\langle i|$ is self-adjoint and $k_i \neq k_j$ for different i and j. For a pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, we find that

$$\begin{aligned} &C_{\mathrm{skew}}(|\psi\rangle\langle\psi|,\boldsymbol{K}) \\ &= \langle\psi|\boldsymbol{K}^2|\psi\rangle - (\langle\psi|\boldsymbol{K}|\psi\rangle)^2 \\ &= \sum_{i=0}^{d-1} k_i^2|\langle i|\psi\rangle|^2 - \left(\sum_{i=0}^{d-1} k_i|\langle i|\psi\rangle|^2\right)^2 \\ &= \frac{1}{2}\sum_{\substack{i,j=0\\ i\neq j}}^{d-1}|\langle i|\psi\rangle|^2(k_i-k_j)^2|\langle j|\psi\rangle|^2. \end{aligned} \tag{B9}$$

We can now see that $C_{\mathrm{skew}}(|\psi\rangle\langle\psi|,\boldsymbol{K})$ would not be conserved under a unitary incoherent operation, i.e., a relabeling of the base vectors up to some phases, given $d \geqslant 3$. Therefore, Theorem 3 and Criterion 5′ would be violated. Also, we know that Theorem 3 is a consequence of Criterion 2, one of the original four criteria, while Criterion 5′ is the consequence of Criterion 5, our additional criterion. Hence, neither Criterion 2 nor 5 would be fulfilled.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[2] P. W. Shor, SIAM J. Comput. **26**, 1484 (1997).
[3] C. H. Bennett and G. Brassard, in *IEEE International Conference on Computers, Systems and Signal Processing* (IEEE, New York, 1984), pp. 175–179.
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[5] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[6] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[7] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[8] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
[9] F. G. Brandão and M. B. Plenio, Nat. Phys. **4**, 873 (2008).
[10] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[11] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[12] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).
[13] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[14] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Phys. Rev. A **91**, 042120 (2015).
[15] Z. Bai and S. Du, Quantum Inf. Comput. **15**, 1355 (2015).
[16] S. Cheng and M. J. W. Hall, Phys. Rev. A **92**, 042101 (2015).
[17] U. Singh, M. N. Bera, H. S. Dhar, and A. K. Pati, Phys. Rev. A **91**, 052115 (2015).
[18] S. Du and Z. Bai, Ann. Phys. (NY) **359**, 136 (2015).
[19] A. Mani and V. Karimipour, Phys. Rev. A **92**, 032331 (2015).
[20] S. Du, Z. Bai, and X. Qi, Quantum Inf. Comput. **15**, 1307 (2015).
[21] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A **92**, 022124 (2015).
[22] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[23] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Phys. Rev. A **92**, 022112 (2015).
[24] T. R. Bromley, M. Cianciaruso, and G. Adesso, Phys. Rev. Lett. **114**, 210401 (2015).
[25] Z. Xi, Y. Li, and H. Fan, Sci. Rep. **5**, 10922 (2015).
[26] J.-J. Chen, J. Cui, and H. Fan, arXiv:1509.03576.
[27] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[28] E. P. Wigner and M. M. Yanase, Proc. Natl. Acad. Sci. USA **49**, 910 (1963).